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NONHOMOGENEOUS, NONAUTONOMOUS RESONANT SINGULAR EQUATIONS

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Abstract. We consider a nonlinear Dirichlet problem driven by a nonautonomous (p,q)-differential operator and with a reaction having the competing effects of a parametric singular term and a $(p-1)$-linear perturbation which can be resonant as $x \to \infty$ with respect to the principal eigenvalue of the relevant operator. If the resonance is from the left, then we demonstrate that the problem has a positive solution for all values of the parameter and if the driving differential operator is only the nonautonomous p-Laplacian, then the positive solution is unique. On the other hand, if the resonance is from the right, then we prove an existence and multiplicity theorem which is global with respect to the parameter (a bifurcation-type theorem). Also, we conduct a detailed study of the continuity properties of solution multifunction.

Keywords. Nonautonomous (p,q)-operator; Resonance; Hardy's inequality; Singular and superlinear terms; Multiple solutions; Continuity of the solution multifunction.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear singular Dirichlet problem

$$\begin{cases} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) = \lambda u(z)^{-\eta} + f(z,u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, 0 < \eta < 1 < q < p, \lambda > 0, u > 0. \end{cases} \qquad (p_\lambda)$$

For $a \in C^{0,1}(\overline{\Omega})$ (the space of all Lipschitz continuous functions defined on $\overline{\Omega}$), with $a(z) \geq \hat{c} > 0$ for all $z \in \overline{\Omega}$ and for $s \in (1,\infty)$, by Δ_s^a we denote the nonautonomous s-Laplace differential operator (the weighted s-Laplacian) defined by

$$\Delta_s^a u = \operatorname{div}(a(z)|Du|^{s-2}Du) \ \forall u \in W_0^{1,s}(\Omega).$$

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Problem (p_λ) is driven by the sum of two such operators with different exponents and possibly different weights. So the differential operator in (p_λ) is nonautonomous and nonhomogeneous. In the reaction (right hand side) of (p_λ), we have the combined effects of a parametric singular term $x \to \lambda x^{-\eta}$ with $0 < \eta < 1$ and $\lambda > 0$ being the parameter and of Caratheodory perturbation $f(z,x)$ (that is, the function $f(z,x)$ is measurable in z, continuous in x, hence jointly measurable, see Papageorgiou-Winkert [22, p.108]). We assume that $f(z,\cdot)$ exhibits $(p-1)$-linear growth as $x \to +\infty$. Our aim is to prove the existence and multiplicity of positive solutions. Our analysis reveals that the existence and multiplicity of positive solutions depends on whether asymptotically as $x \to +\infty$, the quotient $\frac{f(z,x)}{x^{p-1}}$ stays below or above $\widehat{\lambda}_1^{a_1}(p) > 0$, the principal eigenvalue of $((-\Delta_p^{a_1}, W_0^{1,p}(\Omega))$. If the $\limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}}$ stays below $\widehat{\lambda}_1^{a_1}(p)$ with possible full resonance, then we can produce one positive smooth solution which is actually unique for equations driven by the $-\Delta_p^{a_1}$ differential operator and the quotient function $x \to \frac{f(z,x)}{x^{p-1}}$ is nonincreasing on $\overset{o}{R}_+ = (0,+\infty)$. On the other hand, if $\liminf_{x\to+\infty} \frac{f(z,x)}{x^{p-1}}$ stays above $\hat{\lambda}_1^{a_1}(p) > 0$ with possible full resonance, we can have existence and multiplicity of positive solutions for all $\lambda > 0$ small. The result is global in $\lambda > 0$ (a bifurcation-type theorem). For this case we study the dependence of the solution set on the parameter $\lambda > 0$.

In the past, most of the works on singular equations assumed that $f(z,\cdot)$ is $(p-1)$-superlinear as $x \to +\infty$. We refer to the works Haitao [7], Hirano-Saccon-Shioji [9], Sun-Wu-Long [27] (semilinear equations), Giacomoni-Schindler-Takac [6], Papageorgiou-Qin-Rădulescu [16], Papageorgiou-Rădulescu-Zhang [19], and Papageorgiou-Zhang [23, 24] (nonlinear equations). Problems with a $(p-1)$-linear perturbation were considered only recently by Papageorgiou-Vetro-Zhang [21] and Papageorgiou-Zhang [25], for the equations driven by the autonomous (p,q)-Laplacian and under conditions on $f(z,\cdot)$ which exclude the possibility of resonance with respect to $\hat{\lambda}_1(p) > 0$, the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Our work here extends the results of the aforementioned papers.

2. Mathematical Background

The main spaces in the analysis of problem (p_λ) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. On account of the Poincaré inequality, on $W_0^{1,p}(\Omega)$ we can use the equivalent norm $\|\cdot\|$ defined by $\|u\| = \|Du\|_p$ for all $u \in W_0^{1,p}(\Omega)$. The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : 0 \le u(z) \text{ for all } z \in \overline{\Omega}\}.$$

This cone has nonempty interior given by

$$\text{int}C_+ = \{u \in C_+ : 0 < u(z), \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\},$$

where $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

For $a \in C^{0,1}(\overline{\Omega})$ with $a(z) \ge \hat{c} > 0$ for all $z \in \overline{\Omega}$ and $s \in (1,\infty)$, we consider the following nonlinear eigenvalue problem

$$-\Delta_s^a u(z) = \hat{\lambda}|u(z)|^{s-2}u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{2.1}$$

By $\sigma_a(s)$, we denote the spectrum (that is, the set of eigenvalues of (2.1)). From Liu-Papageorgiou [13], we know that (2.1) admits a smallest eigenvalue $\hat{\lambda}_1^a(s)$ such that

- $\hat{\lambda}_1^a(s) > 0$;
- $\hat{\lambda}_1^a(s)$ is isolated, that is, we can find $\varepsilon > 0$ such that $(\hat{\lambda}_1^a(s), \hat{\lambda}_1^a(s)+\varepsilon) \cap \sigma_a(s) = \emptyset$;
- $\hat{\lambda}_1^a$ is simple, that is, if $\hat{u}$, $\hat{v}$ are eigenfunctions for $\hat{\lambda}_1^a(s)$, then $\hat{u} = \theta\hat{v}$ for some $\theta \in \mathbb{R}\setminus\{0\}$;
- $$\hat{\lambda}_1^a(s) = \inf\{\frac{\int_\Omega a(z)|Du|^p\,dz}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\}. \tag{2.2}$$

The infimum in (2.2) is realized on the corresponding one dimensional eigenspace, and the elements of which do not change sign. If $\hat{u}$ is any eigenfunction of (2.1), then the nonlinear regularity theory of Lieberman [12] implies that $\hat{u} \in C_0^1(\bar{\Omega})$. For the eigenfunctions of $\hat{\lambda}_1^a(s) > 0$, using the nonlinear maximum principle (see Pucci-Serrin [26]), we have that they belong in intC_+ or in -intC_+. We mention that only $\hat{\lambda}_1^a(s) > 0$ has eigenfunctions of constant sign. All the other eigenvalues have nodal (sign-changing) eigenfunctions. Moreover, $\hat{u}_1(s) \in \text{int}C_+$ is the eigenfunction with $\|\hat{u}_1(s)\|_s = 1$. We also use a weighted version of (2.1). So, let $m \in L^\infty(\Omega)\setminus\{0\}$ with $m(z) \geq 0$ for a.a $z \in \Omega$ and consider the following nonlinear eigenvalue problem

$$-\Delta_s^a u(z) = \widetilde{\lambda} m(z)|u(z)|^{s-2}u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0.$$

This problem has a smallest eigenvalue $\widetilde{\lambda}_1^a(s,m) > 0$ which has the same properties as $\hat{\lambda}_1^a(s)$. The same is true for the eigenfunctions. In this case, the variational characterization of $\widetilde{\lambda}_1^a(s,m) > 0$ has the following form

$$\widetilde{\lambda}_1^a(s,m) = \inf\{\frac{\int_\Omega a(z)|Du|^p\,dz}{\int_\Omega m(z)|u|^p\,dz} : u \in W_0^{1,p}(\Omega), u \neq 0\}. \tag{2.3}$$

In what follows, for convenience, by $\rho_{a_1,p} : W_0^{1,p}(\Omega) \to \mathbb{R}$ and $\rho_{a_2,q} : W_0^{1,q}(\Omega) \to \mathbb{R}$, we denote the modular functions defined by

$$\rho_{a_1,p}(Du) = \int_\Omega a_1(z)|Du|^p\,dz \text{ for all } u \in W_0^{1,p}(\Omega),$$

and

$$\rho_{a_2,q}(Du) = \int_\Omega a_2(z)|Du|^q\,dz \text{ for all } u \in W_0^{1,q}(\Omega).$$

Both are continuous and convex functions, and hence they are also weakly lower semicontinuous (Mazur's lemma).

Again all the other eigenvalues have nodal eigenfuntions. Using (2.2), (2.3), and the properties of $\hat{\lambda}_1^a(s)$ and of $\widetilde{\lambda}_1(s,m)$ mentioned above, we have the following two useful propositions (see Liu-Papageorgiou [13, 14]).

Proposition 2.1. *If $\theta \in L^\infty(\Omega)$, $\theta(z) \leq \hat{\lambda}_1^a(s)$ for a.a $z \in \Omega$ and $\theta \not\equiv \hat{\lambda}_1^a(s)$, then there exists $c_0 > 0$ such that*

$$c_0\|u\|_{1,s}^s \leq \rho_{a,s}(Du) - \int_\Omega \theta(z)|u|^s\,dz \text{ for all } u \in W_0^{1,s}(\Omega).$$

Proposition 2.2. *If $m, \hat{m} \in L^\infty(\Omega) \setminus \{0\}$, $0 \leq m(z) \leq \hat{m}(z)$ for a.a $z \in \Omega$ and $m \neq \hat{m}$, then $\widetilde{\lambda}_1^a(s,\hat{m}) < \widetilde{\lambda}_1^a(s,m)$.*

Our hypotheses on the weights $a_1(\cdot), a_2(\cdot)$ are the following:
$\underline{H_0}$: $a_1, a_2 \in C^{0,1}(\overline{\Omega})$, $0 < \hat{c} \leq a_1(z), a_2(z)$ for all $z \in \overline{\Omega}$.

We introduce the operators $A_p^{a_1} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* (\frac{1}{p} + \frac{1}{p'} = 1)$ and $A_q^{a_2} : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)^* (\frac{1}{q} + \frac{1}{q'} = 1)$ defined by

$$\langle A_p^{a_1}(u), h \rangle = \int_\Omega a_1(z) |Du|^{p-2} (Du, Dh)_{\mathbb{R}^N} \, dz \text{ for all } u, h \in W_0^{1,p}(\Omega),$$

and

$$\langle A_q^{a_2}(u), h \rangle = \int_\Omega a_2(z) |Du|^{q-2} (Du, Dh)_{\mathbb{R}^N} \, dz \text{ for all } u, h \in W_0^{1,q}(\Omega).$$

We have $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$ continuously and densely and so $W^{-1,q'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ continuously and densely (see Gasinski-Papageorgiou [4], p.46). We can define $V = A_p^{a_1} + A_q^{a_2} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$. For these operators, we have the following properties (see Gasinski-Papageorgiou [4], p.279).

Proposition 2.3. *If hypotheses H_0 hold and $K : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is equal to $A_p^{a_1}$ or $A_q^{a_2}$ or V, then $K(\cdot)$ is bounded (that is, maps bounded sets to bounded ones), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, that is,*
"if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty 0} \langle K(u_n), u_n - u \rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$."

If $u : \Omega \to \mathbb{R}$ is a measurable function, we set

$$u^+ = \max\{u, 0\} \text{ and } u^- = \max\{-u, 0\}.$$

Then $u = u^+ - u^-$, $|u| = u^+ + u^-$. If $u \in W_0^{1,p}(\Omega)$, then $u^\pm \in W_0^{1,p}(\Omega)$. If $u, v : \Omega \to \mathbb{R}$ are two measurable functions and $u \leq v$, then

$$[u,v] = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a } z \in \Omega\},$$
$$\text{int}_{C_0^1(\overline{\Omega})}[u,v] = \text{ interior in } C_0^1(\overline{\Omega}) \text{ of } [u,v] \cap C_0^1(\overline{\Omega}),$$
$$[u) = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \text{ for a.a } z \in \Omega\}.$$

We write that $0 \prec u$ if, for all $K \subseteq \Omega$ compact, $0 < c_K \leq u(z)$ for a.a $z \in K$. Evidently if $0 \prec u$, then $0 < u(z)$ for a.a $z \in \Omega$ and if $u \in C(\Omega)$, $u(z) > 0$ for all $z \in \Omega$, then $0 \prec u$.

3. Existence and Multiplicity of Solutions

In the first part, we investigate the case that $\limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}}$ is below $\hat{\lambda}_1^{a_1}(p) > 0$. In this case, we can only show the existence of a positive solution. In fact, if the differential operator is only $-\Delta_p^{a_1}$ (homogeneous case), then we prove that the positive solution is unique.

