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LARGE TIME BEHAVIOR OF SOLUTIONS TO THE HALL-MHD EQUATIONS WITH ION-SLIP EFFECTS AND HORIZONTAL DISSIPATION

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Abstract. This paper investigates the large time behavior of global solutions to the 3D Hall-MHD equations with ion-slip effects and horizontal dissipation. We establish the global well-posedness of solutions under the assumption of small Sobolev space initial values. The optimal time-decay rates of the global solution and its higher order derivatives are also obtained under additional assumptions on the negative-index Besov space initial value.

Keywords. Global well-posedness; Hall-MHD equations with ion-slip effects; Horizontal dissipation; Time-decay rates.

1. Introduction

This paper considers the initial value problem for the 3D Hall-magnetohydrodynamics (MHD) equations with ion-slip effects and horizontal dissipation

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta_h u = -\nabla p + b\cdot\nabla b, \\ \partial_t b + u\cdot\nabla b + \nabla\times((\nabla\times b)\times b) - \nabla\times(((\nabla\times b)\times b)\times b) - \nu\Delta b = b\cdot\nabla u, \\ \nabla\cdot u = 0, \nabla\cdot b = 0 \end{cases} \tag{1.1}$$

with the initial value

$$u(x,0) = u_0(x), b(x,0) = b_0(x). \tag{1.2}$$

Here $x\in\mathbb{R}^3$, $t>0$, $u=u(x,t)$, $b=b(x,t)$, and $p=p(x,t)$ represents the velocity field of the fluid, the magnetic field, and the pressure of fluid, respectively, the positive constants μ and ν are viscosity and resistivity coefficients, respectively, and $\Delta_h = \partial_1^2+\partial_2^2$ denotes the horizontal Laplacian.

Since Alfvén's [1] initial derivation, MHD equations have governed the dynamics of fluids that transport electric currents in the presence of magnetic fields. They play a crucial role in the study of numerous phenomena across geophysics, astrophysics, and engineering. The MHD equations attracted substantial interest from the mathematical community, owing to their more complex structure relative to the Navier-Stokes equations. This complexity arises from the fact that MHD equations constitute a coupled system, integrating the fluid dynamics governed by

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the Navier-Stokes equations with the electromagnetic field dynamics described by Maxwell's equations. For an electrically conducting fluid, the Hall effect and ion-slip currents have a pronounced impact on the flow dynamics under the influence of a strong magnetic field. Earlier, Elshehawey et al. [2] investigated the impact of Hall and ion-slip currents on MHD flow with variable thermal conductivity by using the Chebyshev finite difference method. Subsequently, Elgazery [3] discovered the role of Hall and ion-slip currents in the MHD flow of micro-polar fluids through porous media with variable viscosity and thermal diffusivity under the influence of chemical reactions. Recently, Krishna [4] studied the effects of Hall and ion-slip on the MHD flow of Casson hybrid nanofluids through an infinite exponentially accelerating vertical porous surface.

For the 3D Hall-MHD equations with ion-slip effects with full dissipation, namely $\Delta_h u$ replaced by Δu in (1.1), Zhao and Zhu [5] established the temporal decay estimates for the weak solutions and the algebraic time decay for higher-order Sobolev norms of small initial data solutions. Later, Zhao [6] proved the local well-posedness of strong solutions. Moreover, the existence of global smooth solutions for large data was proved by Zhang [7]. For more results on 3D Hall-MHD equations with ion-slip effects and full dissipation, we refer to [8, 9, 10].

The study on the theory of well-posedness of solutions to the Hall-MHD equations has grown enormously in recent years. We refer to [11, 12, 13, 14, 15, 16, 17, 18, 19, 20] for the viscous and resistive Hall-MHD equations. When the 3D Hall-MHD equations have only partial dissipation, due to the complex structure, the global well-posedness problem and large time behavior can be quite difficult. For (1.1) without ion-slip effects, Fei and Xiang [21] proved the global in time existence of classical solutions for small initial data in H^3. Later, Li [22] investigated the Cauchy problem for the 3D incompressible Hall-MHD equations under horizontal dissipation and demonstrated the global well-posedness of the system under the condition of axisymmetric initial data, where both the velocity vorticity and the magnetic vorticity components are vanishing. For the classical 3D MHD equations, well-posedness and stability of solutions has attracted the attention of many mathematicians and numerous interesting results were established; see, e.g., [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. We only recall closely related work with partial dissipation and anisotropic dissipation for our purpose. Pan, Zhu and Zhou [31] demonstrated solutions to MHD equations without magnetic diffusion by some time-weighted energy estimation. It is assumed that the initial magnetic field is sufficiently close to equilibrium and that the initial data exhibits some symmetry. Global well-posedness and stability of solutions to 3D MHD equations with horizontal dissipation and vertical magnetic diffusion near a background magnetic field were established by Wu and Zhu [33]. Lin, Wu and Zhu [36] investigated the 3D MHD equations with velocity dissipation in only one direction and magnetic diffusion in two directions. Global stability of solutions near a suitable background magnetic field was proved. For small initial data in H^1 space, Shang, Wu and Zhang [37] proved the global existence and stability of solutions to 3D MHD equations with only horizontal dissipation. Moreover, the large-time behavior of solutions was also obtained by using the MHD equations in an integral form, cancellations and other properties such as the incompressibility in order to control terms involving vertical derivatives.

Inspired by the recent results [38] for 3D incompressible Navier-Stokes equations and [39] and [40] for 3D incompressible MHD equations with horizontal dissipation, the main aim of this paper is to establish large time behavior and global well-posedness of solutions to problem

(1.1)-(1.2). Firstly, we prove the global well-posedness of solutions to problem (1.1)-(1.2) under the assumption of small initial values. For the details, we refer to Theorem 3.1. Secondly, we prove the time-decay rates of global solutions and their higher order derivatives when the initial value belongs to homogeneous negative Besov spaces and H^3, Theorem 4.1. Finally, if the initial value belongs to homogeneous negative Besov spaces and H^4, more decay estimates are established, Theorem 4.2. The paper is divided into four sections. Section 1 outlines the necessary background and provides a review of the current state of research on (1.1) and related models. Section 2 introduces the preliminary conditions. Section 3 presents global well-posedness of solutions to problem (1.1)-(1.2). Finally, Section 4 focuses on establishing large time behavior of global solutions to problem (1.1)-(1.2).

2. Preliminaries

This section presents various notations for functional spaces and several useful calculus inequalities that are commonly employed in this work.

2.1. **Functional spaces.** In this subsection, we recall the Littlewood-Paley operators and their elementary properties and the anisotropic version of the dyadic decomposition of the Fourier space, which allow us to define the Besov spaces. Let $\mathscr{S}$ be the usual Schwarz class and $\mathscr{S}'$ its dual, the space of tempered distributions. For $(j,k,l) \in \mathbb{Z}^3$, one defines

$$\Delta_j f = \mathscr{F}^{-1}(\varphi(2^{-j}|\xi|)\hat{f}) \text{ and}$$
$$\Delta_k^h f = \mathscr{F}^{-1}(\varphi(2^{-k}|\xi_h|)\hat{f}), \Delta_l^v f = \mathscr{F}^{-1}(\varphi(2^{-l}|\xi_3|)\hat{f}),$$

where $\xi = (\xi_1,\xi_2,\xi_3)$, $\xi_h = (\xi_1,\xi_2)$, $\widehat{f}$ denotes the Fourier transform of the tempered distribution $f \in \mathscr{S}'$ over $\mathbb{R}^3$, $\mathscr{F}^{-1}$ designates the inverse Fourier transform of f, and $\varphi(\tau)$ is a smooth function such that

$$\text{supp } \varphi \subset \left\{ \tau \in \mathbb{R} : \frac{3}{4} \le |\tau| \le \frac{8}{3} \right\}, \text{ and } \forall \tau > 0, \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\tau) = 1.$$

Definition 2.1. The homogeneous Besov spaces $\dot{B}^s_{p,q}$ and $\dot{B}^{s_1,s_2}_{p,q}$ with $s,s_1,s_2 \in \mathbb{R}$ and $p,q \in [1,\infty]$ consist of $f \in \mathscr{S}'/\mathscr{P}$ with $\mathscr{P}$ being the set of polynomials, satisfying

$$\|f\|_{\dot{B}^s_{p,q}} \equiv \|2^{js}\|\dot{\Delta}_j f\|_{L^p}\|_{l^q_j} < \infty,$$

$$\|f\|_{\dot{B}^{s_1,s_2}_{p,q}} \equiv \Big(\sum_{(k,l)\in\mathbb{Z}^2} 2^{s_1 qk} 2^{s_2 ql} \|\Delta_k^h \Delta_l^v f\|_{L^p}^q \Big)^{\frac{1}{q}} < \infty.$$

In particular,

$$\|f\|_{\dot{B}^s_{2,\infty}} \equiv \sup_{j\in\mathbb{Z}} \big(2^{sj}\|\Delta_j f\|_{L^2}\big) < \infty,$$

$$\|f\|_{\dot{B}^{s_1,0}_{2,\infty}} \equiv \sup_{k\in\mathbb{Z}} \big(2^{s_1 k}\|\Delta_k^h f\|_{L^2}\big) < \infty,$$

amd

$$\|f\|_{\dot{B}^{s_1,s_2}_{2,\infty}} \equiv \sup_{(k,l)\in\mathbb{Z}^2} (2^{s_1 k} 2^{s_2 l} \|\Delta_k^h \Delta_l^v f\|_{L^2}) < \infty.$$

The following lemma provides Bernstein inequalities for fractional derivatives. The upper bounds also hold when the fractional operators are replaced by partial derivatives.

Lemma 2.1. *Let $\alpha \geq 0$ and $1 \leq p \leq q \leq \infty$.*

(1) If f satisfies supp $\widehat{f} \subset \{\xi \in \mathbb{R}^3 : |\xi| \leq K2^j\}$, for some integer j and a constant $K > 0$, then

$$\left\|(-\Delta)^\alpha f\right\|_{L^q(\mathbb{R}^3)} \leq C_1 2^{2\alpha j + j3\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^3)}.$$

(2) If f satisfies supp $\widehat{f} \subset \{\xi \in \mathbb{R}^3 : \ K_1 2^j \leq |\xi| \leq K_2 2^j\}$ for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^3)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^3)} \leq C_2 2^{2\alpha j + j3\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^3)},$$

where C_1 and C_2 are constants depending on α, p and q.

2.2. **Calculus inequalities.** This subsection gives several useful calculus inequalities. The first fact provides an upper bound for the L^p-norm of a one-dimensional function, which serves as a basic ingredient for anisotropic upper bounds.

Lemma 2.2. *([41]) Let $2 \leq p \leq \infty$ and $s > \frac{1}{2} - \frac{1}{p}$. Then, there exists a constant $C = C(p,s)$ such that, for any 1D functions $f \in H^s(\mathbb{R})$,*

$$\|f\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)} \|\Lambda^s f\|_{L^2(\mathbb{R})}^{\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}.$$

In particular, if $p = \infty$ and $s = 1$, then any $f = f(x_3) \in H^1(\mathbb{R})$ satisfies

$$\|f\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \|\partial_3 f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}.$$

The second fact is an exact L^p-L^q decay estimate for the generalized heat operator associated with a fractional Laplacian.

Lemma 2.3. *([42]) Let $d \geq 1, \beta \geq 0, \alpha > 0, \nu > 0$, and $1 \leq p \leq q \leq \infty$. Then*

$$\|\Lambda^\beta e^{-\nu(-\Delta)^\alpha t} f\|_{L^q(\mathbb{R}^d)} \leq Ct^{-\frac{\beta}{2\alpha}-\frac{d}{2\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}.$$

The last fact is an anisotropic upper bound for the integral of a triple product. It is a powerful tool in investigating anisotropic equations.

