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EXISTENCE OF GLOBAL AXISYMMETRIC SOLUTIONS FOR A 3D INHOMOGENEOUS INCOMPRESSIBLE HALL-MAGNETOHYDRODYNAMIC SYSTEM

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Abstract. We study the global well-posedness of an inhomogeneous incompressible Hall-MHD system in the whole space $\mathbb{R}^3$. Let ρ_0 be the initial density of the fluids. Under certain appropriate smallness assumptions on a_0/r, where $a_0 = (1/\rho_0) - 1$ and $r = (x_1^2 + x_2^2)^{1/2}$, we demonstrate the global regularity of the solutions to the Cauchy problem of the inhomogeneous Hall-MHD system with axisymmetric initial data, where the swirl component of the velocity field and magnetic vorticity field vanish.

Keywords. Axisymmetric; Inhomogeneous incompressible hall-magnetohydrodynamic; Global regularity.

1. Introduction

In this paper, we consider the global well-posedness result to the Cauchy problem of three dimensional density dependent incompressible Hall-magnetohydrodynamic (Hall-MHD) with axisymmetric initial data

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla P = B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B - \Delta B + h \nabla \times ((\nabla \times B) \times B) = B \cdot \nabla u, \\ \operatorname{div} u = 0, \ \operatorname{div} B = 0, \\ \rho(0,x) = \rho_0, \ u(0,x) = u_0, \ B(0,x) = B_0. \end{cases} \tag{1.1}$$

In the following context, we denote $P = p + \frac{1}{2}|B|^2$, the unknown functions $\rho(t,x)$, $u(t,x)$, $p(t,x)$, $B(t,x)$ denote the density, velocity field, pressure, magnetic field of the fluid, respectively, and h is the Hall's constant.

When the initial magnetic field B_0 is identically zero, system (1.1) is nothing but the inhomogeneous incompressible Navier-Stokes (N-S) system. In addition, there are numerous well-posedness results with axisymmetric conditions on the initial data. For the homogeneous N-S system, Ukhovskii and Iudovich [1], Ladyženskaja [2], and Leonardi et al. [3] proved the global existence, uniqueness, and regularity of the generalized solutions when the swirl component of the velocity field is trivial. For the inhomogeneous N-S system, Abidi and Zhang [4] proved the

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global existence of the solutions when $\|\frac{a_0}{r}\|_{L^\infty}$ is sufficiently small, where $a_0 = \frac{1}{\rho_0} - 1$. Moreover, when the initial velocity belongs to L^q for some $q \in [1,2)$, Abidi and Zhang [4] also proved that the velocity field decays to zero with exactly the same rate as the classical N-S system. A similar result for the case that $\|u_0^\theta\|_{L^3}$ is nontrivial but sufficiently small was proved by Chen et al. [5]. For more global well-posedness results with axisymmetric initial data, we refer to [6, 7] and the references therein.

When $h = 0$, (1.1) is the classical MHD system with magnetic diffusion. In what follows, let us briefly recall some known results on the MHD system. Firstly, in the case of $\mathbb{R}^d$(d represents the dimensionality), for the viscous and resistive homogeneous MHD system, Duvaut and Lions [8] established the global existence and uniqueness of the solutions in classical Sobolev spaces for small initial data. The local well-posedness of classical solutions for fully viscous MHD system was established by Sermange and Temam [9], in which the global well-posedness was also proved in $\mathbb{R}^2$.

For the viscous and non-resistive problem, Lin, Xu and Zhang [10] constructed the global smooth solutions around the equilibrium by imposing some admissible conditions in the $\mathbb{R}^2$ case. Later on, the global existence of small solutions without imposing such admissible conditions on the initial magnetic field was obtained by Ren, Wu, Xiang and Zhang [11] (see [12] for a simplified proof).

For the non-resistive MHD system in the $\mathbb{R}^3$ case, the global well-posedness result was obtained by Xu and Zhang [13] by introducing the Lagrangian reformulation of the problem, and by imposing some admissible conditions on the initial magnetic field in [10]. Such admissible conditions were removed by Abidi and Zhang [14] under a more intrinsic Lagrangian reformulation. The existence of global solutions in a periodic domain was obtained by Pan, Zhou and Zhu [15]. The global regularity of the axisymmetric solutions was proved by Lei [16]: If u_0, B_0 are both axisymmetric divergence-free vectors with $u_0^\theta = B_0^r = B_0^z = 0$, and $(u_0, B_0) \in H^s$, $s \geq 2$, $\frac{B_0^\theta}{r} \in L^\infty$, then the MHD system satisfies

$$\|u(t,\cdot)\|_{H^2}^2 + \|B(t,\cdot)\|_{H^2}^2 + \int_0^t \|\nabla u\|_{H^2}^2 \, \mathrm{d}s \lesssim \exp\{e^{(1+t)^{\frac{7}{4}} e^{t^{\frac{5}{4}}}}\}.$$

For more studies on MHD system, we refer to [17]-[33] and the references therein.

Let us now briefly recall some known results on the homogeneous Hall-MHD system (the case of $\rho = 1$ in (1.1)). The global existence of weak solutions and local well-posedness with initial data $(u_0, B_0) \in H^s \times H^s(\mathbb{R}^3)$ when $s > \frac{5}{2}$ were obtained by Chae, Degond and Liu [34]. Later on, Benvenutti and Ferreira [35] improved the results to $H^2(\mathbb{R}^3)$ and Dai [36] showed the local well-posedness when $(u_0, B_0) \in H^s \times H^{s+1-\varepsilon}(\mathbb{R}^3)$ with $s > \frac{1}{2}$ and small constant $\varepsilon > 0$. More recently, the global well-posedness of small initial conditions with (u_0, B_0) in critical space was obtained by Danchin in [37].

Under the assumption of axisymmetric data, motivated by [16], Fan, Huang and Nakamura [38] obtained the global well-posedness result to the viscous Hall-MHD system. Recently, Li and Cui [39] established the global well-posedness for the horizontal dissipation Hall-MHD system. For more studies on Hall-MHD system, we refer to [40]-[52] and the references therein.

The aim of this paper is to establish the global solutions of the inhomogeneous Hall-MHD system (1.1) with axisymmetric initial data. Without loss of generality, we assume $h = 1$. For that, let $x=(x_1,x_2,x_3)\in\mathbb{R}^3$,

$$r=\sqrt{x_1^2+x_2^2},\quad \theta=\arctan\frac{x_2}{x_1},\quad z=x_3,$$

and

$$\begin{cases}\rho(t,x)=\rho(t,r,z),\\ u(t,x)=u^r(t,r,z)e_r+u^z(t,r,z)e_z,\\ p(t,x)=p(t,r,z),\\ B(t,x)=B^\theta(t,r,z)e_\theta,\end{cases}$$

where the basis vectors e_r,e_θ,e_z are given by

$$e_r=(x_1/r,x_2/r,0),\quad e_\theta=(-x_2/r,x_1/r,0),\quad e_z=(0,0,1),$$

and we have assumed that $u^\theta(t,r,z)=B^r(t,r,z)=B^z(t,r,z)=0$. In these settings, we find that

$$u\cdot\nabla=u^r\partial_r+u^z\partial_z,\nabla\times((\nabla\times B)\times B)=-2\frac{B^\theta}{r}\partial_zB^\theta e_\theta=-\partial_z\frac{(B^\theta)^2}{r}e_\theta.$$

Then (1.1) can be rewritten as

$$\begin{cases}\partial_t\rho+u\cdot\nabla\rho=0,\\ \rho(\partial_tu^r+u\cdot\nabla u^r)-\widetilde{\Delta}u^r+\partial_rP=-\frac{(B^\theta)^2}{r},\\ \rho(\partial_tu^z+u\cdot\nabla u^z)-\Delta u^z+\partial_zP=0,\\ \partial_tB^\theta+u\cdot\nabla B^\theta-\widetilde{\Delta}B^\theta=\frac{u^rB^\theta}{r}+\partial_z\frac{(B^\theta)^2}{r},\\ \partial_ru^r+\frac{u^r}{r}+\partial_zu^z=0,\\ \rho|_{t=0}=\rho_0,\ (u^r,u^z)|_{t=0}=(u_0^r,u_0^z),\ B^\theta|_{t=0}=B_0^\theta,\end{cases}\tag{1.2}$$

where

$$\Delta=\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\theta^2+\partial_z^2,\quad \widetilde{\Delta}=\Delta-\frac{1}{r^2}.$$

Denote

$$\omega=\partial_zu^r-\partial_ru^z,\quad \Gamma=\frac{\omega}{r},\quad \Pi=\frac{B^\theta}{r}.\tag{1.3}$$

By elementary analysis, one can see that Γ and Π satisfy

$$\begin{aligned}\partial_t\Gamma+u^r\partial_r\Gamma+u^z\partial_z\Gamma+\frac{1}{r}\left[\partial_z\left(\frac{\partial_rP}{\rho}\right)-\partial_r\left(\frac{\partial_zP}{\rho}\right)-\partial_r\left(\frac{r\partial_r\Gamma+2\Gamma}{\rho}\right)\right]\\ -\partial_z\left(\frac{\partial_z\Gamma}{\rho}\right)-\partial_z\left(\frac{\Pi^2}{\rho}\right)=0,\end{aligned}\tag{1.4}$$

and

$$\partial_t\Pi+u\cdot\nabla\Pi=(\Delta+\frac{2}{r}\partial_r)\Pi+\partial_z\Pi^2.\tag{1.5}$$

We now state our main result in the following.