The hypotheses on the perturbation $f(z,x)$, are the following:
$\underline{H_1}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that
(i) for every $\rho > 0$, there exists $\hat{a}_\rho \in L^\infty(\Omega)$ such that

$$|f(z,x)| \leq \hat{a}_\rho(z) \text{ for a.a } z \in \Omega, \text{ all } |x| \leq \rho;$$

(ii) $\limsup_{x\to+\infty}\frac{f(z,x)}{x^{p-1}} \leq \hat{\lambda}_1^{a_1}(p)$ uniformly for a.a $z\in\Omega$ and if $F(z,x)=\int_0^x f(z,s)\,ds$, then

$$-\beta_0 \leq f(z,x)x - pF(z,x) \text{ for a.a } z\in\Omega, \text{ all } x\geq 0, \text{ some } \beta_0>0;$$

(iii) there exist a function $\theta\in L^\infty(\Omega)$ and $\delta>0$ such that

$$\theta(z)\leq \hat{\lambda}_1^{a_2}(q) \text{ for a.a } z\in\Omega,\ \theta\not\equiv\hat{\lambda}_1^{a_2}(q),$$
$$\limsup_{x\to 0^+}\frac{f(z,x)}{x^{q-1}}\leq\theta(z) \text{ uniformly for a.a } z\in\Omega,$$
$$0\leq f(z,x) \text{ for a.a } z\in\Omega,\ \text{all } 0\leq x\leq\delta.$$

Remark 3.1. Hypothesis H_1(iii) implies that $f(z,0)=0$ for a.a $z\in\Omega$. Since we are interested in positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+=[0,\infty)$, we may assume that $f(z,x)=0$ for a.a $z\in\Omega$, all $x\leq 0$. Hypothesis H_1(ii) permits for resonance with respect to $\hat{\lambda}_1^{a_1}(p)$ to occur as $x\to+\infty$. None of the earlier works on the subject allowed resonance (see [21, 25]). Also, we do not assume that f is globally positive (as is the case in [21]) or that it necessary changes sign near zero (as the case in [25]). Hypotheses H_1 are more general and include both cases.

First we consider the following auxiliary purely singular problem

$$\begin{cases} -\Delta_p^{a_1}u(z)-\Delta_q^{a_2}u(z)=\lambda u(z)^{-\eta} & \text{in } \Omega,\\ u|_{\partial\Omega}=0,\ \lambda>0, u>0. \end{cases} \tag{Q_λ}$$

From Papageorgiou-Zhang [25] (see the proof of Proposition 3.5), we have the following result.

Proposition 3.1. *If hypotheses H_0 hold and $\lambda>0$, then problem (Q_λ) has a unique positive solution*

$$\bar{u}_\lambda\in\operatorname{int}C_+,$$
$$\{\bar{u}_\lambda\}_{\lambda>0} \textit{ is nondecreasing and } \bar{u}_\lambda\to 0 \textit{ in } C_0^1(\overline{\Omega}) \textit{ as } \lambda\to 0^+.$$

We introduce the following two sets:

$$\mathcal{L}=\{\lambda>0:\ \text{problem } (p_\lambda) \text{ has a positive solution}\}$$
$$\text{(set of admissible parameters)},$$
$$S_\lambda=\ \text{set of positive solutions of } (p_\lambda).$$

Proposition 3.2. *If hypotheses H_0 and H_1 hold, then $\mathcal{L}=\overset{\circ}{\mathbb{R}}_+=(0,+\infty)$ and, for every $\lambda>0$, $\emptyset\neq S_\lambda\subseteq \operatorname{int}C_+$.*

Proof. Let $\lambda>0$. Using Proposition 3.1, we can find $\mu\in(0,\lambda)$ small such that $0\leq\bar{u}_\mu(z)\leq\delta$ for all $z\in\overline{\Omega}$, where $\delta>0$ is as in hypotheses H_1(iii). We introduce the Carathéodory function $g_\lambda(z,x)$ defined by

$$g_\lambda(z,x)=\begin{cases}\lambda\bar{u}_\mu(z)^{-\eta}+f(z,x^+) & \text{if } x\leq\bar{u}_\mu(z)\\ \lambda x^{-\eta}+f(z,x) & \text{if } \bar{u}_\mu(z)<x.\end{cases} \tag{3.1}$$

We set $G_\lambda(z,x)=\int_0^x g_\lambda(z,s)\,ds$ and consider the functional $\psi_\lambda: W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\psi_\lambda(u)=\frac{1}{p}\rho_{a_1,p}(Du)+\frac{1}{q}\rho_{a_2,q}(Du)-\int_\Omega G_\lambda(z,u)\,dz \text{ for all } u\in W_0^{1,p}(\Omega).$$

We see that $\psi_\lambda \in C^1(W_0^{1,p}(\Omega))$ (see Papageorgiou-Smyrlis [20, Proposition 3]).
Claim: $\psi_\lambda(\cdot)$ is coercive.

For a.a $z \in \Omega$ and all $x > 0$, we have

$$\begin{aligned} \frac{d}{dx}\left(\frac{F(z,x)}{x^p}\right) &= \frac{f(z,x)x^p - px^{p-1}F(z,x)}{x^{2p}} \\ &= \frac{f(z,x)x - pF(z,x)}{x^{p+1}} \\ &\geq -\frac{\beta_0}{x^{p-1}} \quad \text{(see hypothesis } H_1(\text{iii})), \end{aligned}$$

$$\Rightarrow \frac{F(z,x)}{x^p} - \frac{F(z,v)}{v^p} \geq \frac{\beta_0}{p}\left[\frac{1}{x^p} - \frac{1}{v^p}\right] \text{ for a.a } z \in \Omega, \text{ all } x \geq v > 0. \tag{3.2}$$

On account of hypothesis H_1(ii), we have

$$\limsup_{x \to +\infty} \frac{pF(z,x)}{x^p} \leq \hat{\lambda}_1^{a_1}(p) \quad \text{uniformly for a.a } z \in \Omega. \tag{3.3}$$

If (3.2), we let $x \to +\infty$ and use (3.3) to find

$$\begin{aligned} &\frac{\hat{\lambda}_1^{a_1}(p)}{p} - \frac{F(z,v)}{v^p} \geq -\frac{\beta_0}{p}\frac{1}{v^p} \quad \text{for a.a } z \in \Omega, \text{ all } v > 0 \\ &\Rightarrow -\beta_0 \leq \hat{\lambda}_1^{a_1}(p)v^p - pF(z,v) \quad \text{for a.a } z \in \Omega, \text{ all } v > 0. \end{aligned} \tag{3.4}$$

Suppose that the Claim is not true. We can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$\|u_n\| \to \infty \text{ and } \psi_\lambda(u_n) \leq M \text{ for all } n \in \mathbb{N}, \text{ some } M > 0. \tag{3.5}$$

On account of (3.1), (3.4), and (3.5), we may assume $u_n \geq 0$ for all $n \in \mathbb{N}$. We set $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega),\ y_n \to y \text{ in } L^p(\Omega),\ y \geq 0.$$

From (3.5), we have

$$\frac{1}{p}\rho_{a_1,p}(Dy_n) + \frac{1}{q}\frac{1}{\|u_n\|^{p-q}}\rho_{a_2,q}(Dy_n) - \int_\Omega \frac{G_\lambda(z,u_n)}{\|u_n\|^p}\,dz \leq \frac{M}{\|u\|^p} \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

Note that

$$\begin{aligned} \int_\Omega \frac{G_\lambda(z,u_n)}{\|u_n\|^p}\,dz = &\int_{\{u_n \leq \bar{u}_\mu\}} \frac{\lambda}{\|u_n\|^p}\bar{u}_\mu^{-\eta}u_n\,dz + \int_{\{\bar{u}_\mu < u_n\}} \frac{\lambda}{(1-\eta)\|u_n\|^p}[u_n^{1-\eta} - \bar{u}_\mu^{1-\eta}]\,dz \\ &+ \int_{\{\bar{u}_\mu < u_n\}} \frac{\lambda}{\|u_n\|^p}\bar{u}_\mu^{1-\eta}\,dz - \int_\Omega \frac{F(z,u_n)}{\|u_n\|^p}\,dz \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.7}$$

Observe that

$$\begin{aligned} &\left|\int_{\{u_n \leq \bar{u}_\mu\}} \frac{\lambda}{\|u_n\|^p}\bar{u}_\mu^{-\eta}u_n\,dz\right| \leq \frac{\lambda}{\|u_n\|^p}\int_\Omega \bar{u}_\mu^{1-\eta}\,dz \leq \frac{\lambda c_1}{\|u_n\|^p} \\ &\qquad \text{for some } c_1 > 0, \text{ all } n \in \mathbb{N} \text{ (recall } \bar{u}_\mu \in \operatorname{int}C_+), \\ &\Rightarrow \int_{\{u_n \leq \bar{u}_\mu\}} \frac{\lambda}{\|u_n\|^p}\bar{u}_\mu^{-\eta}u_n\,dz \to 0 \quad \text{as} \quad n \to \infty \quad \text{(see (3.5))}. \end{aligned} \tag{3.8}$$

We also have

$$\left|\int_{\{\bar{u}_\mu<u_n\}}\frac{\lambda}{(1-\eta)\|u_n\|^p}[u_n^{1-\eta}-\bar{u}_\mu^{1-\eta}]\,dz\right|\le\frac{\lambda c_2}{(1-\eta)\|u_n\|^p}\ \text{for some } c_2>0,\ \text{all } n\in\mathbb{N}$$
(using Hewitt-Stromberg [8, Theorem 13.17]),

$$\Rightarrow\int_{\{\bar{u}_\mu<u_n\}}\frac{\lambda}{(1-\eta)\|u_n\|^p}[u_n^{1-\eta}-\bar{u}_\mu^{1-\eta}]\,dz\to0\ \text{as } n\to\infty. \tag{3.9}$$

Moreover, from (3.5) and [8, p.196], we have

$$\frac{\lambda}{\|u_n\|^p}\int_{\{\bar{u}_\mu<u_n\}}\bar{u}_\mu^{1-\eta}\,dz\to0\ \text{as } n\to\infty. \tag{3.10}$$

Finally, on account of hypothesis H_1(ii), we have

$$\int_\Omega\frac{F(z,u_n)}{\|u_n\|^p}\,dz\to\int_\Omega\frac{1}{p}\hat{\theta}(z)y^p\,dz\quad\text{as}\quad n\to\infty \tag{3.11}$$
with $\hat{\theta}\in L^\infty(\Omega),\ \theta(z)\le\hat{\lambda}_1^{a_1}(p)$ for a.a $z\in\Omega$
(see Aizicovici-Papageorgiou-Staicu [2, Proposition 16]).

If we return to (3.6), pass to the limit as $n\to\infty$ and use (3.5) and (3.7)−(3.11) and the fact that $\rho_{a_1,p}(\cdot)$ is sequentially weakly lower semicontinuous, we obtain

$$\rho_{a_1,p}(Dy)\le\int_\Omega\hat{\theta}(z)y^p\,dz\le\hat{\lambda}_1^a(p)\|y\|_p^p\quad(\text{see (2.2)})$$
$$\Rightarrow y=0\ \text{or}\ y=\xi\hat{u}_1(p)\ \text{with}\ \xi>0.$$

If $y=0$, then

$$\rho_{a_1,p}(Dy_n)\to0\ \text{as } n\to\infty\Rightarrow y_n\to0\ \text{in } W_0^{1,p}(\Omega)\ \text{as } n\to\infty\quad(\text{see hypotheses } H_0).$$

This contradicts the fact that $\|y_n\|=1$ for all $n\in\mathbb{N}$. If $y=\xi\hat{u}_1(p)$ with $\xi>0$, then $y\in\operatorname{int}C_+$. Thus

$$u_n(z)\to+\infty\quad\text{for a.a } z\in\Omega. \tag{3.12}$$

It follows that

$$\begin{aligned}&\int_\Omega[\hat{\lambda}_1^{a_1}(p)u_n^p-pF(z,u_n)]\,dz+\frac{1}{q}\rho_{a_2,q}(Du_n)\\&\le\int_{\{u_n\le\bar{u}_\mu\}}\lambda\bar{u}_\mu^{-\eta}u_n\,dz+\int_{\{\bar{u}_\mu<u_n\}}\frac{\lambda}{1-\eta}[u_n^{1-\eta}-\bar{u}_\mu^{1-\eta}]\,dz+\int_{\{\bar{u}_\mu<u_n\}}\lambda\bar{u}_\mu^{1-\eta}\,dz+M\\&\qquad(\text{see (3.5) and (3.1)})\\&\le\lambda c_3\|u_n\|_q^{1-\eta}+M\quad\text{for some}\quad c_3>0,\ \text{all}\ n\in\mathbb{N}\\\Rightarrow&\frac{1}{q}\hat{\lambda}_1^{a_2}(q)\|u_n\|_q^q\le\lambda c_3\|u_n\|_q^{1-\eta}+\hat{M}\quad\text{for some } \hat{M}>0,\ \text{all } n\in\mathbb{N}\ (\text{see (3.4)})\\\Rightarrow&\{u_n\}_{n\in\mathbb{N}}\subseteq L^q(\Omega)\ \text{is bounded.}\end{aligned} \tag{3.13}$$

On the other hand, from (3.12) and Fatou's lemma, we have

$$\int_\Omega u_n^q\,dz\to+\infty\quad\text{as}\quad n\to\infty. \tag{3.14}$$

Comparing (3.13) and (3.14), we have a contradiction. So, $\psi_\lambda(\cdot)$ is coercive and this proves the Claim. Using the Sobolev Embedding Theorem, we see that $\psi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. Combining this with the Claim and using the Weierstrass-Tonelli theorem, we infer that there exists $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\psi_\lambda(u_\lambda) = \inf\{\psi_\lambda(u) : u \in W_0^{1,p}(\Omega)\}. \tag{3.15}$$

On account of hypothesis H_1(iii), given $\varepsilon > 0$, we can find $\hat{\delta} = \hat{\delta}(\varepsilon) > 0$ such that

$$F(z,x) \le \frac{1}{q}[\theta(z)+\varepsilon]x^q \text{ for a.a } z \in \Omega, \text{ all } 0 \le x \le \hat{\delta}. \tag{3.16}$$

Recall that $\bar{u}_\mu \in \mathrm{int}C_+$. Using Hu-Papageorgiou [11, Proposition 2.86], we can find $t \in (0,1)$ small such that

$$0 \le t\hat{u}_1(q)(s) \le \bar{u}_\mu(z) \quad \text{for all } z \in \overline{\Omega}. \tag{3.17}$$