Lemma 2.4. *([33]) The following estimates*

$$\int_{\mathbb{R}^3} |fgh|dx \leq C\|f\|_{L^2}^{\frac{1}{2}} \|\partial_1 f\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_3 h\|_{L^2}^{\frac{1}{2}},$$
$$\int_{\mathbb{R}^3} |fgh|dx \leq C\|f\|_{L^2}^{\frac{1}{4}} \|\partial_1 f\|_{L^2}^{\frac{1}{4}} \|\partial_2 f\|_{L^2}^{\frac{1}{4}} \|\partial_1\partial_2 f\|_{L^2}^{\frac{1}{4}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_3 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}$$

hold, provided that the right-hand sides are all bounded.

We conclude this section with some notations that are frequently used throughout the paper. To simplify the notations, we define $f_h = (f_1, f_2)$ and $\nabla_h = (\partial_1, \partial_2)$. We write $\int f dx$ for $\int_{\mathbb{R}^3} f(x)dx$, $\|f\|_{L^p}$ for $\|f\|_{L^p(\mathbb{R}^3)}$ and $\|f\|_{H^s}$ for $\|f\|_{H^s(\mathbb{R}^3)}$. We write $\|f\|_{L^p_{x_j}}$ with $j = 1,2,3$ for the L^p-norm with respect to x_j on $\mathbb{R}$, and $\|f\|_{L^p_{x_j x_k}}$ with $j,k = 1,2,3$ for the L^p-norm with respect to (x_j, x_k) on $\mathbb{R}^2$. We also separately write $\|f\|_{L^q_h}$ and $\|f\|_{L^q_h L^p_{x_3}}$ for the $\|f\|_{L^q_{x_1 x_2}}$ and $\| \|f\|_{L^q_{x_1 x_2}} \|_{L^p_{x_3}}$ to shorten the notations.

3. Global Well-Posedness

This section is devoted to proving global well-posedness of solutions to problem (1.1)-(1.2) in $H^k (k \geq 2)$, provided that the H^k norm of the initial value is small enough. More precisely, we have the following theorem.

Theorem 3.1. *Let $k \geq 2$ be an integer. Assume $(u_0, b_0) \in H^k$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then there exists a constant $\varepsilon > 0$ such that if $\|u_0\|_{H^k} + \|b_0\|_{H^k} \leq \varepsilon$, then problem (1.1)-(1.2) has a unique global solution (u,b) satisfying $(u,b) \in L^\infty(0,\infty;H^k)$, $\nabla_h u, \nabla b \in L^2(0,\infty;H^k)$ and for some constants $C > 0$ and any $t > 0$, it holds that*

$$\begin{aligned}
&\|u(t)\|_{H^k}^2 + \|b(t)\|_{H^k}^2 + \int_0^t (\|\nabla_h u(\tau)\|_{H^k}^2 + \|\nabla b(\tau)\|_{H^k}^2 + \\
&\|(\nabla \times b) \times b(\tau)\|_{L^2}^2 + \|\nabla^k (\nabla \times b) \times b(\tau)\|_{L^2}^2) d\tau \\
&\leq C(\|u_0\|_{H^k}^2 + \|b_0\|_{H^k}^2).
\end{aligned} \tag{3.1}$$

Proof. Since the local well-posedness of solutions to problem (1.1)-(1.2) in H^k follows from a standard approach such as Friedrichs' method, this proof focuses on the global a priori H^k-bounds. Taking the L^2-inner product of (u,b) with the first two equations of (1.1) and using $\nabla \cdot u = \nabla \cdot b = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2 + \|b\|_{L^2}^2) + \mu\|\nabla_h u\|_{L^2}^2 + \nu\|\nabla b\|_{L^2}^2 + \|(\nabla \times b) \times b\|_{L^2}^2 = 0. \tag{3.2}$$

Integrating (3.2) with respect to time yields

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \left(\mu \|\nabla_h u(\tau)\|_{L^2}^2 + \nu \|\nabla b(\tau)\|_{L^2}^2 + \|(\nabla \times b) \times b\|_{L^2}^2\right) d\tau \\
&= \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 .
\end{aligned}$$

Applying $\partial_i^2 (i = 1,2,3)$ to the first two equations of (1.1), dotting the results by $(\partial_i^2 u, \partial_i^2 b)$, respectively, integrating over $\mathbb{R}^3$, and adding them up, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3 \left(\|\partial_i^2 u\|_{L^2}^2 + \|\partial_i^2 b\|_{L^2}^2\right) + \mu \sum_{i=1}^3 \|\partial_i^2 \nabla_h u\|_{L^2}^2 + \nu \sum_{i=1}^3 \|\partial_i^2 \nabla b\|_{L^2}^2 \\
&= -\sum_{i=1}^3 \int \partial_i^2 (u \cdot \nabla u) \cdot \partial_i^2 u dx + \sum_{i=1}^3 \int \partial_i^2 (b \cdot \nabla b) \cdot \partial_i^2 u dx - \sum_{i=1}^3 \int \partial_i^2 (u \cdot \nabla b) \cdot \partial_i^2 b dx \\
&\quad + \sum_{i=1}^3 \int \partial_i^2 (b \cdot \nabla u) \cdot \partial_i^2 b dx - \sum_{i=1}^3 \int \partial_i^2 (\nabla \times ((\nabla \times b) \times b)) \cdot \partial_i^2 b dx \\
&\quad + \sum_{i=1}^3 \int \partial_i^2 (\nabla \times (((\nabla \times b) \times b) \times b)) \cdot \partial_i^2 b dx \\
&=: H_1 + H_2 + H_3 + H_4 + H_5 + H_6.
\end{aligned} \tag{3.3}$$

To bound H_1, using $\nabla\cdot u=0$, we write it into components

$$\begin{aligned}H_1&=-\int\partial_1^2u\cdot\nabla u\cdot\partial_1^2udx-2\int\partial_1u\cdot\nabla\partial_1u\cdot\partial_1^2udx-\int\partial_2^2u\cdot\nabla u\cdot\partial_2^2udx\\&\quad-2\int\partial_2u\cdot\nabla\partial_2u\cdot\partial_2^2udx-\int\partial_3^2u\cdot\nabla u\cdot\partial_3^2udx-2\int\partial_3u\cdot\nabla\partial_3u\cdot\partial_3^2udx\\&=:H_{11}+H_{12}+H_{13}+H_{14}+H_{15}+H_{16}.\end{aligned}$$

By Lemma 2.4, we obtain

$$\begin{aligned}H_{11}&\le C\|\partial_1^2u\|_{L^2}^{\frac12}\|\partial_3\partial_1^2u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_2\nabla u\|_{L^2}^{\frac12}\|\partial_1^2u\|_{L^2}^{\frac12}\|\partial_1\partial_1^2u\|_{L^2}^{\frac12}\\&\le C\|u\|_{H^2}\|\nabla_hu\|_{H^2}^2.\end{aligned}$$

Similarly, we have $H_{12}+H_{13}+H_{14}\le C\|u\|_{H^2}\|\nabla_hu\|_{H^2}^2$. We cannot estimate H_{15} directly. To bound it, we use $\nabla\cdot u=0$ and Lemma 2.4 to obtain

$$\begin{aligned}H_{15}&=-\int\partial_3^2u_h\cdot\nabla_hu\cdot\partial_3^2udx+\int\partial_3\nabla_h\cdot u_h\cdot\partial_3u\cdot\partial_3^2udx\\&\le C\|u\|_{H^2}\|\nabla_hu\|_{H^2}^2.\end{aligned}$$

Similarly, we have $H_{16}\le C\|u\|_{H^2}\|\nabla_hu\|_{H^2}^2$. Combining the above bounds up, we obtain $H_1\le C\|u\|_{H^2}\|\nabla_hu\|_{H^2}^2$. Similarly, we have $H_2+H_4\le C\|u\|_{H^2}\|\nabla b\|_{H^2}^2$ and

$$H_3\le C(\|u\|_{H^2}+\|b\|_{H^2})(\|\nabla_hu\|_{H^2}^2+\|\nabla b\|_{H^2}^2).$$

Using Hölder's inequality and commutator estimate yields

$$\begin{aligned}H_5&=-\sum_{i=1}^3\int\partial_i^2(\nabla\cdot(b\otimes b))\cdot\partial_i^2(\nabla\times b)dx\\&\le C\sum_{i=1}^3\|b\|_{L^\infty}\|\partial_i^2\nabla b\|_{L^2}\|\partial_i^2(\nabla\times b)\|_{L^2}\\&\le C\|b\|_{H^2}\|\nabla b\|_{H^2}^2.\end{aligned}$$

To bound H_6, we write it into components

$$\begin{aligned}&H_6=\\&\sum_{i=1}^3\int((\partial_i^2(\nabla\times b)\times b)\times b)\cdot\partial_i^2(\nabla\times b)dx+2\sum_{i=1}^3\int((\partial_i(\nabla\times b)\times\partial_ib)\times b)\cdot\partial_i^2(\nabla\times b)dx\\&\quad+2\sum_{i=1}^3\int((\partial_i(\nabla\times b)\times b)\times\partial_ib)\cdot\partial_i^2(\nabla\times b)dx\\&\quad+2\sum_{i=1}^3\int(((\nabla\times b)\times\partial_ib)\times\partial_ib)\cdot\partial_i^2(\nabla\times b)dx\\&\quad+\sum_{i=1}^3\int(((\nabla\times b)\times\partial_i^2b)\times b)\cdot\partial_i^2(\nabla\times b)dx+\sum_{i=1}^3\int(((\nabla\times b)\times b)\times\partial_i^2b)\cdot\partial_i^2(\nabla\times b)dx\\&=:H_{61}+H_{62}+H_{63}+H_{64}+H_{65}+H_{66}.\end{aligned}$$

Using Hölder's inequality, we obtain

$$H_{61}=-\sum_{i=1}^{3}\int\partial_i^2(\nabla\times b)\times b\cdot\partial_i^2(\nabla\times b)\times bdx=-\|\nabla^2(\nabla\times b)\times b\|_{L^2}^2$$

and

$$H_{62}\le C\|\nabla^2 b\|_{L^2}\|\nabla b\|_{L^\infty}\|b\|_{L^\infty}\|\nabla^3 b\|_{L^2}\le C\|b\|_{H^2}^2\|\nabla b\|_{H^2}^2.$$

As in the estimate of H_{62}, $H_{63}+H_{65}+H_{66}\le C\|b\|_{H^2}^2\|\nabla b\|_{H^2}^2$. By Hölder's inequality and Sobolev imbedding theorem, we obtain

$$H_{64}\le C\|\nabla b\|_{L^\infty}\|\nabla b\|_{L^4}^2\|\nabla^3 b\|_{L^2}\le C\|b\|_{H^2}^2\|\nabla b\|_{H^2}^2.$$

Incorporating the above estimates yields $H_6\le-\|\nabla^2(\nabla\times b)\times b\|_{L^2}^2+C\|b\|_{H^2}^2\|\nabla b\|_{H^2}^2$. Substituting the bounds of H_i $(i=1,\cdots,6)$ into (3.3) and adding the result to (3.2), one infers that

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{H^2}^2+\|b\|_{H^2}^2)+\mu\|\nabla_h u\|_{H^2}^2+\nu\|\nabla b\|_{H^2}^2+\|(\nabla\times b)\times b\|_{L^2}^2+\|\nabla^2(\nabla\times b)\times b\|_{L^2}^2\\&\le C(\|u\|_{H^2}+\|b\|_{H^2}+\|b\|_{H^2}^2)(\|\nabla_h u\|_{H^2}^2+\|\nabla b\|_{H^2}^2),\end{aligned}$$

which together with the bootstrap argument immediately yields (3.1) with $k=2$.