Theorem 1.1. *Let $a_0=\frac{1}{\rho_0}-1$ with $(\rho_0)^{\pm1}\in L^\infty$, $\frac{a_0}{r}\in L^\infty$, and assume that there exist two constants m, M such that $0<m\le\rho_0\le M$. For the axisymmetric initial data, let $u_0=u_0^re_r+u_0^ze_z\in H^s$ and $B_0=B_0^\theta e_\theta\in H^s$, $s\ge2$, and assume that $\frac{u_0^r}{r}\in L^2$, $\Gamma_0=\frac{\omega_0}{r}\in L^2$, $\Pi_0=\frac{B_0^\theta}{r}\in L^q$ with $q\in[2,\infty]$. In addition, if*

$$\left\|\frac{a_0}{r}\right\|_{L^\infty}\le\varepsilon_0,\tag{1.6}$$

where ε_0 denotes a sufficiently small positive constant, then there exists a global solution u of (1.1) *such that for all $t>0$*

$$\|\nabla u\|^2_{L^\infty_t(L^2)}+\|\partial_t u\|^2_{L^2_t(L^2)}+\|\nabla^2 u\|^2_{L^2_t(L^2)}+\|\nabla P\|^2_{L^2_t(L^2)}\le C\mathscr{H}_0+\eta_1^2, \tag{1.7}$$

where

$$\mathscr{H}_0=\mathscr{G}_0\left(\mathscr{C}_0+\frac{2}{c_0}(c_0\|\Gamma_0\|^2_{L^2}+\|\Pi_0\|^2_{L^2})\right), \tag{1.8}$$

with

$$\mathscr{G}_0=\exp\Big((\|u_0\|^2_{L^2}+\|B_0\|^2_{L^2})\big(1+\|u_0\|^6_{L^2}+\|B_0\|^6_{L^2}\big)\Big), \tag{1.9}$$

and

$$c_0=\frac{1}{2\|\Pi_0\|^2_{L^3}},\quad \mathscr{C}_0=\|\nabla u_0\|^2_{L^2}+\left\|\frac{u_0^r}{r}\right\|^2_{L^2}+\|\Pi_0\|^2_{L^3}(\|u_0\|^2_{L^2}+\|B_0\|^2_{L^2}). \tag{1.10}$$

Furthermore,

$$\|u(t,\cdot)\|_{H^2}+\|B(t,\cdot)\|_{H^2}\le C\mathscr{E}(t),\ \forall\, t\ge 0,$$

where $\mathscr{E}(t)$ denotes a bounded function of t.

Before ending this section, we present some notations which will be used in this paper.

Notations. For any $1\le q\le\infty$ and any measurable scalar or vector function f, we use $\|f\|_{L^q}$ to denote the usual L^q norm. We denote $\mathbb{R}^2_+=(0,\infty)\times\mathbb{R}$ and $\mathscr{E}(t)$ represents a function about t. For any two quantities X and Y, we denote $X\lesssim Y$ if $X\le CY$ for a constant $C>0$. Similarly $X\gtrsim Y$ if $X\ge CY$ for $C>0$. We denote $X\sim Y$ if $X\lesssim Y$ and $Y\lesssim X$. The dependence of the constant C and function $\mathscr{E}(t)$ on other parameters or constants are usually clear from the context and we usually suppress this dependence. Finally, we denote $\widetilde{\nabla}=(\partial_r,\partial_z)$.

The rest of this paper is organized as follows. In Section 2, we provide some preliminary lemmas that are needed in this paper. In Section 3, we present the basic energy estimates for the solutions under the axisymmetric case. In Section 4, the last section, we first construct the approximate solutions and give the *a priori* uniform bound of the smooth solutions, and the proof of the global well-posedness result is also given in the last section by using a standard compactness argument.

2. Preliminaries

In this section, we provide some preliminary lemmas that are used through out this paper. First, we recall some maximal principle results. The proof of them are referred to [53]. For the sake of completeness, we give the details below.

Lemma 2.1. *Let ρ and Π be satisfy* (1.2)$_1$ *and* (1.5)*, respectively. Then the following estimates hold for any $t>0$,*

$$\|\rho(t)\|_{L^q}\le C\|\rho_0\|_{L^q},\quad \forall\, q\in[2,\infty], \tag{2.1}$$

$$\|\Pi(t)\|_{L^q}\le C\|\Pi_0\|_{L^q},\quad \forall\, q\in[2,\infty], \tag{2.2}$$

and

$$\|\Pi\|^2_{L^\infty_t(L^2)}+\|\nabla\Pi\|^2_{L^2_t(L^2)}\le C\|\Pi_0\|^2_{L^2}. \tag{2.3}$$

Proof. By using characteristic argument, one can directly obtain that the ρ in $(1.2)_1$ satisfies the maximal principle which read as $\|\rho(t)\|_{L^q} \le C\|\rho_0\|_{L^q}$, for all $t>0$.

For the proof of (2.2), multiplying (1.5) by $|\Pi|^{q-2}\Pi$, and then taking $L^2(\mathbb{R}^2_+; r\,\mathrm{d}r\mathrm{d}z)$ inner product, we write

$$
\begin{aligned}
\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\Pi(t)\|_{L^q}^q &= \int_{\mathbb{R}^2_+}(\Delta+\frac{2}{r})\Pi\cdot|\Pi|^{q-2}\Pi r\,\mathrm{d}r\mathrm{d}z + \int_{\mathbb{R}^2_+}\partial_z\Pi^2\cdot|\Pi|^{q-2}\Pi r\,\mathrm{d}r\mathrm{d}z\\
&= \int_{\mathbb{R}^2_+}(\partial_r^2+\frac{3}{r}+\partial_z^2)\Pi\cdot|\Pi|^{q-2}\Pi r\,\mathrm{d}r\mathrm{d}z + \frac{2}{q+1}\int_{\mathbb{R}^2_+}\partial_z|\Pi|^{q+1} r\,\mathrm{d}r\mathrm{d}z\\
&= -(q-1)\int_{\mathbb{R}^2_+}|\Pi|^{q-2}|\widetilde{\nabla}\Pi|^2 r\,\mathrm{d}r\mathrm{d}z - \frac{2}{q}\int_{\mathbb{R}}|\Pi(t,0,z)|^q\,\mathrm{d}z.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\Pi(t)\|_{L^q}^q + (q-1)\int_{\mathbb{R}^2_+}|\Pi|^{q-2}|\widetilde{\nabla}\Pi|^2 r\,\mathrm{d}r\mathrm{d}z\\
&= -\frac{2}{q}\int_{\mathbb{R}}|\Pi(t,0,z)|^q\,\mathrm{d}z \le 0.
\end{aligned}
\tag{2.4}
$$

Integrating it with respect to time gives $\|\Pi(t)\|_{L^q} \le C\|\Pi_0\|_{L^q}$ for all $q\in[2,\infty)$. Taking $q\to\infty$, we have $\|\Pi(t)\|_{L^\infty} \le C\|\Pi_0\|_{L^\infty}$. Particularly, when $q=2$, there holds (2.3). □