Then

$$\begin{aligned}\psi_\lambda(t\hat{u}_1(q)) \le{}& \frac{t^p}{p}\rho_{a_1,p}(D\hat{u}_1(p)) + \frac{t^q}{q}[\rho_{a_2,q}(D\hat{u}_1(q)) - \int_\Omega \theta(z)\hat{u}_1(q)^q\,dz - \varepsilon\|\hat{u}_1(q)\|_q^q] \\ & -\lambda t\int_\Omega \bar{u}_\mu^{-\eta}\hat{u}_1(q)\,dz \quad (\text{see (3.16), (3.17), and (3.43)}).\end{aligned} \tag{3.18}$$

Using Proposition 2.1, we have

$$c_4\|\hat{u}(q)\|_{1,q} \le \rho_{a_2,q}(D\hat{u}_1(q)) - \int_\Omega \theta(z)\hat{u}_1(q)^q\,dz \text{ for some } c_4 > 0. \tag{3.19}$$

Also, from (2.2), we know that

$$\varepsilon\|\hat{u}_1(q)\|_q^q \le \frac{\varepsilon}{\hat{\lambda}_1^{a_2}(q)}\rho_{a_2,q}(Du) \le \frac{\varepsilon\|a_2\|_\infty}{\hat{\lambda}_1^{a_2}(q)}\|u\|_{1,q}^q. \tag{3.20}$$

Choosing $\varepsilon \in (0, \frac{c_4\hat{\lambda}_1^{a_2}(q)}{\|a_2\|_\infty})$, from (3.19) and (3.20), we see that

$$\rho_{a_2,q}(D\hat{u}_1(q)) - \int_\Omega \theta(z)\hat{u}_1(q)^q\,dz - \varepsilon\|\hat{u}_1(q)\|_q^q = c_5 > 0. \tag{3.21}$$

Since $\bar{u}_\mu \in \mathrm{int}C_+$, we have that $\hat{c}\hat{d} \le \bar{u}_\mu$ for some $\hat{c} > 0$ and with $\hat{d}(\cdot) = d(\cdot, \partial\Omega)$. So, using Hardy's inequality (see [11, p.479]) we infer that $\bar{u}_\mu^{-\eta}\hat{u}_1(q) \in L^p(\Omega)$. Returning to (3.18), using (3.21) and recalling that $t \in (0,1)$, $q < p$, we obtain $\psi_\lambda(t\hat{u}_1(q)) \le c_6 t^q - c_7 t$ for some $c_6, c_7 > 0$. Taking $t \in (0,1)$ even smaller if necesssary, we have

$$\begin{aligned}\psi_\lambda(t\hat{u}_1(q)) < 0 \Rightarrow & \psi_\lambda(u_\lambda) < 0 = \psi_\lambda(0) \quad (\text{see (3.15)}), \\ \Rightarrow & u_\lambda \ne 0.\end{aligned}$$

From (3.15), we have

$$\begin{aligned}&\langle \psi_\lambda'(u_\lambda), h\rangle = 0 \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow &\langle V(u_\lambda), h\rangle = \int_\Omega g_\lambda(z,u_\lambda)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega).\end{aligned} \tag{3.22}$$

In (3.22), we choose the test function $h=(\bar{u}_\mu-u_\lambda)^+\in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\langle V(u_\lambda),(\bar{u}_\mu-u_\lambda)^+\rangle &= \int_\Omega [\lambda\bar{u}_\mu^{-\eta}+f(z,u_\lambda^+)](\bar{u}_\mu-u_\lambda)^+\,dz \quad \text{(see (3.1))}\\ &\geq \int_\Omega \lambda\bar{u}_\mu^{-\eta}(\bar{u}_\mu-u_\lambda)^+\,dz \quad \text{(hypothesis } H_1\text{(iii))}\\ &\geq \int_\Omega \mu\bar{u}_\mu^{-\eta}(\bar{u}_\mu-u_\lambda)\,dz \quad (\text{since } u<\lambda)\\ &=\langle V(\bar{u}_\mu),(\bar{u}_\mu-u_\lambda)^+\rangle \quad \text{(see Proposiiton 3.1)},\\ &\Rightarrow\langle V(\bar{u}_\mu)-V(u_\lambda),(\bar{u}_\mu-u_\lambda)^+\rangle\leq 0,\end{aligned}$$

$$\Rightarrow \bar{u}_\mu\leq u_\lambda \qquad \text{(see Proposition 2.3).} \tag{3.23}$$

From (3.23) and (3.1), it follows that u_λ is a positive solution of (p_λ). From Marino-Winkert [15], we have that $u_\lambda\in L^\infty(\Omega)$. Then

$$\begin{aligned}|\lambda u_\lambda^{-\eta}+f(z,u_\lambda)| &\leq \lambda\bar{u}_\mu^{-\eta}+c_8 \text{ for some } c_8>0 \text{ (see (3.23) and hypothesis } H_1\text{(i))}\\ &\leq c_9[\lambda\hat{d}^{-\eta}+1] \text{ for some } c_9>0\\ &\text{(with } \hat{d}(z)=d(z,\partial\Omega) \text{ and since } \bar{u}_\mu\in \mathrm{int}C_+)\\ &\leq \lambda c_{10}\hat{d}^{-\eta} \text{ for some } c_{10}>0.\end{aligned}$$

Invoking Giacomoni-Kumar-Sreenadh [5, Theorem 1.7], we have that $u_\lambda\in\mathrm{int}C_+$. Therefore $\mathcal{L}=\overset{\circ}{\mathbb{R}}_+=(0,+\infty)$. If $\hat{u}\in S_\lambda$, then

$$\begin{aligned}\langle V(\hat{u}),(\bar{u}_\mu-\hat{u})^+\rangle &= \int_\Omega [\lambda\bar{u}_\mu^{-\eta}+f(z,\hat{u})](\bar{u}_\mu-u)^+\,dz\\ &\geq \int_\Omega \mu\hat{u}_\mu^{-\eta}(\bar{u}_\mu-\hat{u})^+\,dz\\ &=\langle V(\bar{u}_\mu),(\bar{u}_\mu-u_\lambda)^+\rangle\\ &\Rightarrow \bar{u}_\mu\leq u_\lambda.\end{aligned}$$

Then, as we did for u_λ, we show that $\hat{u}\in\mathrm{int}C_+$. Therefore $\emptyset\neq S_\lambda\subseteq\mathrm{int}C_+$. ☐

If $p=q$, we can have the uniqueness of the positive solution. Thus the problem under consideration is now the following

$$\left\{\begin{array}{l} -\Delta_p^a u(z)=\lambda u(z)^{-\eta}+f(z,u(z)) \quad \text{in } \Omega,\\ u|_{\partial\Omega}=0,\ 1<\eta<1<p,\ \lambda>0,\ u>0.\end{array}\right. \tag{p'_λ}$$

The hypotheses on the perturbation $f(z,x)$ are:

$\underline{H_1'}$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that

(i) for every $\rho>0$, there exists $\hat{a}_\rho\in L^\infty(\Omega)$ such that $|f(z,x)|\leq\hat{a}_\rho(z)$ for a.a $z\in\Omega$ and all $|x|\leq\rho$;

(ii) for a.a $z\in\Omega$, $x\to\frac{f(z,x)}{x^{p-1}}$ is nonincreasing on $\overset{\circ}{\mathbb{R}}_+=(0,+\infty)$;

(iii) $\lim_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}\leq\hat{\lambda}_1^{a_1}(p)$ uniformly for a.a $z\in\Omega$ and

$$-\beta_0\leq f(z,x)x-pF(z,x) \text{ for a.a } z\in\Omega, \text{ all } x\geq 0, \text{ some } \beta_0>0;$$

(iv) there exist a function $\theta \in L^{\infty}(\Omega)$ and $\delta > 0$ such that

$$\theta(z) \leq \hat{\lambda}_1^{a_2}(q) \text{ for a.a } z \in \Omega,\ \theta \not\equiv \hat{\lambda}_1^{a_2}(q),$$

$$\limsup_{x \to 0^+} \frac{f(z,x)}{x^{q-1}} \leq \theta(z) \text{ uniformly for a.a } z \in \Omega,$$

and

$$0 \leq f(z,x) \text{ for a.a } z \in \Omega, \text{ all } 0 \leq x \leq \delta.$$

Proposition 3.3. *If $a \in C^{0,1}(\overline{\Omega})$, $0 < \hat{c} \leq a(z)$ for all $z \in \overline{\Omega}$, and hypotheses H_1' hold, then, for every $\lambda > 0$, problem (p_λ') has a unique positive solution $u_\lambda \in \mathrm{int}C_+$.*

Proof. From Proposition 3.2, we already know that problem (p_λ') has a solution $u_\lambda \in \mathrm{int}C_+$. Let $v_\lambda \in W_0^{1,p}(\Omega)$ be another positive solution of (p_λ'). Again we have $v_\lambda \in \mathrm{int}C_+$. On account of [11, Proposition 2.86], we have

$$\frac{u_\lambda}{v_\lambda} \in L^{\infty}(\Omega) \quad \text{and} \quad \frac{v_\lambda}{u_\lambda} \in L^{\infty}(\Omega). \tag{3.24}$$

We introduce the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\rho_{a,p}(Du^{1/p}) & \text{if } u \geq 0,\ u^{1/p} \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\mathrm{dom}\, j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of $j(\cdot)$). From Diaz-Saa [2], we know that $j(\cdot)$ is convex. Let $h = u_\lambda^p - v_\lambda^p \in C_0^1(\overline{\Omega})$. On account of (3.24), for $t \in (0,1)$ small, we have $u_\lambda^p + th \in \mathrm{dom}\, j$ and $v_\lambda^p + th \in \mathrm{dom}\, j$. Exploiting the convexity of $j(\cdot)$, we can compute the directional derivatives of $j(\cdot)$ at u_λ^p and at v_λ^p in the direction h and we have

$$j'(u_\lambda^p)(h) = \frac{1}{p}\int_\Omega \frac{-\Delta_p^a u_\lambda}{u_\lambda^{p-1}} h\,dz = \frac{1}{p}\int_\Omega [\frac{\lambda}{u^{p+\eta-1}} + \frac{f(z,u_\lambda)}{u_\lambda^{p-1}}]h\,dz$$

and

$$j'(v_\lambda^p)(h) = \frac{1}{p}\int_\Omega \frac{-\Delta_p^a v_\lambda}{v_\lambda^{p-1}} h\,dz = \frac{1}{p}\int_\Omega [\frac{\lambda}{v^{p+\eta-1}} + \frac{f(z,v_\lambda)}{v_\lambda^{p-1}}]h\,dz \text{ (see also Diaz-Saa [2, Lemma 2]).}$$

The convexity of $j(\cdot)$ implies the monotonicity of its directional derivative. Thus

$$\begin{aligned} 0 \leq \int_\Omega [\lambda(\frac{1}{u^{p+\eta-1}} - \frac{1}{v^{p+\eta-1}}) + (\frac{f(z,u_\lambda)}{u_\lambda^{p-1}} - \frac{f(z,v_\lambda)}{v_\lambda^{p-1}})](u_\lambda^p - v_\lambda^p)\,dz \leq 0 \\ \text{(see hypothesis } H_1'\text{(iv)),} \\ \Rightarrow u_\lambda = v_\lambda. \end{aligned}$$

Therefore, for every $\lambda > 0$, problem (p_λ') has a unique positive solution. ☐

Now we turn our attention to the case that the quotient function $x \to \frac{f(z,x)}{x^{p-1}}$ asymptotically as $x \to +\infty$, stays above the principal eigenvalue $\hat{\lambda}_1^{a_1}(p) > 0$. As we can see, now the situation changes and the set at admissible parameters $\mathcal{L}$ is a bounded interval in $\overset{\circ}{\mathbb{R}}_+ = (0, +\infty)$.

The hypotheses on the perturbation $f(z,x)$ are the following:

$\underline{H_2}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function such that

(i) for every $\rho > 0$, there exists $\hat{a}_\rho \in L^\infty(\Omega)$ such that $|f(z,x)| \le \hat{a}_\rho(z)$ for a.a $z \in \Omega$, all $0 \le x \le \rho$;

(ii) there exist a function $\eta_0 \in L^\infty(\Omega)$ and $\tau \in (q,p)$ such that

$$\hat{\lambda}_1^{a_1}(p) \le \liminf_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \le \limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \le \eta_0(z) \text{ uniformly for a.a } z \in \Omega,$$

$$0 < \beta_0 \le \liminf_{x\to+\infty} \frac{pF(z,x) - f(z,x)x}{x^\tau} \text{ uniformly for a.a } z \in \Omega;$$

(iii) there exist a function $\theta \in L^\infty(\Omega)$ and $\delta > 0$ such that

$$\theta(z) \le \hat{\lambda}_1^{a_1}(p) \text{ for a.a } z \in \Omega,\ \theta \not\equiv \hat{\lambda}_1^{a_1}(p),$$
$$\limsup_{x\to 0^+} \frac{f(z,x)}{x^{q-1}} \le \theta(z) \text{ uniformly for a.a } z \in \Omega,$$
$$0 \le f(z,x) \text{ for a.a } z \in \Omega, \text{ all } 0 \le x \le \delta;$$

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that, for a.a $z \in \Omega$, $x \to f(z,x) + \hat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 3.2. As before, we have $f(z,0) = 0$ for a.a $z \in \Omega$ (see hypothesis H_2(iii)). Without any loss of generality, we may assume that $f(z,x) = 0$ for a.a $z \in \Omega$ and all $x \le 0$.