Now we turn to (3.1) with $k\ge 3$. Applying $\partial_i^k(i=1,2,3)$ to the first two equations of (1.1) and taking the L^2-inner products with $(\partial_i^k u,\partial_i^k b)$, respectively, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\left(\|\partial_i^k u\|_{L^2}^2+\|\partial_i^k b\|_{L^2}^2\right)+\mu\sum_{i=1}^{3}\|\partial_i^k\nabla_h u\|_{L^2}^2+\nu\sum_{i=1}^{3}\|\partial_i^k\nabla b\|_{L^2}^2\\&=-\sum_{i=1}^{3}\int\partial_i^k(u\cdot\nabla u)\cdot\partial_i^k udx+\sum_{i=1}^{3}\int\partial_i^k(b\cdot\nabla b)\cdot\partial_i^k udx-\sum_{i=1}^{3}\int\partial_i^k(u\cdot\nabla b)\cdot\partial_i^k bdx\\&\quad+\sum_{i=1}^{3}\int\partial_i^k(b\cdot\nabla u)\cdot\partial_i^k bdx-\sum_{i=1}^{3}\int\partial_i^k(\nabla\times((\nabla\times b)\times b))\cdot\partial_i^k bdx\\&\quad+\sum_{i=1}^{3}\int\partial_i^k(\nabla\times(((\nabla\times b)\times b)\times b))\cdot\partial_i^k bdx\\&=:I_1+I_2+I_3+I_4+I_5+I_6.\end{aligned}\tag{3.4}$$

Set $C_k^j=\frac{k!}{j!(k-j)!}$. By the divergence free condition $\nabla\cdot u=0$, we have

$$\begin{aligned}I_1&=-\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j u_h\cdot\nabla_h\partial_i^{k-j}u\cdot\partial_i^k udx-\sum_{i=1}^{3}\sum_{j=1}^{k}C_k^j\int\partial_i^j u_3\partial_3\partial_i^{k-j}u\cdot\partial_i^k udx\\&=:I_{11}+I_{12}.\end{aligned}$$

Using Lemma 2.4 and the Young inequality yields

$$\begin{aligned}I_{11}&\le C\sum_{i=1}^{3}\sum_{j=1}^{k}\|\partial_i^j u_h\|_{L^2}^{\frac12}\|\partial_1\partial_i^j u_h\|_{L^2}^{\frac12}\|\nabla_h\partial_i^{k-j}u\|_{L^2}^{\frac12}\|\partial_3\nabla_h\partial_i^{k-j}u\|_{L^2}^{\frac12}\|\partial_i^k u\|_{L^2}^{\frac12}\|\partial_2\partial_i^k u\|_{L^2}^{\frac12}\\&\le C\|u\|_{H^k}\|\nabla_h u\|_{H^{k-1}}\|\nabla_h u\|_{H^k}\\&\le\frac{\mu}{16}\|\nabla_h u\|_{H^k}^2+C\|u\|_{H^k}^2\|\nabla_h u\|_{H^{k-1}}^2\end{aligned}$$

and $I_{12} \leq \frac{\mu}{16}\|\nabla_h u\|_{H^k}^2 + C\|u\|_{H^k}^2\|\nabla_h u\|_{H^{k-1}}^2$. Thus we obtain

$$I_1 \leq \frac{\mu}{8}\|\nabla_h u\|_{H^k}^2 + C\|u\|_{H^k}^2\|\nabla_h u\|_{H^{k-1}}^2.$$

Similarly, we have

$$\begin{aligned}
I_2 + I_4 &\leq \frac{\mu}{16}\|\nabla_h u\|_{H^k}^2 + \frac{\nu}{16}\|\nabla b\|_{H^k}^2 + C(\|u\|_{H^k}^2 + \|b\|_{H^k}^2)(\|\nabla_h u\|_{H^{k-1}}^2 + \|\nabla b\|_{H^{k-1}}^2),\\
I_3 &\leq \frac{\mu}{16}\|\nabla_h u\|_{H^k}^2 + \frac{\nu}{16}\|\nabla b\|_{H^k}^2 + C(\|u\|_{H^k}^2 + \|b\|_{H^k}^2)(\|\nabla_h u\|_{H^{k-1}}^2 + \|\nabla_h b\|_{H^{k-1}}^2).
\end{aligned}$$

Using Hölder's inequality, commutator estimate, and the Young inequality, we infers that

$$\begin{aligned}
I_5 &= -\sum_{i=1}^{3}\int (\partial_i^k((\nabla\times b)\times b) - \partial_i^k(\nabla\times b)\times b)\cdot\partial_i^k(\nabla\times b)dx\\
&\leq C\sum_{i=1}^{3}\|\nabla b\|_{L^\infty}\|\partial_i^{k-1}\nabla b\|_{L^2}\|\partial_i^k(\nabla\times b)\|_{L^2}\\
&\leq \frac{\nu}{16}\|\nabla b\|_{H^k}^2 + C\|b\|_{H^k}^2\|\nabla b\|_{H^{k-1}}^2.
\end{aligned}$$

By the standard calculus inequality, we obtain

$$\begin{aligned}
I_6 &= \sum_{i=1}^{3}\int\Big[\partial_i^k(((\nabla\times b)\times b)\times b) - ((\partial_i^k(\nabla\times b)\times b)\times b)\Big]\cdot\partial_i^k(\nabla\times b)dx\\
&\quad + \sum_{i=1}^{3}\int((\partial_i^k(\nabla\times b)\times b)\times b)\cdot\partial_i^k(\nabla\times b)dx\\
&\leq C(\|\nabla b\|_{H_{k-1}}\|b\|_{L^\infty}\|\nabla b\|_{L^\infty} + \|\nabla b\|_{L^\infty}\|b\|_{L^\infty}\|b\|_{H^k})\|\nabla b\|_{H^k} - \|\nabla^k(\nabla\times b)\times b\|_{L^2}^2\\
&\leq \frac{\nu}{16}\|\nabla b\|_{H^k}^2 + C\|b\|_{H^k}^4\|\nabla b\|_{H_{k-1}}^2 - \|\nabla^k(\nabla\times b)\times b\|_{L^2}^2.
\end{aligned}$$

Inserting the above bounds into (3.4), and adding the result to (3.2), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|u\|_{H^k}^2 + \|b\|_{H^k}^2) + min\{\mu,\nu\}(\|\nabla_h u\|_{H^k}^2 + \|\nabla b\|_{H^k}^2) + \sum_{j=0,k}\|\nabla^j(\nabla\times b)\times b\|_{L^2}^2\\
&\leq C(\|u\|_{H^k}^2 + \|b\|_{H^k}^2 + \|b\|_{H^k}^4)(\|\nabla_h u\|_{H^{k-1}}^2 + \|\nabla b\|_{H^{k-1}}^2),
\end{aligned}$$

which together with the bootstrap argument yields (3.1) with $k \geq 3$. This completes the proof of Theorem 3.1. □

4. Time-Decay Rate of Global Solutions

In this section, our main aim is to establish time-decay rate of global solutions obtained in Theorem 3.1. Meanwhile, time-decay rates of some spatial partial derivatives of global solutions are also obtained.

4.1. **Time-decay rate.** The main goal of this subsection is to establish time-decay rate of global solutions to the problem (1.1)-(1.2) with H^3 initial value. The result is stated as follows.

Theorem 4.1. *Assume that $(u_0,b_0)\in H^3(\mathbb{R}^3)$. Then there exists $\varepsilon>0$ such that if*

$$\|(u_0,\partial_3u_0)\|_{\dot{B}^{-1,0}_{2,\infty}(\mathbb{R}^3)}+\|b_0\|_{\dot{B}^{-1}_{2,\infty}(\mathbb{R}^3)}+\|(u_0,b_0)\|_{H^3(\mathbb{R}^3)}\le\varepsilon, \tag{4.1}$$

then the global solution (u,b) to problem (1.1)-(1.2) obeys

$$\|(u,b,\partial_3u)(t)\|_{L^2(\mathbb{R}^3)}\le C\big(1+t\big)^{-\frac{1}{2}}, \tag{4.2}$$

$$\|(\nabla_hu,\nabla b)\,(t)\|_{L^2(\mathbb{R}^3)}\le C\,(1+t)^{-1}, \tag{4.3}$$

$$\|(\nabla_h^2u,\nabla_h^2b)(t)\|_{L^2(\mathbb{R}^3)}\le C\big(1+t\big)^{-\frac{3}{2}}. \tag{4.4}$$

To prove Theorem 4.1, we first prove the following H^1-estimates.

Lemma 4.1. *Assume that (u,b) is a solution to problem (1.1)-(1.2). Then, for any $0\le s<t$,*

$$\|u(t)\|_{H^1}^2+\|b(t)\|_{H^1}^2+\int_0^t(\|\nabla_hu(\tau)\|_{H^1}^2+\|\nabla b(\tau)\|_{H^1}^2)d\tau\le C(\|u(s)\|_{H^1}^2+\|b(s)\|_{H^1}^2). \tag{4.5}$$

Proof. Applying $\partial_i(i=1,2,3)$ to the first two equations of (1.1), dotting the results by ∂_iu and ∂_ib, respectively, integrating over $\mathbb{R}^3$, and adding them up, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3\big(\|\partial_iu\|_{L^2}^2+\|\partial_ib\|_{L^2}^2\big)+\mu\sum_{i=1}^3\|\partial_i\nabla_hu\|_{L^2}^2+\nu\sum_{i=1}^3\|\partial_i\nabla b\|_{L^2}^2\\&=-\sum_{i=1}^3\int\partial_i(u\cdot\nabla u)\cdot\partial_iudx+\sum_{i=1}^3\int\partial_i(b\cdot\nabla b)\cdot\partial_iudx-\sum_{i=1}^3\int\partial_i(u\cdot\nabla b)\cdot\partial_ibdx\\&\quad+\sum_{i=1}^3\int\partial_i(b\cdot\nabla u)\cdot\partial_ibdx-\sum_{i=1}^3\int\partial_i\big(\nabla\times\big(\big(\nabla\times b\big)\times b\big)\big)\cdot\partial_ibdx\\&\quad+\sum_{i=1}^3\int\partial_i\big(\nabla\times\big(\big(\big(\nabla\times b\big)\times b\big)\times b\big)\big)\cdot\partial_ibdx\\&=:\sum_{i=1}^6J_i.\end{aligned} \tag{4.6}$$

Analogous to the earlier estimates of H_{11} and I_1, we have

$$J_1=-\sum_{i=1}^3\int\partial_iu_h\cdot\nabla_hu\cdot\partial_iudx-\sum_{i=1}^3\int\partial_iu_3\partial_3u\cdot\partial_iudx\le C\|u\|_{H^1}\|\nabla_hu\|_{H^1}^2.$$

Applying Hölder's inequality and Sobolev imbedding theorem, we obtain

$$J_3=\sum_{i=1}^3\int\partial_iu\cdot\nabla b\cdot\partial_ibdx\le C\|\nabla u\|_{L^3}\|\nabla b\|_{L^2}\|\nabla b\|_{L^6}\le C\|u\|_{H^2}\|\nabla b\|_{H^1}^2.$$

Similarly, we have

$$J_2+J_4=\sum_{i=1}^3\int\partial_ib\cdot\nabla b\cdot\partial_iudx+\sum_{i=1}^3\int\partial_ib\cdot\nabla u\cdot\partial_ibdx\le C\|u\|_{H^2}\|\nabla b\|_{H^1}^2.$$

Moreover, by integrating by parts, $J_5 \leq C\|\nabla b\|_{L^3}\|\nabla b\|_{L^6}\|\nabla(\nabla\times b)\|_{L^2} \leq C\|b\|_{H^2}\|\nabla b\|_{H^1}^2$. Similarly, J_6 can also be partially integrated

$$\begin{aligned}
J_6 =& \sum_{i=1}^{3}\int(\partial_i(\nabla\times b)\times b)\times b\cdot\partial_i(\nabla\times b)dx + \sum_{i=1}^{3}\int((\nabla\times b)\times\partial_i b)\times b\cdot\partial_i(\nabla\times b)dx \\
&+\sum_{i=1}^{3}\int((\nabla\times b)\times b)\times\partial_i b\cdot\partial_i(\nabla\times b)dx \\
\leq& \sum_{i=1}^{3}\int b\times\partial_i(\nabla\times b)\cdot\partial_i(\nabla\times b)\times b dx + C\|(\nabla\times b)\times\nabla b\|_{L^3}\|b\|_{L^6}\|\nabla^2 b\|_{L^2} \\
&+C||(\nabla\times b)\times b||_{L^3}||\nabla b||_{L^6}||\nabla^2 b||_{L^2} \\
\leq& -||\nabla(\nabla\times b)\times b||_{L^2}^2 + C||\nabla b||_{L^6}^2||b||_{L^6}||\nabla^2 b||_{L^2} \\
\leq& C\|b\|_{H^2}^2\|\nabla b\|_{H^1}^2 - \|\nabla(\nabla\times b)\times b\|_{L^2}^2.
\end{aligned}$$

