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COEFFICIENT BOUNDS FOR SOME ANALYTIC AND BI-UNIVALENT FUNCTIONS WITH THEIR INVERSES IN DIFFERENT SUBCLASSES

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Abstract. The present investigation in Geometric Function Theory of Complex Analysis is essentially motivated by the importance and usefulness of the coefficient bounds and the coefficient estimates for functions belonging to several analytic and univalent (and bi-univalent) function classes, such as the classes of starlike and convex functions as well as their bi-univalent associates. In this paper, the coefficient bounds are determined for the moduli $|a_2|$, $|a_3|$, and $|a_4|$ of the initial Taylor-Maclaurin coefficients a_2, a_3, and a_4 for some normalized analytic and bi-univalent functions where the functions and their inverses belong to distinct subclasses of analytic and bi-univalent functions. The these coefficient estimates are obtained by applying the familiar bound for the initial coefficient of the Carathéodory functions.

Keywords. Analytic functions; Bi-univalent functions; Fekete-Szegö functional; Henkel and Toeplitz determinants; Starlike and convex functions.

1. Introduction, Definitions and Preliminaries

Let $\mathscr{A}$ denote the family of functions $f(z)$ represented by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ and analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. A function $f \in \mathscr{A}$ is said to be *univalent* in $\mathbb{U}$ if $f(z)$ is one-to-one in $\mathbb{U}$. As usual, we denote by $\mathscr{S}$ the subclass of functions in $\mathscr{A}$ which are univalent in $\mathbb{U}$. On the other hand, a function f in $\mathscr{S}$ is called starlike of order α $(0 \leqq \alpha < 1)$, denoted by $f \in$

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$\mathscr{S}^*(\alpha)$, if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$. A function f in $\mathscr{S}$ is called convex of order α $(0 \leqq \alpha < 1)$, denoted by $f \in \mathscr{K}(\alpha)$, if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U})$. As usual, we write $\mathscr{S}^*(0) =: \mathscr{S}^*$ and $\mathscr{K}(0) =: \mathscr{K}$ for the classes of starlike and convex functions, respectively.

The class of δ-convex functions, denoted by $\mathscr{M}(\delta)$ $(0 \leqq \delta \leqq 1)$, was introduced by Mocanu [18] as follows:

$$\Re\left((1-\delta)\left(\frac{zf'(z)}{f(z)}\right) + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0 \quad (z \in \mathbb{U}).$$

We note that all δ-convex functions are univalent and starlike. In particular, $\mathscr{M}(0) \equiv \mathscr{S}^*$ and $\mathscr{M}(1) \equiv \mathscr{K}$ (see also [16]). We note also that the *extended* class of δ-convex functions of order β for $1 < \beta < 1$ and $(-\infty < \delta < \infty)$ was introduced and studied by Fukui *et al.* [9]. For each $f \in \mathscr{S}$, there exists an inverse function f^{-1} in some neighborhood of the origin. In accordance with the one-quarter theorem in [8], we can define f^{-1} in a neighborhood of the origin that contains a disk with a radius of $\frac{1}{4}$, which includes the disk $|z| < \frac{1}{4}$. In some cases, f^{-1} can be extended to the whole open disk $\mathbb{U}$. A function $f \in \mathscr{S}$ has an inverse f^{-1}, which is also an univalent function and defined by $f^{-1}\big(f(z)\big) = z$, $z \in \mathbb{U}$ and $f\big(f^{-1}(w)\big) = w$, $w \in$ range of f. Moreover, the function $f^{-1}(w)$ has the Taylor-Maclaurin series expansion of the form:

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n \quad (|w| < r_0(f), r_0(f) \geqq \frac{1}{4}). \tag{1.2}$$

For initial values of n, one can easily see $b_2 = -a_2$, $b_3 = 2a_2^2 - a_3$, $b_4 = 5a_2a_3 - 5a_2^3 - a_4$, and so on.