Proposition 3.4. *If hypotheses H_0, H_2 hold, then $\mathcal{L} \neq \emptyset$ and, for every $\lambda \in \mathcal{L}$, $\emptyset \neq S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. Using Proposition 3.1, we can find $\mu \in (0,\lambda)$ small such that $0 \le \bar{u}_\mu(z) \le \delta$ for all $z \in \overline{\Omega}$. We introduce the Carathéodory function $g_\lambda(z,x)$ defined by

$$g_\lambda(z,x) = \begin{cases} \lambda \bar{u}_\mu(z)^{-\eta} + f(z,x^+) & \text{if } \quad x \le \bar{u}_\mu(z) \\ \lambda x^{-\eta} + f(z,x) & \text{if } \quad \bar{u}_\mu(z) < x. \end{cases} \tag{3.25}$$

We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)\,ds$ and consider the C^1-functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega G_\lambda(z,u)\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Claim 1: For every $\lambda > 0$, $\psi_\lambda(\cdot)$ satisfies the C-condition.

We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\psi_\lambda(u_n)| \le c_{11} \quad \text{for some } c_{11} > 0, \text{ all } n \in \mathbb{N}, \tag{3.26}$$

$$(1 + \|u_n\|)\psi_\lambda'(u_n) \to 0 \quad \text{in } W_0^{1,p'}(\Omega) \text{ as } n \to \infty. \tag{3.27}$$

From (3.27), we have

$$|\langle V(u_n), h\rangle - \int_\Omega g_\lambda(z,u_n)h\,dz| \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \text{ for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+. \tag{3.28}$$

In (3.28), we choose the test function $h = -u_n^- \in W_0^{1,p}(\Omega)$ and then

$$\hat{c}\|Du_n^-\|_p^p \le \varepsilon_n \text{ for all } n \in \mathbb{N} \Rightarrow u_n^- \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{3.29}$$

We show that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. If this is not true, then by passing to a subsequence if necessary, we may assume that

$$\|u_n^+\| \to \infty \quad \text{as } n \to \infty. \tag{3.30}$$

Setting $y_n = \frac{u_n^+}{\|u_n^+\|}$ for all $n \in \mathbb{N}$, one sees that $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. Thus we may assume that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(\Omega)$ and $y_n \to y$ in $L^p(\Omega)$, $y \geq 0$. From (3.27) and (3.29), we have

$$|\langle V(u_n^+), h\rangle - \int_\Omega g_\lambda(z, u_n) h dz| \leq \varepsilon_n' \|h\| \text{ for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n' \to 0^+; \tag{3.31}$$

$$\Rightarrow \langle A_p^{a_1}(y_n), h\rangle + \frac{1}{\|u_n^+\|^{p-q}} \langle A_q^{a_2}(y_n), h\rangle \leq \frac{\varepsilon_n'}{\|u_n^+\|^p}\|h\| + \int_\Omega \frac{g_\lambda(z,u_n)}{\|u_n^+\|^{p-1}} h dz \tag{3.32}$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

Using (3.25), we have

$$|\int_\Omega \frac{g_\lambda(z,u_n)}{\|u_n^+\|^{p-1}} h dz| = \int_{\{u_n^+ \leq \bar{u}_\mu\}} \frac{\lambda}{\|u_n^+\|^{p-1}} \bar{u}_\mu^{-\eta} h dz + \int_{\{\bar{u}_\mu < u_n^+\}} \frac{\lambda}{\|u_n^+\|^{p-1}} (u_n^+)^{-\eta} h dz$$
$$+ \int_\Omega \frac{f(z,u_n^+)}{\|u_n^+\|^{p-1}} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}.$$

Note that

$$\begin{aligned}
&|\int_{\{u_n^+ \leq \bar{u}_\mu\}} \frac{\lambda}{\|u_n^+\|^{p-1}} \bar{u}_\mu^{-\eta} h dz| \\
\leq & \frac{\lambda}{\|u_n^+\|^{p-1}} \int_\Omega \bar{u}_\mu^{1-\eta} \frac{|h|}{\bar{u}_\mu} dz \\
\leq & \frac{\lambda c_{12}}{\|u_n^+\|^{p-1}} \int_\Omega \frac{|h|}{\hat{d}} dz \quad \text{for some } c_{12} > 0 \\
& (\text{recall that since } \bar{u}_\mu \in \operatorname{int} C_+, \text{ we have } c_0 \hat{d} \leq \bar{u}_\mu \text{ for some } c_0 > 0) \\
\leq & \frac{\lambda c_{13}}{\|u_n^+\|^{p-1}} \|Dh\|_p \quad \text{for some } c_{13} > 0, \text{ all } n \in \mathbb{N} \\
& (\text{using Hardy's inequality, see [11, p.479]}),
\end{aligned}$$

$$\Rightarrow \int_{\{u_n^+ \leq \bar{u}_\mu\}} \frac{\lambda}{\|u_n^+\|^{p-1}} \bar{u}_\mu^{-\eta} h dz \to 0 \quad \text{as} \quad n \to \infty \quad (\text{see (3.30)}). \tag{3.33}$$

Similarly, we have

$$\int_{\{\bar{u}_\mu < u_n^+\}} \frac{\lambda}{\|u_n^+\|^{p-1}} (u_n^+)^{-\eta} h dz \to 0 \quad \text{as} \quad n \to \infty. \tag{3.34}$$

Moreover, using hypothesis H_2(iii), we have

$$\frac{f(\cdot, u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \hat{\eta}(\cdot) y(\cdot)^{p-1} \text{ in } L^{p'}(\Omega) \text{ as } n \to \infty \text{ with } \hat{\lambda}_1^{a_1}(p) \leq \hat{\eta}(z) \leq \eta_0(z) \text{ for a.a } z \in \Omega \tag{3.35}$$

(see Aizicovici-Papageorgiou-Staicu [1, Proposition 16]). Using (3.35), we have

$$\int_\Omega \frac{f(z,u_n^+)}{\|u_n^+\|^{p-1}} h dz \to \int_\Omega \hat{\eta}(z) y^{p-1} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.36}$$

From (3.33), (3.33), (3.34), and (3.36), we obtain

$$\int_\Omega \frac{g_\lambda(z,u_n^+)}{\|u_n^+\|^{p-1}} h dz \to \int_\Omega \hat{\eta}(z) y^{p-1} dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

Using (3.32) with $h = y_n - y \in W_0^{1,p}(\Omega)$ and (3.30), we see that

$$\lim_{n\to\infty} \langle A_p^{a_1}(y_n), y_n - y\rangle = 0,$$
$$\Rightarrow y_n \to y \text{ in } W_0^{1,p}(\Omega),\ \|y\| = 1,\ y \geq 0 \text{ (see Proposition 2.3).} \tag{3.37}$$

Taking the limit as $n \to \infty$ in (3.32), we obtain

$$\langle A_p^{a_1}(y), h\rangle = \int_\Omega \hat{\eta}(z) y^{p-1} h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$
$$\Rightarrow -\Delta_p^{a_1} y(z) = \hat{\eta}(z) y(z)^{p-1} \text{ in } \Omega,\ y|_{\partial\Omega} = 0. \tag{3.38}$$

If $\hat{\eta} \not\equiv \hat{\lambda}_1^{a_1}(p)$ (see (3.35)), then we see from (3.35) and Proposition 2.2 that

$$\widetilde{\lambda}_1^{a_1}(p, \hat{\eta}) < \widetilde{\lambda}_1^{a_1}(p, \hat{\lambda}_1^{a_1}(p)) = 1, \Rightarrow y \text{ must be nodal (see (3.38)),}$$

which contradicts the fact that $y \geq 0$ (see (3.37)).

Next, one supposes that $\hat{\eta}(z) = \hat{\lambda}_1^{a_1}(p)$ for a.a $z \in \Omega$ (see (3.35)). From (3.38), it follows that

$$y = \theta \hat{u}_1(p) \quad \text{for some } \theta > 0 \text{ (see (3.37))} \Rightarrow y \in \operatorname{int} C_+.$$

We infer that

$$u_n^+(z) \to +\infty \quad \text{for a.a } z \in \Omega, \text{ as } n \to \infty. \tag{3.39}$$

From (3.26) and (3.29), we see that

$$-\rho_{a_1,p}(Du_n^+) - \frac{p}{q}\rho_{a_2,q}(Du_n^+) + \int_\Omega pG_\lambda(z, u_n)\,dz \leq c_{14} \text{ for some } c_{14} > 0 \text{ and all } n \in \mathbb{N}. \tag{3.40}$$

In (3.31), we use the test function $h = u_n^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\rho_{a_1,p}(Du_n^+) + \rho_{a_2,q}(Du_n^+) - \int_\Omega g_\lambda(z, u_n) u_n^+\,dz \leq \varepsilon_n' \|u_n^+\| \text{ for all } n \in \mathbb{N}. \tag{3.41}$$

Adding (3.40) and (3.41), we obtain

$$\int_\Omega [pG_\lambda(z, u_n) - g_\lambda(z, u_n) u_n^+]\,dz \leq c_{14} + \varepsilon_n' \|u_n^+\| + (\frac{p}{q} - 1)\rho_{a_2,q}(Du_n^+),$$
$$\Rightarrow \int_\Omega [pF(z, u_n^+) - f(z, u_n^+) u_n^+]\,dz \leq c_{15}(1 + \|u_n^+\|^q) + \varepsilon_n' \|u_n^+\|$$

for some $c_{15} > 0$, all $n \in \mathbb{N}$ (see (3.25)),

$$\Rightarrow \int_\Omega \frac{pF(z, u_n^+) - f(z, u_n^+) u_n^+}{\|u_n^+\|^\tau}\,dz \leq c_{15}[\frac{1}{\|u_n^+\|^\tau} + \frac{1}{\|u_n^+\|^{\tau-q}}] + \frac{\varepsilon_n'}{\|u_n^+\|^{\tau-1}}. \tag{3.42}$$

On account of hypothesis H_2(ii), (3.39), and $y = \theta \hat{u}_1(p)$, we have by Fatou's lemma that

$$0 < \liminf \int_\Omega \frac{pF(z, u_n^+) - f(z, u_n^+) u_n^+}{\|u_n^+\|^\tau}\,dz,$$

a contradiction (see (3.30)). Therefore $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. From (3.29), it follows that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$, $u_n \to u$ in $L^p(\Omega)$. In (3.28), we use the test function $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.25). Then

$$\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0 \ \Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty.$$

This shows that $\psi_\lambda(\cdot)$ satisfies the C-condition. This proves Claim 1.
Claim 2: There exists $\lambda_0 > 0$ such that, for all $\lambda \in (0,\lambda_0)$, there exists $\rho_\lambda > 0$ such that $\psi_\lambda(u) \geq m_\lambda > 0$ for all $\|u\| = \rho_\lambda$.

Let $r > p$. For given $\varepsilon > 0$, we can find $c_{16} = c_{16}(\varepsilon, r) > o$ such that

$$f(z,x) \leq (\theta(z)+\varepsilon)x^{p-1} + c_{16}x^{r-1} \quad \text{for a.a } z \in \Omega, \text{ all } x \geq 0,$$
$$\Rightarrow F(z,x) \leq \frac{1}{p}(\theta(z)+\varepsilon)x^p + \frac{c_{16}}{r}x^r \quad \text{for a.a } z \in \Omega, \text{ all } x \geq 0, \tag{3.43}$$

For $u \in W_0^{1,p}(\Omega)$, we have

$$\psi_\lambda(u) \geq \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}[\rho_{a_2,q}(Du) - \int_\Omega \theta(z)|u|^q\,dz - \varepsilon\|u\|_q^q] - c_{17}\|u\|^r$$
$$- \int_{\{u\leq \bar{u}_\mu\}} \lambda \bar{u}_\mu^{-\eta} u\,dz - \int_{\{\bar{u}_\mu < u\}} \frac{\lambda}{1-\eta}(u^{1-\eta} - \bar{u}_\mu^{1-\eta})\,dz - \int_{\{\bar{u}_\mu < u\}} \lambda \bar{u}_\mu^{1-\eta}\,dz.$$

Using Proposition 2.1 and choosing $\varepsilon \in (0,1)$ small, we have $\rho_{a_2,q}(Du) - \int_\Omega \theta(z)|u|^q\,dz - \varepsilon\|u\|_q^q \geq 0$. In view of Hewitt-Stromberg [8, Theorem 13.17], we have

$$\psi_\lambda(u) \geq \frac{\hat{c}}{p}\|u\|^p - c_{18}(\lambda\|u\|^{1-\eta} + \|u\|^r) \quad \text{for some } c_{18} > 0$$
$$= [\frac{\hat{c}}{p} - c_{18}(\lambda\|u\|^{1-(p+\eta)} + \|u\|^{r-p})]\|u\|^p. \tag{3.44}$$

Let $\gamma_\lambda(t) = \lambda t^{1-(p+\eta)} + t^{r-p}$ for all $t > 0$. Evidently, $\gamma_\lambda(t) \to +\infty$ as $t \to 0^+$ and $t \to +\infty$. So, we can find $t_0 > 0$ such that $\gamma_\lambda(t_0) = \inf_{t>0}\gamma_\lambda(t)$, which implies

$$\gamma_\lambda'(t_0) = 0 \Rightarrow (r-p)t_0^{r-p-1} = \lambda(p+\eta-1)t_0^{-(p+\eta)},$$
$$\Rightarrow t_0 = t_0(\lambda) = (\frac{\lambda(p+\eta-1)}{r-p})^{\frac{1}{r+\eta-1}}.$$

Note that

$$\gamma_\lambda(t_0) = \lambda(\frac{r-p}{\lambda(p+\eta-1)})^{\frac{p+\eta-1}{r+\eta-1}} + (\frac{\lambda(p+\eta-1)}{r-p})^{\frac{r-p}{r+\eta-1}}.$$

Since $p < r$, we see that $\gamma_\lambda(t_0) \to 0^+$ as $\lambda \to 0^+$. So, we can find $\lambda_0 > 0$ such that

$$0 < \gamma_\lambda(t_0) < \frac{\hat{c}}{pc_{18}} \text{ for all } 0 < \lambda < \lambda_0.$$

From (3.44), it follows that $\psi_\lambda(u) \geq m_\lambda > 0$ for all $\|u\| = \rho_\lambda = t_0(\lambda)$ and all $0 < \lambda < \lambda_0$. This proves Claim 2.
Claim 3: $\psi_\lambda(t\hat{u}_1(p)) \to -\infty$ as $t \to +\infty$.