Combining the estimates above with (4.6), and adding the result to (3.2), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{H^1}^2+\|b\|_{H^1}^2\right)+2\mu\|\nabla_h u\|_{H^1}^2+2\nu\|\nabla b\|_{H^1}^2+\sum_{j=0,1}\|\nabla^j(\nabla\times b)\times b\|_{L^2}^2 \\
&\quad\leq C(\|u\|_{H^2}+\|b\|_{H^2}+\|b\|_{H^2}^2)(\|\nabla_h u\|_{H^1}^2+\|\nabla b\|_{H^1}^2).
\end{aligned}$$

Choosing $\varepsilon < C^{-1}\min\{\mu,\nu\}$ in (4.1), we derive that

$$\frac{d}{dt}\left(\|u\|_{H^1}^2+\|b\|_{H^1}^2\right)+\min\{\mu,\nu\}\left(\|\nabla_h u\|_{H^1}^2+\|\nabla b\|_{H^1}^2\right)\leq 0.$$

Integrating the above inequality from s to t, (4.5) immediately follows. We complete the proof of Lemma 4.1. □

In what follows, we prove Theorem 4.1.

Proof. The idea of the proof is as follows.

(i) To establish the desired decay estimates (4.2) and (4.3), we introduce the suitable time-weighted energy functional, namely

$$E(t) = \sup_{0\leq\tau\leq t}\Big((1+\tau)^{\frac{1}{2}}\|(u,b,\partial_3 u)(\tau)\|_{L^2}+(1+\tau)\|(\nabla_h u,\nabla b)(\tau)\|_{L^2}\Big).$$

The main effort is to use delicate energy estimates to prove

$$E(t)\leq CE(0)+C\Big(E^2(t)+E^{\frac{3}{2}}(t)+E^{\frac{5}{3}}(t)+E^{\frac{5}{4}}(t)+E^{\frac{9}{4}}(t)+E^3(t)\Big). \tag{4.7}$$

Once this is established, an application of the bootstrapping argument would imply the desired decay estimates (4.2) and (4.3).

(ii) We use the a priori H^1-estimate to obtain a coarse decay estimate for $\|(\nabla_h^2 u,\nabla_h^2 b)(t)\|_{L^2}$. Then we improve this decay rate by an iterative process.

The proof is slightly long. For the sake of clarity, we divide the proof into two steps.

Step 1. Decay estimates for $\|(u,b,\partial_3 u)(t)\|_{L^2}$ and $\|(\nabla_h u,\nabla b)(t)\|_{L^2}$.

To do this, we rewrite problem (1.1)-(1.2) in the integral form, namely

$$u(x,t)=e^{\mu\Delta_h t}u_0-\int_0^t e^{\mu\Delta_h(t-\tau)}\mathbb{P}(u\cdot\nabla u)(\tau)d\tau+\int_0^t e^{\mu\Delta_h(t-\tau)}\mathbb{P}(b\cdot\nabla b)(\tau)d\tau, \tag{4.8}$$

and

$$\begin{aligned} b(x,t) =& e^{\nu\Delta t}b_0 - \int_0^t e^{\nu\Delta(t-\tau)}(u\cdot\nabla b)(\tau)d\tau + \int_0^t e^{\nu\Delta(t-\tau)}(b\cdot\nabla u)(\tau)d\tau \\ &- \int_0^t e^{\nu\Delta(t-\tau)}(\nabla\times((\nabla\times b)\times b))(\tau)d\tau \\ &+ \int_0^t e^{\nu\Delta(t-\tau)}(\nabla\times(((\nabla\times b)\times b)\times b))(\tau)d\tau, \end{aligned} \tag{4.9}$$

where $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ denotes the Leray projection onto divergence-free vector fields. Applying ∂_3 to (4.8) yields

$$\partial_3 u(x,t) = \partial_3 e^{\mu\Delta_h t}u_0 - \int_0^t \partial_3 e^{\mu\Delta_h(t-\tau)}\mathbb{P}(u\cdot\nabla u)(\tau)d\tau \int_0^t \partial_3 e^{\mu\Delta_h(t-\tau)}\mathbb{P}(b\cdot\nabla b)(\tau)d\tau. \tag{4.10}$$

Taking the L^2-norm to (4.8)-(4.10), then adding them up, and noticing the boundedness of $\mathbb{P}$ on L^2 functions, we obtain

$$\begin{aligned} \|(u,b,\partial_3 u)(t)\|_{L^2} \le& \|e^{\mu\Delta_h t}u_0\|_{L^2} + \|e^{\nu\Delta t}b_0\|_{L^2} + \|e^{\mu\Delta_h t}\partial_3 u_0\|_{L^2} \\ &+ \int_0^t \|e^{\mu\Delta_h(t-\tau)}(u\cdot\nabla u)(\tau)\|_{L^2}d\tau + \int_0^t \|e^{\mu\Delta_h(t-\tau)}(b\cdot\nabla b)(\tau)\|_{L^2}d\tau \\ &+ \int_0^t \|e^{\nu\Delta(t-\tau)}(u\cdot\nabla b)(\tau)\|_{L^2}d\tau + \int_0^t \|e^{\nu\Delta(t-\tau)}(b\cdot\nabla u)(\tau)\|_{L^2}d\tau \\ &+ \int_0^t \|e^{\nu\Delta(t-\tau)}(\nabla\times((\nabla\times b)\times b))(\tau)\|_{L^2}d\tau \\ &+ \int_0^t \|e^{\nu\Delta(t-\tau)}(\nabla\times(((\nabla\times b)\times b)\times b))(\tau)\|_{L^2}d\tau \\ &+ \int_0^t \|e^{\mu\Delta_h(t-\tau)}\partial_3(u\cdot\nabla u)(\tau)\|_{L^2}d\tau + \int_0^t \|e^{\nu\Delta(t-\tau)}\partial_3(b\cdot\nabla b)(\tau)\|_{L^2}d\tau \\ =:& \sum_{i=1}^{11} K_i. \end{aligned} \tag{4.11}$$

Next, we estimate K_1 through K_{11}. By the Littlewood-Paley decomposition and Bernstein's inequality, we arrive at

$$\begin{aligned} K_1 &\le C\Big(\sum_{k\in\mathbb{Z}} e^{-2\mu 2^{2k}t}\|\Delta_k^h u_0\|_{L^2}^2\Big)^{\frac{1}{2}} \\ &\le C\big(1+t\big)^{-\frac{1}{2}}\big(\|u_0\|_{\dot{B}^{-1,0}_{2,\infty}} + \|u_0\|_{L^2}\big) \\ &\le C(1+t)^{-\frac{1}{2}}E(0), \end{aligned}$$

where $\sum_{k\in\mathbb{Z}}(2^{2k}te^{-2\mu 2^{2k}t}) \le C$ is used. Similarly, we have $K_2 + K_3 \le C(1+t)^{-\frac{1}{2}}E(0)$. To estimate K_4, we further write it into two terms

$$\begin{aligned} K_4 &\le \int_0^t \|e^{\mu\Delta_h(t-\tau)}(u_h\cdot\nabla_h u)(\tau)\|_{L^2}d\tau + \int_0^t \|e^{\mu\Delta_h(t-\tau)}(u_3\partial_3 u)(\tau)\|_{L^2}d\tau \\ &=: K_{41} + K_{42}. \end{aligned}$$

Using Lemmas 2.2 and 2.3, we arrive at

$$\begin{aligned}
K_{41} &\le C\int_0^t (t-\tau)^{-\frac{1}{2}} \| \|(u_h\cdot\nabla_h u)(\tau)\|_{L^1_h}\|_{L^2_{x_3}}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}} \|u_h(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3 u_h(\tau)\|_{L^2}^{\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}\, d\tau \\
&\le CE^2(t)\int_0^t (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{3}{2}}\, d\tau \\
&\le C(1+t)^{-\frac{1}{2}}E^2(t).
\end{aligned}$$

Similar, $K_{42}\le C(1+t)^{-\frac{1}{2}}E^2(t)$. It follows that $K_4\le C(1+t)^{-\frac{1}{2}}E^2(t)$. Similarly, we can derive that K_5 has the same bound as K_4. Using Lemma 2.3 and the Gagliardo-Nirenberg inequality, it holds that, for $\frac{3}{4}<\alpha<1$,

$$\begin{aligned}
K_6+K_7 &\le C\int_0^t (t-\tau)^{-\frac{1+\alpha}{2}} \|(u\otimes b)(\tau)\|_{L^{\frac{6}{3+2\alpha}}}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1+\alpha}{2}} \|b(\tau)\|_{L^2}^{\alpha-\frac{1}{2}} \|\nabla b(\tau)\|_{L^2}^{\frac{3}{2}-\alpha} \|u(\tau)\|_{L^2} d\tau \\
&\le C(1+t)^{-\frac{1}{2}}E^2(t).
\end{aligned}$$

To bound K_8, we again use Lemma 2.3 together with the Gagliardo-Nirenberg inequality to obtain

$$\begin{aligned}
K_8 &\le C\int_0^t (t-\tau)^{-\frac{1+\alpha}{2}} \|((\nabla\times b)\times b)(\tau)\|_{L^{\frac{6}{3+2\alpha}}}\, d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1+\alpha}{2}} \|b(\tau)\|_{L^2}^{\alpha-\frac{1}{2}} \|\nabla b(\tau)\|_{L^2}^{\frac{5}{2}-\alpha}\, d\tau \\
&\le C(1+t)^{-\frac{1}{2}}E^2(t).
\end{aligned}$$

Similarly, it holds that $K_9\le C(1+t)^{-\frac{1}{2}}E^3(t)$. As in the estimate of K_4, we have

$$K_{10}\le C(1+t)^{-\frac{1}{2}}(E^{\frac{3}{2}}(t)+E^{\frac{5}{3}}(t)).$$

Similarly as K_8, we obtain $K_{11}\le C(1+t)^{-\frac{1}{2}}E^2(t)$. Substituting the above bounds into (4.11), we find that

$$\|(u,b,\partial_3 u)(t)\|_{L^2}\le C(1+t)^{-\frac{1}{2}}E(0)+C(1+t)^{-\frac{1}{2}}\Big(E^2(t)+E^{\frac{3}{2}}(t)+E^{\frac{5}{3}}(t)+E^3(t)\Big). \quad (4.12)$$

Applying ∇_h and ∇ to (4.8) and (4.9), respectively, taking the L^2-norm to the resulting equation and then adding them up, we derive that

$$\begin{aligned}
&\|(\nabla_h u, \nabla b)(t)\|_{L^2} \\
&\le \|\nabla_h e^{\mu\Delta_h t}u_0\|_{L^2} + \|\nabla e^{\nu\Delta t}b_0\|_{L^2} + \int_0^t \|\nabla_h e^{\mu\Delta_h(t-\tau)}(u\cdot\nabla u)(\tau)\|_{L^2}d\tau \\
&\quad + \int_0^t \|\nabla_h e^{\mu\Delta_h(t-\tau)}(b\cdot\nabla b)(\tau)\|_{L^2}d\tau + \int_0^t \|\nabla e^{\nu\Delta(t-\tau)}(u\cdot\nabla b)(\tau)\|_{L^2}d\tau \\
&\quad + \int_0^t \|\nabla e^{\nu\Delta(t-\tau)}(b\cdot\nabla u)(\tau)\|_{L^2}d\tau + \int_0^t \|\nabla e^{\nu\Delta(t-\tau)}(\nabla\times((\nabla\times b)\times b))(\tau)\|_{L^2}d\tau \\
&\quad + \int_0^t \|\nabla e^{\nu\Delta(t-\tau)}(\nabla\times(((\nabla\times b)\times b)\times b))(\tau)\|_{L^2}d\tau \\
&=: \sum_{i=1}^{8} L_i.
\end{aligned} \tag{4.13}$$