Lemma 2.2. *Let ω and Γ be defined in (1.3). Then the following estimates hold:*

$$
\begin{aligned}
\|\widetilde{\nabla}\omega\|_{L^2}^2 + \|\Gamma\|_{L^2}^2 \le C\bigg(&\|u_t^r\|_{L^2}^2 + \|u_t^z\|_{L^2}^2 + \|u^r\partial_r u^r\|_{L^2}^2 + \|u^z\partial_z u^r\|_{L^2}^2\\
&+ \|u^r\partial_r u^z\|_{L^2}^2 + \|u^z\partial_z u^z\|_{L^2}^2 + \left\|\frac{(B^\theta)^2}{r}\right\|_{L^2}^2\bigg),
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
\|\widetilde{\nabla}P\|_{L^2}^2 \le C\bigg(&\|u_t^r\|_{L^2}^2 + \|u_t^z\|_{L^2}^2 + \|u^r\partial_r u^r\|_{L^2}^2 + \|u^z\partial_z u^r\|_{L^2}^2\\
&+ \|u^r\partial_r u^z\|_{L^2}^2 + \|u^z\partial_z u^z\|_{L^2}^2 + \left\|\frac{(B^\theta)^2}{r}\right\|_{L^2}^2\bigg).
\end{aligned}
\tag{2.6}
$$

Proof. One can refer to [53] for the proof. For the sake of simplicity, we omit the proof. □

3. Energy Estimate

In this section, we obtain the H^1 energy estimate of (1.1). To achieve it, we start with L^2 energy estimate, and then obtain local in time $\dot{H}^1$ estimate. Finally, by using the standard continuity argument, the global in time $\dot{H}^1$ estimate holds true. Before going any further, we first deduce from (2.1) that there exist two absolute positive constants m, M such that

$$
m \le \rho(t,r,z) \le M,
\tag{3.1}
$$

provided that $0 < m \le \rho_0 \le M$.

Lemma 3.1. [43] *(L^2 energy estimate) Let (ρ,u,B) be a smooth solution to (1.1) with $(u_0,B_0)\in H^2$. Then there holds for all $t>0$*

$$\|u\|_{L_t^\infty(L^2)}^2+\|B\|_{L_t^\infty(L^2)}^2+\|\nabla u\|_{L_t^2(L^2)}^2+\|\nabla B\|_{L_t^2(L^2)}^2\lesssim \|u_0\|_{L^2}^2+\|B_0\|_{L^2}^2. \tag{3.2}$$

Lemma 3.2. *($\dot H^1$ energy estimate)*

Let (ρ,u,B) be a smooth solution to (1.1) with $(u_0,B_0)\in H^2$. Then there holds for all $t>0$

$$\begin{aligned}&\|\nabla u\|_{L_t^\infty(L^2)}^2+\left\|\frac{u^r}{r}\right\|_{L_t^\infty(L^2)}^2+\|\partial_t u\|_{L_t^2(L^2)}^2+\|u\|_{L_t^2(\dot H^2)}^2+\|\Gamma\|_{L_t^2(L^2)}^2+\|\nabla P\|_{L_t^2(L^2)}^2\\&\lesssim\mathscr{G}_0\left(\mathscr{C}_0+\|\Gamma\|_{L_t^\infty(L^2)}^2+\|\widetilde\nabla\Gamma\|_{L_t^2(L^2)}^2\right),\end{aligned} \tag{3.3}$$

where $\mathscr{G}_0$ and $\mathscr{C}_0$ are given in (1.9) and (1.10).

Proof. By taking $L^2(\mathbb{R}_+^2, r\,\mathrm{d}r\mathrm{d}z)$ inner product of $(1.2)_{2,3}$ with u_t^r and u_t^z, respectively, and using integration by parts, we have

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}\int_{\mathbb{R}_+^2}\left((\partial_r u^r)^2+(\partial_z u^r)^2+\frac{(u^r)^2}{r^2}\right)r\,\mathrm dr\mathrm dz+\int_{\mathbb{R}_+^2}\rho(\partial_t u^r)^2 r\,\mathrm dr\mathrm dz\\&=-\int_{\mathbb{R}_+^2}\rho\left(u^r\partial_r u^r+u^z\partial_z u^r\right)\partial_t u^r r\,\mathrm dr\mathrm dz+\int_{\mathbb{R}_+^2}P\partial_r(\partial_t u^r r)\,\mathrm dr\mathrm dz-\int_{\mathbb{R}_+^2}\frac{(B^\theta)^2}{r}\partial_t u^r r\,\mathrm dr\mathrm dz,\end{aligned}$$

and

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}\int_{\mathbb{R}_+^2}\left((\partial_r u^z)^2+(\partial_z u^z)^2\right)r\,\mathrm dr\mathrm dz+\int_{\mathbb{R}_+^2}\rho(\partial_t u^z)^2 r\,\mathrm dr\mathrm dz\\&=-\int_{\mathbb{R}_+^2}\rho\left(u^r\partial_r u^z+u^z\partial_z u^z\right)\partial_t u^z r\,\mathrm dr\mathrm dz+\int_{\mathbb{R}_+^2}P\partial_z(\partial_t u^z r)\,\mathrm dr\mathrm dz.\end{aligned}$$

Using the incompressibility condition and the maximal principle (3.1), we have

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}\int_{\mathbb{R}_+^2}\left(|\widetilde\nabla u|^2+\frac{(u^r)^2}{r^2}\right)r\,\mathrm dr\mathrm dz+\|\partial_t u\|_{L^2}^2\\&\lesssim\|u^r\partial_r u^r\|_{L^2}^2+\|u^z\partial_z u^r\|_{L^2}^2+\|u^r\partial_r u^z\|_{L^2}^2+\|u^z\partial_z u^z\|_{L^2}^2+\left\|\frac{(B^\theta)^2}{r}\right\|_{L^2}^2.\end{aligned} \tag{3.4}$$

Let $\varepsilon>0$ be a small positive constant, which will be chosen later. Summing up (3.4) with $\varepsilon\times\big((2.5)+(2.6)\big)$ and choosing $\varepsilon=\frac{1}{4C}$, one has

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}\left(\|\widetilde\nabla u(t)\|_{L^2}^2+\left\|\frac{u^r}{r}(t)\right\|_{L^2}^2\right)+\|\partial_t u\|_{L^2}^2+\|\widetilde\nabla\omega\|_{L^2}^2+\|\Gamma\|_{L^2}^2+\|\widetilde\nabla P\|_{L^2}^2\\&\le C\left(\|u^r\partial_r u^r\|_{L^2}^2+\|u^z\partial_z u^r\|_{L^2}^2+\|u^r\partial_r u^z\|_{L^2}^2+\|u^z\partial_z u^z\|_{L^2}^2+\left\|\frac{(B^\theta)^2}{r}\right\|_{L^2}^2\right).\end{aligned}$$

According to the standard calculation in [4], for any $\delta > 0$, we see that

$$\|u^r\partial_r u\|_{L^2}^2 \le C_\delta \|u^r\|_{L^2}^2 \left(\|\widetilde{\nabla} u^r\|_{L^2}^2 + \left\| \frac{u^r}{r} \right\|_{L^2}^2 \right) \left(\|\omega\|_{L^2}^2 + \|\Gamma\|_{L^2}^2 \right)$$
$$+ \delta \left(\|\omega\|_{L^2}^2 + \|\widetilde{\nabla}\omega\|_{L^2}^2 + \|\Gamma\|_{L^2}^2 + \|\widetilde{\nabla}\Gamma\|_{L^2}^2 \right),$$

and

$$\|u^z\partial_z u\|_{L^2}^2 \le C_\delta \Big(\left(1 + \|u^z\|_{L^2}^6\right) \|\widetilde{\nabla} u^z\|_{L^2}^2 \left(\|\widetilde{\nabla} u\|_{L^2}^2 + \|\Gamma\|_{L^2}^2 \right)$$
$$+ (1 + \|u^z\|_{L^2}^4)\|\partial_z u\|_{L^2}^2 \Big) + \delta \left(\|\Gamma\|_{L^2}^2 + \|\widetilde{\nabla}\partial_z u\|_{L^2}^2 + \|\widetilde{\nabla}\Gamma\|_{L^2}^2 \right).$$

