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ON POSITIVE SOLUTIONS FOR A CLASS OF DOUBLE PHASE PROBLEMS WITH STRONG SINGULAR WEIGHTS AND NONLINEARITIES

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Abstract. We study the existence and nonexistence of positive solutions for the singular double phase problem:

$$\begin{cases} -(\alpha(t)\varphi_p(u')+\beta(t)\varphi_q(u'))' = \lambda h(t)f(u),\ t\in(0,1), \\ u(0)=0=u(1), \end{cases}$$

where $\lambda>0$, $1<p<q<\infty$, and $\varphi_m(s):=|s|^{m-2}s$. This model can contain strong singular weights and nonlinearities such as $h(t)=t^{-\delta}$ with $\delta\geq 1$ and $f(u)=1+u^{-\gamma}$ with $\gamma\geq 1$. Firstly, we provide sufficient conditions on α, β, and h such that positive solutions belong to $C[0,1]$, which generalizes previous results. Secondly, we establish various existence results, including the existence of three positive solutions, which are new results even for strong singular p-Laplacian problems. We prove the existence results by applying approximation techniques, resulting in approximation solutions that are not in $C^1[0,1]$, stemming from the degeneracy of α and β.

Keywords. Double phase problem; Multiplicity; Positive solution; Singular boundary value problem.

1. Introduction

In this paper, we are concerned with the existence of multiple positive solutions for one-dimensional double phase problems of the form:

$$\begin{cases} -\big(\alpha(t)\varphi_p(u')+\beta(t)\varphi_q(u')\big)' = \lambda h(t)f(u),\ t\in(0,1), \\ u(0)=0=u(1), \end{cases} \tag{1.1}$$

where $\lambda>0$, $1<p<q<\infty$, $\varphi_m(s):=|s|^{m-2}s$, and α, β, f, and h satisfy the following conditions:

(H_1) $\alpha:[0,1]\to[0,\infty)$ and $\beta:[0,1]\to[0,\infty)$ are continuous such that $\alpha(t)+\beta(t)>0$ almost everywhere and $K_{\alpha\beta}:=\int_0^1 \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)ds<\infty$, where

$$\alpha_*(t):=\begin{cases}\alpha(t) & \text{for } \alpha(t)>0,\\ 1 & \text{for } \alpha(t)=0,\end{cases} \text{ and } \beta_*(t):=\begin{cases}\beta(t) & \text{for } \beta(t)>0,\\ 1 & \text{for } \beta(t)=0,\end{cases}$$

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(H_2) $f:(0,\infty)\to(0,\infty)$ is continuous and there exist $a>0$, $a^*>0$, and $\gamma\geq 0$ such that $0<f(s)\leq\frac{a}{s^\gamma}$ for $s<1$ and $f(s)\geq a^*$ for $s>1$,

(H_3) $h:(0,1)\to(0,\infty)$ is continuous and satisfies

$$\max\left\{\int_0^{\frac{1}{2}}\alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\varphi_p^{-1}(H(s))\,ds,\int_{\frac{1}{2}}^1\alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\varphi_p^{-1}(H(s))\,ds\right\}<\infty,$$

where $H(s)$ is defined as $\int_s^{\frac{1}{2}}h(r)dr$ for $s\in[0,\frac{1}{2}]$ and as $\int_{\frac{1}{2}}^s h(r)dr$ for $s\in[\frac{1}{2},1]$.

Since Fulks-Maybee's seminal work on the problem of an electrical conductor in which electrical resistance is a singular function (see [1]), the study of singular problems has been extended in various directions. One of them is to study singular problems with pure singular terms given by:

$$\begin{cases}-\Delta u=\frac{g(x)}{u^\gamma},\ x\in\Omega,\\ u=0,\ x\in\partial\Omega,\end{cases}\tag{1.2}$$

where $\gamma>0$, Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N\geq 1$, and g is Hölder continuous on Ω. For example, Crandall-Rabinowitz-Tartar [2] studied (1.2) with more general linear elliptic operators. Lazer-McKenna [3] showed that if $\gamma>1$, then a solution u is not in $C^1(\overline{\Omega})$. They also discussed the uniqueness of positive solutions by applying the fact that approximation solutions $u_n\in C^1(\overline{\Omega})$ to nonsingular nonlinearity converge to $u\in C(\overline{\Omega})$. Further, Taliaferro [4] considered (1.2) with $N=1$, $\Omega=(0,1)$ and a continuous function g. He proved that

$$\int_0^1 x(1-x)g(x)dx<\infty\tag{1.3}$$

is a necessary and sufficient condition for (1.2) to have a unique positive solution. We note that (1.3) implies that g may be non-integrable on $(0,1)$.

Another interesting research is to study singular problems with perturbed singular terms as follows:

$$\begin{cases}-\Delta_p u=\frac{\lambda}{u^\gamma}+u^q,\ x\in\Omega,\\ u=0,\ x\in\partial\Omega,\end{cases}\tag{1.4}$$

where $0<\gamma<1$ and $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p>1$. Haitao [5] dealt with the case when $p=2$ and $1<q<\frac{N+2}{N-2}$, while Giacomoni-Schindler-Takáč [6] explored scenarios for $1<p<N$ and $1<q<\frac{N+p}{N-p}$. Both studies showed that (1.4) has two positive solutions (one obtained by the mountain pass theorem and the other by the sub-supersolution method) for a certain range of λ. In [7], these studies were extended from constant exponents p, q, and γ to variable exponents $p(x)$, $q(x)$, and $\gamma(x)$. It is noteworthy that condition $0<\gamma(x)<1$ is essential for applying variational arguments.

Recently, the authors [8] investigated the positive solutions of (1.1), which we refer to as the generalized double phase problem, as the term 'double phase problem' typically denotes the case when $\alpha\equiv 1$ (see [8, 9] for its historical context). Under weaker singular nonlinear terms f and h:

(F) $f:(0,\infty)\to\mathbb{R}$ is continuous and there exist $a>0$, $a^*>0$, $\hat{a}>0$, $\varepsilon\geq 0$, and $0\leq\gamma<1$ such that $(s-\varepsilon)f(s)>0$ for $s\in(0,\infty)\setminus\{\varepsilon\}$, $f(s)\leq\frac{a}{s^\gamma}$ for $s<1$ and $f(s)\geq-\frac{a^*}{s^\gamma}+\hat{a}$ for $s>0$,

(H) $h:(0,1)\to(0,\infty)$ is continuous and there exist $b>0$ and $\delta\geq 0$ such that $\gamma+\delta<1$ and $h(t)\leq\frac{b}{d(t)^{\delta}}$, where $d(t):=\min\{t,1-t\}$,

they explored various existence results depending on the behaviors of f near 0 and ∞. They stated that (H_1) may weaken the regularities of solutions so that the solutions may not be differentiable at $t=t_0$ where $\alpha(t_0)+\beta(t_0)=0$. Thus they investigated positive solutions belonging to $C[0,1]$.

Following the study in [8], we examine (1.1) with a focus on the strong singular nonlinear term f and the strong singular weight h. Here, f is strong singular means $\limsup_{s\to 0} s^{\gamma}f(s)\in(0,\infty)$ for some $\gamma\geq 1$, and h is strong singular means that h is non-integrable on $(0,1)$. The strong singular f and h may further weaken the regularities of solutions. This makes it difficult to use the ideas in [8] to obtain the existence and multiplicity results.

The first goal of this paper is to establish a sufficient condition (H_3), which ensures that the solutions of (1.1) belong to $C[0,1]$. The second goal is to demonstrate various existence results, including the existence of three positive solutions of (1.1). To overcome the difficulties caused by regularity, we combine the ideas in [8, 10]. We use the ideas in [8] to find positive solutions to approximation problems, and use the approximation technique in [10] to find positive solutions of (1.1) from the solutions of the approximation problems. We note that positive solutions to approximation problems in [10] are actually in $C^1[0,1]$ but they worked it on $C[0,1]$. Meanwhile, solutions to the approximation problems of (1.1) (see Section 2) cannot necessarily be expected to belong to $C^1[0,1]$ due to the degeneracy of α and β at $t=t_0$ with $\alpha(t_0)+\beta(t_0)=0$.

We define a function u as a positive solution of (1.1) if u is positive on $(0,1)$, satisfies (1.1) almost everywhere, and adheres to the boundary conditions. As noted in [8], when either $\alpha\equiv 0$ or $\beta\equiv 0$ on some subintervals of $(0,1)$, four distinct cases emerge:

Case A: $|\Omega_\alpha|=0$ and $|\Omega_\beta|=0$
Case B: $|\Omega_\alpha|>0$ and $|\Omega_\beta|=0$
Case C: $|\Omega_\alpha|=0$ and $|\Omega_\beta|>0$
Case D: $|\Omega_\alpha|>0$ and $|\Omega_\beta|>0$

where $\Omega_\alpha:=\{t\in(0,1)\mid\alpha(t)>0 \text{ and } \beta(t)=0\}$ and $\Omega_\beta:=\{t\in(0,1)\mid\alpha(t)=0 \text{ and } \beta(t)>0\}$.

The existence or nonexistence of positive solutions to (1.1) depends on the behaviors of f near 0 and ∞. We consider the behavior of f near 0:

(H_{4a}) $\lim\limits_{s\to 0}\frac{f(s)}{\Phi_\beta(s)}=\infty$,

(H_{4b}) $\lim\limits_{s\to 0}\frac{f(s)}{\Phi_\beta(s)}=0$,

and the behavior of f near ∞:

(H_{5a}) $\lim\limits_{s\to\infty}\frac{f(s)}{\Phi_\alpha(s)}=\infty$,

(H_{5b}) $\lim\limits_{s\to\infty}\frac{f(s)}{\Phi_\alpha(s)}=0$,

where $\Phi_\alpha(s):=\varphi_p(s)\chi(|\Omega_\alpha|)+\varphi_q(s)\chi^c(|\Omega_\alpha|)$, $\Phi_\beta(s):=\varphi_p(s)\chi^c(|\Omega_\beta|)+\varphi_q(s)\chi(|\Omega_\beta|)$, $\chi:[0,\infty)\to\{0,1\}$ is such that $\chi(0)=0$ and $\chi(s)=1$ for $s>0$, and $\chi^c(s):=1-\chi(s)$.

Let $L_{\alpha\beta} := \max\left\{1, \|\alpha\|_\infty^{\frac{1}{p-1}}, \|\beta\|_\infty^{\frac{1}{q-1}}\right\}$, $M_{\alpha\beta h} := \max\left\{M^0_{\alpha\beta h}, M^1_{\alpha\beta h}\right\}$ and

$$(c,c^*) := \begin{cases} (0,1) & \text{for Case } A, \\ (c_\alpha, c^*_\alpha) & \text{for Case } B, \\ (c_\beta, c^*_\beta) & \text{for Case } C, \\ (c_\alpha, c^*_\alpha) & \text{for Case } D, \end{cases}$$

where (c_α, c^*_α) is a subinterval of Ω_α such that $c^*_\alpha - c_\alpha \geq \tilde{c}^*_\alpha - \tilde{c}_\alpha$ for any interval $(\tilde{c}_\alpha, \tilde{c}^*_\alpha) \subset \Omega_\alpha$, (c_β, c^*_β) is a subinterval of Ω_β such that $c^*_\beta - c_\beta \geq \tilde{c}^*_\beta - \tilde{c}_\beta$ for any interval $(\tilde{c}_\beta, \tilde{c}^*_\beta) \subset \Omega_\beta$,

$$M^0_{\alpha\beta h} := \max\left\{\int_0^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\varphi_p^{-1}(H(s))\,ds, \int_0^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\varphi_q^{-1}(H(s))\,ds\right\}$$

and

$$M^1_{\alpha\beta h} := \max\left\{\int_{\frac{1}{2}}^1 \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\varphi_p^{-1}(H(s))\,ds, \int_{\frac{1}{2}}^1 \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\varphi_q^{-1}(H(s))\,ds\right\}.$$

We note that $c^* - c$ is uniquely determined because it is the maximum of the lengths of subintervals of Ω_α or Ω_β (see Remark 1.1 in [8]). Thus we can define

$$h_* := \min_{(\tilde{c},\tilde{c}^*)}\left\{\int_{\frac{3\tilde{c}+\tilde{c}^*}{4}}^{\frac{\tilde{c}+\tilde{c}^*}{2}} h(r)dr, \int_{\frac{\tilde{c}+\tilde{c}^*}{2}}^{\frac{\tilde{c}+3\tilde{c}^*}{4}} h(r)dr\right\}$$

for any interval $(\tilde{c},\tilde{c}^*) \subset (0,1)$ such that $\tilde{c}^* - \tilde{c} = c^* - c$. To discuss the existence of three positive solutions, we assume f satisfies the following conditions:

- (H_6) $f(s) := \frac{f_\gamma(s)}{s^\gamma}$, where $\gamma \geq 0$ is the constant in (H_2) and $f_\gamma : (0,\infty) \to (0,\infty)$ is nondecreasing and continuous,
- (H_7) there exist $\eta \geq \frac{c^*-c}{4L_{\alpha\beta}}$ and $\theta \geq 8L_{\alpha\beta}M_{\alpha\beta h}$ such that

$$\frac{f(\eta)}{\Phi_\alpha(\eta)} \Big/ \frac{f(\theta)}{\Phi_\alpha(\theta)} > \frac{2(32K_{\alpha\beta}L^2_{\alpha\beta})^\gamma}{h_*(c^*-c)^\gamma}\Phi_\alpha\Big(\frac{256K_{\alpha\beta}L^4_{\alpha\beta}M_{\alpha\beta h}}{(c^*-c)^2}\Big).$$

Let

$$\lambda_* := \Big(\frac{a}{(2L_{\alpha\beta}M_{\alpha\beta h})^\gamma} + \bar{f}(1+4L_{\alpha\beta}M_{\alpha\beta h})\Big)^{-1},$$

$$\lambda^* := \frac{1}{h_*\tilde{f}(1)}\phi\Big(\frac{32K_{\alpha\beta}L^3_{\alpha\beta}}{(c^*-c)^2}\Big),$$

$$\lambda_\eta := \frac{2(8K_{\alpha\beta}L^2_{\alpha\beta})^\gamma}{h_*(c^*-c)^\gamma f(\eta)}\Phi_\alpha\Big(\frac{32\eta K_{\alpha\beta}L^3_{\alpha\beta}}{(c^*-c)^2}\Big),$$

$$\lambda_\theta := \frac{\Phi_\alpha(\theta)}{4^\gamma\Phi_\alpha(8L_{\alpha\beta}M_{\alpha\beta h})f(\theta)},$$

where $\phi(s) := \varphi_p(s) + \varphi_q(s)$, $\tilde{f}(s) := \inf_{r\in(s,\infty)} f(r)$ and $\bar{f} : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function such that $\bar{f}(0) = 0$, and $\bar{f}(s) = \max_{r\in[1,s]} f(r)$ for $s \geq 1$. Then we establish the following results.