As in the estimate of K_1, we obtain $L_1+L_2 \le C(1+t)^{-1}E(0)$. By Lemma 2.3, we have

$$\begin{aligned}
L_3 &\le C\int_0^{\frac{t}{2}}(t-\tau)^{-1}\|\,\|(u\cdot\nabla u)(\tau)\|_{L_h^1}\|_{L_{x_3}^2}d\tau + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1+\alpha}{2}}\|\,\|(u\cdot\nabla u)(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L_{x_3}^2}d\tau \\
&=: L_{31}+L_{32}.
\end{aligned}$$

As in the estimate of K_4, we arrive at $L_{31} \le C(1+t)^{-1}E^2(t)$. Due to $\frac{3}{4}<\alpha<1$, direct calculation implies that

$$\begin{aligned}
\|\,\|f\cdot\nabla g\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L_{x_3}^2} &\le C\|\,\|f\|_{L_{x_3}^2}^{\frac12}\|\partial_3 f\|_{L_{x_3}^2}^{\frac12}\|_{L_h^{\frac{2}{\alpha}}}\|\nabla g\|_{L^2} \\
&\le C\|\,\|f\|_{L_h^2}^{2\alpha-1}\|\nabla_h f\|_{L_h^2}^{2-2\alpha}\|_{L_{x_3}^2}^{\frac12}\|\partial_3 f\|_{L^2}^{\frac12}\|\nabla g\|_{L^2} \\
&\le C\|f\|_{L^2}^{\alpha-\frac12}\|\nabla_h f\|_{L^2}^{1-\alpha}\|\partial_3 f\|_{L^2}^{\frac12}\|\nabla g\|_{L^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
L_{32} &\le C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1+\alpha}{2}}\|\,\|(u_h\cdot\nabla_h u)(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L_{x_3}^2}d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1+\alpha}{2}}\|\,\|(u_3\partial_3 u)(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L_{x_3}^2}d\tau \\
&\le C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1+\alpha}{2}}\|u_h(\tau)\|_{L^2}^{\alpha-\frac12}\|\nabla_h u_h(\tau)\|_{L^2}^{1-\alpha}\|\partial_3 u_h(\tau)\|_{L^2}^{\frac12}\|\nabla_h u(\tau)\|_{L^2}d\tau \\
&\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1+\alpha}{2}}\|u_3(\tau)\|_{L^2}^{\alpha-\frac12}\|\nabla_h u_3(\tau)\|_{L^2}^{1-\alpha}\|\partial_3 u_3(\tau)\|_{L^2}^{\frac12}\|\partial_3 u(\tau)\|_{L^2}d\tau \\
&\le C(1+t)^{-1}E^2(t).
\end{aligned}$$

Hence, $L_3 \le C(1+t)^{-1}E^2(t)$. Similarly, we have $L_4 \le C(1+t)^{-1}E^2(t)$. Applying Lemma 2.3 and the Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned}L_5 &\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{7}{4}}\|(u\otimes b)(\tau)\|_{L^1}d\tau + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1+\alpha}{2}}\|(u\cdot\nabla b)(\tau)\|_{L^{\frac{6}{3+2\alpha}}}d\tau \\ &\le C(1+t)^{-1}E^2(t).\end{aligned}$$

Similarly, it holds that $L_6 \le C(1+t)^{-1}E^2(t)$.

Now we turn to bound L_7 as follows. For terms L_7 and L_8, we apply the same method as L_5, which are

$$\begin{aligned}L_7 &\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{7}{4}}\|((\nabla\times b)\times b)(\tau)\|_{L^1}d\tau \\ &\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1+\alpha}{2}}\|\left(\nabla\times((\nabla\times b)\times b)\right)(\tau)\|_{L^{\frac{6}{3+2\alpha}}}d\tau \\ &\le C(1+t)^{-1}\big(E^2(t)+E^{\frac{3}{2}}(t)+E^{\frac{5}{4}}(t)\big)\end{aligned}$$

and

$$\begin{aligned}L_8 &\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{7}{4}}\|(((\nabla\times b)\times b)\times b)(\tau)\|_{L^1}d\tau \\ &\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1+\alpha}{2}}\|(\nabla\times(((\nabla\times b)\times b)\times b))(\tau)\|_{L^{\frac{6}{3+2\alpha}}}d\tau \\ &\le C(1+t)^{-1}\big(E^3(t)+E^{\frac{3}{2}}(t)+E^{\frac{9}{4}}(t)\big),\end{aligned}$$

where $\frac{3}{4} < \alpha < 1$. Inserting the above bounds into (4.13), we see that

$$\begin{aligned}&\|(\nabla_h u,\nabla b)(t)\|_{L^2} \\ &\le C(1+t)^{-1}E(0) + C(1+t)^{-1}\Big(E^2(t)+E^{\frac{3}{2}}(t)+E^{\frac{5}{3}}(t)+E^{\frac{5}{4}}(t)+E^{\frac{9}{4}}(t)+E^3(t)\Big).\end{aligned}$$

Adding this to (4.12) immediately yields (4.7). The stand bootstrapping argument would imply the desired decay estimates (4.2) and (4.3).

Step 2. Decay estimates of $\|\nabla_h^2 u(t)\|_{L^2}$ and $\|\nabla_h^2 b(t)\|_{L^2}$.

Applying ∇_h^2 to the first two equations in (1.1), and taking the L^2-inner product with $\nabla_h^2 u$ and $\nabla_h^2 b$, respectively, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\nabla_h^2 u\|_{L^2}^2+\|\nabla_h^2 b\|_{L^2}^2\big)+\mu\|\nabla_h^3 u\|_{L^2}^2+\nu\|\nabla_h^2\nabla b\|_{L^2}^2 \\ &= -\int\nabla_h^2(u\cdot\nabla u)\cdot\nabla_h^2 u dx + \int\nabla_h^2(b\cdot\nabla b)\cdot\nabla_h^2 u dx - \int\nabla_h^2(u\cdot\nabla b)\cdot\nabla_h^2 b dx + \int\nabla_h^2(b\cdot\nabla u)\cdot\nabla_h^2 b dx \\ &\quad - \int\nabla_h^2(\nabla\times((\nabla\times b)\times b))\cdot\nabla_h^2 b dx + \int\nabla_h^2(\nabla\times(((\nabla\times b)\times b)\times b))\cdot\nabla_h^2 b dx \\ &=: \sum_{i=1}^6 M_i.\end{aligned} \tag{4.14}$$

Using the divergence free condition $\nabla\cdot u=0$, we rewrite M_1 as

$$\begin{aligned}M_1 &= -2\int \nabla_h u_h\cdot\nabla_h^2 u\cdot\nabla_h^2 u dx-2\int \nabla_h u_3\cdot\nabla_h\partial_3 u\cdot\nabla_h^2 u dx\\ &\quad-\int \nabla_h^2 u_h\cdot\nabla_h u\cdot\nabla_h^2 u dx-\int \nabla_h^2 u_3\partial_3 u\cdot\nabla_h^2 u dx\\ &=: M_{11}+M_{12}+M_{13}+M_{14}.\end{aligned}$$

In the following, we bound the terms M_{11} through M_{14}. Analogous to the earlier estimates of H_1, we have

$$\begin{aligned}M_{11}&\le \frac{\mu}{16}\|\nabla_h^3 u\|_{L^2}^2+C\|\nabla_h u\|_{L^2}\|\nabla_h^2 u\|_{L^2}(\|\partial_3\nabla_h u\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2),\\ M_{13}&\le \frac{\nu}{16}\|\nabla_h^3 u\|_{L^2}^2+C\|\nabla_h u\|_{L^2}\|\nabla_h^2 u\|_{L^2}(\|\partial_3\nabla_h u\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2),\\ M_{14}&\le \frac{\mu}{16}\|\nabla_h^3 u\|_{L^2}^2+C\|\partial_3 u\|_{L^2}\|\nabla_h^2 u\|_{L^2}\big(\|\partial_3\nabla_h u\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2\big).\end{aligned}$$

We cannot estimate M_{12} directly. To bound it, we apply integration by parts and obtain

$$\begin{aligned}M_{12}&=2\int \nabla_h^2 u_3\cdot\partial_3 u\cdot\nabla_h^2 u dx+2\int \nabla_h u_3\cdot\partial_3 u\cdot\nabla_h^3 u dx\\ &=: M_{121}+M_{122}.\end{aligned}$$

Clearly, M_{121} has the same bound as M_{14}. Applying Lemma 2.4 and the Young inequality, we derive that

$$\begin{aligned}M_{122}&\le C\|\nabla_h u_3\|_{L^2}^{\frac14}\|\partial_1\nabla_h u_3\|_{L^2}^{\frac14}\|\partial_3\nabla_h u_3\|_{L^2}^{\frac14}\|\partial_1\partial_3\nabla_h u_3\|_{L^2}^{\frac14}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_2\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h^3 u\|_{L^2}\\ &\le \frac{\mu}{16}\|\nabla_h^3 u\|_{L^2}^2+C\|\nabla_h^2 u\|_{L^2}^{\frac43}\|\partial_3 u\|_{L^2}^{\frac43}\|\nabla_h u\|_{L^2}^{\frac23}(\|\partial_3\nabla_h u\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2).\end{aligned}$$

Therefore, we have

$$M_{12}\le \frac{\mu}{16}\|\nabla_h^3 u\|_{L^2}^2+C\Big(\|\partial_3 u\|_{L^2}\|\nabla_h^2 u\|_{L^2}+\|\nabla_h^2 u\|_{L^2}^{\frac43}\|\partial_3 u\|_{L^2}^{\frac43}\|\nabla_h u\|_{L^2}^{\frac23}\Big)(\|\partial_3\nabla_h u\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2).$$

Inserting the boundness of $M_{1i}(i=1,\cdots,4)$ into M_1 yields

$$\begin{aligned}M_1&\le \frac{\mu}{4}\|\nabla_h^3 u\|_{L^2}^2+C\Big(\|\nabla_h u\|_{L^2}\|\nabla_h^2 u\|_{L^2}+\|\partial_3 u\|_{L^2}\|\nabla_h^2 u\|_{L^2}+\|\nabla_h^2 u\|_{L^2}^{\frac43}\|\partial_3 u\|_{L^2}^{\frac43}\|\nabla_h u\|_{L^2}^{\frac23}\Big)\\ &\quad\times(\|\partial_3\nabla_h u\|_{L^2}^2+\|\nabla_h^2 u\|_{L^2}^2).\end{aligned}$$

Similar as the estimate of H_1 and M_1, one has

$$\begin{aligned}M_3&=-\int \nabla_h^2 u\cdot\nabla b\cdot\nabla_h^2 b dx-2\int \nabla_h u\cdot\nabla_h\nabla b\cdot\nabla_h^2 b dx\\ &\le \frac{\mu}{16}\|\nabla_h^3 u\|_{L^2}^2+\frac{\nu}{8}\|\nabla_h^2\nabla b\|_{L^2}^2+C(\|\nabla_h u\|_{L^2}+\|\nabla b\|_{L^2})\|\nabla_h^2 u\|_{L^2}\|\Delta b\|_{L^2}^2.\end{aligned}$$