Note that, for the axisymmetric flow, we have the conclusions that, for $1 < q < \infty$

$$\|\widetilde{\nabla} u\|_{L^q} + \left\| \frac{u^r}{r} \right\|_{L^q} \sim \|\nabla u\|_{L^q}, \ \|\omega\|_{L^q} \sim \|\nabla u\|_{L^q}, \ \|\nabla\omega\|_{L^q} + \left\| \frac{\omega}{r} \right\|_{L^q} \sim \|\nabla^2 u\|_{L^q}.$$

Furthermore, from (1.3), we have

$$\left\| \frac{(B^\theta)^2}{r} \right\|_{L^2}^2 \le C\|\Pi\|_{L^3}^2 \|B^\theta\|_{L^6}^2 \le C\|\Pi\|_{L^3}^2 \|\nabla B^\theta\|_{L^2}^2.$$

Combining the above estimates, we take δ to be sufficiently small and apply Gronwall's inequality. It follows that

$$\|\nabla u\|_{L_t^\infty(L^2)}^2 + \left\| \frac{u^r}{r} \right\|_{L_t^\infty(L^2)}^2 + \|\partial_t u\|_{L_t^2(L^2)}^2 + \|u\|_{L_t^2(\dot{H}^2)}^2 + \|\Gamma\|_{L_t^2(L^2)}^2 + \|\nabla P\|_{L_t^2(L^2)}^2$$
$$\lesssim \exp\left\{ \left(1 + \|u\|_{L_t^\infty(L^2)}^6\right) \left(\|\nabla u\|_{L_t^2(L^2)}^2 + \left\| \frac{u^r}{r} \right\|_{L_t^2(L^2)}^2 \right) \right\}$$
$$\times \Bigg(\|\nabla u_0\|_{L^2}^2 + \left\| \frac{u_0^r}{r} \right\|_{L^2}^2 + \left(1 + \|u^z\|_{L_t^\infty(L^2)}^4\right) \|\widetilde{\nabla} u\|_{L_t^2(L^2)}^2$$
$$+ \|\Gamma\|_{L_t^\infty(L^2)}^2 + \|\widetilde{\nabla}\Gamma\|_{L_t^2(L^2)}^2 + \|\Pi\|_{L_t^\infty(L^3)}^2 \|\nabla B^\theta\|_{L_t^2(L^2)}^2 \Bigg),$$

from which (2.2) and (3.2), Lemma 3.2 follows. □

Lemma 3.3. *(The estimate of Γ) If (ρ, u, B) is a smooth solution to (1.1) with $(u_0, B_0) \in H^2$, then there holds, for all $t > 0$,*

$$\|\Pi\|_{L_t^\infty(L^2)}^2 + \|\nabla\Pi\|_{L_t^2(L^2)}^2 + c_0\|\Gamma\|_{L_t^\infty(L^2)}^2 + c_0\|\nabla\Gamma\|_{L_t^2(L^2)}^2$$
$$\lesssim c_0\|\Gamma_0\|_{L^2}^2 + \|\Pi_0\|_{L^2}^2 + \mathscr{G}_0 \left\| \frac{a_0}{r} \right\|_{L^\infty}^2 \exp\left(t^{\frac{3}{4}} \|\Gamma\|_{L_t^\infty(L^2)}^{\frac{1}{2}} \|\nabla\Gamma\|_{L_t^2(L^2)}^{\frac{1}{2}} \right) \tag{3.5}$$
$$\times \left(\mathscr{C}_0 + \|\Gamma\|_{L_t^\infty(L^2)}^2 + \|\widetilde{\nabla}\Gamma\|_{L_t^2(L^2)}^2 \right),$$

where $\mathscr{G}_0$ and $\mathscr{C}_0$ are given in (1.9) and (1.10).

Proof. By taking L^2 inner product of (1.4) with Γ and using integration by parts, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma(t)\|_{L^2}^2+\int_{\mathbb{R}^2_+}\frac{1}{\rho}|\widetilde{\nabla}\Gamma|^2 r\mathrm{d}r\mathrm{d}z-2\int_{\mathbb{R}^2_+}\partial_r\left(\frac{\Gamma}{\rho}\right)\Gamma\mathrm{d}r\mathrm{d}z\\
&=\int_{\mathbb{R}^2_+}a(\partial_r P\partial_z\Gamma-\partial_z P\partial_r\Gamma)\mathrm{d}r\mathrm{d}z-\int_{\mathbb{R}^2_+}\partial_z(\Gamma r)\frac{\Pi^2}{\rho}\mathrm{d}r\mathrm{d}z\\
&\lesssim\left\|\frac{a}{r}\right\|_{L^\infty}\|\widetilde{\nabla}P\|_{L^2}\|\widetilde{\nabla}\Gamma\|_{L^2}+\left\|\frac{1}{\rho}\right\|_{L^\infty}\|\partial_z\Gamma\|_{L^2}\|\Pi\|_{L^3}\|\Pi\|_{L^6}.
\end{aligned}$$

From [5, Lemma 3.2], we have $a(t,0,z)=0$. By integration by parts, we have

$$-2\int_{\mathbb{R}^2_+}\partial_r\left(\frac{\Gamma}{\rho}\right)\Gamma\mathrm{d}r\mathrm{d}z\geq -C\left\|\frac{a}{r}\right\|_{L^\infty}^2\|\Gamma\|_{L^2}^2-\frac{1}{4}\left\|\frac{\partial_r\Gamma}{\sqrt{\rho}}\right\|_{L^2}^2.$$

Using condition (3.1) again, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma(t)\|_{L^2}^2+\|\widetilde{\nabla}\Gamma\|_{L^2}^2\lesssim\left\|\frac{a}{r}\right\|_{L^\infty}^2(\|\widetilde{\nabla}P\|_{L^2}^2+\|\Gamma\|_{L^2}^2)+\|\Pi\|_{L^3}^2\|\nabla\Pi\|_{L^2}^2. \tag{3.6}$$

Moreover, when $q=2$, (2.4) shows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Pi(t)\|_{L^2}^2+\|\nabla\Pi\|_{L^2}^2\leq 0. \tag{3.7}$$

In addition, by taking $c_0=\frac{1}{2\|\Pi_0\|_{L^3}^2}$ and summing up (3.7) with $c_0\times$(3.6), one has

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(c_0\|\Gamma(t)\|_{L^2}^2+\|\Pi(t)\|_{L^2}^2\right)+c_0\|\widetilde{\nabla}\Gamma\|_{L^2}^2+\|\nabla\Pi\|_{L^2}^2\\
&\quad\lesssim\left\|\frac{a}{r}\right\|_{L^\infty}^2(\|\widetilde{\nabla}P\|_{L^2}^2+\|\Gamma\|_{L^2}^2).
\end{aligned}\tag{3.8}$$

On the other hand, as [4, (2.26)], we have

$$\left\|\frac{a}{r}(t)\right\|_{L^\infty}\leq\left\|\frac{a_0}{r}\right\|_{L^\infty}\exp\left(\left\|\frac{u^r}{r}\right\|_{L^1_t(L^\infty)}\right). \tag{3.9}$$

From [54, 55], we obtain

$$\left\|\frac{u^r}{r}\right\|_{L^1_t(L^\infty)}\lesssim\|\Gamma\|_{L^1_t(L^{3,1})}\lesssim t^{\frac{3}{4}}\|\Gamma\|_{L^\infty_t(L^2)}^{\frac{1}{2}}\|\nabla\Gamma\|_{L^2_t(L^2)}^{\frac{1}{2}}. \tag{3.10}$$

By integrating (3.8) over [0,t], and combining (3.9)-(3.10), we have

$$\begin{aligned}
&\|\Pi\|_{L^\infty_t(L^2)}^2+\|\nabla\Pi\|_{L^2_t(L^2)}^2+c_0\|\Gamma\|_{L^\infty_t(L^2)}^2+c_0\|\nabla\Gamma\|_{L^2_t(L^2)}^2\\
&\lesssim c_0\|\Gamma_0\|_{L^2}^2+\|\Pi_0\|_{L^2}^2\\
&\quad+\left\|\frac{a_0}{r}\right\|_{L^\infty}^2\exp\left(t^{\frac{3}{4}}\|\Gamma\|_{L^\infty_t(L^2)}^{\frac{1}{2}}\|\nabla\Gamma\|_{L^2_t(L^2)}^{\frac{1}{2}}\right)(\|\widetilde{\nabla}P\|_{L^2_t(L^2)}+\|\Gamma\|_{L^2_t(L^2)}^2).
\end{aligned}$$