Theorem 1.1. *Assume $(H_1)-(H_3)$, (H_{4a}) and (H_{5a}). Then (1.1) has no positive solution for $\lambda \gg 1$ and has two positive solutions u_1 and u_2 for $\lambda < \lambda_*$ such that $\|u_1\|_\infty \to 0$ and $\|u_2\|_\infty \to \infty$ as $\lambda \to 0$.*

Theorem 1.2. *Assume $(H_1)-(H_3)$, (H_{4a}) and (H_{5b}). Then (1.1) has a positive solution u for $\lambda > 0$ such that $\|u\|_\infty \to 0$ as $\lambda \to 0$ and $\|u\|_\infty \to \infty$ as $\lambda \to \infty$. If (H_6) and (H_7) are additionally satisfied with $\theta < \frac{8\eta K_{\alpha\beta}L^2_{\alpha\beta}}{c^*-c}$, then (1.1) has three positive solutions u_1, u_2 and u_3 for $\lambda \in (\lambda_\eta, \lambda_\theta)$ such that $\|u_1\|_\infty < \frac{\theta}{2} < \|u_2\|_\infty < \frac{4\eta K_{\alpha\beta}L^2_{\alpha\beta}}{c^*-c} < \|u_3\|_\infty$.*

Theorem 1.3. *Assume $(H_1)-(H_3)$, (H_{4b}) and (H_{5a}). Then (1.1) has a positive solution u for $\lambda > 0$ such that $\|u\|_\infty \to \infty$ as $\lambda \to 0$ and $\|u\|_\infty \to 0$ as $\lambda \to \infty$. If (H_6) and (H_7) are additionally satisfied with $\theta > \frac{8\eta K_{\alpha\beta}L^2_{\alpha\beta}}{c^*-c}$, then (1.1) has three positive solutions u_1, u_2 and u_3 for $\lambda \in (\lambda_\eta, \lambda_\theta)$ such that $\|u_1\|_\infty < \frac{4\eta K_{\alpha\beta}L^2_{\alpha\beta}}{c^*-c} < \|u_2\|_\infty < \frac{\theta}{2} < \|u_3\|_\infty$.*

Theorem 1.4. *Assume $(H_1)-(H_3)$, (H_{4b}) and (H_{5b}). Then (1.1) has no positive solution for $\lambda \approx 0$ and has two positive solutions u_1 and u_2 for $\lambda > \lambda^*$ such that $\|u_1\|_\infty \to 0$ and $\|u_2\|_\infty \to \infty$ as $\lambda \to \infty$.*

This paper is organized as follows. In Section 2, we discuss the existence and multiplicity of positive solutions to approximation problems with nonsingular nonlinear term and weight to derive estimates of positive solutions of (1.1). Section 3 presents preliminary results that will be utilized in subsequent sections. Sections 4 and 5 are dedicated to proving Lemmas 2.1 - 2.4 and showing that positive solutions of (1.1) can be obtained from a sequence of positive solutions to the approximation problems, respectively. Sections 6 and 7 contain the proofs and examples of Theorems 1.1 - 1.4, respectively. In the Appendix, we calculate useful estimates for integrals involving α, β, and h. Throughout this paper, we use the notations $\lambda \approx 0$ and $\lambda \gg 1$ for sufficiently small $\lambda > 0$ and sufficiently large $\lambda > 0$, respectively.

2. Positive Solutions to Approximation Problems

In this section, we introduce the existence and multiplicity results for approximation problems with nonsingular nonlinear term and weight.

For the case (H_{4a}), we consider the following approximation problem:

$$\begin{cases} -\big(\alpha(t)\varphi_p(v') + \beta(t)\varphi_q(v')\big)' = \lambda h_n(t) f_n(v),\ t \in (0,1), \\ v(0) = 0 = v(1), \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}$, $f_n(v(t)) := f(\frac{1}{n} + v(t))$ and

$$h_n(t) := \begin{cases} \inf_{r\in(t,\frac{1}{n+2})} h(r), & t \in (0, \frac{1}{n+2}), \\ h(t), & t \in [\frac{1}{n+2}, 1 - \frac{1}{n+2}], \\ \inf_{r\in(1-\frac{1}{n+2},t)} h(r), & t \in (1 - \frac{1}{n+2}, 1). \end{cases}$$

We note that there exists $n^* \in \mathbb{N}$ such that $h_n(t) = h(t)$ for $t \in (\frac{3c+c^*}{4}, \frac{c+3c^*}{4})$ and $n \geq n^*$. We also note that f_n satisfies (H_{4a}) since $f_n(0) = f(\frac{1}{n}) > 0$. By (H_{4a}), there exists a positive constant $r_\lambda < \min\{1, L_{\alpha\beta}M_{\alpha\beta h}\}$ such that

$$\inf_{s\in(0,r_\lambda)} \frac{\tilde{f}(s)}{\Phi_\beta(s)} > \frac{1}{\lambda h_*}\phi\Big(\frac{8L_{\alpha\beta}}{\bar{c}^* - \bar{c}}\Big). \tag{2.2}$$

If (H_{5a}) is assumed, then we can find a constant $R_\lambda > \max\big\{4L_{\alpha\beta}M_{\alpha\beta h}, \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}\big\}$ such that

$$\inf_{s\in(R_\lambda,\infty)} \frac{\tilde{f}\Big(\frac{s(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)}{\Phi_\alpha\Big(\frac{s(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)} > \frac{1}{\lambda h_*}\phi\Big(\frac{32K_{\alpha\beta}L_{\alpha\beta}^3}{(c^*-c)^2}\Big). \tag{2.3}$$

If (H_{5b}) is assumed, then we can find a constant $R_\lambda^* > 4L_{\alpha\beta}M_{\alpha\beta h}$ such that

$$\sup_{s\in(R_\lambda^*,\infty)} \Big(\frac{2^\gamma a}{s^\gamma \Phi_\alpha(s)} + \frac{\bar{f}(1+s)}{\Phi_\alpha(s)}\Big) < \frac{1}{\lambda \Phi_\alpha(4L_{\alpha\beta}M_{\alpha\beta h})}. \tag{2.4}$$

Let $n^{**} = \max\big\{n^*, \frac{2}{\theta}, \frac{c^*-c}{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}\big\}$. Following the arguments as in [8], we have the following two lemmas.

Lemma 2.1. *Assume $(H_1)-(H_3)$, (H_{4a}), (H_{5a}) and $n \geq n^*$. Then (2.1) has two positive solutions v_n^1 and v_n^2 for $\lambda < \lambda_*$ such that $r_\lambda < \|v_n^1\|_\infty < 4L_{\alpha\beta}M_{\alpha\beta h} < \|v_n^2\|_\infty < R_\lambda$.*

Lemma 2.2. *Assume $(H_1)-(H_3)$, (H_{4a}), (H_{5b}) and $n \geq n^*$ Then (2.1) has a positive solution v_n for $\lambda > 0$ such that $r_\lambda < \|v_n\|_\infty < R_\lambda^*$. If (H_6) and (H_7) are additionally satisfied with $\theta < \frac{8\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ and $n \geq n^{**}$, then (2.1) has three positive solutions v_n^1, v_n^2 and v_n^3 for $\lambda \in (\lambda_\eta, \lambda_\theta)$ such that $r_\lambda < \|v_n^1\|_\infty < \frac{\theta}{2} < \|v_n^2\|_\infty < \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \|v_n^3\|_\infty < R_\lambda^*$.*

We give the proofs of Lemmas 2.1 and 2.2 in Section 4.

For the case (H_{4b}), we consider the approximation problem:

$$\begin{cases} -\big(\alpha(t)\varphi_p(v') + \beta(t)\varphi_q(v')\big)' = \lambda h_n(t) f(v),\ t \in (0,1), \\ v(0) = 0 = v(1). \end{cases} \tag{2.5}$$

Since (H_{4b}) implies $\lim_{s\to 0} f(s) = 0$, we can define $\hat{f}(s) := \sup_{r\in(0,s)} f(r)$ and find a positive constant $\bar{r}_\lambda < L_{\alpha\beta}M_{\alpha\beta h}$ such that

$$\sup_{s\in(0,\bar{r}_\lambda)} \frac{\hat{f}(s)}{\Phi_\beta(s)} < \frac{1}{\lambda \Phi_\beta(4L_{\alpha\beta}M_{\alpha\beta h})}. \tag{2.6}$$

If (H_{5b}) is assumed, then we can find a constant $\overline{R}_\lambda > \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ such that

$$\sup_{s\in(\overline{R}_\lambda,\infty)} \frac{\hat{f}(s)}{\Phi_\alpha(s)} < \frac{1}{\lambda \Phi_\alpha(4L_{\alpha\beta}M_{\alpha\beta h})}. \tag{2.7}$$

Following the arguments as in [8], we obtain the following two lemmas.

Lemma 2.3. *Assume $(H_1)-(H_3)$, (H_{4b}), (H_{5a}) and $n \geq n^*$. Then (2.5) has a positive solution v_n for $\lambda > 0$ such that $\bar{r}_\lambda < \|v_n\|_\infty < R_\lambda$. If (H_6) and (H_7) are additionally satisfied with $\theta > \frac{8\eta K_{\alpha\beta}L^2_{\alpha\beta}}{c^*-c}$ and $n \geq n^{**}$, then (2.5) has three positive solutions v^1_n, v^2_n and v^3_n for $\lambda \in (\lambda_\eta, \lambda_\theta)$ such that $\bar{r}_\lambda < \|v^1_n\|_\infty < \frac{4\eta K_{\alpha\beta}L^2_{\alpha\beta}}{c^*-c} < \|v^2_n\|_\infty < \frac{\theta}{2} < \|v^3_n\|_\infty < R_\lambda$.*

Lemma 2.4. *Assume $(H_1)-(H_3)$, (H_{4b}), (H_{5b}) and $n \geq n^*$. Then (2.5) has two positive solutions v^1_n and v^2_n for $\lambda > \lambda^*$ such that $\bar{r}_\lambda < \|v^1_n\|_\infty < \frac{4K_{\alpha\beta}L^2_{\alpha\beta}}{c^*-c} < \|v^2_n\|_\infty < \overline{R}_\lambda$.*

We give the proofs of Lemmas 2.3 and 2.4 in Section 4.

3. Preliminaries

In this section, we list preliminary results to be used in the next sections.

First, we consider a function $x \in C[0,1]$ such that

$$x \text{ is continuously differentiable on } \Omega_+ := \{t \in [0,1] \mid \alpha(t)+\beta(t) > 0\}, \tag{3.1}$$

$$\lim_{t \to \tilde{t}} \big(\alpha(t)\varphi_p(x'(t)) + \beta(t)\varphi_q(x'(t))\big) \text{ exists for } \tilde{t} \in [0,1], \tag{3.2}$$

and

$$\begin{cases} -\big(\alpha(t)\varphi_p(x') + \beta(t)\varphi_q(x')\big)' \geq 0, \; t \in \Omega_\alpha \cup \Omega_\beta \cup \Omega_{\alpha\beta}, \\ x(0) \geq 0, \; x(1) \geq 0, \end{cases} \tag{3.3}$$

where $\Omega_{\alpha\beta} := \{t \in (0,1) \mid \alpha(t) > 0 \text{ and } \beta(t) > 0\}$. Then x satisfies the following properties.