As in the estimates of M_1, we have

$$
\begin{aligned}
M_2+M_4 =& \int \nabla_h^2 b\cdot\nabla b\cdot\nabla_h^2 u dx + 2\int \nabla_h b\cdot\nabla\nabla_h b\cdot\nabla_h^2 u dx\\
&+\int \nabla_h^2 b\cdot\nabla u\cdot\nabla_h^2 b dx + 2\int \nabla_h b\cdot\nabla\nabla_h u\cdot\nabla_h^2 b dx\\
\le& \frac{\mu}{16}\|\nabla_h^3 u\|_{L^2}^2 + \frac{\nu}{16}\|\nabla_h^2\nabla b\|_{L^2}^2 + C\Big(\|\nabla_h u\|_{L^2}\|\nabla_h^2 b\|_{L^2}\\
&+\|\nabla b\|_{L^2}\|\nabla_h^2 u\|_{L^2} + \|\partial_3 u\|_{L^2}\|\nabla_h^2 b\|_{L^2}\\
&+\|\nabla_h u\|_{L^2}^{\frac43}\|\nabla_h b\|_{L^2}^{\frac23}\|\nabla_h^2 u\|_{L^2}^{\frac23}\Big)(\|\partial_3\nabla_h u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2).
\end{aligned}
$$

Next, we turn to bound M_5. Integration by parts yields

$$
\begin{aligned}
M_5 =& -\int (\nabla\times b\times\nabla_h^2 b)\cdot\nabla_h^2\nabla\times b dx - 2\int (\nabla_h\nabla\times b\times\nabla_h b)\cdot\nabla_h^2\nabla\times b dx\\
\le& \frac{\nu}{16}\|\nabla_h^2\nabla b\|_{L^2}^2 + C\|\nabla b\|_{L^2}^2\|\Delta b\|_{L^2}^4\|\nabla_h^2 b\|_{L^2}^4.
\end{aligned}
$$

As in the estimate of H_6, we derive that

$$
\begin{aligned}
M_6 =& \int ((\nabla_h^2(\nabla\times b)\times b)\times b)\cdot\nabla_h^2(\nabla\times b)dx + 2\int ((\nabla_h(\nabla\times b)\times\nabla_h b)\times b)\cdot\nabla_h^2(\nabla\times b)dx\\
&+2\int ((\nabla_h(\nabla\times b)\times b)\times\nabla_h b)\cdot\nabla_h^2(\nabla\times b)dx + \int (((\nabla\times b)\times\nabla_h^2 b)\times b)\cdot\nabla_h^2(\nabla\times b)dx\\
&+2\int (((\nabla\times b)\times\nabla_h b)\times\nabla_h b)\cdot\nabla_h^2(\nabla\times b)dx + \int (((\nabla\times b)\times b)\times\nabla_h^2 b)\cdot\nabla_h^2(\nabla\times b)dx\\
\le& -||\nabla_h^2(\nabla\times b)\times b||_{L^2}^2 + \frac{5\nu}{16}||\nabla_h^2\nabla b||_{L^2}^2 + C\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}^3\|\nabla^3 b\|_{L^2}^2\\
&+C\|\nabla b\|_{L^2}^{\frac32}\|\nabla^2 b\|_{L^2}^3\|\nabla^3 b\|_{L^2}^{\frac32}.
\end{aligned}
$$

Inserting the above bounds into (4.14) yields

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla_h^2 u\|_{L^2}^2 + \|\nabla_h^2 b\|_{L^2}^2) + \mu\|\nabla_h^3 u\|_{L^2}^2 + \nu\|\nabla_h^2\nabla b\|_{L^2}^2 + ||\nabla_h^2(\nabla\times b)\times b||_{L^2}^2\\
&\le C\Big((\|\nabla_h u\|_{L^2} + \|\nabla b\|_{L^2} + \|\partial_3 u\|_{L^2})(\|\nabla_h^2 u\|_{L^2} + \|\nabla_h^2 b\|_{L^2})\\
&\quad+\|\nabla_h^2 u\|_{L^2}^{\frac43}\|\partial_3 u\|_{L^2}^{\frac43}\|\nabla_h u\|_{L^2}^{\frac23} + \|\nabla_h u\|_{L^2}^{\frac43}\|\nabla_h b\|_{L^2}^{\frac23}\|\nabla_h^2 u\|_{L^2}^{\frac23}\\
&\quad+\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla^3 b\|_{L^2}^2 + \|\nabla b\|_{L^2}^{\frac32}\|\nabla^2 b\|_{L^2}\|\nabla^3 b\|_{L^2}^{\frac32}\Big)(\|\nabla_h u\|_{H^1}^2 + \|\nabla b\|_{H^1}^2).
\end{aligned}
$$

Multiplying $(t-s)^2$ to this inequality, we deduce that

$$
\begin{aligned}
&\frac{d}{dt}((t-s)^2(\|\nabla_h^2 u(t)\|_{L^2}^2 + \|\nabla_h^2 b(t)\|_{L^2}^2))\\
&\le 2(t-s)(\|\nabla_h^2 u\|_{L^2}^2 + \|\nabla_h^2 b\|_{L^2}^2)\\
&\quad+C(t-s)^2\Big((\|\nabla_h u\|_{L^2} + \|\nabla b\|_{L^2} + \|\partial_3 u\|_{L^2})(\|\nabla_h^2 u\|_{L^2} + \|\nabla_h^2 b\|_{L^2})\\
&\quad+\|\nabla_h^2 u\|_{L^2}^{\frac43}\|\partial_3 u\|_{L^2}^{\frac43}\|\nabla_h u\|_{L^2}^{\frac23} + \|\nabla_h u\|_{L^2}^{\frac43}\|\nabla_h b\|_{L^2}^{\frac23}\|\nabla_h^2 u\|_{L^2}^{\frac23}\\
&\quad+\|\nabla b\|_{L^2}\|\nabla^2 b\|_{L^2}\|\nabla^3 b\|_{L^2}^2 + \|\nabla b\|_{L^2}^{\frac32}\|\nabla^2 b\|_{L^2}\|\nabla^3 b\|_{L^2}^{\frac32}\Big)(\|\nabla_h u\|_{H^1}^2 + \|\nabla b\|_{H^1}^2).
\end{aligned}
\tag{4.15}
$$

Combining this with (4.2) and (4.3), we integrate the resulting inequality from s to t and obtain

$$\begin{aligned}&(t-s)^2\left(\|\nabla_h^2 u(t)\|_{L^2}^2+\|\nabla_h^2 b(t)\|_{L^2}^2\right)\\&\leq 2\int_s^t(\tau-s)(\|\nabla_h^2 u(\tau)\|_{L^2}^2+\|\nabla_h^2 b(\tau)\|_{L^2}^2)d\tau\\&\quad+C\int_s^t(\tau-s)^2(1+\tau)^{-\frac{1}{2}}\left(\|\nabla_h u(\tau)\|_{H^1}^2+\|\nabla b(\tau)\|_{H^1}^2\right)d\tau.\end{aligned}\tag{4.16}$$

It follows from (4.2), (4.3), and Lemma 4.1 that

$$\int_s^t(\|\nabla_h u(\tau)\|_{H^1}^2+\|\nabla b(\tau)\|_{H^1}^2)d\tau\leq\|u(s)\|_{H^1}^2+\|b(s)\|_{H^1}^2\leq C(1+s)^{-1}.\tag{4.17}$$

Setting $s=\frac{t}{2}$, one sees that (4.16) and (4.17) implies that

$$\begin{aligned}&\frac{t^2}{4}(\|\nabla_h^2 u(t)\|_{L^2}^2+\|\nabla_h^2 b(t)\|_{L^2}^2)\\&\leq 2\int_{\frac{t}{2}}^t(\tau-\frac{t}{2})(\|\nabla_h^2 u(\tau)\|_{L^2}^2+\|\nabla_h^2 b(\tau)\|_{L^2}^2)d\tau+C(1+t)^{\frac{3}{2}}\int_{\frac{t}{2}}^t(\left\|\nabla_h u(\tau)\right\|_{H^1}^2+\left\|\nabla b(\tau)\right\|_{H^1}^2)d\tau\\&\leq C\Big(1+t+(1+t)^{\frac{3}{2}}\Big)\Big(\|u(\frac{t}{2})\|_{H^1}^2+\|b(\frac{t}{2})\|_{H^1}^2\Big)\\&\leq C(1+t)^{\frac{1}{2}}.\end{aligned}$$

Thus

$$\|\nabla_h^2 u(t)\|_{L^2}^2\leq C(1+t)^{-\frac{3}{2}}.\tag{4.18}$$

To enhance the decay rate further, we adopt an iterative approach. For notational convenience, we define $a_0=\frac{3}{2}$. Inserting (4.2), (4.3), and (4.18) into (4.15), and following the above procedure, we obtain

$$\|\nabla_h^2 u(t)\|_{L^2}^2+\|\nabla_h^2 b(t)\|_{L^2}^2\leq C(1+t)^{-a_1},$$

where $a_1=\frac{3}{2}+\frac{1}{2}a_0$. Repeating this procedure $n\geq 1$ times yields

$$\|\nabla_h^2 u(t)\|_{L^2}^2+\|\nabla_h^2 b(t)\|_{L^2}^2\leq C(1+t)^{-a_n},\tag{4.19}$$

where $a_n=\frac{3}{2}+\frac{1}{2}a_{n-1}$. By this iterative formula, we have

$$a_n=\frac{3}{2}\Big(1+\frac{1}{2}+\cdots+(\frac{1}{2})^{n-1}\Big)+(\frac{1}{2})^n a_0=\frac{3}{2}\Big(2-(\frac{1}{2})^{n+1}\Big).$$

Clearly, it holds that, as $n\to\infty$, $a_n\to 3$ Thus, by choosing $n\to\infty$, (4.19) implies that

$$\|\nabla_h^2 u(t)\|_{L^2}^2+\|\nabla_h^2 b(t)\|_{L^2}^2\leq C(1+t)^{-3}.$$

This implies that (4.4) holds. This completes the proof of Theorem 4.1. □

4.2. **Decay Estimates of $\|(\nabla_h\partial_3 u,\nabla\partial_3 b)(t)\|_{L^2}$ and $\|\partial_3^2 u(t)\|_{L^2}$.** The main purpose of this subsection is to establish the decay estimates of $\|(\nabla_h\partial_3 u,\nabla\partial_3 b)(t)\|_{L^2}$ and $\|\partial_3^2 u(t)\|_{L^2}$ for H^4 initial value. We state the result as follows.

Theorem 4.2. *Assume that* $(u_0,b_0)\in H^4(\mathbb{R}^3)$. *Then there exists* $\varepsilon>0$ *such that if* (u_0,b_0) *satisfies* $\|(u_0,\partial_3 u_0)\|_{\dot{B}^{-1,0}_{2,\infty}(\mathbb{R}^3)}+\|b_0\|_{\dot{B}^{-1}_{2,\infty}(\mathbb{R}^3)}+\|(u_0,b_0)\|_{H^4(\mathbb{R}^3)}\le\varepsilon$, *then the global solution* (u,b) *to problem* (1.1)-(1.2) *also obeys*

$$\|(u,\partial_3 u)(t)\|_{\dot{B}^{-1,0}_{2,\infty}(\mathbb{R}^3)}+\|b(t)\|_{\dot{B}^{-1}_{2,\infty}(\mathbb{R}^3)}\le C, \tag{4.20}$$

$$\|(\nabla_h\partial_3 u,\nabla\partial_3 b)(t)\|_{L^2(\mathbb{R}^3)}\le C(1+t)^{-1}, \tag{4.21}$$

$$\|\partial_3^2 u(t)\|_{L^2(\mathbb{R}^3)}\le C\big(1+t\big)^{-\frac{1}{3}}. \tag{4.22}$$

Proof. The idea of proof is follows. To obtain the decay $\|(\nabla_h\partial_3 u,\nabla_h\partial_3 b)(t)\|_{L^2}$, we resort to integral form of the problem. The estimate for $\|\partial_3^2 u(t)\|_{L^2}$ is more involved. To deal with it, we also need to introduce the time-weighted functional $\mathscr{M}(t)=\sup_{0\le\tau\le t}(1+\tau)^{\frac{1}{3}}\|\partial_3^2 u(\tau)\|_{L^2}$. In order to clarify the proof idea, we divide the proof into three steps.