Plugging estimate (3.3) into the above inequality leads to (3.5). □

4. Boundedness of the Approximate Solutions

In this section, we construct a sequence of approximate solutions to (1.1). It is well known that if the initial data (ρ_0, u_0, B_0) satisfies the condition, $0 < m \le \rho_0 \le M$, $u_0, B_0 \in H^2$, then system (1.1) possesses a unique local solution (ρ, u, B) on $[0, T^*)$ satisfying

$$\rho \in L^\infty(0,t;\mathbb{R}^3), \quad (u,B) \in C([0,t];H^2) \quad \text{and} \quad (\nabla u, \nabla B) \in L^2(0,t;H^2(\mathbb{R}^3)),$$

for all $t < T^*$, where T^* is the maximal existence time of the solutions. Under the axisymmetric condition, we now mollify the initial data (ρ_0, u_0, B_0) as follows. Let $J^\varepsilon = \varepsilon^{-3} J(\frac{r}{\varepsilon}, \frac{x_3}{\varepsilon})$ be a mollifier, with

$$0 \le J \le 1, \quad \operatorname{supp} J \subset \{0 \le r \le 2, \ -1 \le x_3 \le 1\},$$
$$J = 1 \quad \text{if} \quad x \in \left\{0 \le r \le \frac{1}{2}, \ -\frac{1}{2} \le x_3 \le \frac{1}{2}\right\}, \quad \int J \,\mathrm{d}x = 1,$$

and

$$\rho_0^\varepsilon = J^\varepsilon * \rho_0 - (J^\varepsilon * (\rho_0 - 1))(0, x_3), \quad u_0^\varepsilon = J^\varepsilon * u_0, \quad B_0^\varepsilon = J^\varepsilon * B_0.$$

We then see that $(\rho_0^\varepsilon, u_0^\varepsilon, B_0^\varepsilon)$ is still axisymmetric and thus system (1.1) has a unique global smooth axisymmetric solution $(\rho^\varepsilon, u^\varepsilon, B^\varepsilon)$ in $[0, T^*(\varepsilon))$ with the initial data satisfying the same assumptions in Theorem 1.1. Here $T^*(\varepsilon)$ denotes the maximal lifespan to the approximate system. In what follows, we are going to show that $T^*(\varepsilon)$ has a uniform low bound which only depends on the initial data. Denote $a^\varepsilon = \frac{1}{\rho^\varepsilon} - 1$. For the sake of convenience, we omit the index ε and let (a, u, B) be the local smooth solutions on $[0, T^*)$. By using Lemma 3.2 in [5], we also have $a|_{r=0} = 0$. The following proposition asserts that, for the local solutions constructed above, we can obtain a uniform low bound for T^*.

Proposition 4.1. *Let (ρ, u, B) be a smooth enough solution of (1.2) on $[0, T^*)$, which satisfies (3.1). Then, under the assumption of (1.6) and $c_0 = \frac{1}{2\|\Pi_0\|_{L^3}^2}$, there exists a positive time $t_1 \le T^*$ such that*

$$\|\Pi\|_{L^\infty_{t_1}(L^2)}^2 + \|\nabla\Pi\|_{L^2_{t_1}(L^2)}^2 + c_0\left(\|\Gamma\|_{L^\infty_{t_1}(L^2)}^2 + \|\nabla\Gamma\|_{L^2_{t_1}(L^2)}^2\right) \le 2\left(c_0\|\Gamma_0\|_{L^2}^2 + \|\Pi_0\|_{L^2}^2\right), \tag{4.1}$$

and

$$\|\nabla u\|_{L^\infty_{t_1}(L^2)}^2 + \left\|\frac{u^r}{r}\right\|_{L^\infty_{t_1}(L^2)}^2 + \|\partial_t u\|_{L^2_{t_1}(L^2)}^2 + \|u\|_{L^2_{t_1}(\dot H^2)}^2 + \|\Gamma\|_{L^2_{t_1}(L^2)}^2 + \|\nabla P\|_{L^2_{t_1}(L^2)}^2 \lesssim \mathscr{H}_0, \tag{4.2}$$

where t_1 is given by

$$t_1 \overset{\text{def}}{=} \left(\frac{\sqrt{\frac{c_0}{2}}}{\sqrt{c_0}\|\Gamma_0\|_{L^2} + \|\Pi_0\|_{L^2}} \ln\left(\frac{c_0\|\Gamma_0\|_{L^2}^2 + \|\Pi_0\|_{L^2}^2}{2\|\frac{a_0}{r}\|_{L^\infty}^2 \mathscr{H}_0}\right)\right)^{\frac{4}{3}}, \tag{4.3}$$

and $\mathscr{H}_0$ is given in (1.8).

Proof. To prove (4.1), we first assume that, for all $t \in [0, T^*)$, (4.1) holds. Plugging (4.1) into the right side of (3.5), we have

$$\begin{aligned}
&\|\Pi\|_{L^\infty_t(L^2)}^2 + \|\nabla\Pi\|_{L^2_t(L^2)}^2 + c_0\|\Gamma\|_{L^\infty_t(L^2)}^2 + c_0\|\nabla\Gamma\|_{L^2_t(L^2)}^2 \\
&\lesssim c_0\|\Gamma_0\|_{L^2}^2 + \|\Pi_0\|_{L^2}^2 + \mathscr{H}_0\left\|\frac{a_0}{r}\right\|_{L^\infty}^2 \exp\left(t^{\frac{3}{4}}\sqrt{\frac{2}{c_0}}\left(\sqrt{c_0}\|\Gamma_0\|_{L^2} + \|\Pi_0\|_{L^2}\right)\right).
\end{aligned}$$

Choosing t such that

$$t_1=\left(\frac{\sqrt{\frac{c_0}{2}}}{\sqrt{c_0}\|\Gamma_0\|_{L^2}+\|\Pi_0\|_{L^2}}\ln\left(\frac{c_0\|\Gamma_0\|_{L^2}^2+\|\Pi_0\|_{L^2}^2}{2\|\frac{a_0}{r}\|_{L^\infty}^2\mathscr{H}_0}\right)\right)^{\frac{4}{3}},$$

under the smallness condition on $\|\frac{a_0}{r}\|_{L^\infty}$, we obtain

$$\|\Pi\|_{L^\infty_{t_1}(L^2)}^2+\|\nabla\Pi\|_{L^2_{t_1}(L^2)}^2+c_0\|\Gamma\|_{L^\infty_{t_1}(L^2)}^2+c_0\|\nabla\Gamma\|_{L^2_{t_1}(L^2)}^2\le\frac{3}{2}\left(c_0\|\Gamma_0\|_{L^2}^2+\|\Pi_0\|_{L^2}^2\right).$$

Plugging the above estimate into (3.3) yields (4.2). □

We now have the following global in time $\dot H^1$ energy estimate.

Proposition 4.2. *Let (ρ,u,B) be the local unique smooth solutions to (1.1) on $[0,T^*)$, which satisfies (3.1). Suppose that there exists t_0, such that $\|\nabla u(t_0)\|_{L^2}+\|\nabla B(t_0)\|_{L^2}\le\eta_1$. Then the following inequality holds true*

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2+\|\nabla B(t)\|_{L^2}^2+\int_{t_0}^t m(\|\partial_t u(t')\|_{L^2}^2+\|\partial_t B(t')\|_{L^2}^2)\,\mathrm{d}t'\\&\quad+\int_{t_0}^t\eta_2\left(\|\nabla^2 u(t')\|_{L^2}^2+\|\nabla^2 B(t')\|_{L^2}^2+\|\nabla P(t')\|_{L^2}^2\right)\mathrm{d}t'\\&\le\|\nabla u(t_0)\|_{L^2}^2+\|\nabla B(t_0)\|_{L^2}^2,\end{aligned}\tag{4.4}$$

where η_1 and η_2, depend only on $\|u_0\|_{L^2},\|B_0\|_{L^2}$ and $\left\|\frac{B_0^\theta}{r}\right\|_{L^6}$. Furthermore, take ε_0 in (1.6) so small that $t_0\le t_1$. Then $T^*=\infty$ and (1.7) holds true.