On account of hypothesis H_2(ii), given $\beta_1 \in (0,\beta_0)$, we can find $M = M(\beta_1) > 0$ such that

$$\beta_1 x^\tau \leq pF(z,x) - f(z,x)x \quad \text{for a.a } z \in \Omega, \text{ all } x \geq M. \tag{3.45}$$

Reasoning as in the "Claim" in the proof of Proposition 3.2 and using this time (3.45), we obtain

$$\frac{F(z,x)}{x^p} - \frac{F(z,v)}{v^p} \leq \frac{\beta_1}{p-\tau}[\frac{1}{x^{p-\tau}} - \frac{1}{v^{p-\tau}}] \text{ for a.a } z \in \Omega, \text{ all } x > v \geq M.$$

Passing to the limit as $x \to +\infty$ and using hypothesis H_2(ii), we have

$$\frac{\hat{\lambda}_1^{a_1}(p)}{p} - \frac{F(z,v)}{v^p} \leq -\frac{\beta_1}{p-\tau}\frac{1}{v^{p-\tau}} \text{ for a.a } z \in \Omega, \text{ all } v \geq M$$
(recall that $\tau < p$),
$$\Rightarrow \hat{\lambda}_1^{a_1}(p)v^p - pF(z,v) \leq -\frac{\beta_1 p}{p-\tau}v^\tau \text{ for a.a } z \in \Omega, \text{ all } v \geq M. \tag{3.46}$$

Then, for $t > 0$ and $\hat{u}_1 = \hat{u}_1(p) \in \operatorname{int}C_+$,

$$\psi_\lambda(t\hat{u}_1) \leq \frac{1}{p}\int_\Omega [\hat{\lambda}_1^{a_1}(p)(t\hat{u}_1)^p - pF(z,t\hat{u}_1)]\,dz + \frac{t^p}{q}\rho_{a_2,q}(D\hat{u}_1) + c_{19}$$
for some $c_{19} > 0$ (see (3.16))
$$\leq -\frac{\beta_1 t^\tau}{p-\tau}\int_{\{t\hat{u}_1 \geq M\}} \hat{u}_1^\tau\,dz + \frac{t^q}{q}\rho_{a_2,q}(D\hat{u}_1) + c_{20} \text{ for some } c_{20} > 0. \tag{3.47}$$

By $|\cdot|_N$, we denote the Lebesgue measure on $\mathbb{R}^N$. In view of $\hat{u}_1 \in \operatorname{int}C_+$, we have $|\{t\hat{u}_1 \geq M\}|_N \to |\Omega|_N$ as $t \to +\infty$. If, in (3.47), we let $t \to +\infty$, then $\psi_\lambda(t\hat{u}_1) \to -\infty$ as $t \to +\infty$, due to $\eta < \tau$, This proves Claim 3.

Claims 1, 2, and 3 permit the use of the Mountain Pass Theorem. So, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that $u_\lambda \in K_{\psi_\lambda}$ and $0 < m_\lambda \leq \psi_\lambda(u_\lambda)$ for all $0 < \lambda < \lambda_0$, which implies that

$$\langle V(u_\lambda), h\rangle = \int_\Omega g_\lambda(z,u_\lambda)h\,dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{3.48}$$

In (3.48), if we use the test function $h = (\bar{u}_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$, then

$$\begin{aligned}
&\langle V(u_\lambda), (\bar{u}_\mu - u_\lambda)^+\rangle \\
=&\int_\Omega [\lambda \bar{u}_\mu^{-\eta} + f(z,u_\lambda^+)](\bar{u}_\mu - u_\lambda)^+\,dz \text{ (see (3.25))} \\
\geq&\int_\Omega \lambda \bar{u}_\mu^{-\eta}(\bar{u}_\mu - u_\lambda)^+\,dz \quad \text{(since on } \{u_\lambda < \bar{u}_\mu\} \text{ we have } f(z,u_\lambda^+) \geq 0) \\
=&\langle V(\bar{u}_\mu), (\bar{u}_\mu - u_\lambda)^+\rangle \quad \text{(see Proposition 3.1)} \\
\Rightarrow& \bar{u}_\mu \leq u_\lambda.
\end{aligned}$$

From (3.25) and (3.48), it follows that $u_\lambda \in S_\lambda$ and $s \in (0,\lambda_0) \subseteq \mathcal{L} \neq \emptyset$. Letting $\hat{u} \in S_\lambda$, one has

$$\begin{aligned}
&\langle V(\hat{u}), (\bar{u}_\mu - \hat{u})^+\rangle \\
=&\int_\Omega [\lambda \bar{u}_\mu^{-\eta} + f(z,\hat{u})](\bar{u}_\mu - \hat{u})^+\,dz \quad \text{(see (3.25))} \\
\geq&\int_\Omega \mu \bar{u}_\mu^{-\eta}(\bar{u}_\mu - \hat{u})^+\,dz \text{ (as before on } \{\hat{u} < \bar{u}_\mu\},\ f(z,\hat{u}) \geq 0 \text{ and } \mu < \lambda) \\
=&\langle V(\bar{u}_\mu), (\bar{u}_\mu - \hat{u})^+\rangle \quad \text{(see Proposition 3.1)}, \\
\Rightarrow& \bar{u}_\mu \leq \hat{u}.
\end{aligned} \tag{3.49}$$

We see that $\hat{u} \in L^\infty(\Omega)$ (see Marino-Winkert [15]). Therefore

$$\begin{aligned}
&|\lambda \hat{u}^{-\eta} + f(z,\hat{u})| \\
\leq&\lambda \bar{u}_\mu^{-\eta} + c_{21} \quad \text{for some } c_{21} > 0 \text{ (see hypothesis } H_2\text{(i))} \\
\leq&c_{22}[\lambda \hat{d}^{-\eta} + 1] \quad \text{for some } c_{22} > 0 \;\; (\text{since } \bar{u}_\mu \in \operatorname{int}C_+, \text{ we have } c_0\hat{d} \leq \bar{u}_\mu,\, c_0 > 0) \\
\leq&\lambda c_{23}\hat{d}^{-\eta} \quad \text{for some } c_{23} > 0.
\end{aligned}$$

Invoking Giacomoni-Kumar-Sreenadh [5, Theorem 1.7], we have that $\hat{u} \in C_+ \setminus \{0\}$. Let $\rho = \|\hat{u}\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_2(iv). Then

$$\begin{aligned}
&-\Delta_p^{a_1}\hat{u} - \Delta_q^{a_2}\hat{u} + \hat{\xi}_\rho \hat{u}^{p-1} - \lambda \hat{u}^{-\eta} = f(z,\hat{u}) + \hat{\xi}_\rho \hat{u}^{p-1} \geq 0 \;\text{ in } \Omega, \\
\Rightarrow&\hat{u} \in \operatorname{int}C_+ \text{ (see Papageorgiou-Rădulescu-Zhang [19, Proposition A2]).}
\end{aligned}$$

We conclude that if $\lambda \in \mathcal{L}$, then $\emptyset \neq S_\lambda \subseteq \operatorname{int}C_+$. □

Next, we show that $\mathcal{L}$ is an interval.

Proposition 3.5. *If hypotheses H_0 and H_2 hold, $\lambda \in \mathcal{L}$, and $0 < \sigma < \lambda$, then $\sigma \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$, we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int}C_+$. We choose $\mu \in (0,\sigma)$ small such that $0 \leq \bar{u}_\mu(z) \leq \delta$ for all $z \in \overline{\Omega}$ with $\delta > 0$ as in hypothesis H_2(iii) (see Proposition 3.1). In view of Proposition 3.4, one sees that $\bar{u}_\mu \leq u_\lambda$. So, we can define the Carathéodory function $k_\sigma(z,x)$ by setting

$$k_\sigma(z,x) = \begin{cases} \sigma \bar{u}_\mu(z)^{-\eta} + f(z,\bar{u}_\mu(z)) & \text{if } x < \bar{u}_\mu(z) \\ \sigma x^{-\eta} + f(z,x) & \text{if } \bar{u}_\mu(z) \leq x \leq u_\lambda(z) \\ \sigma u_\lambda(z)^{-\eta} + f(z,u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.50}$$

We set $K_\sigma(z,x) = \int_0^x k_\sigma(z,s)\,ds$ and consider the C^1-functional $l_\sigma : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$l_\sigma(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega K_\sigma(z,u)\,dz \;\text{ for all } u \in W_0^{1,p}(\Omega)$$
(see Papageorgiou-Smyrlis [20, Proposition 3]).

From (3.50), it is clear that $l_\sigma(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we see that $l_\sigma(\cdot)$ is sequentially weakly lower semicontinuous. By the Weierstrass-Tonelli theorem, we can find $u_\sigma \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}
l_\sigma(u_\sigma) = \inf\{l_\sigma(u) : u \in W_0^{1,p}(\Omega)\} \Rightarrow&\langle l_\sigma'(u_\sigma), h\rangle = 0 \;\text{ for all }\; h \in W_0^{1,p}(\Omega), \\
\Rightarrow&\langle V(u_\sigma), h\rangle = \int_\Omega k_\sigma(z,u_\sigma)h\,dz \;\text{ for all }\; h \in W_0^{1,p}(\Omega).
\end{aligned} \tag{3.51}$$

In (3.51), we use the test function $h = (u_\sigma - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
\langle V(u_\sigma), (u_\sigma - u_\lambda)^+\rangle =& \int_\Omega [\sigma u_\lambda^{-\eta} + f(z,u_\lambda)](u_\sigma - u_\lambda)^+\,dz \;\text{ (see (3.50))} \\
\leq& \int_\Omega [\lambda u_\lambda^{-\eta} + f(z,u_\lambda)](u_\sigma - u_\lambda)^+\,dz \;\text{ (see } \sigma < \lambda) \\
=&\langle V(u_\lambda), (u_\sigma - u_\lambda)^+\rangle \;\text{ (since } u_\lambda \in S_\lambda) \\
\Rightarrow& u_\sigma \leq u_\lambda \;\text{ (see Proposition 2.3)}.
\end{aligned}$$

Next, in (3.51), we choose the test function $h = (\bar{u}_\mu - u_\sigma)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\langle V(u_\sigma), (u_\mu - u_\sigma)^+\rangle &= \int_\Omega [\sigma \bar{u}_\mu^{-\eta} + f(z, \bar{u}_\mu)](\bar{u}_\mu - u_\sigma)^+ dz \\ &\geq \int_\Omega \mu \bar{u}_\mu^{-\eta}(\bar{u}_\mu - u_\sigma)^+ dz \ \text{ (since } \mu < \sigma \text{ and } f(z, \bar{u}_\mu) \geq 0) \\ &= \langle V(\bar{u}_\mu), (\bar{u}_\mu - u_\sigma)^+\rangle \ \text{ (see Proposition 3.1)} \\ &\Rightarrow \bar{u}_\mu \leq u_\sigma \ \text{ (see Proposition 2.3).}\end{aligned}$$

It follows from (3.50) and (3.51) that

$$u_\sigma \in [\bar{u}_\mu, u_\lambda] \Rightarrow u_\sigma \in S_\sigma \subseteq \operatorname{int} C_+ \Rightarrow \sigma \in \mathcal{L}.$$

□

Corollary 3.1. *If hypotheses H_0, H_2 hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$, and $0 < \sigma < \lambda$, then $\sigma \in \mathcal{L}$ and there exists $u_\sigma \in S_\sigma \subseteq \operatorname{int} C_+$ such that $u_\sigma \leq u_\lambda$.*

We can improve this monotonicity property as follows.

Proposition 3.6. *If hypotheses H_0, H_2 hold, $\lambda \in \mathcal{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$, and $0 < \sigma < \lambda$, then $\sigma \in \mathcal{L}$ and there exists $u_\sigma \in S_\sigma \subseteq \operatorname{int} C_+$ such that $u_\lambda - u_\sigma \in \operatorname{int} C_+$.*

Proof. From Corollary 3.1, we know that $\sigma \in \mathcal{L}$ and we can find $u_\sigma \in S_\sigma \subseteq \operatorname{int} C_+$ such that

$$\bar{u}_\mu \leq u_\sigma \leq u_\lambda \text{ (with } u \in (0, \sigma)\text{, see the proof of Proposition 3.4).} \tag{3.52}$$

Let $\rho = \|u_\lambda\|_\infty$ and $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_2(iv). Then

$$\begin{aligned} & -\Delta_p^{a_1} u_\sigma - \Delta_q^{a_2} u_\sigma + \hat{\xi}_\rho u_\sigma^{p-1} - \lambda u_\sigma^{-\eta} \\ &= f(z, u_\sigma) + \hat{\xi}_\rho u_\sigma^{p-1} - (\lambda - \sigma) u_\sigma^{-\eta} \\ &\leq f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} \ \text{ (see (3.52), hypothesis } H_2\text{(iv) and recall } \sigma < \lambda) \\ &= -\Delta_p^{a_1} u_\lambda - \Delta_q^{a_2} u_\lambda + \hat{\xi}_\rho u_\lambda^{p-1} - \lambda u_\lambda^{-\eta} \ \text{ in } \Omega. \end{aligned} \tag{3.53}$$

Since $u_\sigma \in \operatorname{int} C_+$, we see that $0 \prec (\lambda - \sigma) u_\sigma^{-\eta}$. From (3.53) and Papageorgiou-Rădulescu-Repovš [18, Proposition 7], we conclude that $u_\lambda - u_\sigma \in \operatorname{int} C_+$. □

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 3.7. *If hypotheses H_0, H_2 hold, then $\lambda^* < \infty$.*

Proof. Hypotheses H_2 imply that, for given $c \in (0, \hat{\lambda}_1^{a_1}(p))$, we can find $\hat{\lambda} = \hat{\lambda}(c) > 0$ such that

$$c x^{p-1} \leq \hat{\lambda} x^{-\eta} + f(z, x) \ \text{ for a.a } z \in \Omega \text{, all } x \geq 0. \tag{3.54}$$

Let $\lambda > \hat{\lambda}$ and $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$. Let $\Omega_0 \subseteq \Omega$ be open with C^2-boundary $\partial\Omega_0$ and $\overline{\Omega}_0 \subseteq \Omega$. Set $m_\lambda = \min_{\overline{\Omega}_0} u_\lambda$. Since $u_\lambda \in \operatorname{int} C_+$, we have $m_\lambda > 0$. For $\varepsilon > 0$, we set $m_\lambda^\varepsilon = m + \varepsilon$. Let $\rho = \|u_\lambda\|_\infty$ and consider $\hat{\xi}_\rho > 0$ as postulated by hypothesis H_2(iv).