Lemma 3.1. [8, Lemma 3.3] *Assume (H_1). If $x \in C[0,1]$ satisfies (3.1), (3.2), and (3.3), then*

$$x(t) \geq \frac{1}{L_{\alpha\beta}} \phi^{-1}\Big(\min\Big\{\varphi_p\Big(\frac{\|x\|_\infty}{K_{\alpha\beta}L_{\alpha\beta}}\Big), \varphi_q\Big(\frac{\|x\|_\infty}{K_{\alpha\beta}L_{\alpha\beta}}\Big)\Big\}\Big) t(1-t).$$

Lemma 3.2. [8, Lemma 3.4] *Assume (H_1). Let $x \in C[0,1]$ satisfy (3.1), (3.2), (3.3), and $\|x\|_\infty \geq K_{\alpha\beta}L_{\alpha\beta}$. Let $t_x \in (0,1)$ be such that $\|x\|_\infty = x(t_x)$. If $t_x \geq \frac{c+c^*}{2}$, then $x(t) \geq \frac{\|x\|_\infty}{K_{\alpha\beta}L^2_{\alpha\beta}}(t-c)$ for $t \in (c, \frac{c+c^*}{2})$. If $t_x < \frac{c+c^*}{2}$, then $x(t) \geq \frac{\|x\|_\infty}{K_{\alpha\beta}L^2_{\alpha\beta}}(c^*-t)$ for $t \in (\frac{c+c^*}{2}, c^*)$.*

Now we construct the solution operator for the following problem:

$$\begin{cases} -\big(\alpha(t)\varphi_p(x') + \beta(t)\varphi_q(x')\big)' = \lambda h_n(t) f^*_m(y), \; t \in \Omega_\alpha \cup \Omega_\beta \cup \Omega_{\alpha\beta}, \\ x(0) = 0 = x(1), \end{cases} \tag{3.4}$$

where $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $y \in C[0,1]$, and

$$f^*_m(y(t)) := \begin{cases} f(\frac{1}{m} + \max\{0, y(t)\}) & \text{for } m \in \mathbb{N}, \\ f(\max\{0, y(t)\}) & \text{for } m = 0. \end{cases}$$

For the case $m = 0$, we only consider f satisfying (H_{4b}) (so $f(0) := \lim_{s \to 0} f(s) = 0$ and $f^*_0(y(t))$ is well-defined). Define $T^{nm}_\lambda : C[0,1] \to C[0,1]$ by

$$T^{nm}_\lambda y(t) := \int_0^t \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\Big(m^{nm}_y + \lambda \int_s^1 h_n(r) f^*_m(y)dr\Big)\Big)ds,$$

where $m_y^{nm} \in \mathbb{R}$ is the constant such that

$$0 = \int_0^1 \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\left(\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\left(m_y^{nm} + \lambda\int_s^1 h_n(r)f_m^*(y)dr\right)\right)ds$$

with

$$\Phi(y(t)) := \begin{cases} \varphi_p(y(t)) & \text{for } t \in \Omega_\alpha, \\ \varphi_q(y(t)) & \text{for } t \in \Omega_\beta, \\ \phi(y(t)) & \text{for } t \in \Omega_{\alpha\beta}. \end{cases}$$

Then we can show that $T_\lambda^{nm} : C[0,1] \to C[0,1]$ is the solution operator of (3.4), and $T_\lambda^{nm}y(t) \geq 0$ for $t \in (0,1)$ by Lemma 3.1. Since $T_\lambda^{nm}y(t) \geq 0$ for $t \in (0,1)$ and $T_\lambda^{nm}y(0) = 0 = T_\lambda^{nm}y(1)$, there exists $t_y^{nm} \in (0,1)$ such that $\|T_\lambda^{nm}y\|_\infty = T_\lambda^{nm}y(t_y^{nm})$. Thus $T_\lambda^{nm}y$ can be written as

$$T_\lambda^{nm}y(t) = \begin{cases} \displaystyle\int_0^t \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\left(\lambda\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\int_s^{t_y^{nm}} h_n(r)f_m^*(y)dr\right)ds, & 0 \leq t \leq t_y^{nm}, \\ \displaystyle\int_t^1 \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\left(\lambda\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\int_{t_y^{nm}}^s h_n(r)f_m^*(y)dr\right)ds, & t_y^{nm} \leq t \leq 1. \end{cases}$$

Further, T_λ^{nm} satisfies the following properties.

Lemma 3.3. [8, Lemmas 2.1-2.2] *Assume* $(H_1) - (H_3)$. *Then* $T_\lambda^{nm} : C[0,1] \to C[0,1]$ *is completely continuous,* $T_\lambda^{nm}y$ *is continuously differentiable on* Ω_+*, and*

$$\lim_{t\to\tilde{t}}\left(\alpha(t)\varphi_p\left((T_\lambda^{nm}y)'(t)\right) + \beta(t)\varphi_q\left((T_\lambda^{nm}y)'(t)\right)\right) = \int_{\tilde{t}}^{t_y^{nm}} h_n(r)f_m^*(y)dr \text{ for } \tilde{t} \in [0,1].$$

Then we can show that $T_\lambda^{nm}y$ is nondecreasing on $[0, t_y^{nm}]$ and nonincreasing on $[t_y^{nm}, 1]$ by Lemma 3.3, and if T_λ^{nn} (or T_λ^{n0}) has a fixed point y, then y is a positive solution of (2.1) (or (2.5)) by Lemma 3.1.

4. Proofs of Lemmas 2.1 - 2.4

In this section, we provide proofs of Lemmas 2.1 - 2.4 (similar to but different from the proofs in [8]) to obtain the information needed to find positive solutions of (1.1). For this, we use the Krasnoselskii-type fixed point theorem.

Proposition 4.1. [11, Lemma A] *Let X be a Banach space, and let $I : X \to X$ be a completely continuous operator. Suppose that there exist a nonzero element $z \in X$ and positive constants r and R with $r \neq R$ such that*

(*a*) *if $y \in X$ satisfies $y = \sigma I y$ for $\sigma \in (0,1]$, then $\|y\|_X \neq r$,*
(*b*) *if $y \in X$ satisfies $y = Iy + \tau z$ for $\tau \geq 0$, then $\|y\|_X \neq R$.*

Then I has a fixed point $y \in X$ with $\min\{r,R\} < \|y\|_X < \max\{r,R\}$.

The proofs of Lemmas 2.1 - 2.4 are as follows.

Proof of Lemma 2.1. Let $\sigma \in (0,1]$ and $v_n \in C[0,1]$ be a nontrivial solution of $v_n = \sigma T_\lambda^{nn} v_n$. We show $\|v_n\|_\infty \neq 4L_{\alpha\beta}M_{\alpha\beta h}$ for $\lambda < \lambda_*$. Let $t_n \in (0,1)$ be such that $\|v_n\|_\infty = v_n(t_n)$. We first consider the case when $t_n \le \frac{1}{2}$. Let $t_n^* \in (0,t_n)$ be such that $\frac{\|v_n\|_\infty}{2} = v_n(t_n^*) < v_n(t) < v_n(t_n) = \|v_n\|_\infty$ for $t \in (t_n^*, t_n)$. Noting that $f(s) \le \frac{a}{s^\gamma} + \bar{f}(s)$ for $s > 0$ by (H_2), f_n^* satisfies

$$f_n^*(s) = f\Big(\frac{1}{n} + \max\{0,s\}\Big) \le \frac{a}{s^\gamma} + \bar{f}(1+s).$$

Thus we have

$$h_n(t) f_n^*(v_n(t)) \le h(t)\Big(\frac{a}{v_n(t)^\gamma} + \bar{f}(1+v_n(t))\Big) \le h(t)\Big(\frac{2^\gamma a}{\|v_n\|_\infty^\gamma} + \bar{f}(1+\|v_n\|_\infty)\Big)$$

for $t \in (t_n^*, t_n)$. Then we obtain

$$\begin{aligned}\frac{\|v_n\|_\infty}{2} &= \sigma \int_{t_n^*}^{t_n} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\int_s^{t_n} h_n(r) f_n^*(v_n)dr\Big)ds \\ &\le \int_0^{\frac{1}{2}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda\Big(\frac{2^\gamma a}{\|v_n\|_\infty^\gamma} + \bar{f}(1+\|v_n\|_\infty)\Big)\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)H(s)\Big)ds.\end{aligned}$$

By Proposition 8.2, we have

$$\frac{\|v_n\|_\infty}{4L_{\alpha\beta}M_{\alpha\beta h}} \le \max\Big\{\varphi_p^{-1}\Big(\lambda\Big(\frac{2^\gamma a}{\|v_n\|_\infty^\gamma} + \bar{f}(1+\|v_n\|_\infty)\Big)\Big), \varphi_q^{-1}\Big(\lambda\Big(\frac{2^\gamma a}{\|v_n\|_\infty^\gamma} + \bar{f}(1+\|v_n\|_\infty)\Big)\Big)\Big\}. \tag{4.1}$$

If $\|v_n\|_\infty = 4L_{\alpha\beta}M_{\alpha\beta h}$, then we have

$$1 \le \max\Big\{\varphi_p^{-1}\Big(\frac{\lambda}{\lambda_*}\Big), \varphi_q^{-1}\Big(\frac{\lambda}{\lambda_*}\Big)\Big\}. \tag{4.2}$$

By similar arguments, we can also show that if $t_n > \frac{1}{2}$ then v_n satisfies (4.2). However, this is a contradiction for $\lambda < \lambda_*$. Hence $\|v_n\|_\infty \ne 4L_{\alpha\beta}M_{\alpha\beta h}$ for $\lambda < \lambda_*$. Let $\tau \ge 0$ and $v_n \in C[0,1]$ be a nontrivial solution of $v_n = T_\lambda^{nn} v_n + \tau$. To apply Proposition 4.1, we find a positive constant greater than $\max\{\|v_n\|_\infty, 4L_{\alpha\beta}M_{\alpha\beta h}\}$ and a positive constant less than $\min\{\|v_n\|_\infty, 4L_{\alpha\beta}M_{\alpha\beta h}\}$. Let

$$\|v_n\|_\infty \ge \max\big\{4L_{\alpha\beta}M_{\alpha\beta h}, \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}\big\}.$$

If $t_n \ge \frac{c+c^*}{2}$, then $v_n(t) = T_\lambda^{nn} v_n(t) + \tau \ge 0$ for $t \in (0,1)$ and

$$v_n(t) \ge \frac{\|v_n\|_\infty}{K_{\alpha\beta}L_{\alpha\beta}^2}(t-c) \ge \frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}$$

for $t \in (\frac{3c+c^*}{4}, \frac{c+3c^*}{4})$ by Lemmas 3.1 - 3.2. We note that $\tilde{f}(s) = \inf_{r\in(s,\infty)} f(r)$ and $h_n(t) = h(t)$ for $t \in (\frac{3c+c^*}{4}, \frac{c+3c^*}{4})$ and $n \ge n^*$. Thus we have

$$\int_t^{t_n} h_n(r) f_n^*(v_n)dr \ge \int_{\frac{3c+c^*}{4}}^{\frac{c+c^*}{2}} h(r)\tilde{f}(v_n)dr \ge h_*\tilde{f}\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)$$

for $t \in (\frac{7c+c^*}{8}, \frac{3c+c^*}{4})$ and $n \geq n^*$. Then, by Proposition 8.1, v_n satisfies

$$\|v_n\|_\infty \geq \int_{\frac{7c+c^*}{8}}^{\frac{3c+c^*}{4}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\int_s^{t_n} h_n(r)f_n^*(v_n)dr\Big)ds$$
$$\geq \begin{cases} \frac{c^*-c}{8L_{\alpha\beta}}\phi^{-1}\Big(\lambda h_*\tilde{f}\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)\Big) & \text{for Case } A, \\ \frac{c^*-c}{8L_{\alpha\beta}}\varphi_p^{-1}\Big(\lambda h_*\tilde{f}\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)\Big) & \text{for Cases } B \text{ and } D, \\ \frac{c^*-c}{8L_{\alpha\beta}}\varphi_q^{-1}\Big(\lambda h_*\tilde{f}\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)\Big) & \text{for Case } C. \end{cases}$$

Since $\|v_n\|_\infty \geq \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{(c^*-c)}$, we have

$$\lambda h_*\tilde{f}\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big) \leq \begin{cases} \phi\Big(\frac{8L_{\alpha\beta}\|v_n\|_\infty}{c^*-c}\Big) & \text{for Case } A, \\ \varphi_p\Big(\frac{8L_{\alpha\beta}\|v_n\|_\infty}{c^*-c}\Big) & \text{for Cases } B \text{ and } D, \\ \varphi_q\Big(\frac{8L_{\alpha\beta}\|v_n\|_\infty}{c^*-c}\Big) & \text{for Case } C, \end{cases}$$
$$\leq \begin{cases} \phi\Big(\frac{32K_{\alpha\beta}L_{\alpha\beta}^3}{(c^*-c)^2}\Big)\varphi_q\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big) & \text{for Cases } A \text{ and } C, \\ \phi\Big(\frac{32K_{\alpha\beta}L_{\alpha\beta}^3}{(c^*-c)^2}\Big)\varphi_p\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big) & \text{for Cases } B \text{ and } D. \end{cases} \tag{4.3}$$

By similar arguments, we can show that if $t_n < \frac{c+c^*}{2}$, then v_n satisfies (4.3). Noting that

$$\Phi_\alpha(s) = \begin{cases} \varphi_q(s) & \text{for Cases } A \text{ and } C, \\ \varphi_p(s) & \text{for Cases } B \text{ and } D, \end{cases}$$

we obtain

$$\frac{\tilde{f}\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)}{\Phi_\alpha\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)} \leq \frac{1}{\lambda h_*}\phi\Big(\frac{32K_{\alpha\beta}L_{\alpha\beta}^3}{(c^*-c)^2}\Big). \tag{4.4}$$