Proof. Firstly, from the proof of [53, Proposition 3.7], one has

$$\|\nabla u(t_0)\|_{L^2}^2+\|\nabla B(t_0)\|_{L^2}^2\le\frac{1}{2N}\|\sqrt{\rho_0}u_0\|_{L^2}^2+\frac{1}{2N}\|B_0\|_{L^2}^2\le\eta_1^2,\tag{4.5}$$

where N denotes a large constant. Then, by taking the L^2 inner product of $(1.1)_2$ with $\partial_t u$ and using integration by parts, one has

$$\begin{aligned}&\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2=-\left(\rho u\cdot\nabla u\mid\partial_t u\right)_{L^2}+\left(B\cdot\nabla B\mid\partial_t u\right)_{L^2}\\&\le C\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^3}\|\nabla u\|_{L^6}\|\sqrt{\rho}\partial_t u\|_{L^2}+C\|B\|_{L^3}\|\nabla B\|_{L^6}\left\|\frac{1}{\sqrt{\rho}}\right\|_{L^\infty}\|\sqrt{\rho}\partial_t u\|_{L^2}\\&\le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2+\frac{1}{4}\|\sqrt{\rho}\partial_t u\|_{L^2}^2+C\|B\|_{L^2}\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}^2,\end{aligned}$$

which gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2+\frac{3}{4}\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2\\&\le C\Big(\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2+\|B\|_{L^2}\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}^2\Big).\end{aligned}\tag{4.6}$$

Similarly, by taking the L^2 inner product of (1.1)$_3$ with $\partial_t B$ and using integration by parts, one has

$$\begin{aligned}\|\partial_t B(t)\|_{L^2}^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla B(t)\|_{L^2}^2 =&-\big(u\cdot\nabla B\mid\partial_t B\big)_{L^2}+\big(B\cdot\nabla u\mid\partial_t B\big)_{L^2}\\&-\big(\nabla\times(\nabla\times B)\times B\mid\partial_t B\big)_{L^2}\\ \le& C\Bigg(\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 B\|_{L^2}^2+\|B\|_{L^2}\|\nabla B\|_{L^2}\|\nabla^2 u\|_{L^2}^2\\&+\left\|\frac{B^\theta}{r}\right\|_{L^6}\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}^2\Bigg)+\frac{1}{4}\|\partial_t B(t)\|_{L^2}^2.\end{aligned}\tag{4.7}$$

The inequality in (4.7) can be proved as follows. First, we note that

$$|\nabla B|^2=|(e_r\partial_r+\frac{1}{r}e_\theta\partial_\theta+e_z\partial_z)B^\theta e_\theta|^2=|\nabla B^\theta|^2+|\Pi|^2.$$

Thus $\|\Pi\|_{L^2}\le\|\nabla B\|_{L^2}$. Therefore, the hall term reads as

$$\begin{aligned}\big(\nabla\times(\nabla\times B)\times B\mid\partial_t B\big)_{L^2}\le& C\int_{\mathbb{R}^3}\frac{\partial_z(B^\theta)^2}{r}\partial_t B\mathrm{d}x\\ \le& C\left\|\frac{B^\theta}{r}\right\|_{L^2}^{\frac{1}{2}}\left\|\frac{B^\theta}{r}\right\|_{L^6}^{\frac{1}{2}}\|\partial_z B^\theta\|_{L^6}\|\partial_t B^\theta\|_{L^2}\\ \le&\left\|\frac{B^\theta}{r}\right\|_{L^6}\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}^2+\frac{1}{12}\|\partial_t B\|_{L^2}^2.\end{aligned}$$

Combining (4.6) and (4.7) gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla B(t)\|_{L^2}^2\right)+\frac{3}{4}\left(\|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2+\|\partial_t B(t)\|_{L^2}^2\right)\\ \le& C\left(\|u\|_{L^2}\|\nabla u\|_{L^2}+\|B\|_{L^2}\|\nabla B\|_{L^2}+\left\|\frac{B^\theta}{r}\right\|_{L^6}\|\nabla B\|_{L^2}\right)\times(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2).\end{aligned}\tag{4.8}$$

On the other hand, from the following equations

$$\begin{cases}-\Delta u+\nabla P=-\rho\partial_t u-\rho u\cdot\nabla u+B\cdot\nabla B,\\ -\Delta B=-\partial_t B-u\cdot\nabla B+B\cdot\nabla u-\nabla\times((\nabla\times B)\times B),\end{cases}\tag{4.9}$$

and the L^q estimate of elliptic equations, we have

$$\begin{aligned}\|\nabla^2 u\|_{L^2}^2+\|\nabla P\|_{L^2}^2\le& C\big(\|\rho\partial_t u\|_{L^2}^2+\|\rho u\cdot\nabla u\|_{L^2}^2+\|B\cdot\nabla B\|_{L^2}^2\big)\\ \le& C\left(\|\rho\partial_t u\|_{L^2}^2+\|\rho\|_{L^\infty}^2\|u\|_{L^3}^2\|\nabla u\|_{L^6}^2+\|B\|_{L^3}^2\|\nabla B\|_{L^6}^2\right)\\ \le& C\left(\|\partial_t u\|_{L^2}^2+\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^2+\|B\|_{L^2}\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}^2\right),\end{aligned}\tag{4.10}$$

and

$$\|\nabla^2 B\|_{L^2}^2 \le C\left(\|\partial_t B\|_{L^2}^2+\|u\|_{L^3}^2\|\nabla B\|_{L^6}^2+\|B\|_{L^3}^2\|\nabla u\|_{L^6}^2+\left\|\frac{\partial_z (B^\theta)^2}{r}\right\|_{L^2}^2\right)$$
$$\le C\Bigg(\|\partial_t B\|_{L^2}^2+\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla^2 B\|_{L^2}^2+\|B\|_{L^2}\|\nabla B\|_{L^2}\|\nabla^2 u\|_{L^2}^2 \tag{4.11}$$
$$+\left\|\frac{B^\theta}{r}\right\|_{L^6}\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}^2\Bigg),$$

where

$$\left\|\frac{\partial_z (B^\theta)^2}{r}\right\|_{L^2}^2 \le C\left\|\frac{B^\theta}{r}\right\|_{L^6}\left\|\frac{B^\theta}{r}\right\|_{L^2}\|\partial_z B^\theta\|_{L^6}^2$$
$$\le C\left\|\frac{B^\theta}{r}\right\|_{L^6}\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}^2.$$

Then, for any $\eta_2>0$, (4.8) $+\,\eta_2((4.10)+(4.11))$, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla B(t)\|_{L^2}^2\right)+\left(\frac{3m}{4}-C\eta_2\right)\left(\|\partial_t u\|_{L^2}^2+\|\partial_t B\|_{L^2}^2\right)$$
$$+\left\{\eta_2-C\left(\|u\|_{L^2}\|\nabla u\|_{L^2}+\|B\|_{L^2}\|\nabla B\|_{L^2}+\left\|\frac{B^\theta}{r}\right\|_{L^6}\|\nabla B\|_{L^2}\right)\right\}$$
$$\times\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2+\|\nabla P\|_{L^2}^2\right)\le 0.$$

This implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla B(t)\|_{L^2}^2\right)+\left(\frac{3m}{4}-C\eta_2\right)\left(\|\partial_t u\|_{L^2}^2+\|\partial_t B\|_{L^2}^2\right)$$
$$+\left\{\eta_2-C\left(\|u_0\|_{L^2}+\|B_0\|_{L^2}+\left\|\frac{B_0^\theta}{r}\right\|_{L^6}\right)\left(\|\nabla u\|_{L^2}+\|\nabla B\|_{L^2}\right)\right\} \tag{4.12}$$
$$\times\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2+\|\nabla P\|_{L^2}^2\right)\le 0.$$

In the following, we use the standard continuity argument to show that the maximal lifespan T^* can be extended to any positive time. For that, we denote

$$\tau^* \stackrel{\text{def}}{=} \sup\left\{\, t\in[t_0,T^*) \,\middle|\, \|\nabla u(t)\|_{L^2}+\|\nabla B(t)\|_{L^2}\le 2\eta_1 \,\right\}. \tag{4.13}$$