Step 1. Decay estimates for $\|(u,\partial_3 u)(t)\|_{\dot{B}^{-1,0}_{2,\infty}}$ and $\|b(t)\|_{\dot{B}^{-1}_{2,\infty}}$.

Applying $(\Delta_k^h,\Delta_k^h\partial_3)$ and Δ_k to the equations u and b of (1.1), taking the L^2-inner product with $(\Delta_k^h u,\Delta_k^h\partial_3 u)$ and $\Delta_k b$, respectively, and adding them together, we obtain an inequality. Then, multiplying this inequality by 2^{-2k} and taking the l_k^∞ over $k\in\mathbb{Z}$, together with the Hölder's inequality yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|u\|^2_{\dot{B}^{-1,0}_{2,\infty}}+\|b\|^2_{\dot{B}^{-1}_{2,\infty}}+\|\partial_3 u\|^2_{\dot{B}^{-1,0}_{2,\infty}}\big)+\mu\|\nabla_h u\|^2_{\dot{B}^{-1,0}_{2,\infty}}+\nu\|\nabla b\|^2_{\dot{B}^{-1}_{2,\infty}}+\mu\|\nabla_h\partial_3 u\|^2_{\dot{B}^{-1,0}_{2,\infty}}\\
&\le\|b\cdot\nabla b\|_{\dot{B}^{-1,0}_{2,\infty}}\|u\|_{\dot{B}^{-1,0}_{2,\infty}}+\|u\cdot\nabla u\|_{\dot{B}^{-1,0}_{2,\infty}}\|u\|_{\dot{B}^{-1,0}_{2,\infty}}\\
&\quad+\|u\cdot\nabla b\|_{\dot{B}^{-1}_{2,\infty}}\|b\|_{\dot{B}^{-1}_{2,\infty}}+\|b\cdot\nabla u\|_{\dot{B}^{-1}_{2,\infty}}\|b\|_{\dot{B}^{-1}_{2,\infty}}\\
&\quad+\|(\nabla\times b)\times b\|_{\dot{B}^{-1}_{2,\infty}}\|\nabla\times b\|_{\dot{B}^{-1}_{2,\infty}}\\
&\quad+\|((\nabla\times b)\times b)\times b\|_{\dot{B}^{-1}_{2,\infty}}\|\nabla\times b\|_{\dot{B}^{-1}_{2,\infty}}\\
&\quad+\|\partial_3(b\cdot\nabla b)\|_{\dot{B}^{-1,0}_{2,\infty}}\|\partial_3 u\|_{\dot{B}^{-1,0}_{2,\infty}}+\|\partial_3(u\cdot\nabla u)\|_{\dot{B}^{-1,0}_{2,\infty}}\|\partial_3 u\|_{\dot{B}^{-1,0}_{2,\infty}}\\
&=:\sum_{i=1}^{8}O_i.
\end{aligned}\tag{4.23}$$

Using the Hardy-Littlewood-Sobolev inequality and Lemma 2.2, we have

$$\|b\cdot\nabla b\|_{\dot{B}^{-1,0}_{2,\infty}}\le C\|\,\|b\cdot\nabla b\|_{L^1_h}\|_{L^2_{x_3}}\le C\|b\|^{\frac{1}{2}}_{L^2}\|\nabla b\|^{\frac{3}{2}}_{L^2}.$$

Therefore, we obtain $O_1\le C\|b\|^{\frac{1}{2}}_{L^2}\|\nabla b\|^{\frac{3}{2}}_{L^2}\|u\|_{\dot{B}^{-1,0}_{2,\infty}}$. As in the estimate of O_1, together with $\nabla\cdot u=0$, we arrive at

$$O_2\le C\Big(\|u\|^{\frac{1}{2}}_{L^2}\|\partial_3 u\|^{\frac{1}{2}}_{L^2}\|\nabla_h u\|_{L^2}+\|u\|^{\frac{1}{2}}_{L^2}\|\nabla_h u\|^{\frac{1}{2}}_{L^2}\|\partial_3 u\|_{L^2}\Big)\|u\|_{\dot{B}^{-1,0}_{2,\infty}}.$$

Applying the Hardy-Littlewood-Sobolev and Gagliardo-Nirenberg inequalities yields

$$\|u\cdot\nabla b\|_{\dot{B}^{-1}_{2,\infty}}\le C\|u\cdot\nabla b\|_{L^{\frac{6}{5}}}\le C\|u\|^{\frac{1}{2}}_{L^2}\|\nabla u\|^{\frac{1}{2}}_{L^2}\|\nabla b\|_{L^2}.$$

Thus $O_3 \leq C\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}\|b\|_{\dot{B}^{-1}_{2,\infty}}$. Similarly, one has $O_4 \leq C\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|b\|_{\dot{B}^{-1}_{2,\infty}}$. Using the Young inequality, one obtains

$$O_5 \leq C\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{3}{2}}\|\nabla b\|_{\dot{B}^{-1}_{2,\infty}} \leq \frac{\nu}{4}\|\nabla b\|^2_{\dot{B}^{-1}_{2,\infty}} + C\|b\|_{L^2}\|\nabla b\|^3_{L^2}.$$

The same procedure yields

$$O_6 \leq C\|(\nabla\times b)\times b\|_{L^2}\|b\|_{L^3}\|\nabla b\|_{\dot{B}^{-1}_{2,\infty}} \leq \frac{\nu}{4}\|\nabla b\|^2_{\dot{B}^{-1}_{2,\infty}} + C\|b\|_{L^2}^{\frac{3}{2}}\|\nabla b\|_{L^2}^{\frac{15}{4}}.$$

As O_1, we obtain

$$O_7 \leq C\Big(\|\nabla b\|_{L^2}^{\frac{3}{2}}\|\partial_3^2 b\|_{L^2}^{\frac{1}{2}} + \|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}\|\nabla\partial_3^2 b\|_{L^2}^{\frac{1}{2}}\Big)\|\partial_3 u\|_{\dot{B}^{-1,0}_{2,\infty}}.$$

To bound O_8, we further write $\|\partial_3(u\cdot\nabla u)\|_{\dot{B}^{-1,0}_{2,\infty}}$ into four terms. More precisely, the Hardy-Littlewood-Sobolev inequality yields

$$\begin{aligned}
O_8 \leq{}& C\Big(\|\ \|\partial_3 u_h\cdot\nabla_h u\|_{L^1_h}\|_{L^2_{x_3}} + C\|\ \|\partial_3 u_3\partial_3 u\|_{L^1_h}\|_{L^2_{x_3}} \\
&+ C\|\ \|u_h\cdot\nabla_h\partial_3 u\|_{L^1_h}\|_{L^2_{x_3}} + C\|\ \|u_3\partial_3^2 u\|_{L^1_h}\|_{L^2_{x_3}}\Big)\|\partial_3 u\|_{\dot{B}^{-1,0}_{2,\infty}} \\
\leq{}& C\Big(\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}\|\partial_3^2 u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{2}{3}}\|\nabla^3\nabla_h u\|_{L^2}^{\frac{1}{3}} \\
&+ \|u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{2}{3}}\|\nabla^3\partial_3 u\|_{L^2}^{\frac{1}{3}}\Big)\|\partial_3 u\|_{\dot{B}^{-1,0}_{2,\infty}}.
\end{aligned}$$

Inserting the bounds of $O_i (i = 1,\cdots,8)$ into (4.23), we obtain

$$\begin{aligned}
&\frac{d}{dt}\big(\|u\|^2_{\dot{B}^{-1,0}_{2,\infty}} + \|b\|^2_{\dot{B}^{-1}_{2,\infty}} + \|\partial_3 u\|^2_{\dot{B}^{-1,0}_{2,\infty}}\big) \\
\leq{}& C\|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{3}{2}}\|u\|_{\dot{B}^{-1,0}_{2,\infty}} + C\|b\|_{L^2}\|\nabla b\|^3_{L^2} + C\|b\|_{L^2}^{\frac{3}{2}}\|\nabla b\|_{L^2}^{\frac{15}{4}} \\
&+ C\Big(\|u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2} + \|u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}\Big)\|u\|_{\dot{B}^{-1,0}_{2,\infty}} \\
&+ C\Big(\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2} + \|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\Big)\|b\|_{\dot{B}^{-1}_{2,\infty}} \\
&+ C\Big(\|\nabla b\|_{L^2}^{\frac{3}{2}}\left\|\partial_3^2 b\right\|_{L^2}^{\frac{1}{2}} + \|b\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}\left\|\nabla\partial_3^2 b\right\|_{L^2}^{\frac{1}{2}}\Big)\|\partial_3 u\|_{\dot{B}^{-1,0}_{2,\infty}} \\
&+ C\Big(\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}\|\partial_3^2 u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{2}{3}}\|\nabla^3\nabla_h u\|_{L^2}^{\frac{1}{3}} \\
&+ \|u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{2}{3}}\|\nabla^3\partial_3 u\|_{L^2}^{\frac{1}{3}}\Big)\|\partial_3 u\|_{\dot{B}^{-1,0}_{2,\infty}}.
\end{aligned}$$

Integrating this over $[0,t]$ with $0<t\le T$ and using (4.2) and (4.3), we obtain

$$
\begin{aligned}
&\|u(t)\|^2_{\dot{B}^{-1,0}_{2,\infty}}+\|b(t)\|^2_{\dot{B}^{-1}_{2,\infty}}+\|\partial_3 u(t)\|^2_{\dot{B}^{-1,0}_{2,\infty}}\\
&\le \|u_0\|^2_{\dot{B}^{-1,0}_{2,\infty}}+\|b_0\|^2_{\dot{B}^{-1}_{2,\infty}}+\|\partial_3 u_0\|^2_{\dot{B}^{-1,0}_{2,\infty}}+C\int_0^t(1+\tau)^{-\frac{7}{2}}\,d\tau+C\int_0^t(1+\tau)^{-\frac{9}{2}}\,d\tau\\
&\quad+C\sup_{0\le\tau\le t}(\|u(\tau)\|_{\dot{B}^{-1,0}_{2,\infty}}+\|b(\tau)\|_{\dot{B}^{-1}_{2,\infty}}+\|\partial_3 u(\tau)\|_{\dot{B}^{-1,0}_{2,\infty}})\int_0^t(1+\tau)^{-\frac{13}{12}}\,d\tau\\
&\le C+C\sup_{0\le\tau\le t}\left(\|u(\tau)\|_{\dot{B}^{-1,0}_{2,\infty}}+\|b(\tau)\|_{\dot{B}^{-1}_{2,\infty}}+\|\partial_3 u(\tau)\|_{\dot{B}^{-1,0}_{2,\infty}}\right).
\end{aligned}
$$

Then this together with the Young inequality yields (4.20).

Step 2. Decay estimates of $\|\nabla_h\partial_3 u(t)\|_{L^2}$ and $\|\nabla\partial_3 b(t)\|_{L^2}$.