Hence $\|v_n\|_\infty < R_\lambda$ for $\lambda > 0$ and $n \geq n^*$ by (2.3). Let $\|v_n\|_\infty \leq \min\{1, L_{\alpha\beta}M_{\alpha\beta h}\}$. Define

$$(\bar{c}, \bar{c}^*) := \begin{cases} (c, c^*) & \text{for Cases } A, B \text{ and } C, \\ (c_\beta, c_\beta^*) & \text{for Case } D. \end{cases}$$

Then $(\bar{c}, \bar{c}^*) \subset \Omega_\alpha$ for Case B and $(\bar{c}, \bar{c}^*) \subset \Omega_\beta$ for Cases C and D. Note that $v_n(t)$ is nondecreasing for $t \in (0, t_n)$ and $h_n(t) = h(t)$ for $t \in (\frac{3\bar{c}+\bar{c}^*}{4}, \frac{\bar{c}+3\bar{c}^*}{4})$ and $n \geq n^*$. If $t_n \geq \frac{\bar{c}+\bar{c}^*}{2}$, then

$$\int_t^{t_n} h_n(r)f_n^*(v_n)dr \geq \int_{\frac{3\bar{c}+\bar{c}^*}{4}}^{\frac{\bar{c}+\bar{c}^*}{2}} h(r)\tilde{f}(v_n)dr \geq h_*\tilde{f}\Big(v_n\Big(\frac{3\bar{c}+\bar{c}^*}{4}\Big)\Big)$$

for $t \in (\frac{7\bar{c}+\bar{c}^*}{8}, \frac{3\bar{c}+\bar{c}^*}{4})$ and $n \geq n^*$. Thus v_n satisfies

$$v_n\Big(\frac{3\bar{c}+\bar{c}^*}{4}\Big) \geq \int_{\frac{7\bar{c}+\bar{c}^*}{8}}^{\frac{3\bar{c}+\bar{c}^*}{4}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\int_s^{t_n} h_n(r)\tilde{f}(v_n)dr\Big)ds$$

$$\geq \begin{cases} \frac{\bar{c}^*-\bar{c}}{8L_{\alpha\beta}}\phi^{-1}\big(\lambda h_*\tilde{f}(v_n(\frac{3\bar{c}+\bar{c}^*}{4}))\big) & \text{for Case } A, \\ \frac{\bar{c}^*-\bar{c}}{8L_{\alpha\beta}}\varphi_p^{-1}\big(\lambda h_*\tilde{f}(v_n(\frac{3\bar{c}+\bar{c}^*}{4}))\big) & \text{for Case } B, \\ \frac{\bar{c}^*-\bar{c}}{8L_{\alpha\beta}}\varphi_q^{-1}\big(\lambda h_*\tilde{f}(v_n(\frac{3\bar{c}+\bar{c}^*}{4}))\big) & \text{for Cases } C \text{ and } D. \end{cases}$$

Since $v_n\left(\frac{3\bar{c}+\bar{c}^*}{4}\right) \leq 1$, we have

$$\lambda h_*\tilde{f}\Big(v_n\Big(\frac{3\bar{c}+\bar{c}^*}{4}\Big)\Big) \leq \begin{cases} \phi\big(\frac{8L_{\alpha\beta}}{\bar{c}^*-\bar{c}}\big)\varphi_p\big(v_n(\frac{3\bar{c}+\bar{c}^*}{4})\big) & \text{for Cases } A \text{ and } B, \\ \phi\big(\frac{8L_{\alpha\beta}}{\bar{c}^*-\bar{c}}\big)\varphi_q\big(v_n(\frac{3\bar{c}+\bar{c}^*}{4})\big) & \text{for Cases } C \text{ and } D. \end{cases}$$

Noting that

$$\Phi_\beta(s) = \begin{cases} \varphi_p(s) & \text{for Cases } A \text{ and } B, \\ \varphi_q(s) & \text{for Cases } C \text{ and } D, \end{cases}$$

we obtain

$$\frac{\tilde{f}(v_n(\frac{3\bar{c}+\bar{c}^*}{4}))}{\Phi_\beta(v_n(\frac{3\bar{c}+\bar{c}^*}{4}))} \leq \frac{1}{\lambda h_*}\phi\Big(\frac{8L_{\alpha\beta}}{\bar{c}^*-\bar{c}}\Big). \tag{4.5}$$

By similar arguments, we can show that if $t_n < \frac{\bar{c}+\bar{c}^*}{2}$,then v_n satisfies (4.5). Thus $v_n(\frac{3\bar{c}+\bar{c}^*}{4}) > r_\lambda$ by (2.2). Hence $\|v_n\|_\infty > r_\lambda$ for $\lambda > 0$ and $n \geq n^*$. By Proposition 4.1, T_λ^{nn} has two fixed points z_n^1 and z_n^2 for $\lambda < \lambda_*$ and $n \geq n^*$ such that $r_\lambda < \|z_n^1\|_\infty < 4L_{\alpha\beta}M_{\alpha\beta h} < \|z_n^2\|_\infty < R_\lambda$. Then z_n^1 and z_n^2 are positive solutions of (2.1) for $\lambda < \lambda_*$ and $n \geq n^*$ by Lemma 3.1.

Proof of Lemma 2.2. We first show the existence result for $\lambda > 0$. Let $\tau \geq 0$ and $v_n \in C[0,1]$ be a nontrivial solution of $v_n = T_\lambda^{nn}v_n + \tau$. Then we can show $\|v_n\|_\infty > r_\lambda$ for $\lambda > 0$ and $n \geq n^*$ following the arguments in the proof of Lemma 2.1.

Let $\sigma \in (0,1]$ and $v_n \in C[0,1]$ be a nontrivial solution of $v_n = \sigma T_\lambda^{nn}v_n$. We find a positive constant greater than $\max\{\|v_n\|_\infty, r_\lambda\}$. Let $\|v_n\|_\infty \geq 4L_{\alpha\beta}M_{\alpha\beta h}$. If $t_n \leq \frac{1}{2}$, we obtain from (4.1) that

$$\lambda\Big(\frac{2^\gamma a}{\|v_n\|_\infty^\gamma} + \bar{f}(1+\|v_n\|_\infty)\Big) \geq 1.$$

Thus

$$\frac{\|v_n\|_\infty}{4L_{\alpha\beta}M_{\alpha\beta h}} \leq \Phi_\alpha^{-1}\Big(\lambda\Big(\frac{2^\gamma a}{\|v_n\|_\infty^\gamma} + \bar{f}(1+\|v_n\|_\infty)\Big)\Big),$$

which implies

$$\frac{1}{\lambda\Phi_\alpha(4L_{\alpha\beta}M_{\alpha\beta h})} \leq \frac{2^\gamma a}{\|v_n\|_\infty^\gamma\Phi_\alpha(\|v_n\|_\infty)} + \frac{\bar{f}(1+\|v_n\|_\infty)}{\Phi_\alpha(\|v_n\|_\infty)}. \tag{4.6}$$

By similar arguments, we can also show that if $t_n > \frac{1}{2}$ then v_n satisfies (4.6). Hence $\|v_n\|_\infty < R_\lambda^*$ for $\lambda > 0$ by (2.4). By Proposition 4.1, T_λ^{nn} has a fixed point z_n for $\lambda > 0$ and $n \geq n^*$ such that $r_\lambda < \|z_n\|_\infty < R_\lambda^*$. By Lemma 3.1, one sees that z_n is a positive solution of (2.1) for $\lambda > 0$ and $n \geq n^*$.

Next we show the multiplicity result for $\lambda \in (\lambda_\eta, \lambda_\theta)$. Let $\sigma \in (0,1]$ and $v_n \in C[0,1]$ be a nontrivial solution of $v_n = \sigma T_\lambda^{nn} v_n$. Assume $\|v_n\|_\infty = \frac{\theta}{2}$ $(\geq 4L_{\alpha\beta}M_{\alpha\beta h})$. We first consider the case that $t_n \leq \frac{1}{2}$. Noting that $\frac{\|v_n\|_\infty}{2} < v_n(t) < \|v_n\|_\infty$ and $\frac{1}{n} + v_n(t) \leq 2\|v_n\|_\infty$ for $t \in (t_n^*, t_n)$ and $n \geq n^{**}$, we have

$$f_n^*(v_n(t)) = \frac{f_\gamma(\frac{1}{n} + v_n(t))}{(\frac{1}{n} + v_n(t))^\gamma} \leq \frac{2^\gamma f_\gamma(2\|v_n\|_\infty)}{\|v_n\|_\infty^\gamma} \leq 4^\gamma f(2\|v_n\|_\infty)$$

for $t \in (t_n^*, t_n)$ and $n \geq n^{**}$. Thus v_n satisfies

$$\begin{aligned} \frac{\|v_n\|_\infty}{2} &= \sigma \int_{t_n^*}^{t_n} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda \alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\int_s^{t_n} h_n(r) f_n^*(v_n)dr\Big)ds \\ &\leq \int_0^{\frac{1}{2}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda 4^\gamma f(2\|v_n\|_\infty)\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)H(s)\Big)ds \\ &\leq 2L_{\alpha\beta}M_{\alpha\beta h}\max\{\varphi_p^{-1}(\lambda 4^\gamma f(2\|v_n\|_\infty)), \varphi_q^{-1}(\lambda 4^\gamma f(2\|v_n\|_\infty))\}. \end{aligned} \tag{4.7}$$

Since $\|v_n\|_\infty \geq 4L_{\alpha\beta}M_{\alpha\beta h}$, we have $\lambda 4^\gamma f(2\|v_n\|_\infty) \geq 1$. Then we obtain

$$\|v_n\|_\infty \leq 4L_{\alpha\beta}M_{\alpha\beta h}\Phi_\alpha^{-1}(\lambda 4^\gamma f(2\|v_n\|_\infty)) \tag{4.8}$$

from (4.7) with Proposition 8.2. By similar arguments, we can also show that if $t_n > \frac{1}{2}$, then v_n satisfies (4.8). This implies $\lambda \geq \frac{\Phi_\alpha(\theta)}{4^\gamma \Phi_\alpha(8L_{\alpha\beta}M_{\alpha\beta h})f(\theta)} = \lambda_\theta$. However, this is a contradiction for $\lambda < \lambda_\theta$. Hence $\|v_n\|_\infty \neq \frac{\theta}{2}$ for $\lambda < \lambda_\theta$ and $n \geq n^{**}$.

Let $\tau \geq 0$ and $v_n \in C[0,1]$ be a nontrivial solution of $v_n = T_\lambda^{nn} v_n + \tau$. Assume $\|v_n\|_\infty = \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ $(\geq K_{\alpha\beta}L_{\alpha\beta})$. If $t_n \geq \frac{c+c^*}{2}$, then $v_n(t) \geq \frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}$ for $t \in (\frac{3c+c^*}{4}, \frac{c+c^*}{2})$ by Lemma 3.2. By (H_6), we have

$$f_n^*(v_n(t)) = \frac{f_\gamma(\frac{1}{n} + v_n(t))}{(\frac{1}{n} + v_n(t))^\gamma} \geq \frac{1}{(2\|v_n\|_\infty)^\gamma} f_\gamma\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big) \geq \frac{(c^*-c)^\gamma}{(8K_{\alpha\beta}L_{\alpha\beta}^2)^\gamma} f\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)$$

for $t \in (\frac{3c+c^*}{4}, \frac{c+c^*}{2})$ and $n \geq n^{**}$. Thus v_n satisfies

$$\begin{aligned} \|v_n\|_\infty &\geq \int_{\frac{7c+c^*}{8}}^{\frac{3c+c^*}{4}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda \alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\int_s^{t_n} h_n(r) f_n^*(v_n)dr\Big)ds \\ &\geq \int_{\frac{7c+c^*}{8}}^{\frac{3c+c^*}{4}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda \alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\int_{\frac{3c+c^*}{4}}^{\frac{c+c^*}{2}} h(r) f_n^*(v_n)dr\Big)ds \\ &\geq \begin{cases} \frac{c^*-c}{8L_{\alpha\beta}}\phi^{-1}\Big(\frac{\lambda h_*(c^*-c)^\gamma}{(8K_{\alpha\beta}L_{\alpha\beta}^2)^\gamma} f\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)\Big) & \text{for Case } A, \\ \frac{c^*-c}{8L_{\alpha\beta}}\varphi_p^{-1}\Big(\frac{\lambda h_*(c^*-c)^\gamma}{(8K_{\alpha\beta}L_{\alpha\beta}^2)^\gamma} f\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)\Big) & \text{for Cases } B \text{ and } D, \\ \frac{c^*-c}{8L_{\alpha\beta}}\varphi_q^{-1}\Big(\frac{\lambda h_*(c^*-c)^\gamma}{(8K_{\alpha\beta}L_{\alpha\beta}^2)^\gamma} f\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big)\Big) & \text{for Case } C, \end{cases} \end{aligned}$$

which implies

$$\frac{\lambda h_*(c^*-c)^\gamma}{(8K_{\alpha\beta}L_{\alpha\beta}^2)^\gamma} f\Big(\frac{\|v_n\|_\infty(c^*-c)}{4K_{\alpha\beta}L_{\alpha\beta}^2}\Big) \leq 2\Phi_\alpha\Big(\frac{8L_{\alpha\beta}\|v_n\|_\infty}{c^*-c}\Big). \tag{4.9}$$

By similar arguments, we can show that if $t_n < \frac{c+c^*}{2}$, then v_n satisfies (4.9). Thus we obtain

$$\lambda \le \frac{2(8K_{\alpha\beta}L_{\alpha\beta}^2)^\gamma}{h_*(c^*-c)^\gamma f(\eta)} \Phi_\alpha\Big(\frac{32\eta K_{\alpha\beta}L_{\alpha\beta}^3}{(c^*-c)^2}\Big) = \lambda_\eta.$$

However, this is a contradiction for $\lambda > \lambda_\eta$. Hence $\|v_n\|_\infty \neq \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ for $\lambda > \lambda_\eta$ and $n \ge n^{**}$.