If η_1 is sufficiently small and η_2 is suitably selected small number, then $\tau^*=T^*$. When $\tau^*<T^*$, taking $\eta_2=\frac{m}{4C}$, and

$$\eta_1\le\frac{\eta_2}{2C\left(\|u_0\|_{L^2}+\|B_0\|_{L^2}+\left\|\frac{B_0^\theta}{r}\right\|_{L^6}\right)},$$

we deduce from (4.12) that, for all $t\in[t_0,\tau^*)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla B(t)\|_{L^2}^2\right)+m\left(\|\partial_t u\|_{L^2}^2+\|\partial_t B\|_{L^2}^2\right)$$
$$+\eta_2\left(\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2+\|\nabla P\|_{L^2}^2\right)\le 0.$$

Combining with (4.5), we have

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2+\|\nabla B(t)\|_{L^2}^2+\int_{t_0}^{\tau^*}\Big(m(\|\partial_t u(t')\|_{L^2}^2+\|\partial_t B(t')\|_{L^2}^2)\\&\quad+\eta_2(\|\nabla^2 u(t')\|_{L^2}^2+\|\nabla^2 B(t')\|_{L^2}^2+\|\nabla P(t')\|_{L^2}^2)\Big)\,\mathrm{d}t'\\&\le\|\nabla u(t_0)\|_{L^2}^2+\|\nabla B(t_0)\|_{L^2}^2\\&\le\eta_1^2.\end{aligned}$$

Thus, it contradicts (4.13). Therefore, $\tau^*=T^*$.

On the other hand, we define t_1 in (4.3). Then, by choosing $\|\frac{a_0}{r}\|_{L^\infty}$ so small that $t_1\ge t_0$. Therefore, by summing up (4.2) and (4.4), we can obtain for $t<T^*$

$$\begin{aligned}&\|\nabla u\|_{L_t^\infty(L^2)}^2+\|\partial_t u\|_{L_t^2(L^2)}^2+\|\nabla^2 u\|_{L_t^2(L^2)}^2+\|\nabla P\|_{L_t^2(L^2)}^2\\&\quad\le\|\nabla u\|_{L^\infty(0,t_0;L^2)}^2+\|\partial_t u\|_{L^2(0,t_0;L^2)}^2+\|\nabla^2 u\|_{L^2(0,t_0;L^2)}^2\\&\qquad+\|\nabla P\|_{L^2(0,t_0;L^2)}^2+\|\nabla u\|_{L^\infty(t_0,t;L^2)}^2+\|\partial_t u\|_{L^2(t_0,t;L^2)}^2+\|\nabla^2 u\|_{L^2(t_0,t;L^2)}^2+\|\nabla P\|_{L^2(t_0,t;L^2)}^2\\&\quad\le C\mathscr{H}_0+\eta_1^2,\end{aligned}\tag{4.14}$$

for $\mathscr{H}_0$ given by (1.8). Thanks to (4.14) and the blow-up criteria in [56], we conclude that $T^*=\infty$. By summing up (3.2) and (4.14), (1.7) holds true. We then finish the proof of Proposition 4.2. □

Before proving Theorem 1.1, we derive the estimates of $\|B^\theta\|_{L^\infty}$ and $\|\nabla B\|_{L^2}$.

Lemma 4.1. *(The estimate of B^θ)*

Under the assumptions of Proposition 4.1. There holds, for all $t>0$

$$\|B^\theta(t)\|_{L^\infty}\lesssim\mathscr{E}(t),\tag{4.15}$$

and

$$\|\nabla B\|_{L_t^\infty(L^2)}^2+\|\nabla^2 B\|_{L_t^2(L^2)}^2\lesssim\mathscr{E}(t),\tag{4.16}$$

each $\mathscr{E}(t)$ denotes different function about t.

Proof. Firstly, multiplying $(1.2)_4$ with $q|B^\theta|^{q-2}B^\theta, 2<q<\infty$ and taking $L^2(\mathbb{R}_+^2;r\mathrm{d}r\mathrm{d}z)$ inner product, one has

$$\|B^\theta(t)\|_{L^q}\le\|B_0^\theta\|_{L^q}+\int_0^t\|B^\theta\|_{L^q}\Big\|\frac{u^r}{r}\Big\|_{L^\infty}\mathrm{d}s.$$

By Gronwall's inequality, (3.10) and (4.1), we have

$$\|B^\theta(t)\|_{L^q}\le\|B_0^\theta\|_{L^q}\exp\int_0^t\Big\|\frac{u^r}{r}\Big\|_{L^\infty}\mathrm{d}s\lesssim\mathscr{E}(t).$$

Taking $q\to\infty$, (4.15) follows immediately. For the proof of (4.16). We first apply ∇ to $(1.1)_3$, and have

$$\partial_t\nabla B+\nabla u\cdot\nabla B+u\cdot\nabla\nabla B-\nabla\Delta B=\nabla B\cdot\nabla u+B\cdot\nabla\nabla u-\nabla\nabla\times((\nabla\times B)\times B).$$

Taking L^2 inner product with ∇B, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla B(t)\|_{L^2}^2+\|\nabla^2 B\|_{L^2}^2 =& -\int_{\mathbb{R}^3}\nabla u\cdot\nabla B\nabla B\,\mathrm{d}x+\int_{\mathbb{R}^3}\nabla B\cdot\nabla u\nabla B\,\mathrm{d}x\\ &+\int_{\mathbb{R}^3}B\cdot\nabla\nabla u\nabla B\,\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\nabla\times((\nabla\times B)\times B)\nabla B\,\mathrm{d}x\\ \lesssim&\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla B\|_{L^2}^2+\frac{1}{2}\|\nabla^2 B\|_{L^2}^2\\ &+\|B\|_{L^\infty}^2\|\nabla B\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2+\left\|\frac{B^\theta}{r}\right\|_{L^\infty}^2\|\nabla B\|_{L^2}^2.\end{aligned}$$

We can use Gronwall's inequality, (1.7), (2.2) and (4.15) to estimate

$$\begin{aligned}&\|\nabla B\|_{L_t^\infty(L^2)}^2+\|\nabla^2 B\|_{L_t^2(L^2)}^2\\ &\lesssim\exp\left(\int_0^t\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}+\|B\|_{L^\infty}^2+\left\|\frac{B^\theta}{r}\right\|_{L^\infty}^2\mathrm{d}s\right)\\ &\quad\times\left(\|\nabla B_0\|_{L^2}^2+\int_0^t\|\nabla^2 u\|_{L^2}^2\,\mathrm{d}s\right)\\ &\lesssim\mathscr{E}(t).\end{aligned}$$

□

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By taking ∂_t to $(1.1)_{2,3}$, we write

$$\rho\left(\partial_t u_t+u\cdot\nabla u_t\right)-\Delta u_t+\nabla P_t=-\rho_t u_t-(\rho u)_t\cdot\nabla u+B_t\cdot\nabla B+B\cdot\nabla B_t,$$

and

$$\partial_{tt}B-\Delta B_t+u_t\cdot\nabla B+u\cdot\nabla B_t=-\partial_t\nabla\times((\nabla\times B)\times B)+B_t\cdot\nabla u+B\cdot\nabla u_t.$$

Taking the L^2 inner product of the above equations with u_t and B_t and combining the equations, respectively, using $(1.1)_{1,4}$, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho}u_t(t)\|_{L^2}^2+\|B_t(t)\|_{L^2}^2\right)+\|\nabla u_t(t)\|_{L^2}^2+\|\nabla B_t(t)\|_{L^2}^2\\ &=-\int_{\mathbb{R}^3}\rho_t|u_t|^2\,\mathrm{d}x-\int_{\mathbb{R}^3}\rho_t u\cdot\nabla u u_t\,\mathrm{d}x-\int_{\mathbb{R}^3}\rho u_t\cdot\nabla u u_t\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^3}B_t\cdot\nabla B u_t\,\mathrm{d}x-\int_{\mathbb{R}^3}\partial_t\nabla\times((\nabla\times B)\times B)B_t\,\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^3}u_t\cdot\nabla B B_t\,\mathrm{d}x+\int_{\mathbb{R}^3}B_t\cdot\nabla u B_t\,\mathrm{d}x.\end{aligned}\tag{4.17}$$

As same as [4], the first term on the right side of (4.17) can be bounded by

$$\left|\int_{\mathbb{R}^3}\rho_t|u_t|^2\,\mathrm{d}x\right|\lesssim\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^2+\frac{1}{20}\|\nabla u_t\|_{L^2}^2.$$