Then

$$\begin{aligned}
&-\Delta_p^{a_1} m_\lambda^\varepsilon - \Delta_q^{a_2} m_\lambda^\varepsilon + \hat{\xi}_\rho (m_\lambda^\varepsilon)^{p-1} - \lambda (m_\lambda^\varepsilon)^{-\eta} \\
\leq& \hat{\xi}_\rho m_\lambda^{p-1} + \chi(\varepsilon) - \lambda m_\lambda^{-\eta} \quad \text{with } \chi(\varepsilon) \to 0^+ \text{ as } \varepsilon \to 0^+ \\
=& [\hat{\xi}_\rho + c] m_\lambda^{p-1} + \chi(\varepsilon) - \hat{\lambda} m_\lambda^{-\eta} - (\lambda - \hat{\lambda}) m_\lambda^{-\eta} \\
\leq& f(z, m_\lambda) + \hat{\xi}_\rho m_\lambda^{p-1} - (\lambda - \hat{\lambda}) m_\lambda^{-\eta} + \chi(\varepsilon) \quad (\text{see } (3.54)) \\
\leq& f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} \quad \text{for } \varepsilon > 0 \text{ small (see hypothesis } H_2(\text{iv}) \text{ and recall } \hat{\lambda} < \lambda) \\
=& -\Delta_p^{a_1} u_\lambda - \Delta_q^{a_2} u_\lambda + \hat{\xi}_\rho u_\lambda^{p-1} - \lambda u_\lambda^{-\eta} \quad \text{in } \Omega_0.
\end{aligned} \tag{3.55}$$

Note that, for $\varepsilon > 0$ small, $0 < c^* = \frac{\lambda - \hat{\lambda}}{m_\lambda \eta} - \chi(\varepsilon)$. From (3.55) and Papageorgiou-Rădulescu-Repovš [18, Proposition 6] (see also Papageorgiou-Rădulescu [17, Theorem 7(b)]), we have $m_\lambda^\varepsilon = m_\lambda + \varepsilon \leq u_\lambda(z)$ for all $z \in \Omega_0$ and all $\varepsilon > 0$ small, which is a contradiction. This means that $\lambda^* \leq \hat{\lambda} < \infty$. □

Proposition 3.8. *If hypotheses H_0 and H_2 hold and $0 < \lambda < \lambda^*$, then problem (p_λ) has at least two positive solutions $u_0, \hat{u} \in \mathrm{int} C_+$.*

Proof. Let $0 < \sigma < \lambda < \gamma < \lambda^*$. On account of Proposition 3.6, we can find $u_\gamma \in S_\gamma \subseteq \mathrm{int} C_+$, $u_0 \in S_\lambda \subseteq \mathrm{int} C_+$, and $u_\sigma \in S_\sigma \subseteq \mathrm{int} C_+$ such that

$$u_\gamma - u_0 \in \mathrm{int} C_+ \text{ and } u_0 - u_\sigma \in \mathrm{int} C_+ \Rightarrow u_0 \in \mathrm{int}_{C_0^1(\overline{\Omega})} [u_\sigma, u_\gamma]. \tag{3.56}$$

We introduce the Carathéodory function $\hat{w}_\lambda(z,x)$ defined by

$$\hat{w}_\lambda(z,x) = \begin{cases} \lambda u_\sigma(z)^{-\eta} + f(z, u_\sigma(z)) & \text{if } \quad x < u_\sigma(z), \\ \lambda x^{-\eta} + f(z,x) & \text{if } \quad u_\sigma(z) \leq x \leq u_\gamma(z), \\ \lambda u_\gamma(z)^{-\eta} + f(z, u_\gamma(z)) & \text{if } \quad u_\gamma(z) < x. \end{cases} \tag{3.57}$$

We set $\hat{W}_\lambda(z,x) = \int_0^x \hat{w}_\lambda(z,s)\, ds$ and consider the C^1-functional $\hat{\varphi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \frac{1}{p} \rho_{a_1,p}(Du) + \frac{1}{q} \rho_{a_2,q}(Du) - \int_\Omega \hat{W}_\lambda(z,u)\, dz \ \text{ for all } u \in W_0^{1-p}(\Omega).$$

Let $w_\lambda(z,x)$ be the Carathéodory function defined by

$$w_\lambda(z,x) = \begin{cases} \lambda u_\sigma(z)^{-\eta} + f(z, u_\sigma(z)) & \text{if } \quad x \leq u_\sigma(z), \\ \lambda x^{-\eta} + f(z,x) & \text{if } \quad u_\sigma(z) < x. \end{cases} \tag{3.58}$$

We set $W_\lambda(z,x) = \int_0^x w_\lambda(z,s)\, ds$ and consider the C^1-functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p} \rho_{a_1,p}(Du) + \frac{1}{q} \rho_{a_2,q}(Du) - \int_\Omega W_\lambda(z,u)\, dz \ \text{ for all } u \in W_0^{1-p}(\Omega).$$

Consider the critical sets

$$K_{\hat{\varphi}_\lambda} = \{u \in W_0^{1,p}(\Omega) : \hat{\varphi}_\lambda'(u) = 0\} \text{ and } K_{\varphi_\lambda} = \{u \in W_0^{1,p}(\Omega) : \varphi_\lambda'(u) = 0\}.$$

Using (3.57) and (3.58), we can easily show that

$$K_{\hat{\varphi}_\lambda} \subseteq [u_\sigma, u_\gamma] \cap \mathrm{int} C_+ \quad \text{and} \quad K_{\varphi_\lambda} \subseteq [u_\sigma) \cap \mathrm{int} C_+. \tag{3.59}$$

Moreover, from (3.57), it follows that $\hat{\varphi}_\lambda(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\bar{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda(\bar{u}_0) = \inf\{\hat{\varphi}_\lambda(u) : u \in W_0^{1,p}(\Omega)\}. \tag{3.60}$$

Then $\bar{u}_0 \in K_{\hat{\varphi}_\lambda}$. From (3.59) and (3.57), we may assume that $\bar{u}_0 = u_0$, or otherwise we already have a second positive smooth solution and so we are done. From (3.56) and (3.57), we see that $\hat{\varphi}_\lambda|_{[u_\sigma,u_\gamma]} = \varphi_\lambda|_{[u_z,u_\gamma]}$, which together with (3.60), and (3.56) follows that

$$\begin{aligned} &u_0 \text{ is a local } C_0^1(\bar{\Omega})\text{-minimizer of } \varphi_\lambda(\cdot), \\ \Rightarrow &u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda(\cdot) \text{ (see [19, Proposition A3]).} \end{aligned}$$

From (3.59), we may assume that K_{φ_λ} is finite (otherwise, on account of (3.59) and (3.58), we already have an infinite number of positive smooth solutions. Thus it is done). Then Hu-Papageorgiou [11, Proposition 3.132] implies that there exists $\rho \in (0,1)$ small such that

$$\varphi_\lambda(u_0) < \inf\{\varphi_\lambda(u) : \|u - u_0\| = \rho\} = m_\lambda^*. \tag{3.61}$$

As in the Claims 1 and 3 of Proposition 3.4, we can find that

$$\varphi_\lambda(\cdot) \text{ satisfies the C-condition and } \varphi_\lambda(t\hat{u}_1(p)) \to -\infty \text{ as } t \to +\infty. \tag{3.62}$$

From (3.61) and the Mountain Pass Theorem, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that $\hat{u} \in K_{\varphi_\lambda}$ and $\varphi_\lambda(u_0) < m_\lambda^* \leq \varphi_\lambda(\hat{u})$. In view of (3.59) and (3.58), we conclude that $\hat{u} \in S_\lambda \subseteq \mathrm{int}C_+$ and $\hat{u} \neq u_0$. □

Note that $(0,\lambda^*) \subseteq \mathcal{L} \subseteq (0,\lambda^*]$. For $\lambda \in (0,\lambda^*)$, we have multiplicity of the positive solutions. We have to decide about the admissibility of the critical parameter value λ^*. We can show the admissibility of $\lambda^* > 0$ only under nonuniform nonresonance as $x \to +\infty$. When we have uniform resonance with respect to $\hat{\lambda}_1^{a_1}(p) > 0$ (this is the case of hypothesis H_2(ii)), we do not know if we have $\lambda^* \in \mathcal{L}$. This is an interesting open problem. The more restrictive hypotheses on the perturbation $f(z,x)$ are the following.

<u>H_2'</u>: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that hypotheses H_2'(i), (iii), and (iv) are the same as the corresponding hypotheses H_2(i), (iii), (iv), and

(ii) there exist functions $\hat{\eta}_0, \eta \in L^\infty(\Omega)$ such that

$$\begin{aligned} &\hat{\lambda}_1^{a_1}(p) \leq \hat{\eta}_0(z) \text{ for a.a } z \in \Omega,\ \hat{\eta}_0 \not\equiv \hat{\lambda}_1^{a_1}(p) \\ &\hat{\eta}_0(z) \leq \liminf_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leq \limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leq \eta_0(z) \text{ uniformly for a.a } z \in \Omega. \end{aligned}$$

Proposition 3.9. *If hypotheses H_0 and H_2' hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Let $\lambda_n \in \mathcal{L}$, $n \in \mathbb{N}$ such that $\lambda_n \uparrow \lambda^*$ as $n \to \infty$. We choose $\mu \in (0,\lambda_1)$ small such that

$$0 \leq \bar{u}_\mu(z) \leq \delta \text{ for all } z \in \overline{\Omega} \text{ (see Proposition 3.1).}$$

Since $\lambda_n \in \mathcal{L}$, we can find $u_n \in S_{\lambda_n} \subseteq \mathrm{int}C_+$, $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, $\bar{u}_\mu \leq u_n$. We show that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. We argue indirectly. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is not bounded. We may assume $\|u_n\| \to \infty$ as $n \to \infty$. Setting $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$, one has

$\|y_n\| = 1,\ y_n \geq 0$ for all $n \in \mathbb{N}$. We may assume that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(\Omega)$ and $y_n \to y$ in $L^p(\Omega)$ as $n \to \infty$. Note that

$$\langle V(u_n), h\rangle = \int_\Omega [\lambda_n u_n^{-\eta} + f(z,u_n)]h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{3.63}$$

$$\Rightarrow \langle A_p^{a_1}(y_n), h\rangle + \frac{1}{\|u_n\|^{p-q}}\langle A_q^{a_2}(y_n), h\rangle = \frac{\lambda_n}{\|u_n\|^{p-1}}\int_\Omega u_n^{-\eta}h\,dz + \int_\Omega \frac{f(z,u_n)}{\|u_n\|^{p-1}}h\,dz \tag{3.64}$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

In (3.64), we choose the test function $h = y_n - y \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to \infty$. Then

$$\lim_{n\to\infty}\langle A_p^{a_1}(y_n), y_n - y\rangle = 0 \Rightarrow y_n \to y \text{ in } W_0^{1,p}(\Omega),\ \|y\| = 1,\ y \geq 0 \ \text{(see Proposition 2.3)}. \tag{3.65}$$

If we pass to the limit as $n \to \infty$ in (3.64) and use (3.65), then

$$\langle A_p^{a_1}(y), h\rangle = \int_\Omega \hat{\eta}(z) y^{p-1} h\,dz$$

for all $h \in W_0^{1,p}(\Omega)$ and with $\hat{\eta}_0(z) \leq \hat{\eta}(z) \leq \eta_0(z)$ for a.a $z \in \Omega$ (see (3.35)),

$$\Rightarrow -\Delta_p^{a_1} y(z) = \hat{\eta}(z) y(z)^{p-1} \text{ in } \Omega,\ y|_{\partial\Omega} = 0. \tag{3.66}$$

From Proposition 2.2, we see that

$$\widetilde{\lambda}_1^{a_1}(p,\hat{\eta}) \leq \widetilde{\lambda}_1^{a_1}(p,\hat{\eta}_0) < \widetilde{\lambda}_1^{a_1}(p,\hat{\lambda}_1^{a_1}(p)) = 1.$$

It follows from (3.66) that $y = 0$ or $y =$ nodal. Both cases contradict (3.65). Thus $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded and we may assume that $u_n \xrightarrow{w} u^*$ in $W_0^{1,p}(\Omega)$ and $u_n \to u^*$ in $L^p(\Omega)$ as $n \to \infty$. In (3.63), we choose the test function $h = u_n - u^* \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to \infty$ to see that $\lim_{n\to\infty}\langle V(u_n), u_n - u^*\rangle = 0$, which implies

$$u_n \to u^* \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 2.3)},\ \bar{u}_\mu \leq u^*. \tag{3.67}$$