Note that we can choose $r_\lambda \approx 0$ and $R_\lambda^* \gg 1$ such that $r_\lambda < \frac{\theta}{2} < \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < R_\lambda^*$. Then T_λ^{nn} has three fixed points z_n^1, z_n^2 and z_n^3 for $\lambda \in (\lambda_\eta, \lambda_\theta)$ and $n \ge n^{**}$ such that $r_\lambda < \|z_n^1\|_\infty < \frac{\theta}{2} < \|z_n^2\|_\infty < \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \|z_n^3\|_\infty < R_\lambda^*$ by Proposition 4.1. Hence z_n^1, z_n^2 and z_n^3 are positive solutions of (2.1) for $\lambda \in (\lambda_\eta, \lambda_\theta)$ and $n \ge n^{**}$.

Proof of Lemma 2.3. We first show the existence result for $\lambda > 0$. Let $\tau \ge 0$ and $v_n \in C[0,1]$ be a nontrivial solution of $v_n = T_\lambda^{n0} v_n + \tau$. Then we can show $\|v_n\|_\infty < R_\lambda$ for $\lambda > 0$ and $n \ge n^*$ following the arguments in the proof of Lemma 2.1.

Let $\sigma \in (0,1]$ and $v_n \in C[0,1]$ be a nontrivial solution of $v_n = \sigma T_\lambda^{n0} v_n$. If $t_n \le \frac{1}{2}$, then v_n satisfies

$$\begin{aligned}\frac{\|v_n\|_\infty}{2} &= \sigma \int_{t_n^*}^{t_n} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda \alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\int_s^{t_n} h_n(r) f_0^*(v_n)dr\Big)ds \\ &\le \int_0^{\frac{1}{2}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda \hat f(\|v_n\|_\infty)\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)H(s)\Big)ds.\end{aligned}$$

Then, by Proposition 8.2, we obtain

$$\frac{\|v_n\|_\infty}{2} \le \begin{cases} 2L_{\alpha\beta}M_{\alpha\beta h}\min\{\varphi_p^{-1}(\lambda \hat f(\|v_n\|_\infty)), \varphi_q^{-1}(\lambda \hat f(\|v_n\|_\infty))\} & \text{for Case } A, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_p^{-1}(\lambda \hat f(\|v_n\|_\infty)) & \text{for Case } B, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_q^{-1}(\lambda \hat f(\|v_n\|_\infty)) & \text{for Case } C, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\max\{\varphi_p^{-1}(\lambda \hat f(\|v_n\|_\infty)), \varphi_q^{-1}(\lambda \hat f(\|v_n\|_\infty))\} & \text{for Case } D. \end{cases} \tag{4.10}$$

By similar arguments, we can also show that if $t_n > \frac{1}{2}$ then v_n satisfies (4.10). Let $\|v_n\|_\infty \le 4L_{\alpha\beta}M_{\alpha\beta h}$. Then

$$\frac{\|v_n\|_\infty}{2} \le \begin{cases} 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_p^{-1}(\lambda \hat f(\|v_n\|_\infty)) & \text{for Cases } A \text{ and } B, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_q^{-1}(\lambda \hat f(\|v_n\|_\infty)) & \text{for Cases } C \text{ and } D. \end{cases}$$

It is easy to show the inequality for Cases A - C. For Case D, if $\lambda \hat f(\|v_n\|_\infty) \le 1$, then it is clear since $\max\{\varphi_p^{-1}(\lambda \hat f(\|v_n\|_\infty)), \varphi_q^{-1}(\lambda \hat f(\|v_n\|_\infty))\} = \varphi_q^{-1}(\lambda \hat f(\|v_n\|_\infty))$. If $\lambda \hat f(\|v_n\|_\infty) > 1$, then the inequality holds since

$$\frac{\|v_n\|_\infty}{2} \le 2L_{\alpha\beta}M_{\alpha\beta h} \le 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_q^{-1}(\lambda \hat f(\|v_n\|_\infty)).$$

Recalling that

$$\Phi_\beta(s) = \begin{cases} \varphi_p(s) & \text{for Cases } A \text{ and } B, \\ \varphi_q(s) & \text{for Cases } C \text{ and } D, \end{cases}$$

we have

$$\frac{1}{\lambda\Phi_\beta(4L_{\alpha\beta}M_{\alpha\beta h})} \leq \frac{\hat{f}(\|v_n\|_\infty)}{\Phi_\beta(\|v_n\|_\infty)}. \tag{4.11}$$

Hence $\|v_n\|_\infty \neq \bar{r}_\lambda$ for $\lambda > 0$ by (2.6). By Proposition 4.1 and Lemma 3.1, there exists a positive solution z_n of (2.5) for $\lambda > 0$ and $n \geq n^*$ such that $\bar{r}_\lambda < \|z_n\|_\infty < R_\lambda$.

Next we show the multiplicity result for $\lambda \in (\lambda_\eta, \lambda_\theta)$. Following the arguments in the proof of Lemma 2.2, we can show that if $v_n \in C[0,1]$ is a nontrivial solution of $v_n = \sigma T_\lambda^{n0} v_n$ then $\|v_n\|_\infty \neq \frac{\theta}{2}$ for $\lambda < \lambda_\theta$ and $n \geq n^{**}$, and if $v_n \in C[0,1]$ is a nontrivial solution of $v_n = T_\lambda^{n0} v_n + \tau$, then $\|v_n\|_\infty \neq \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ for $\lambda > \lambda_\eta$ and $n \geq n^{**}$. Note that we can choose $\bar{r}_\lambda \approx 0$ and $R_\lambda \gg 1$ such that $\bar{r}_\lambda < \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \frac{\theta}{2} < R_\lambda$. Then T_λ^{n0} has three fixed points z_n^1, z_n^2 and z_n^3 for $\lambda \in (\lambda_\eta, \lambda_\theta)$ and $n \geq n^{**}$ such that

$$\bar{r}_\lambda < \|z_n^1\|_\infty < \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \|z_n^2\|_\infty < \frac{\theta}{2} < \|z_n^3\|_\infty < R_\lambda.$$

By Lemma 3.1, z_n^1, z_n^2 and z_n^3 are positive solutions of (2.5) for $\lambda \in (\lambda_\eta, \lambda_\theta)$ and $n \geq n^{**}$.

Proof of Lemma 2.4. Let $\tau \geq 0$ and $v_n \in C[0,1]$ be a nontrivial solution of $v_n = T_\lambda^{n0} v_n + \tau$. If $\|v_n\|_\infty = \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$, then v_n satisfies (4.4) for $n \geq n^*$. Thus

$$\tilde{f}(1) \leq \frac{1}{\lambda h_*}\phi\Big(\frac{32K_{\alpha\beta}L_{\alpha\beta}^3}{(c^*-c)^2}\Big). \tag{4.12}$$

However, this is a contradiction for $\lambda > \lambda^*$. Hence $\|v_n\|_\infty \neq \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ for $\lambda > \lambda^*$ and $n \geq n^*$.

Let $\sigma \in (0,1]$ and $v_n \in C[0,1]$ be a nontrivial solution of $v_n = \sigma T_\lambda^{n0} v_n$. If $\|v_n\|_\infty \leq 4L_{\alpha\beta}M_{\alpha\beta h}$, then v_n satisfies (4.11). Thus $\|v_n\|_\infty \neq \bar{r}_\lambda$ for $\lambda > 0$. If $\|v_n\|_\infty > 4L_{\alpha\beta}M_{\alpha\beta h}$, then v_n satisfies (4.10) and $\lambda\hat{f}(\|v_n\|_\infty) \geq 1$. Thus

$$\|v_n\|_\infty \leq 4L_{\alpha\beta}M_{\alpha\beta h}\Phi_\alpha^{-1}(\lambda\hat{f}(\|v_n\|_\infty))$$

from (4.10) with Proposition 8.2. It follows that

$$\frac{1}{\lambda\Phi_\alpha(4L_{\alpha\beta}M_{\alpha\beta h})} \leq \frac{\hat{f}(\|v_n\|_\infty)}{\Phi_\alpha(\|v_n\|_\infty)}. \tag{4.13}$$

Hence $\|v_n\|_\infty < \overline{R}_\lambda$ for $\lambda > 0$ by (2.7).

By Proposition 4.1 and Lemma 3.1, there exist positive solutions z_n^1 and z_n^2 of (2.5) for $\lambda > \lambda^*$ and $n \geq n^*$ such that $\bar{r}_\lambda < \|z_n^1\|_\infty < \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \|z_n^2\|_\infty < \overline{R}_\lambda$.

5. Limit of Solutions to the Approximation Problems

In this section, we demonstrate that the positive solutions of (1.1) can be obtained from a sequence of positive solutions to approximation problems (2.1) or (2.5).

Lemma 5.1. *Assume* $(H_1)-(H_3)$. *For a given* $\lambda > 0$, *let* $\{v_n\}$ *be a sequence of positive solutions of* (2.1) *(or* (2.5)*) such that* $0 < \underline{v} < \inf\|v_n\|_\infty \leq \sup\|v_n\|_\infty < \overline{v} < \infty$. *Then there exists a* *positive solution v of (1.1) such that $\underline{v} \le \|v\|_\infty \le \bar{v}$ and $\lim_{k\to\infty} \|v_{n_k} - v\|_\infty = 0$, where $\{v_{n_k}\}$ is a subsequence of $\{v_n\}$.*

Proof. Let $\{v_n\}$ be a sequence of positive solutions of (2.1) (or (2.5)) such that

$$0 < \underline{v} < \inf \|v_n\|_\infty \le \sup \|v_n\|_\infty < \bar{v} < \infty.$$

It is clear that $\{v_n\}$ is uniformly bounded. Now we prove that $\{v_n\}$ is equicontinuous on $[0,1]$. Since $f(s) \le \frac{a}{s^\gamma} + \bar{f}(s)$ for $s > 0$ and $v_n(t)$ is nondecreasing for $t \in (0,t_n)$, we have

$$f_n(v_n(r)) \le \frac{a}{v_n(r)^\gamma} + \bar{f}(1+v_n(r)) = \frac{a + v_n(r)^\gamma \bar{f}(1+v_n(r))}{v_n(r)^\gamma} \le \frac{a + \bar{v}^\gamma \bar{f}(1+\bar{v})}{v_n(t)^\gamma}. \tag{5.1}$$

for $r \in (t,t_n)$. By [8, Proposition 3.1], one sees that v_n' satisfies

$$\begin{aligned} v_n'(t) &= \alpha_*^{\frac{1}{q-p}}(t)\beta_*^{-\frac{1}{q-p}}(t)\Phi^{-1}\Big(\lambda \alpha_*^{-\frac{q-1}{q-p}}(t)\beta_*^{\frac{p-1}{q-p}}(t)\int_t^{t_n} h_n(r) f_n(v_n) dr\Big) \\ &\le L_{\alpha\beta}\alpha_*^{-\frac{1}{p-1}}(t)\beta_*^{-\frac{1}{q-1}}(t)\Phi^{-1}\Big(\lambda \int_t^{t_n} h(r) f_n(v_n) dr\Big) \\ &\le L_{\alpha\beta}\alpha_*^{-\frac{1}{p-1}}(t)\beta_*^{-\frac{1}{q-1}}(t)\Phi^{-1}\Big(\frac{\lambda(a+\bar{v}^\gamma \bar{f}(1+\bar{v}))}{v_n(t)^\gamma}\int_t^{t_n} h(r) dr\Big) \\ &\le \frac{v_\lambda^* L_{\alpha\beta}\alpha_*^{-\frac{1}{p-1}}(t)\beta_*^{-\frac{1}{q-1}}(t)}{\min\{\varphi_p^{-1}(v_n(t)^\gamma), \varphi_q^{-1}(v_n(t)^\gamma)\}}\Phi^{-1}\Big(\int_t^{t_n} h(r) dr\Big) \end{aligned}$$

for $t \in (0,t_n)$, where $v_\lambda^* := \max\{\varphi_p^{-1}(\lambda(a+\bar{v}^\gamma \bar{f}(1+\bar{v}))), \varphi_q^{-1}(\lambda(a+\bar{v}^\gamma \bar{f}(1+\bar{v})))\}$. Let $t_I := \inf t_n$ and $t_S := \sup t_n$. Then