Applying $\nabla_h\partial_3$ and $\nabla\partial_3$ to (4.8) and (4.9), respectively, and taking the L^2-norm, we obtain

$$
\begin{aligned}
&\|\nabla_h\partial_3 u(t)\|_{L^2}+\|\nabla\partial_3 b(t)\|_{L^2}\\
&\le \|\nabla_h e^{\mu\Delta_h t}\partial_3 u_0\|_{L^2}+\|\nabla\partial_3 e^{\nu\Delta_h t}b_0\|_{L^2}+\int_0^t\|\nabla_h e^{\mu\Delta_h(t-\tau)}\partial_3(u\cdot\nabla u)(\tau)\|_{L^2}d\tau\\
&\quad+\int_0^t\|\nabla_h e^{\mu\Delta_h(t-\tau)}\partial_3(b\cdot\nabla b)(\tau)\|_{L^2}d\tau+\int_0^t\|\nabla e^{\nu\Delta(t-\tau)}\partial_3(b\cdot\nabla u)(\tau)\|_{L^2}d\tau\\
&\quad+\int_0^t\|\nabla e^{\nu\Delta(t-\tau)}\partial_3(u\cdot\nabla b)(\tau)\|_{L^2}d\tau+\int_0^t\|\nabla e^{\nu\Delta(t-\tau)}\partial_3(\nabla\times((\nabla\times b)\times b))(\tau)\|_{L^2}d\tau\\
&\quad+\int_0^t\|\nabla e^{\nu\Delta(t-\tau)}\partial_3(\nabla\times(((\nabla\times b)\times b)\times b))(\tau)\|_{L^2}d\tau\\
&=:\sum_{i=1}^{8}P_i.
\end{aligned}
\tag{4.24}
$$

By Lemma 2.3, we see that

$$
P_1\le C(1+t)^{-1}(\|\partial_3 u_0\|_{\dot{B}^{-1,0}_{2,\infty}}+\|u_0\|_{L^2})\le C(1+t)^{-1},
$$

where we used $\sum_{k\in\mathbb{Z}}(2^{4k}t^2e^{-2\mu 2^{2k}t})\le C$. Similarly, we have $P_2\le C(1+t)^{-1}$. To bound P_3, we rewrite it as

$$
\begin{aligned}
P_3&=\int_0^{\frac{t}{2}}\|\nabla_h e^{\mu\Delta_h(t-\tau)}\partial_3(u\cdot\nabla u)(\tau)\|_{L^2}d\tau+\int_{\frac{t}{2}}^t\|\nabla_h e^{\mu\Delta_h(t-\tau)}\partial_3(u\cdot\nabla u)(\tau)\|_{L^2}d\tau\\
&=:P_{31}+P_{32}.
\end{aligned}
$$

Using the Hardy-Littlewood-Sobolev inequality, one sees that

$$
\begin{aligned}
P_{31}&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-1}\Big(\|\,\|\partial_3 u_h\cdot\nabla_h u(\tau)\|_{L^1_h}\|_{L^2_{x_3}}+\|\,\|\partial_3 u_3\partial_3 u(\tau)\|_{L^1_h}\|_{L^2_{x_3}}\\
&\qquad+\|\,\|u_h\cdot\nabla_h\partial_3 u(\tau)\|_{L^1_h}\|_{L^2_{x_3}}+\|\,\|u_3\partial_3^2 u(\tau)\|_{L^1_h}\|_{L^2_{x_3}}\Big)d\tau\\
&\le C(1+t)^{-1}.
\end{aligned}
$$

Applying $\nabla\cdot u=0$ and Lemma 2.3 , and following the same procedure that leads to L_3 and K_{10}, we obtain

$$\begin{aligned}P_{32}\leq{}& C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1+\alpha}{2}}\Big(\|\,\|\partial_3 u_h\cdot\nabla_h u(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L^2_{x_3}}+\|\,\|\partial_3 u_3\partial_3 u(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L^2_{x_3}}\\&+\|\,\|u_h\cdot\nabla_h\partial_3 u(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L^2_{x_3}}+\|\,\|u_3\partial_3^2 u(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L^2_{x_3}}\Big)d\tau\\ \leq{}& C(1+t)^{-1}\end{aligned}$$

with $\frac{3}{4}<\alpha<1$. Thus $P_3\leq C(1+t)^{-1}$. Similarly as the estimate of P_3, we obatin $P_4\leq C(1+t)^{-1}$. To bound P_5, we rewrite it as

$$\begin{aligned}P_5={}&\int_0^{\frac{t}{2}}\|\nabla e^{\nu\Delta(t-\tau)}\partial_3(b\cdot\nabla u)(\tau)\|_{L^2}d\tau+\int_{\frac{t}{2}}^{t}\|\nabla e^{\nu\Delta(t-\tau)}\partial_3(b\cdot\nabla u)(\tau)\|_{L^2}d\tau\\=:{}&P_{51}+P_{52}.\end{aligned}$$

Using Lemma 2.3, we obtain

$$P_{51}\leq C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{9}{4}}\|(b\otimes u)(\tau)\|_{L^1}d\tau\leq C(1+t)^{-1}.$$

Applying the Hardy-Littlewood-Sobolev and Gagliardo-Nirenberg inequalities yields

$$\begin{aligned}P_{52}\leq{}& C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1+\alpha}{2}}\Big(\|\partial_3 b\cdot\nabla u(\tau)\|_{L^{\frac{6}{3+2\alpha}}}+\|b\cdot\nabla\partial_3 u(\tau)\|_{L^{\frac{6}{3+2\alpha}}}\Big)d\tau\\ \leq{}& C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1+\alpha}{2}}\Big(\|\partial_3 b(\tau)\|_{L^2}^{\frac{\alpha}{2}+\frac14}\|\nabla_h u(\tau)\|_{L^2}+\|\partial_3 b(\tau)\|_{L^2}^{\frac{\alpha}{2}+\frac14}\|\partial_3 u(\tau)\|_{L^2}\\&+\|b(\tau)\|_{L^2}^{\alpha-\frac12}\|\nabla b(\tau)\|_{L^2}^{\frac32-\alpha}\|\partial_3 u(\tau)\|_{L^2}^{\frac12}\Big)d\tau\\ \leq{}& C(1+t)^{-1}.\end{aligned}$$

Thus $P_5\leq C(1+t)^{-1}$. Similarly, we can show that P_6 and P_7 have the same bound as P_5. Applying the Hölder's inequality and Sobolev imbedding theorem, we derive that

$$\begin{aligned}P_8\leq{}&\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{9}{4}}\|(((\nabla\times b)\times b)\times b)(\tau)\|_{L^1}d\tau\\&+\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1+\alpha}{2}}\|\partial_3(\nabla\times(((\nabla\times b)\times b)\times b))(\tau)\|_{L^{\frac{6}{3+2\alpha}}}d\tau\\ \leq{}& C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{9}{4}}\|\nabla b(\tau)\|_{L^2}\|b(\tau)\|_{L^4}^2d\tau+C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1+\alpha}{2}}\Big(\|b(\tau)\|_{L^\infty}\|\nabla b(\tau)\|_{L^{\frac{3}{\alpha}}}\|\nabla^2 b(\tau)\|_{L^2}\\&+\|b(\tau)\|_{L^{\frac{3}{\alpha}}}\|b(\tau)\|_{L^\infty}\|\nabla^3 b(\tau)\|_{L^2}+\|\nabla b(\tau)\|_{L^4}^2\|\nabla b(\tau)\|_{L^{\frac{3}{\alpha}}}\Big)d\tau\\ \leq{}& C(1+t)^{-1}.\end{aligned}$$

Inserting the bounds of $P_i(i=1,\cdots,8)$ into (4.24), (4.21) immediately follows.

Step 3. Decay estimates of $\|\partial_3^2 u(t)\|_{L^2}$.

Let $\mathscr{M}(t)=\sup_{0\le\tau\le t}(1+\tau)^{\frac{1}{3}}\|\partial_3^2 u(\tau)\|_{L^2}$. Applying ∂_3^2 to (4.8), and taking the L^2 norm, one has

$$\begin{aligned}\|\partial_3^2 u(t)\|_{L^2} &\le \|\partial_3^2 e^{\mu\Delta_h t}u_0\|_{L^2}+\int_0^t \|\partial_3^2 e^{\mu\Delta_h(t-\tau)}(u\cdot\nabla u)(\tau)\|_{L^2}d\tau\\ &\quad+\int_0^t \|\partial_3^2 e^{\mu\Delta_h(t-\tau)}(b\cdot\nabla b)(\tau)\|_{L^2}d\tau\\ &=: Q_1+Q_2+Q_3.\end{aligned}\tag{4.25}$$

Taking advantage of the Gagliardo-Nirenberg inequality and Lemma 2.3 yields

$$Q_1\le C\|\partial_3 e^{\mu\Delta_h t}u_0\|_{L^2}^{\frac{2}{3}}\|\partial_3^4 e^{\mu\Delta_h t}u_0\|_{L^2}^{\frac{1}{3}}\le C(1+t)^{-\frac{1}{3}}.$$

To bound Q_2, we rewrite it as

$$\begin{aligned}Q_2&\le\int_0^t \|e^{\mu\Delta_h(t-\tau)}(\partial_3^2 u\cdot\nabla u)(\tau)\|_{L^2}d\tau+\int_0^t \|e^{\mu\Delta_h(t-\tau)}(u\cdot\nabla\partial_3^2 u)(\tau)\|_{L^2}d\tau\\ &\quad+2\int_0^t \|e^{\mu\Delta_h(t-\tau)}(\partial_3 u\cdot\nabla\partial_3 u)(\tau)\|_{L^2}d\tau\\ &=: Q_{21}+Q_{22}+Q_{23}.\end{aligned}$$

Similar, we can obtain

$$\begin{aligned}Q_{21}&\le C\int_0^t (t-\tau)^{-\frac{\alpha}{2}}\|\,\|(\partial_3^2 u_h\cdot\nabla_h u)(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L^2_{x_3}}d\tau\\ &\quad+C\int_0^t (t-\tau)^{-\frac{\alpha}{2}}\|\,\|(\partial_3^2 u_3\partial_3 u)(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L^2_{x_3}}d\tau\\ &\le C(1+t)^{-\frac{1}{3}}\mathscr{M}^{\alpha-\frac{1}{2}}(t)+C(1+t)^{-\frac{1}{3}}\mathscr{M}^{\frac{1}{2}}(t)\end{aligned}$$

and

$$\begin{aligned}Q_{22}&\le C\int_0^t (t-\tau)^{-\frac{\alpha}{2}}\|\,\|(u_h\cdot\nabla_h\partial_3^2 u)(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L^2_{x_3}}d\tau\\ &\quad+C\int_0^t (t-\tau)^{-\frac{\alpha}{2}}\|\,\|(u_3\partial_3^3 u)(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L^2_{x_3}}d\tau\\ &\le C(1+t)^{-\frac{1}{3}}+C(1+t)^{-\frac{1}{3}}\mathscr{M}^{\frac{1}{2}}(t),\end{aligned}$$

where $\frac{3}{4}<\alpha<1$. In view of the estimate of Q_{21} together with $\nabla\cdot u=0$, we have

$$\begin{aligned}Q_{23}&\le C\int_0^t (t-\tau)^{-\frac{\alpha}{2}}\|\,\|(\partial_3 u_h\cdot\nabla_h\partial_3 u)(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L^2_{x_3}}d\tau\\ &\quad+C\int_0^t (t-\tau)^{-\frac{\alpha}{2}}\|\,\|(\partial_3 u_3\partial_3^2 u)(\tau)\|_{L_h^{\frac{2}{1+\alpha}}}\|_{L^2_{x_3}}d\tau\\ &\le C(1+t)^{-\frac{1}{3}}\mathscr{M}^{\alpha-\frac{1}{2}}(t)+C(1+t)^{-\frac{1}{3}}\mathscr{M}^{\frac{1}{2}}(t).\end{aligned}$$

Incorporating the above bounds yields

$$Q_2\le C(1+t)^{-\frac{1}{3}}\mathscr{M}^{\alpha-\frac{1}{2}}(t)+C(1+t)^{-\frac{1}{3}}\mathscr{M}^{\frac{1}{2}}(t)+C(1+t)^{-\frac{1}{3}}.$$

Similarly, we can derive that Q_3 has the same bound as Q_2. Substituting the bounds of Q_1, Q_2, and Q_3 into (4.25), we find that $\mathscr{M}(t)\le C+C\mathscr{M}^{\alpha-\frac{1}{2}}(t)+C\mathscr{M}^{\frac{1}{2}}(t)$, which together with the Young inequality immediately yields (4.22). Thus Theorem 4.2 is proved. □

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