Along the same line, we also have

$$\left|\int_{\mathbb{R}^3}\rho_t u\cdot\nabla u u_t\,\mathrm{d}x\right|\lesssim\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2+\|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2+\frac{1}{20}\|\nabla u_t\|_{L^2}^2,$$

and

$$\left|\int_{\mathbb{R}^3} \rho u_t \cdot \nabla u u_t \,\mathrm{d}x\right| \leq \sqrt{M}\|u_t\|_{L^6}\|\nabla u\|_{L^3}\|\sqrt{\rho}u_t\|_{L^2}$$
$$\lesssim \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{1}{20}\|\nabla u_t\|_{L^2}^2.$$

Similarly, the Hall term has the following estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \partial_t \nabla \times ((\nabla \times B) \times B) B_t \,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}^3} \partial_{tz} \frac{(B^\theta)^2}{r} B_t \,\mathrm{d}x\right| \\ &= \left|\int_{\mathbb{R}^3} \partial_t \frac{(B^\theta)^2}{r} \partial_z B_t \,\mathrm{d}x\right| \\ &\lesssim \|\Pi\|_{L^\infty}^2 \|B_t\|_{L^2}^2 + \frac{1}{8}\|\nabla B_t\|_{L^2}^2.\end{aligned}$$

Finally, the last three terms can be tackled in the same way, which read as

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} B_t \cdot \nabla B u_t \,\mathrm{d}x - \int_{\mathbb{R}^3} u_t \cdot \nabla B B_t \,\mathrm{d}x + \int_{\mathbb{R}^3} B_t \cdot \nabla u B_t \,\mathrm{d}x\right| \\ &\lesssim \|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}\|B_t\|_{L^2}^2 + \frac{1}{10}\|\nabla u_t\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|B_t\|_{L^2}^2 + \frac{1}{8}\|\nabla B_t\|_{L^2}^2.\end{aligned}$$

Plugging the above inequalities into (4.17), we finally have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \|B_t(t)\|_{L^2}^2\right) + \|\nabla u_t(t)\|_{L^2}^2 + \|\nabla B_t(t)\|_{L^2}^2 \\ &\lesssim \frac{1}{4}\|\nabla u_t\|_{L^2}^2 + \frac{1}{4}\|\nabla B_t\|_{L^2}^2 + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^2 \\ &\quad + \|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}\|B_t\|_{L^2}^2 \\ &\quad + \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|B_t\|_{L^2}^2 + \|\Pi\|_{L^\infty}^2\|B_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2.\end{aligned} \tag{4.18}$$

On the other hand, we deduce from the Stokes system (4.9) that

$$\|\nabla^2 u(t)\|_{L^6} + \|\nabla P(t)\|_{L^6} \leq C\left(\|\rho \partial_t u\|_{L^6} + \|\rho u \cdot \nabla u\|_{L^6} + \|B \cdot \nabla B\|_{L^6}\right), \tag{4.19}$$

and

$$\|\nabla^2 B(t)\|_{L^6} \leq C(\|B_t\|_{L^6} + \|u \cdot \nabla B\|_{L^6} + \|B \cdot \nabla u\|_{L^6} + \|\nabla \times ((\nabla \times B) \times B)\|_{L^6}). \tag{4.20}$$

We deduce

$$\begin{aligned}&\|\rho u \cdot \nabla u\|_{L^6} + \|B \cdot \nabla B\|_{L^6} + \|u \cdot \nabla B\|_{L^6} + \|B \cdot \nabla u\|_{L^6} \\ &\leq C(\|u\|_{L^6}\|\nabla u\|_{L^\infty} + \|B\|_{L^6}\|\nabla B\|_{L^\infty} + \|u\|_{L^6}\|\nabla B\|_{L^\infty} + \|B\|_{L^6}\|\nabla u\|_{L^\infty}) \\ &\lesssim \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^6}^{\frac{1}{2}} + \|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}^{\frac{1}{2}}\|\nabla^2 B\|_{L^6}^{\frac{1}{2}} \\ &\quad + \|\nabla u\|_{L^2}\|\nabla^2 B\|_{L^2}^{\frac{1}{2}}\|\nabla^2 B\|_{L^6}^{\frac{1}{2}} + \|\nabla B\|_{L^2}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^6}^{\frac{1}{2}} \\ &\lesssim \|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2} + \|\nabla B\|_{L^2}^2\|\nabla^2 B\|_{L^2} + \|\nabla B\|_{L^2}^2\|\nabla^2 u\|_{L^2} \\ &\quad + \|\nabla u\|_{L^2}^2\|\nabla^2 B\|_{L^2} + \frac{1}{2}\|\nabla^2 u\|_{L^6} + \frac{1}{2}\|\nabla^2 B\|_{L^6},\end{aligned} \tag{4.21}$$

and

$$\|\nabla\times((\nabla\times B)\times B)\|_{L^6}\le C\left\|\partial_z\frac{(B^\theta)^2}{r}\right\|_{L^6}\lesssim \|\Pi\|_{L^\infty}\|\nabla^2 B\|_{L^2}. \tag{4.22}$$

Then, together with (4.19) – (4.22), (4.18) can be read as

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho}u_t(t)\|_{L^2}^2+\|B_t(t)\|_{L^2}^2\right)+\frac{1}{2}\|\nabla u_t(t)\|_{L^2}^2+\frac{1}{2}\|\nabla B_t(t)\|_{L^2}^2\\&\quad+\|\nabla^2 u(t)\|_{L^6}^2+\|\nabla^2 B(t)\|_{L^6}^2+\|\nabla P(t)\|_{L^6}^2\\&\lesssim\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2\|\sqrt{\rho}u_t\|_{L^2}^2\\&\quad+\|\nabla B\|_{L^2}\|\nabla^2 B\|_{L^2}\|B_t\|_{L^2}^2+\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|B_t\|_{L^2}^2+\|\Pi\|_{L^\infty}^2\|B_t\|_{L^2}^2\\&\quad+\|\nabla B\|_{L^2}^4\|\nabla^2 B\|_{L^2}^2+\|\nabla B\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2+\|\nabla u\|_{L^2}^4\|\nabla^2 B\|_{L^2}^2\\&\quad+\|\nabla^2 u\|_{L^2}^2+\|\nabla u\|_{L^2}^4\|\nabla^2 u\|_{L^2}^2+\|\Pi\|_{L^\infty}^2\|\nabla^2 B\|_{L^2}^2.\end{aligned}$$

Finally, by using Gronwall's inequality, (1.7), (2.2), (3.1), (3.2), (4.15), and (4.16), we have the following estimate

$$\begin{aligned}&\|u_t\|_{L^\infty(L^2)}^2+\|B_t\|_{L^\infty(L^2)}^2+\|\nabla u_t\|_{L^2(L^2)}^2+\|\nabla B_t\|_{L^2(L^2)}^2\\&\quad+\|\nabla^2 u\|_{L^2(L^6)}^2+\|\nabla^2 B\|_{L^2(L^6)}^2+\|\nabla P\|_{L^2(L^6)}^2\lesssim\mathscr{E}(t),\ \forall t\ge 0.\end{aligned} \tag{4.23}$$

By virtue of (1.7), (4.16), and (4.23), we infer

$$\int_0^\infty\|\nabla u(t)\|_{L^\infty}\,\mathrm{d}t\le C\int_0^\infty\|\nabla^2 u(t)\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u(t)\|_{L^6}^{\frac{1}{2}}\,\mathrm{d}t\lesssim\mathscr{E}(t), \tag{4.24}$$

and

$$\int_0^\infty\|\nabla B(t)\|_{L^\infty}\,\mathrm{d}t\le C\int_0^\infty\|\nabla^2 B(t)\|_{L^2}^{\frac{1}{2}}\|\nabla^2 B(t)\|_{L^6}^{\frac{1}{2}}\,\mathrm{d}t\lesssim\mathscr{E}(t). \tag{4.25}$$

(4.24) and (4.25) give the global in time Lipschitz estimates of u and B. Moreover, the estimates (4.23), (4.24) and (4.25) are sufficient for the global regularity of the inhomogeneous incompressible Hall-MHD system (1.1). Consequently, one has

$$\|u(t,\cdot)\|_{H^2}+\|B(t,\cdot)\|_{H^2}\lesssim\mathscr{E}(t),\quad\forall t\ge 0.$$

This finishes the proof of the theorem. □

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