$$\begin{aligned} v_n'(t)\min\{\varphi_p^{-1}(v_n(t)^\gamma), \varphi_q^{-1}(v_n(t)^\gamma)\} &\le v_\lambda^* L_{\alpha\beta}\alpha_*^{-\frac{1}{p-1}}(t)\beta_*^{-\frac{1}{q-1}}(t)\Phi^{-1}\Big(\int_t^{t_n} h(r) dr\Big) \\ &\le v_\lambda^* L_{\alpha\beta}\alpha_*^{-\frac{1}{p-1}}(t)\beta_*^{-\frac{1}{q-1}}(t)\Phi^{-1}\Big(\int_t^{t_S} h(r) dr\Big) \end{aligned} \tag{5.2}$$

for $t \in (0,t_n)$. Further, we have

$$\begin{aligned} \int_0^{\underline{v}} \min\{\varphi_p^{-1}(k^\gamma), \varphi_q^{-1}(k^\gamma)\} dk &\le \int_0^{v_n(t_n)} \min\{\varphi_p^{-1}(k^\gamma), \varphi_q^{-1}(k^\gamma)\} dk \\ &\le \int_0^{t_n} v_n'(s)\min\{\varphi_p^{-1}(v_n(s)^\gamma), \varphi_q^{-1}(v_n(s)^\gamma)\} ds \\ &\le v_\lambda^* L_{\alpha\beta}\int_0^{t_n} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_s^{t_n} h(r) dr\Big) ds, \end{aligned}$$

which implies $t_I > 0$. By similar arguments, we can show $t_S < 1$ and

$$-v_n'(t)\min\{\varphi_p^{-1}(v_n(t)^\gamma), \varphi_q^{-1}(v_n(t)^\gamma)\} \le v_\lambda^* L_{\alpha\beta}\alpha_*^{-\frac{1}{p-1}}(t)\beta_*^{-\frac{1}{q-1}}(t)\Phi^{-1}\Big(\int_{t_I}^t h(r) dr\Big) \tag{5.3}$$

for $t \in (t_n, 1)$. Let $J(l) := \int_0^l \min\{\varphi_p^{-1}(k^\gamma), \varphi_q^{-1}(k^\gamma)\}dk$. By (5.2) and (5.3), if $0 \le a \le b \le t_n$, then

$$|J(v_n(b)) - J(v_n(a))| = \Big| \int_{v_n(a)}^{v_n(b)} \min\{\varphi_p^{-1}(k^\gamma), \varphi_q^{-1}(k^\gamma)\}dk \Big| \le v_\lambda^* L_{\alpha\beta} \int_a^b \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_s^{t_S} h(r)dr\Big)ds,$$

if $t_n \le a \le b \le 1$, then

$$|J(v_n(b)) - J(v_n(a))| = \Big| \int_{v_n(a)}^{v_n(b)} \min\{\varphi_p^{-1}(k^\gamma), \varphi_q^{-1}(k^\gamma)\}dk \Big| \le v_\lambda^* L_{\alpha\beta} \int_a^b \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_{t_I}^{s} h(r)dr\Big)ds,$$

and if $a < t_n < b$, then

$$\begin{aligned}
|J(v_n(b)) - J(v_n(a))| &\le |J(v_n(b)) - J(v_n(t_n))| + |J(v_n(t_n)) - J(v_n(a))| \\
&\le \int_{v_n(b)}^{v_n(t_n)} \min\{\varphi_p^{-1}(k^\gamma), \varphi_q^{-1}(k^\gamma)\}dk + \int_{v_n(a)}^{v_n(t_n)} \min\{\varphi_p^{-1}(k^\gamma), \varphi_q^{-1}(k^\gamma)\}dk \\
&\le v_\lambda^* L_{\alpha\beta} \int_{t_n}^b \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_{t_I}^{s} h(r)dr\Big)ds \\
&\quad + v_\lambda^* L_{\alpha\beta} \int_a^{t_n} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_s^{t_S} h(r)dr\Big)ds \\
&\le v_\lambda^* L_{\alpha\beta} \int_{\max\{a,t_I\}}^b \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_{t_I}^{s} h(r)dr\Big)ds \\
&\quad + v_\lambda^* L_{\alpha\beta} \int_a^{\min\{b,t_S\}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_s^{t_S} h(r)dr\Big)ds.
\end{aligned}$$

This implies that $\{J(v_n)\}$ is equicontinuous on $[0,1]$ by Proposition 8.3. Since $J(0) = 0$ and J is nondecreasing, $J^{-1}(l)$ is uniformly continuous on $[0, J(\overline{v})]$. Hence $\{v_n\} = \{J^{-1}(J(v_n))\}$ is equicontinuous on $[0,1]$. Then, by Arzela-Ascoli theorem, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ that converges uniformly to a function, say v, on $[0,1]$. Then $\underline{v} \le \|v\|_\infty \le \overline{v}$ since $\lim_{k\to\infty} \|v_{n_k} - v\|_\infty = 0$ and $\underline{v} < \inf \|v_n\|_\infty \le \sup \|v_n\|_\infty < \overline{v}$.

Let $\hat{t} \in (0,1)$ be such that $\alpha(\hat{t}) + \beta(\hat{t}) > 0$ (independent on n_k). Then v_{n_k} satisfies

$$v_{n_k}(t) = v_{n_k}(\hat{t}) - \int_t^{\hat{t}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\Big(m_{n_k} + \lambda \int_s^{\hat{t}} h_{n_k}(r)f_{n_k}(v_{n_k})dr\Big)\Big)ds$$

for $t \in [0, \hat{t}]$ and

$$v_{n_k}(t) = v_{n_k}(\hat{t}) + \int_{\hat{t}}^{t} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\Big(m_{n_k} - \lambda \int_{\hat{t}}^{s} h_{n_k}(r)f_{n_k}(v_{n_k})dr\Big)\Big)ds$$

for $t \in [\hat{t}, 1]$, where $m_{n_k} := \lim_{t\to\hat{t}} \big(\alpha(t)\varphi_p(v'_{n_k}(t)) + \beta(t)\varphi_q(v'_{n_k}(t))\big)$. We note from Lemma 3.1 and Lemma 3.3 that $v'_{n_k}(\hat{t})$ exists and there exists $\hat{v} > 0$ (independent on n_k) such that $v_{n_k}(t) \ge \hat{v}$

for $t \in [\min\{\hat{t}, t_I\}, \max\{\hat{t}, t_S\}]$. Then, from (5.1), we have

$$f_{n_k}(v_{n_k}(t)) \leq \frac{a+\bar{v}^\gamma \bar{f}(1+\bar{v})}{v_{n_k}(t)^\gamma} \leq \frac{a+\bar{v}^\gamma \bar{f}(1+\bar{v})}{\hat{v}^\gamma}$$

for $t \in [\min\{\hat{t}, t_I\}, \max\{\hat{t}, t_S\}]$. Thus we obtain

$$\begin{aligned}\left|\alpha(\hat{t})\varphi_p\left(v'_{n_k}(\hat{t})\right)+\beta(\hat{t})\varphi_q\left(v'_{n_k}(\hat{t})\right)\right| &= \left|\int_{\hat{t}}^{t_{n_k}} h_{n_k}(r) f_{n_k}(v_{n_k}) dr\right| \\ &\leq \frac{a+\bar{v}^\gamma \bar{f}(1+\bar{v})}{\hat{v}^\gamma}\int_{\min\{\hat{t},t_I\}}^{\max\{\hat{t},t_S\}} h(r)dr.\end{aligned}$$

This implies that $\{v'_{n_k}(\hat{t})\}$ is uniformly bounded. Hence $\{v'_{n_k}(\hat{t})\}$ has a convergent subsequence. Without loss of generality, we assume $\lim_{k\to\infty} v'_{n_k}(\hat{t}) = v_*$. Let $[t_*, t^*]$ be any subinterval of $(0,1)$ containing $\hat{t}$, t_I and t_S. Then we obtain $m = \lim_{k\to\infty} m_{n_k} = \alpha(\hat{t})\varphi_p(v_*) + \beta(\hat{t})\varphi_q(v_*)$ and

$$v(t) =$$
$$\begin{cases} v(\hat{t}) - \displaystyle\int_t^{\hat{t}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\left(\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\left(m+\lambda\int_s^{\hat{t}} h(r)f(v)dr\right)\right)ds, & t_* \leq t \leq \hat{t}, \\ v(\hat{t}) + \displaystyle\int_{\hat{t}}^{t} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\left(\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\left(m-\lambda\int_{\hat{t}}^{s} h(r)f(v)dr\right)\right)ds, & \hat{t} \leq t \leq t^*. \end{cases}$$

This implies $v_* = v'(\hat{t})$. Then $v(t)$ satisfies the equation in (1.1) almost everywhere for $t \in [t_*, t^*]$. Hence v satisfies the equation in (1.1) almost everywhere for $t \in (0,1)$. This implies v is a positive solution of (1.1). □

6. Proofs of Theorems 1.1 - 1.4

In this section, we prove Theorems 1.1 - 1.4. Approximation techniques are used to obtain the existence and multiplicity results.

Proof of Theorem 1.1. By Lemma 2.1, (2.1) has positive solutions z_n^1 and z_n^2 for $\lambda < \lambda_*$ and $n \geq n^*$ such that $r_\lambda < \|z_n^1\|_\infty < 4L_{\alpha\beta}M_{\alpha\beta h} < \|z_n^2\|_\infty < R_\lambda$. Then, by Lemma 5.1, there exist positive solutions z^1 and z^2 of (1.1) for $\lambda < \lambda_*$ such that $r_\lambda \leq \|z^1\|_\infty \leq 4L_{\alpha\beta}M_{\alpha\beta h} \leq \|z^2\|_\infty \leq R_\lambda$, $\lim_{k\to\infty} \|z_{n_k}^1 - z^1\|_\infty = 0$ and $\lim_{l\to\infty} \|z_{n_l}^2 - z^2\|_\infty = 0$, where $\{z_{n_k}^1\}$ and $\{z_{n_l}^2\}$ are subsequences of $\{z_n^1\}$ and $\{z_n^2\}$, respectively.

For each $i \in \{1,2\}$, if $\|z^i\|_\infty = 4L_{\alpha\beta}M_{\alpha\beta h}$, then z^i satisfies (4.1). Thus we obtain (4.2). However, this is a contradiction since $\lambda < \lambda_*$. Thus $\|z^i\|_\infty \neq 4L_{\alpha\beta}M_{\alpha\beta h}$ for $i \in \{1,2\}$. Hence z^1 and z^2 are positive solutions of (1.1) for $\lambda < \lambda_*$ such that $r_\lambda \leq \|z^1\|_\infty < 4L_{\alpha\beta}M_{\alpha\beta h} < \|z^2\|_\infty \leq R_\lambda$.

Further, we have

$$\begin{aligned}&\frac{\|z^1\|_\infty}{4L_{\alpha\beta}M_{\alpha\beta h}} \\ &\leq \max\left\{\varphi_p^{-1}\left(\lambda\left(\frac{2^\gamma a}{\|z^1\|_\infty^\gamma}+\bar{f}(1+4L_{\alpha\beta}M_{\alpha\beta h})\right)\right), \varphi_q^{-1}\left(\lambda\left(\frac{2^\gamma a}{\|z^1\|_\infty^\gamma}+\bar{f}(1+4L_{\alpha\beta}M_{\alpha\beta h})\right)\right)\right\}\end{aligned} \tag{6.1}$$

and

$$\frac{1}{\lambda \Phi_\alpha(4L_{\alpha\beta}M_{\alpha\beta h})} \le \frac{2^\gamma a}{\|z^2\|_\infty^\gamma \Phi_\alpha(\|z^2\|_\infty)} + \frac{\bar{f}(1+\|z^2\|_\infty)}{\Phi_\alpha(\|z^2\|_\infty)} \tag{6.2}$$

from (4.1) and (4.6). This implies $\|z^1\|_\infty \to 0$ and $\|z^2\|_\infty \to \infty$ as $\lambda \to 0$.