Note that $\bar{u}_\mu \in \mathrm{int}C_+$. Thus we can find $c_{24} > 0$ such that

$$c_{24}\hat{d} \leq \bar{u}_\mu \quad \text{(recall } \hat{d}(z) = d(z,\partial\Omega) \text{ for all } z \in \Omega). \tag{3.68}$$

For all $n \in \mathbb{N}$ and all $h \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \frac{|h|}{u_n^\eta} &\leq c_{25}\frac{|h|}{\bar{u}_\mu} \quad \text{for some } c_{25} > 0 \text{ (recall } \bar{u}_\mu \in \mathrm{int}C_+) \\ &\leq c_{26}\frac{|h|}{\hat{d}} \quad \text{for some } c_{26} > 0 \text{ (see (3.68))}. \end{aligned}$$

By Hardy's inequality, it follows that

$$\{\frac{h}{u_n^\eta}\}_{n\in\mathbb{N}} \subseteq L^p(\Omega) \text{ is bounded} \Rightarrow \{\frac{h}{u_n^\eta}\}_{n\in\mathbb{N}} \text{ is uniformly integrable.}$$

On account of (3.67), we have (at least for a subsequence), that

$$(\frac{h}{u_n^\eta})(z) \to (\frac{h}{(u^*)^\eta})(z) \ \text{ for a.a } z \in \Omega.$$

By Vitali's theorem (see [10, p.91]), we infer that

$$\int_\Omega \lambda_n \frac{h}{u_n^\eta}\,dz \to \int_\Omega \lambda^* \frac{h}{(u^*)^\eta}\,dz. \tag{3.69}$$

If, in (3.63), we pass to the limit as $n\to\infty$ and use (3.55) and (3.69), then

$$\langle V(u^*),h\rangle = \int_\Omega [\lambda^*(u^*)^{-\eta} + f(z,u^*)]h\,dz \ \text{ for all } h\in W_0^{1,p}(\Omega),$$
$$\bar{u}_\mu \le u^* \Rightarrow u^* \in S_{\lambda^*} \subseteq \mathrm{int}C_+ \ \text{ and so } \lambda^*\in\mathcal{L}.$$

□

We can state the following existence and multiplicity result for problem (p_λ) when we have resonance from the right with respect to $\hat{\lambda}_1^{a_1}(p) > 0$.

Theorem 3.1. *(1) If hypotheses H_0 and H_2 hold, then there exists $\lambda^*>0$ such that*
(a) for all $\lambda\in(0,\lambda^*)$, problem (p_λ) has at least two solutions $u_0,\hat{u}\in\mathrm{int}C_+$, $u_0\neq\hat{u}$;
(b) for all $\lambda>\lambda^*$ problem (p_λ) has no solution.
(2) If hypotheses H_0 and H_2' hold, then, for $\lambda=\lambda^*$, problem (p_λ) has at least one solution (that is, $\mathcal{L}=(0,\lambda^*]$).

Remark 3.3. There is a difference between resonance from the left and from the right of $\hat{\lambda}_1^{a_1}(p)>0$. In the first case, the energy functional is coercive and we have the existence for all $\lambda>0$ but no multiplicity of solutions. In fact, if the equation is driven only by $\Delta_p^{a_1}$, then we can have the uniqueness of the solution (see Proposition 3.2 and 3.3). In the second case (resonance from the right), the set of admissible parameters is a bounded interval in $\overset{\circ}{\mathbb{R}}_+ = (0,+\infty)$ and we have a bifurcation-type situation (see Theorem 3.1).

4. Minimal Solutions-Solution Multifunction

First we show that, for every $\lambda\in\mathcal{L}$, S_λ has a minimal element.

Proposition 4.1. *If hypotheses H_0 and H_2 hold, then*
(a) for every $\lambda\in\mathcal{L}$, there exists $u_\lambda^*\in S_\lambda\subseteq \mathrm{int}C_+$ such that $u_\lambda^*\le u$ for all $u\in S_\lambda$;
(b) if $\sigma,\lambda\in\mathcal{L}$ and $0<\sigma<\lambda$, then $u_\lambda^*-u_\sigma^*\in\mathrm{int}C_+$;
(c) $u_\lambda^*\to 0$ in $C_0^1(\bar{\Omega})$ as $\lambda\to 0^+$.

Proof. (a) From Filippakis-Papageorgiou [3], we see that S_λ is downward directed (that is, if $u_1,u_2\in S_\lambda$, then there exists $u\in S_\lambda$ such that $u\le u_1$, $u\le u_2$). In view of Hu-Papageorgiou [10, Theorem 5.109], we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq S_\lambda$ such that $\inf S_\lambda = \inf_{n\in\mathbb{N}} u_n$. Note that

$$\langle V(u_n),h\rangle = \int_\Omega [\lambda u_n^{-\eta} + f(z,u_n)]h\,dz \ \text{ for all } h\in W_0^{1,p}(\Omega), \text{ all } n\in\mathbb{N}, \tag{4.1}$$
$$\bar{u}_\mu \le u_n \le u_1 \ \text{ for all } n\in\mathbb{N} \text{ and for } \mu\in(0,1) \text{ small.} \tag{4.2}$$

We test (4.1) with $h=u_n\in W_0^{1,p}(\Omega)$. Using (4.2) and hypothesis H_2(i), we obtain

$$\hat{c}\|u_n\|^p \le c_{27} \ \text{ for some } c_{27}>0, \text{ all } n\in\mathbb{N} \text{ (see hypotheses } H_0),$$
$$\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \ \text{ is bounded.}$$

So, we may assume that $u_n \xrightarrow{w} u_\lambda^*$ in $W_0^{1,p}(\Omega)$, $u_n \to u_\lambda^*$ in $L^p(\Omega)$. In (4.1), we choose the test function $h = u_n - u_\lambda^* \in W_0^{1,p}(\Omega)$. It follows that

$$\lim_{n\to\infty} \langle V(u_n), u_n - u_\lambda^* \rangle = 0,$$

$$\Rightarrow u_n \to u_\lambda^* \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see Proposition 2.3).} \tag{4.3}$$

If, in (4.1), we take the limit as $n \to \infty$ and use (4.3), then

$$\langle V(u_\lambda^*), h \rangle = \int_\Omega [\lambda (u_\lambda^*)^{-\eta} + f(z, u_\lambda^*)] h dz \text{ for all } h \in W_0^{1,p}(\Omega),$$

$$\bar{u}_\mu \leq u_\lambda^* \text{ (see (3.69)).}$$

Therefore $u_\lambda^* \in S_\lambda \subseteq \mathrm{int} C_+$ and $u_\lambda^* = \inf S_\lambda$.

(b) According to Proposition 3.6, we can find $u_\sigma \in S_\sigma \subseteq \mathrm{int} C_+$ such that

$$u_\lambda^* - u_\sigma \in \mathrm{int} C_+ \Rightarrow u_\lambda^* - u_\sigma^* \in \mathrm{int} C_+ \quad (\text{since } u_\sigma^* \leq u_\sigma).$$

(c) We have

$$\langle V(u_\lambda^*), h \rangle = \int_\Omega [\lambda (u_\lambda^*)^{-\eta} + f(z, u_\lambda^*)] h dz \text{ for all } h \in W_0^{1,p}(\Omega), \text{ all } 0 < \lambda \leq 1. \tag{4.4}$$

In (4.4), we choose the test function $h = u_\lambda^* \in W_0^{1,p}(\Omega)$. Since $u_\lambda^* \leq u_1^*$ for all $0 < \lambda \leq 1$, hypothesis H_2(i) and the fact that $u_1^* \in \mathrm{int} C_+$, we obtain

$$\hat{c} \|u_\lambda^*\|^p \leq \lambda c_{28} \quad \text{for some } c_{28} > 0, \text{ all } 0 < \lambda \leq 1,$$

$$\Rightarrow u_\lambda^* \to 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ as } \lambda \to 0^+. \tag{4.5}$$

In view of the regularity result of Giacomoni-Kumar-Sreenadh [5], we can find $\alpha \in (0,1)$ and $c_{29} > 0$ such that $u_\lambda^* \in C_0^{1,\alpha}(\overline{\Omega})$ and $\|u_\lambda^*\|_{C_0^1(\overline{\Omega})} \leq c_{29}$ for all $0 < \lambda \leq 1$. The compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ (Arzela-Ascoli Theorem) and (4.5) imply that $u_\lambda^* \to 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to 0^+$. ☐

For the solution set $S_\lambda \subseteq \mathrm{int} C_+$, we have the following topological property.

Proposition 4.2. *If hypotheses H_0 and H_2' hold and $\lambda \in \mathcal{L}$, then S_λ is nonempty and compact in $C_0^1(\overline{\Omega})$.*

Proof. We claim that $S_\lambda \subseteq W_0^{1,p}(\Omega)$ is bounded. Arguing by contradiction, we suppose that there exists $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda$ such that $\|u_n\| \to \infty$ as $n \to \infty$. Setting $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$, one sees that $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. Thus one may assume that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(\Omega)$ and $y_n \to y$ in $L^p(\Omega)$, $y \geq 0$. Note that $\langle V(u_n), h \rangle = \int_\Omega [\lambda u_n^{-\eta} + f(z, u_n)] h dz$ for all $h \in W_0^{1,p}(\Omega)$ and all $n \in \mathbb{N}$. This implies, for all $h \in W_0^{1,p}(\Omega)$ and all $n \in \mathbb{N}$,

$$\langle A_p^{a_1}(y_n), h \rangle + \frac{1}{\|u_n\|^{p-q}} \langle A_q^{a_2}(y_n), h \rangle = \frac{\lambda}{\|u_n\|^{p-1}} \int_\Omega u_n^{-\eta} h dz + \int_\Omega \frac{f(z, u_n)}{\|u_n\|^{p-1}} h dz. \tag{4.6}$$

We test (4.6) with $h = y_n - y \in W_0^{1,p}(\Omega)$. It follows that

$$\lim_{n\to\infty} \langle A_p^{a_1}(y_n), y_n - y \rangle = 0$$

$$\Rightarrow y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ and } \|y\| = 1, \ y \geq 0 \text{ (see Proposition 2.3).} \tag{4.7}$$

If, in (4.6), we take the limit as $n \to \infty$ and use hypothesis H_2'(ii), then

$$\langle A_p^{a_1}(y), h\rangle = \int_\Omega \hat{\eta}(z) y^{p-1} h\,dz \text{ for all } h \in W_0^{1,p}(\Omega), \text{ with } \hat{\eta}_0(z) \le \hat{\eta}(z) \le \eta_0(z) \text{ for a.a } z \in \Omega,$$
$$\Rightarrow -\Delta_p^{a_1} y(z) = \hat{\eta}(z) y(z)^{p-1} \text{ in } \Omega,\ y|_{\partial\Omega} = 0,$$
$$\Rightarrow y = 0 \text{ or } y = \text{nodal (see Proposition 2.2)}.$$

Both contradict (4.7). This proves the boundedness of $S_\lambda \subseteq W_0^{1,p}(\Omega)$ and as before. In view of the nonlinear regularity result of Giacomoni-Kumar-Sreenadh [5], we have that $S_\lambda \subseteq C_0^1(\bar{\Omega})$ is relatively compact.

Finally, we show that $S_\lambda \subseteq C_0^1(\bar{\Omega})$ is closed. To this end, one assumes that $u_n \in S_\lambda$, $n \in \mathbb{N}$ and $u_n \to u$ in $C_0^1(\bar{\Omega})$. Note that $\bar{u}_\mu \le u$, due to $\bar{u}_\mu \le u_n$ for all $n \in \mathbb{N}$ and $\mu \in (0,1)$ small. One also has

$$\langle V(u_n), h\rangle = \int_\Omega [\lambda u_n^{-\eta} + f(z,u_n)] h\,dz \text{ for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \langle V(u), h\rangle = \int_\Omega [\lambda u^{-\eta} + f(z,u)] h\,dz \text{ for all } h \in W_0^{1,p}(\Omega) \ \Rightarrow u \in S_\lambda$$

Therefore $\overline{S} \subseteq C_0^1(\overline{\Omega})$ is closed, hence compact. ☐

Remark 4.1. The above proof reveals that if $D \subseteq \mathbb{R}_+$ is bounded, then $\overline{\cup_{\lambda\in D} S_\lambda}^{C_0^1(\overline{\Omega})} \subseteq C_0^1(\overline{\Omega})$ is compact.

Next, we study the continuity properties of the solution multifunction $\lambda \to S_\lambda$. First, we recall some notions and results from multivalued analysis that we need for this study. Details can be found in Hu-Papageorgiou [10, Chapter 5].

Let X, Y be two Hausdorff topological spaces and $S : X \to 2^Y \setminus \{\emptyset\}$ a multifunction. We say

(a) $S(\cdot)$ is "lower semicontinuous" (abbreviated "lsc") if, for all $U \subseteq Y$ open, $S^-(U) = \{x \in X : S(x) \cap U \neq 0\}$ is open in X.

(b) $S(\cdot)$ is "upper semicontinuous" (abbreviated "usc") if, for all $U \subseteq Y$ open, $S^+(U) = \{x \in X : S(x) \subseteq U\}$ is open in X.

(c) $S(\cdot)$ is "continuous" or "Vietoris continuous" if it is both lsc and usc.

Remark 4.2. If $S(\cdot)$ is single valued, then both notions of lower and upper semicontinuity, coincide with that of continuity.

If Y is a metric with metric $d(\cdot,\cdot)$ and $A, C \subseteq Y$ nonempty, we set

$$h^*(A,C) = \sup\{d(a,C) : a \in A\} = \inf\{\varepsilon > 0 : A \subseteq C_\varepsilon\}$$

with $C_\varepsilon = \{x \in X : d(x,C) < \varepsilon\}$ (the open ε-enlargement of C).