Next we show the nonexistence result for $\lambda \gg 1$. Assume to the contrary that (1.1) has a positive solution z for $\lambda \gg 1$. If $\|z\|_\infty \ge \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$, then z satisfies (4.4). Thus we have

$$\inf_{s\in(1,\infty)} \frac{\tilde{f}(s)}{\Phi_\alpha(s)} \le \frac{1}{\lambda h_*}\phi\Big(\frac{32K_{\alpha\beta}L_{\alpha\beta}^3}{(c^*-c)^2}\Big). \tag{6.3}$$

However, this is a contradiction for $\lambda \gg 1$ by (H_{5a}). Let $N_{\alpha\beta h} := \min\{1, L_{\alpha\beta}M_{\alpha\beta h}\}$. If $\|z\|_\infty \le N_{\alpha\beta h}$, then z satisfies (4.5). Thus we have

$$\inf_{s\in(0,1)} \frac{\tilde{f}(s)}{\Phi_\beta(s)} \le \frac{1}{\lambda h_*}\phi\Big(\frac{8L_{\alpha\beta}}{\bar{c}^*-\bar{c}}\Big). \tag{6.4}$$

However, this is a contradiction for $\lambda \gg 1$ by (H_{4a}). This implies $N_{\alpha\beta h} < \|z\|_\infty < \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ for $\lambda \gg 1$. Let $t_z \in (0,1)$ be such that $\|z\|_\infty = z(t_z)$. By Lemma 3.1, if $t_z \ge \frac{1}{2}$, then

$$\begin{aligned}\int_t^{t_z} h(r)f(z)dr &\ge \int_{\frac{1}{4}}^{\frac{1}{2}} h(r)\tilde{f}(z)dr \\ &\ge \bar{h}\tilde{f}\Big(\frac{1}{4L_{\alpha\beta}}\phi^{-1}\Big(\min\Big\{\varphi_p\Big(\frac{N_{\alpha\beta h}}{K_{\alpha\beta}L_{\alpha\beta}}\Big), \varphi_q\Big(\frac{N_{\alpha\beta h}}{K_{\alpha\beta}L_{\alpha\beta}}\Big)\Big\}\Big)\Big)\end{aligned}$$

for $t \in (0, \frac{1}{4})$, where $\bar{h} := \min\Big\{\int_{\frac{1}{4}}^{\frac{1}{2}} h(r)dr, \int_{\frac{1}{2}}^{\frac{3}{4}} h(r)dr\Big\}$. Thus z satisfies

$$\begin{aligned}\|z\|_\infty &\ge \int_0^{\frac{1}{4}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\lambda\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)\int_s^{t_z} h(r)f(z)dr\Big)ds \\ &\ge \frac{1}{4L_{\alpha\beta}}\phi^{-1}\Big(\lambda\bar{h}\tilde{f}\Big(\frac{1}{4L_{\alpha\beta}}\phi^{-1}\Big(\min\Big\{\varphi_p\Big(\frac{N_{\alpha\beta h}}{K_{\alpha\beta}L_{\alpha\beta}}\Big), \varphi_q\Big(\frac{N_{\alpha\beta h}}{K_{\alpha\beta}L_{\alpha\beta}}\Big)\Big\}\Big)\Big)\Big)\end{aligned}$$

by Proposition 8.1. By similar arguments, we can show that if $t_z < \frac{1}{2}$ then z satisfies

$$\|z\|_\infty \ge \frac{1}{4L_{\alpha\beta}}\phi^{-1}\Big(\lambda\bar{h}\tilde{f}\Big(\frac{1}{4L_{\alpha\beta}}\phi^{-1}\Big(\min\Big\{\varphi_p\Big(\frac{N_{\alpha\beta h}}{K_{\alpha\beta}L_{\alpha\beta}}\Big), \varphi_q\Big(\frac{N_{\alpha\beta h}}{K_{\alpha\beta}L_{\alpha\beta}}\Big)\Big\}\Big)\Big)\Big). \tag{6.5}$$

However, this is a contradiction for $\lambda \gg 1$ since $N_{\alpha\beta h} < \|z\|_\infty < \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$. Hence (1.1) has no positive solution for $\lambda \gg 1$.

Proof of Theorem 1.2. By Lemma 2.2, (2.1) has a positive solution z_n for $\lambda > 0$ and $n \ge n^*$ such that $r_\lambda < \|z_n\|_\infty < R_\lambda^*$. Then, by Lemma 5.1, there exists a positive solution z of (1.1) for $\lambda > 0$ such that $r_\lambda \le \|z\|_\infty \le R_\lambda^*$ and $\lim_{k\to\infty} \|z_{n_k} - z\|_\infty = 0$, where $\{z_{n_k}\}$ is a subsequence of $\{z_n\}$.

If $\|z\|_\infty \ge 4L_{\alpha\beta}M_{\alpha\beta h}$, then z satisfies (6.2). However, this is a contradiction for $\lambda \approx 0$ by (H_{5b}). Thus $\|z\|_\infty < 4L_{\alpha\beta}M_{\alpha\beta h}$ for $\lambda \approx 0$. Then z satisfies (6.1). This implies $\|z\|_\infty \to 0$ as $\lambda \to 0$.

If $\|z\|_\infty < \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$, then z satisfies either (6.4) or (6.5). However, this is a contradiction for $\lambda \gg 1$. Thus $\|z\|_\infty \geq \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ for $\lambda \gg 1$. Then z satisfies (4.4). This implies $\|z\|_\infty \to \infty$ as $\lambda \to \infty$ by (H_{5b}).

Next we show the multiplicity result for $\lambda \in (\lambda_\eta, \lambda_\theta)$. Let z_n^1, z_n^2 and z_n^3 be positive solutions of (2.1) for $\lambda \in (\lambda_\eta, \lambda_\theta)$ and $n \geq n^{**}$ such that $r_\lambda < \|z_n^1\|_\infty < \frac{\theta}{2} < \|z_n^2\|_\infty < \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \|z_n^3\|_\infty < R_\lambda^*$. Then we can show that (1.1) has positive solutions z^1, z^2 and z^3 for $\lambda \in (\lambda_\eta, \lambda_\theta)$ such that $r_\lambda \leq \|z^1\|_\infty \leq \frac{\theta}{2} \leq \|z^2\|_\infty \leq \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} \leq \|z^3\|_\infty \leq R_\lambda^*$ by Lemma 5.1.

If $\|z^1\|_\infty = \frac{\theta}{2}$, then z^1 satisfies (4.8). However, this is a contradiction for $\lambda < \lambda_\theta$. Thus $\|z^1\|_\infty \neq \frac{\theta}{2}$ for $\lambda < \lambda_\theta$. By similar arguments, we can show $\|z^2\|_\infty \neq \frac{\theta}{2}$ for $\lambda < \lambda_\theta$.

If $\|z^2\|_\infty = \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$, then z^2 satisfies (4.9). However, this is a contradiction for $\lambda > \lambda_\eta$. Thus $\|z^2\|_\infty \neq \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ for $\lambda > \lambda_\eta$. By similar arguments, we can show $\|z^3\|_\infty \neq \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ for $\lambda > \lambda_\eta$.

Hence z^1, z^2 and z^3 are positive solutions of (1.1) for $\lambda \in (\lambda_\eta, \lambda_\theta)$ such that $r_\lambda \leq \|z^1\|_\infty < \frac{\theta}{2} < \|z^2\|_\infty < \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \|z^3\|_\infty \leq R_\lambda^*$.

Proof of Theorem 1.3. By Lemma 2.3, (2.5) has a positive solution z_n for $\lambda > 0$ and $n \geq n^*$ such that $\bar{r}_\lambda < \|z_n\|_\infty < R_\lambda$. Then, by Lemma 5.1, there exists a positive solution z of (1.1) for $\lambda > 0$ such that $\bar{r}_\lambda \leq \|z\|_\infty \leq R_\lambda$ and $\lim_{k\to\infty} \|z_{n_k} - z\|_\infty = 0$, where $\{z_{n_k}\}$ is a subsequence of $\{z_n\}$.

If $\|z\|_\infty \leq 4L_{\alpha\beta}M_{\alpha\beta h}$, then z satisfies (4.11). Thus we have

$$\frac{1}{\lambda \Phi_\beta(4L_{\alpha\beta}M_{\alpha\beta h})} \leq \sup_{s\in(0,4L_{\alpha\beta}M_{\alpha\beta h})} \frac{\hat{f}(s)}{\Phi_\beta(s)}. \tag{6.6}$$

However, this is a contradiction for $\lambda \approx 0$ by (H_{4b}). Thus $\|z\|_\infty > 4L_{\alpha\beta}M_{\alpha\beta h}$ for $\lambda \approx 0$. Then z satisfies (4.13). Thus $\|z\|_\infty \to \infty$ as $\lambda \to 0$.

If $\|z\|_\infty \geq \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$, then z satisfies (4.4). Thus (6.3) is satisfied. However, this is a contradiction for $\lambda \gg 1$ by (H_{5a}). Therefore $\|z\|_\infty < \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ for $\lambda \gg 1$. Then we obtain

$$\begin{aligned}
\frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} &> \|z\|_\infty \\
&\geq \frac{1}{4L_{\alpha\beta}} \max\left\{\phi^{-1}\left(\lambda \bar{h}\tilde{f}\left(z\left(\frac{1}{4}\right)\right)\right), \phi^{-1}\left(\lambda \bar{h}\tilde{f}\left(z\left(\frac{3}{4}\right)\right)\right)\right\} \\
&\geq \frac{1}{4L_{\alpha\beta}} \phi^{-1}\left(\lambda \bar{h}\tilde{f}\left(\frac{1}{4L_{\alpha\beta}}\phi^{-1}\left(\min\left\{\varphi_p\left(\frac{\|z\|_\infty}{K_{\alpha\beta}L_{\alpha\beta}}\right), \varphi_q\left(\frac{\|z\|_\infty}{K_{\alpha\beta}L_{\alpha\beta}}\right)\right\}\right)\right)\right)
\end{aligned} \tag{6.7}$$

by similar arguments in (6.5). Hence $\|z\|_\infty \to 0$ as $\lambda \to \infty$.

Now we show the multiplicity result for $\lambda \in (\lambda_\eta, \lambda_\theta)$. Let z_n^1, z_n^2 and z_n^3 be positive solutions of (2.5) for $\lambda \in (\lambda_\eta, \lambda_\theta)$ and $n \geq n^{**}$ such that $\bar{r}_\lambda < \|z_n^1\|_\infty < \frac{\theta}{2} < \|z_n^2\|_\infty < \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \|z_n^3\|_\infty < R_\lambda$. Then we can show that (1.1) has three solutions z^1, z^2 and z^3 for $\lambda \in (\lambda_\eta, \lambda_\theta)$ such that $\bar{r}_\lambda \leq \|z^1\|_\infty < \frac{\theta}{2} < \|z^2\|_\infty < \frac{4\eta K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \|z^3\|_\infty \leq R_\lambda$ following the arguments in the proof of

Theorem 1.2.

Proof of Theorem 1.4. By Lemma 2.4, (2.5) has positive solutions z_n^1 and z_n^2 for $\lambda > \lambda^*$ and $n \geq n^*$ such that $\bar{r}_\lambda < \|z_n^1\|_\infty < \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \|z_n^2\|_\infty < \overline{R}_\lambda$. Then, by Lemma 5.1, there exist positive solutions z^1 and z^2 of (1.1) for $\lambda > \lambda^*$ such that $\bar{r}_\lambda \leq \|z^1\|_\infty \leq \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} \leq \|z^2\|_\infty \leq \overline{R}_\lambda$, $\lim_{k\to\infty} \|z_{n_k}^1 - z^1\|_\infty = 0$ and $\lim_{l\to\infty} \|z_{n_l}^2 - z^2\|_\infty = 0$, where $\{z_{n_k}^1\}$ and $\{z_{n_l}^2\}$ are subsequences of $\{z_n^1\}$ and $\{z_n^2\}$, respectively.

For each $i \in \{1,2\}$, if $\|z^i\|_\infty = \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$, then z^i satisfies (4.4). Thus we obtain (4.12). However, this is a contradiction since $\lambda > \lambda^*$. This implies $\|z^i\|_\infty \neq \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c}$ for $\lambda > \lambda^*$ and $i \in \{1,2\}$. Hence z^1 and z^2 are positive solutions of (1.1) for $\lambda > \lambda^*$ such that $r_\lambda \leq \|z^1\|_\infty < \frac{4K_{\alpha\beta}L_{\alpha\beta}^2}{c^*-c} < \|z^2\|_\infty \leq R_\lambda$.

Further, z^1 and z^2 satisfy (6.7) and (4.4), respectivley. Thus $\|z^1\|_\infty \to 0$ and $\|z^2\|_\infty \to \infty$ as $\lambda \to \infty$ by (H_{4b}) and (H_{5b}).

Next we show the nonexistence result for $\lambda \approx 0$. Assume to the contrary that (1.1) has a positive solution z for $\lambda \approx 0$. If $\|z\|_\infty \leq 4L_{\alpha\beta}M_{\alpha\beta h}$, then z satisfies (4.11). Thus z satisfies (6.6). However, this is a contradiction for $\lambda \approx 0$ by (H_{4b}). Thus $\|z\|_\infty > 4L_{\alpha\beta}M_{\alpha\beta h}$ for $\lambda \approx 0$. Then z satisfies (4.13). This implies

$$\frac{1}{\lambda \Phi_\alpha(4L_{\alpha\beta}M_{\alpha\beta h})} \leq \sup_{s\in(4L_{\alpha\beta}M_{\alpha\beta h},\infty)} \frac{\hat{f}(s)}{\Phi_\alpha(s)}.$$

However, this is also a contradiction for $\lambda \approx 0$ by (H_{5b}). Hence (1.1) has no positive solution for $\lambda \approx 0$.

7. Examples

In this section, we discuss examples of Theorems 1.1 - 1.4. We consider the double phase problem (1.1) with

$$\alpha(t) := \begin{cases} \left(\frac{1}{2} - t\right)^{c_1}, & t \in [0, \frac{1}{2}], \\ 0, & t \in [\frac{1}{2}, 1], \end{cases}$$

$$\beta(t) := \begin{cases} 0, & t \in [0, \frac{1}{2}], \\ \left(t - \frac{1}{2}\right)^{c_2}, & t \in [\frac{1}{2}, 1], \end{cases}$$

and

$$h(t) := \frac{1}{t^{c_3}(1-t)^{c_4}},\ t \in (0,1),$$

where c_1, c_2, c_3, and c_4 are constants such that $0 < c_1 < p-1$, $0 < c_2 < q-1$ and $0 \leq c_3, c_4 < p$. Then α, β and h satisfy (H_1) and (H_3). We note that if $c_3 \geq 1$ or $c_4 \geq 1$, then h is non-integrable on $(0,1)$.

1. Let

$$f(u) = \frac{1}{u^{\gamma_1}} + u^{\gamma_2},$$

where $\gamma_1 > 0$ and $\gamma_2 > p-1$. Then f satisfies (H_2), (H_{4a}), and (H_{5a}), and if $\gamma_1 \geq 1$, then f is strong singular. By Theorem 1.1, (1.1) has no positive solution for $\lambda \gg 1$ and has two positive solutions u_1 and u_2 for $\lambda < \lambda_*$ such that $\|u_1\|_\infty \to 0$ and $\|u_2\|_\infty \to \infty$ as $\lambda \to 0$.

2. Let
$$f(u) = \frac{1}{u^{\gamma_1}} + e^{\frac{\gamma_2 u}{\gamma_2+u}},$$
where $\gamma_1 > 0$ and $\gamma_2 > 0$. Then f satisfies (H_2), (H_{4a}), (H_{5b}), and (H_6), and if $\gamma_1 \geq 1$, then f is strong singular. By Theorem 1.2, (1.1) has a positive solution u for $\lambda > 0$ such that $\|u\|_\infty \to 0$ as $\lambda \to 0$ and $\|u\|_\infty \to \infty$ as $\lambda \to \infty$.

   If we choose $\eta = \gamma_2$ and $\theta = 8M_{\alpha\beta h}$ $(= 8L_{\alpha\beta}M_{\alpha\beta h})$, then (H_7) is satisfied for $\gamma_2 \gg 1$ since
$$\frac{f(\eta)}{\Phi_\alpha(\eta)} \Big/ \frac{f(\theta)}{\Phi_\alpha(\theta)} = \frac{\frac{1}{\gamma_2^{\gamma_1}} + e^{\frac{\gamma_2}{2}}}{\gamma_2^{p-1}} \Big/ \frac{\frac{1}{(8M_{\alpha\beta h})^{\gamma_1}} + e^{\frac{8M_{\alpha\beta h}\gamma_2}{\gamma_2+8M_{\alpha\beta h}}}}{(8M_{\alpha\beta h})^{p-1}} \gg 1$$
for $\gamma_2 \gg 1$. Hence, if $\gamma_2 \gg 1$, then (1.1) has three positive solutions for $\lambda \in (\lambda_\eta, \lambda_\theta)$.

3. Let
$$f(u) = A\min\{u^{\gamma_1}, u^{\gamma_3}\} + \min\{u^{\gamma_2}, u^{\gamma_3}\},$$
where $A > 0$ and $\gamma_1 < p-1 < \gamma_2 < q-1 < \gamma_3$. Then f satisfies (H_2), (H_{4b}), (H_{5a}), and (H_6). Therefore, by Theorem 1.3, (1.1) has a positive solution u for $\lambda > 0$ such that $\|u\|_\infty \to \infty$ as $\lambda \to 0$ and $\|u\|_\infty \to 0$ as $\lambda \to \infty$.

   If we choose $\eta = 1$ $(\geq \frac{c^*-c}{4L_{\alpha\beta}})$ and $\theta = A^\kappa$ with $0 < \kappa < \frac{1}{\gamma_2-p+1}$, then (H_7) is satisfied for $A \gg 1$ since
$$\frac{f(\eta)}{\Phi_\alpha(\eta)} \Big/ \frac{f(\theta)}{\Phi_\alpha(\theta)} = \frac{A+1}{1} \Big/ \frac{A^{1+\kappa\gamma_1} + A^{\kappa\gamma_2}}{A^{\kappa(p-1)}} = \frac{1+\frac{1}{A}}{A^{\kappa(\gamma_1-p+1)} + A^{\kappa(\gamma_2-p+1)-1}} \gg 1$$
for $A \gg 1$. Hence, if $A \gg 1$, then (1.1) has three positive solutions for $\lambda \in (\lambda_\eta, \lambda_\theta)$.

4. Let
$$f(u) = \min\{u^{\gamma_1}, u^{\gamma_2}\},$$
where $0 < \gamma_1 < p-1 < q-1 < \gamma_2$. Then f satisfies (H_2), (H_{4b}), and (H_{5b}). Therefore, by Theorem 1.4, (1.1) has no positive solution for $\lambda \approx 0$ and has two positive solutions u_1 and u_2 for $\lambda > \lambda^*$ such that $\|u_1\|_\infty \to 0$ and $\|u_2\|_\infty \to \infty$ as $\lambda \to \infty$.

8. Appendix

In this section, we provide useful lower and upper estimates for integrals involving α, β and h.

Proposition 8.1. [8, Proposition 3.2] *Assume* (H_1). *If* $k(t) \geq k_* > 0$ *for* $t \in (a,b) \subset (0,1)$, *then*
$$\int_a^b \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)k(s)\Big)ds \geq \begin{cases} \frac{\phi^{-1}(k_*)}{L_{\alpha\beta}}(b-a) & \text{for any } (a,b), \\ \frac{\varphi_p^{-1}(k_*)}{L_{\alpha\beta}}(b-a) & \text{for } (a,b) \subset \Omega_\alpha, \\ \frac{\varphi_q^{-1}(k_*)}{L_{\alpha\beta}}(b-a) & \text{for } (a,b) \subset \Omega_\beta. \end{cases}$$

Proposition 8.2. *Assume* (H_1) *and* (H_3)*. If* $C \geq 0$*, then*

$$\begin{aligned}\int_0^{\frac{1}{2}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(C\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)H(s)\Big)ds \\ \leq \begin{cases} 2L_{\alpha\beta}M_{\alpha\beta h}\min\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\} & \text{for Case } A, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_p^{-1}(C) & \text{for Case } B, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_q^{-1}(C) & \text{for Case } C, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\max\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\} & \text{for Case } D, \end{cases} \\ \leq 2L_{\alpha\beta}M_{\alpha\beta h}\max\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\}.\end{aligned}$$

Further, we have

$$\int_0^{\frac{1}{2}} \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(C\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)H(s)\Big)ds \leq \begin{cases} 2L_{\alpha\beta}M_{\alpha\beta h}\Phi_\alpha^{-1}(C) & \text{for } C \geq 1, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\Phi_\beta^{-1}(C) & \text{for } C \leq 1. \end{cases}$$

Similarly, if $C \geq 0$*, then*

$$\begin{aligned}\int_{\frac{1}{2}}^1 \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(C\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)H(s)\Big)ds \\ \leq \begin{cases} 2L_{\alpha\beta}M_{\alpha\beta h}\min\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\} & \text{for Case } A, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_p^{-1}(C) & \text{for Case } B, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_q^{-1}(C) & \text{for Case } C, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\max\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\} & \text{for Case } D, \end{cases} \\ \leq 2L_{\alpha\beta}M_{\alpha\beta h}\max\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\}.\end{aligned}$$

We also have

$$\int_{\frac{1}{2}}^1 \alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(C\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)H(s)\Big)ds \leq \begin{cases} 2L_{\alpha\beta}M_{\alpha\beta h}\Phi_\alpha^{-1}(C) & \text{for } C \geq 1, \\ 2L_{\alpha\beta}M_{\alpha\beta h}\Phi_\beta^{-1}(C) & \text{for } C \leq 1. \end{cases}$$

Proof. Let $C \geq 0$. If $(a,b) \subset \Omega_{\alpha\beta}$, then

$$\begin{aligned}\Phi^{-1}\Big(C\int_a^b h(r)dr\Big) &= \phi^{-1}\Big(C\int_a^b h(r)dr\Big) \\ &\leq \min\Big\{\varphi_p^{-1}\Big(C\int_a^b h(r)dr\Big),\varphi_q^{-1}\Big(C\int_a^b h(r)dr\Big)\Big\} \\ &\leq \min\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\}\Big(\varphi_p^{-1}\Big(\int_a^b h(r)dr\Big)+\varphi_q^{-1}\Big(\int_a^b h(r)dr\Big)\Big).\end{aligned}$$

If $(a,b) \subset \Omega_\alpha$, then

$$\Phi^{-1}\Big(C\int_a^b h(r)dr\Big) = \varphi_p^{-1}\Big(C\int_a^b h(r)dr\Big) = \varphi_p^{-1}(C)\varphi_p^{-1}\Big(\int_a^b h(r)dr\Big).$$

If $(a,b) \subset \Omega_\beta$, then

$$\Phi^{-1}\Big(C\int_a^b h(r)dr\Big) = \varphi_q^{-1}\Big(C\int_a^b h(r)dr\Big) = \varphi_q^{-1}(C)\varphi_q^{-1}\Big(\int_a^b h(r)dr\Big).$$

For $(a,b)\subset(0,1)$, we have

$$\Phi^{-1}\Big(C\int_a^b h(r)dr\Big)\le\varphi_p^{-1}\Big(C\int_a^b h(r)dr\Big)+\varphi_q^{-1}\Big(C\int_a^b h(r)dr\Big)$$
$$\le\max\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\}\Big(\varphi_p^{-1}\Big(\int_a^b h(r)dr\Big)+\varphi_q^{-1}\Big(\int_a^b h(r)dr\Big)\Big).$$

Then, by [8, Proposition 3.1], we obtain

$$\int_0^{\frac12}\alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(C\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)H(s)\Big)ds$$
$$\le L_{\alpha\beta}\int_0^{\frac12}\alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}(CH(s))\,ds$$
$$\le\begin{cases}2L_{\alpha\beta}M_{\alpha\beta h}\min\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\} & \text{for Case } A,\\ 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_p^{-1}(C) & \text{for Case } B,\\ 2L_{\alpha\beta}M_{\alpha\beta h}\varphi_q^{-1}(C) & \text{for Case } C,\\ 2L_{\alpha\beta}M_{\alpha\beta h}\max\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\} & \text{for Case } D,\end{cases}$$
$$\le 2L_{\alpha\beta}M_{\alpha\beta h}\max\{\varphi_p^{-1}(C),\varphi_q^{-1}(C)\}.$$

Noting that

$$\Phi_\alpha(s)=\begin{cases}\varphi_q(s) & \text{for Cases } A \text{ and } C,\\ \varphi_p(s) & \text{for Cases } B \text{ and } D,\end{cases}$$

and

$$\Phi_\beta(s)=\begin{cases}\varphi_p(s) & \text{for Cases } A \text{ and } B,\\ \varphi_q(s) & \text{for Cases } C \text{ and } D,\end{cases}$$

we have

$$\int_0^{\frac12}\alpha_*^{\frac{1}{q-p}}(s)\beta_*^{-\frac{1}{q-p}}(s)\Phi^{-1}\Big(C\alpha_*^{-\frac{q-1}{q-p}}(s)\beta_*^{\frac{p-1}{q-p}}(s)H(s)\Big)ds\le\begin{cases}2L_{\alpha\beta}M_{\alpha\beta h}\Phi_\alpha^{-1}(C) & \text{for } C\ge1,\\ 2L_{\alpha\beta}M_{\alpha\beta h}\Phi_\beta^{-1}(C) & \text{for } C\le1.\end{cases}$$

By similar arguments, we can show the remaining parts. □

Proposition 8.3. *Assume* (H_1) *and* (H_3). *Let* $c^*\in(0,1)$. *Then*

$$\int_0^{c^*}\alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_s^{c^*}h(r)dr\Big)ds<\infty \tag{8.1}$$

and

$$\int_{c^*}^1\alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_{c^*}^s h(r)dr\Big)ds<\infty. \tag{8.2}$$

Proof. We provide the proof of (8.1). A similar argument can be used to show (8.2).

If $c^* \le \frac{1}{2}$, then it is clear that (8.1) is satisfied by (H_3). Let $c^* > \frac{1}{2}$. If $\int_0^{\frac{1}{2}} h(r)dr \le \int_{\frac{1}{2}}^{c^*} h(r)dr$, then

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_s^{c^*} h(r)dr\Big)ds \\
&\qquad = \int_0^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_s^{\frac{1}{2}} h(r)dr + \int_{\frac{1}{2}}^{c^*} h(r)dr\Big)ds \\
&\qquad \le \int_0^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(2\int_{\frac{1}{2}}^{c^*} h(r)dr\Big)ds \\
&\qquad = \Phi^{-1}\Big(2\int_{\frac{1}{2}}^{c^*} h(r)dr\Big)\int_0^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)ds < \infty.
\end{aligned}$$

If $\int_{s_*}^{\frac{1}{2}} h(r)dr = \int_{\frac{1}{2}}^{c^*} h(r)dr$ for some $s_* \in (0, \frac{1}{2})$, then

$$\int_s^{c^*} h(r)dr = \int_s^{\frac{1}{2}} h(r)dr + \int_{\frac{1}{2}}^{c^*} h(r)dr \le 2\int_s^{\frac{1}{2}} h(r)dr = 2H(s)$$

for $s \in (0, s_*)$. Thus

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_s^{c^*} h(r)dr\Big)ds \\
&= \int_0^{s_*} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_s^{c^*} h(r)dr\Big)ds + \int_{s_*}^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_s^{c^*} h(r)dr\Big)ds \\
&\le \int_0^{s_*} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}(2H(s))\,ds + \int_{s_*}^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}\Big(\int_{s_*}^{c^*} h(r)dr\Big)ds \\
&\le \varphi_p^{-1}(2)\int_0^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)\Phi^{-1}(H(s))\,ds + \Phi^{-1}(2H(c^*))\int_0^{\frac{1}{2}} \alpha_*^{-\frac{1}{p-1}}(s)\beta_*^{-\frac{1}{q-1}}(s)ds \\
&< \infty.
\end{aligned}$$

□

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