Using h^*, we can define the "Hausdorff distance" between A and C by

$$h(A,C) = \max\{h^*(A,C),\ h^*(C,A)\} = \inf\{\varepsilon > 0 : A \subseteq C_\varepsilon,\ C \subseteq A_\varepsilon\}.$$

Remark 4.3. Observe that $h^*(A,C) = \sup\{d(y,C) - d(y,A) : y \in Y\}$ and $h(A,C) = \sup\{|d(y,C) - d(y,A)| : y \in Y\}$.

The space of bounded and closed subsets of Y equipped with the Hausdorff distance $h(\cdot,\cdot)$ is a metric space and $\emptyset$ is an isolated point in that space. Moreover, if Y is complete, then so is this hyperspace with the Hausdorff metric. Finally, by $P_k(Y)$, we denote the family of nonempty and compact subset of Y and by $P_{bf}(Y)$ the family of nonempty, bounded closed subset of Y.

Suppose that X is a Hausdorff topological space, (Y,d) a metric space and $S : X \to 2^Y \setminus \{\emptyset\}$. We say

(a)$'$ $S(\cdot)$ is "h-lower semicontinuous" (abbreviated "h-lsc") if, for all $x \in X$, $u \to h^*(S(x), S(u))$ is continuous on X.

(b)$'$ $S(\cdot)$ is "h-upper semicontinuous" (abbreviated "h-usc") if, for all $x \in X$, $u \to h^*(S(u), S(x))$ is continuous on X.

(c)$'$ $S(\cdot)$ is "h-continuous" if it is both h-lsc and h-usc.

Remark 4.4. If $S(\cdot)$ has values in $P_{bf}(Y)$, then h-continuity is continuity from X into $(P_{bf}(Y), h)$.

The next statements establish the relations between these notions.

(1) h-lsc $\Rightarrow$ lsc and usc $\Rightarrow$ h-usc.

(2) If $S(\cdot)$ is $P_k(Y)$-valued, then h-lsc $\Leftrightarrow$ lsc and usc $\Leftrightarrow$ h-usc.

Now we can state the first continuity result for the solution multifunction.

Proposition 4.3. *If hypotheses H_0 and H_2' hold, then the solution multifunction $\lambda \to S_\lambda$ from $\mathcal{L}$ into $P_k(C_0^1(\bar{\Omega}))$ is both lsc and h-lsc.*

Proof. According to Hu-Papageorgiou [10, Proposition 5.6], it suffices to demonstrate that if $\{\lambda_n, \lambda\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ and $\lambda_n \to \lambda$, then $S_\lambda \subseteq \liminf_{n\to\infty} S_\lambda$ in $C_0^1(\overline{\Omega})$, where we recall that

$$\begin{aligned}\liminf_{n\to\infty} S_\lambda &= \{u \in C_+ : u = \lim_{n\to\infty} u_n \text{ in } C_0^1(\overline{\Omega}),\ u_n \in S_{\lambda_n} \text{ for all } n \in \mathbb{N}\}\\ &= \{u \in C_+ : \lim_{n\to\infty} d(u, S_{\lambda_n}) = 0\}.\end{aligned}$$

Let $u \in S_\lambda \subseteq \mathrm{int}C_+$ and consider the Dirichlet problem

$$-\Delta_p^{a_1} v - \Delta_q^{a_2} v = \lambda_n u^{-\eta} + f(z,u) \ \text{ in } \Omega,\ v|_{\partial\Omega} = 0. \tag{4.8}$$

Using Hardy's inequality, we see that $\lambda_n u^{-\eta} \in W^{-1,p'}(\Omega) \cap L^s(\Omega)$, $1 \le s < \frac{1}{\eta}$. Recall that $W^{-1,p'}(\Omega) = W^{1,p}(\Omega)^*$, $\frac{1}{p} + \frac{1}{p'} = 1$. On account of hypothesis H_2'(i) and $u \in S_\lambda \subseteq \mathrm{int}C_+$, we have $f(\cdot, u(\cdot)) \in L^\infty(\Omega)$. From Proposition 2.3, we see that $V(\cdot)$ is maximal monotone and clearly it is coercive. Therefore $V(\cdot)$ is surjective (see Hu-Papageorgiou [10, p.444]). So, we can find $v_n \in W_0^{1,p}(\Omega)$ solution of (4.8). This solution is unique because of the strict monotonicity of $V(\cdot)$. We know that $u \in \mathrm{int}C_+$, $\bar{u}_\mu \le u$ for $\mu \in (0, \inf\{\lambda_n\}_{n\in\mathbb{N}})$, and $\lambda_n u(\cdot)^{-\eta} + f(\cdot, u(\cdot)) \in L^\infty_{loc}(\Omega)$. Moreover, we have

$$\begin{aligned}|\lambda_n u^{-\eta} + f(z,u)| &\le c_{30}(\bar{u}_\mu^{-\eta} + 1) \quad \text{for some } c_{30} > 0\\ &\le c_{31}\hat{d}^{-\eta} \quad \text{for some } c_{31} > 0, \text{ all } n \in \mathbb{N}\end{aligned}$$

(recall that $\bar{u}_\mu \in \mathrm{int}C_+$ and so $c^*\hat{d} \le \bar{u}_\mu$ for some $c^* > 0$ and $\hat{d}(x) = d(z, \partial\Omega)$, $z \in \Omega$). Therefore $\{v_n\}_{n\in\mathbb{N}} \subseteq L^\infty(\Omega)$ is bounded and we can find $\alpha \in (0,1)$ and $c_{32} > 0$ such that $v_n \in C_0^{1,\alpha}(\overline{\Omega})$ and $\|v_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le c_{32}$ for all $n \in \mathbb{N}$. (see Papageorgiou-Rădulescu [17, Proposition 4] and Giacomoni-Kumar-Sreencidh [5, Theorem 1.7]). As before, exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$, we may assume that

$$v_n \to v \ \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty, \tag{4.9}$$

$$\Rightarrow -\Delta_p^{a_1} v - \Delta_q^{a_2} v = \lambda u^{-\eta} + f(z,u) \ \text{ in } \Omega,\ v|_{\partial\Omega} = 0 \tag{4.10}$$

We see that problem (4.10) has a unique solution and $u \in S_\lambda \subseteq \text{int}C_+$ is a solution to (4.10). Therefore $v = u \in \text{int}C_+$. From (4.9), it follows that we may assume that $v_n \in \text{int}C_+$ for all $n \in \mathbb{N}$. Let $v_n^0 = v_n \in \text{int}C_+$ and consider the Dirichlet problem

$$-\Delta_p^{a_1} v - \Delta_q^{a_2} v = \lambda_n (v_n^0)^{-\eta} + f(z, v_n^0) \ \text{ in } \Omega,\ v|_{\partial\Omega} = 0.$$

As above, this problem has a unique solution $v_n^1 \in C_0^{1,\alpha}(\overline{\Omega})$ and $v_n^1 \to u$ in $C_0^1(\overline{\Omega})$ as $n \to \infty$. Since $u \in \text{int}C_+$, we can say that $v_n^1 \in \text{int}C_+$ for all $n \in \mathbb{N}$. Continuing this way, we generate a sequence $\{v_n^k\}_{n\in\mathbb{N},k\in\mathbb{N}_0} \subseteq \text{int}C_+$ such that

$$\begin{cases} -\Delta_p^{a_1} v_n^k - \Delta_q^{a_2} v_n^k = \lambda (v_n^{k-1})^{-\eta} + f(z, v_n^{k-1}) \ \text{ in } \Omega, \\ v_n^k|_{\partial\Omega} = 0, \\ v_n^k \to u \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty, \text{ for all } n \in \mathbb{N}_0. \end{cases} \tag{4.11}$$

<u>Claim</u>: For every $n \in \mathbb{N}$, $\{v_n^k\}_{k\in\mathbb{N}_0} \subseteq W_0^{1,p}(\Omega)$ is bounded.

We argue by contradiction. Suppose that $\|v_n^k\| \to \infty$ as $k \to +\infty$. Setting $y_k = \frac{v_n^k}{\|v_n^k\|}$ for all $k \in \mathbb{N}_0$, one sees that $\|y_k\| = 1,\ y_k \geq 0$ for all $k \in \mathbb{N}_0$ and so we may assume that $y_k \xrightarrow{w} y$ in $W_0^{1,p}(\Omega)$ and $y_k \to y$ in $L^p(\Omega)$ as $k \to \infty$. It follows from (4.11) that

$$\langle A_p^{a_1}(y_k), h\rangle + \frac{1}{\|v_n^k\|^{p-q}} \langle A_q^{a_2}(y_k), h\rangle = \frac{\lambda_n}{\|v_n^k\|^{p-1}} \int_\Omega (v_n^{k-1})^{-\eta} h\,dz + \int_\Omega \frac{f(z, v_n^{k-1})}{\|v_n^k\|^{p-1}} h\,dz \tag{4.12}$$

for all $h \in W_0^{1,p}(\Omega)$, all $k \in \mathbb{N}_0$.

In (4.12), we choose the test function $h = y_k - y \in W_0^{1,p}(\Omega)$ and pass to the limit as $k \to \infty$. Then $\lim_{k\to\infty} \langle A_p^{a_1}(y_k), y_k - y\rangle = 0$, which implies that

$$y_k \to y \text{ in } W_0^{1,p}(\Omega),\ \|y\| = 1,\ y \geq 0. \tag{4.13}$$

In (4.12), letting $k \to \infty$ and using (4.13), we obtain

$$\langle A_p^{a_1}(y), h\rangle = \int_\Omega \hat{\eta}(z) y^{p-1} h\,dz \ \text{ for all } h \in W_0^{1,p}(\Omega), \text{where } \hat{\eta}_0(z) \leq \hat{\eta}(z) \leq \eta_0(z) \text{ for a.a } z \in \Omega,$$

$$\Rightarrow -\Delta_p^{a_1} y(z) = \hat{\eta}(z) y(z)^{p-1} \ \text{ in } \Omega,\ y|_{\partial\Omega} = 0.$$

As before, we have that $y = 0$ or $y =$nodal, both contradicting (4.13). This means that $\{y_k\}_{k\in\mathbb{N}_0} \subseteq W_0^{1,p}(\Omega)$ bounded. This proves the Claim. Using the Claim and the regularity result of Giacomoni-Kumar-Sreenadh [5], we can find $\alpha \in (0,1)$ and $c_{33} > 0$ such that $v_n^k \in C_0^{1,\alpha}(\overline{\Omega})$ and $\|v_n^k\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_{33}$ for all $k \in \mathbb{N}_0$. So, we may assume $v_n^k \to v_n$ in $C_0^1(\overline{\Omega})$ as $k \to \infty$. From (4.11), we obtain

$$-\Delta_p^{a_1} v_n - \Delta_q^{a_2} v_n = \lambda_n v_n^{-\eta} + f(z, v_n) \text{ in } \Omega,\ v_n|_{\partial\Omega} = 0,\ n \in \mathbb{N},$$

$$\Rightarrow v_n \in S_{\lambda_n} \ \text{ for all } n \in \mathbb{N}. \tag{4.14}$$

Moreover, $\{v_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. Using the double limit lemma (see Hu-Papageorgiou [10, Lemma 1.193]), we have $v_n \to u$ in $C_0^1(\overline{\Omega})$, which implies

$$u \in \liminf_{n\to\infty} S_\lambda \text{ (see (4.14))} \Rightarrow S_\lambda \subseteq \liminf_{n\to\infty} S_\lambda \Rightarrow \lambda \to S_\lambda \text{ is lsc.}$$

Since the multifunction is $P_k(C_0^1(\overline{\Omega}))$-valued, it follows that $\lambda \to S_\lambda$ is h-lsc. □

In fact, the solution multifunction has more continuity properties.

Proposition 4.4. *If hypotheses H_2' hold, then the solution multifunction $\lambda \to S_\lambda$ from $\mathcal{L}$ into $P_k(C_0^1(\overline{\Omega}))$ is both usc and h-usc.*

Proof. From Proposition 3.12 and its proof (see also the Remark following that proposition), we have that $\lambda \to S_\lambda$ is locally compact. By Hu-Papageorgiou [10, Proposition 5.13], to show the upper semicontinuity of the solution multifunction, it suffices to show that it has closed graph. To this end, let $\{\lambda_n, \lambda\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$, $\{u_n\}_{n\in\mathbb{N}} \in \mathrm{int}C_+$ and

$$\lambda_n \to \lambda,\ u_n \to u \text{ in } C_0^1(\overline{\Omega}),\ u_n \in S_{\lambda_n} \text{ for all } n \in \mathbb{N}. \tag{4.15}$$

For $\mu \in (0, \inf\{\lambda_n\}_{n\in\mathbb{N}})$ small, we have

$$\bar{u}_\mu \le u_n \text{ for all } n \in \mathbb{N}. \tag{4.16}$$

and

$$\langle V(u_n), h\rangle = \int_\Omega [\lambda_n u_n^{-\eta} + f(z,u_n)]h\,dz \text{ for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{4.17}$$

Passing to the limit as $n \to \infty$ in (4.17) and using (4.15), we obtain

$$\begin{aligned}&\langle V(u), h\rangle = \int_\Omega [\lambda u^{-\eta} + f(z,u)]h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega),\\ &\bar{u}_\mu \le u \text{ (see (4.16))}.\end{aligned}$$

It follows that $u \in S_\lambda$, so $\lambda \to S_\lambda$ is usc, and hence h-usc too. □

Propositions 4.3 and 4.4 lead to the following strong continuity property of the solution multifunction.

Theorem 4.1. *If hypotheses H_0 and H_2' hold, then the solution multifunction $\lambda \to S_\lambda$ from $\mathcal{L}$ into $P_k(C_0^1(\overline{\Omega}))$ is continuous and h-continuous.*

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