The function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{U}$ if $f \in \mathscr{S}$ and f^{-1} has univalent analytic continuation to the disk $\mathbb{U}$. Let Σ denote the class of bi-univalent analytic functions in $\mathbb{U}$ of the form (1.1). Lewin [13] investigated the class Σ of bi-univalent analytic functions and demonstrated that the inequality $|a_2| \leqq 1.51$ is satisfied by the second coefficient of each $f \in \Sigma$. Let Σ_1 be the class all functions $f = \phi \circ \psi^{-1}$, where ϕ and ψ map $\mathbb{U}$ onto a domain containing $\mathbb{U}$ and $\phi'(0) = \psi'(0)$. A function in $\Sigma_1 \subset \Sigma$ that satisfies $a_2 = 4/3$ was provided by Suffridge [33], who also conjectured that $|a_2| \leqq 4/3$ for all functions in Σ. For the subclass Σ_1, Netanyahu [19] refuted this conjecture in 1969. In 1981, Styer and Wright [32] proved that $a_2 > 4/3$ for some function in Σ, thereby refuting Suffridge's conjecture. For another example demonstrating that $\Sigma \neq \Sigma_1$, we refer to [7]. Smith [21] and Kedzierawski and Waniurski [12] derived results on bi-univalent polynomials. For $f \in \Sigma$, Brannan and Clunie [5] conjectured in 1967 that $|a_2| \leqq \sqrt{2}$. Kedzierawski [11, Theorem 2] proved this conjecture for a special case where the functions f and f^{-1} are starlike functions. The bound $|a_2| \leqq 1.485$ found by Tan [35] happens to be the best estimate currently available for functions in the class Σ; see [10] and [22] for a survey and a list of the related open problems.

Kedzierawski [11] established the following results in 1985:

$$|a_2| \leqq \begin{cases} 1.5894 & \left(f \in \mathscr{S};\ f^{-1} \in \mathscr{S}\right), \\ \sqrt{2} & \left(f \in \mathscr{S}^*;\ f^{-1} \in \mathscr{S}^*\right), \\ 1.507 & \left(f \in \mathscr{S}^*;\ f^{-1} \in \mathscr{S}\right), \\ 1.224 & \left(f \in \mathscr{K};\ f^{-1} \in \mathscr{S}\right), \end{cases}$$

where $\mathscr{S}^*$ and $\mathscr{K}$ denote the classes of starlike and convex functions in $\mathscr{S}$.

A brief history of the developments regarding the function class Σ can be found in the pioneering work by Srivastava *et al.* [29] which apparently revived the study of the analytic and bi-univalent function class Σ. One can find outstanding works various subclasses of the analytic and bi-univalent function class Σ in, for example, [6, 15, 20, 23, 24, 25, 26, 27, 28, 30, 31, 34, 36]. In the class $\mathscr{S}^*_\Sigma(\beta)$ for $0 \leqq \beta < 1$, there exists a function $f \in \Sigma$ of bi-starlike function of order β, or $\mathscr{K}_\Sigma(\beta)$ of bi-convex function of order β if both f and f^{-1} are starlike or convex functions of order β, respectively. Function $f \in \Sigma$ for $0 < \alpha \leqq 1$ belongs to the class $\mathscr{S}^{*,\alpha}_\Sigma$ of strongly bi-starlike functions of order α, or $\mathscr{K}^\alpha_\Sigma$ of strongly bi-convex functions of order α if both f and f^{-1} are, respectively, strongly starlike or strongly convex functions of order α. Brannan and Taha [6] introduced and analyzed these classes and estimated the initial coefficients a_2 and a_3 for functions in these classes. In this connection, Kumar *et al.* [15] derived the following results similar to those of Kedezierawski [11]:

$$|a_2| \leqq \begin{cases} 0.867 & (f \in \mathscr{K};\ f^{-1} \in \mathscr{C}) \\ 1.054 & (f \in \mathscr{S}^*;\ f^{-1} \in \mathscr{C}) \end{cases}$$

and

$$|a_3| \leqq \begin{cases} 0.833 & (f \in \mathscr{K};\ f^{-1} \in \mathscr{C}) \\ 1.56 & (f \in \mathscr{S}^*;\ f^{-1} \in \mathscr{C}), \end{cases}$$

where $\mathscr{S}^*$, $\mathscr{K}$, and $\mathscr{C}$ denote, respectively, the classes of starlike, convex, and close-to-convex functions in $\mathscr{S}$.

The works of Kedezierawski [11] and Kumar *et al.* [15] incite us to estimate the bounds on the coefficients a_2, a_3, and a_4 when f is in some subclasses of the class of univalent functions and its inverse f^{-1} is in some other subclasses of univalent functions.

Each of the following definitions is needed to prove our present investigation.

Definition 1.1. (see [6]) A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{S}^*_\Sigma(\eta)$ $(0 \leqq \eta < 1)$ if $f \in \Sigma$ and $\Re\left(\frac{zf'(z)}{f(z)}\right) > \eta$, $z \in \mathbb{U}$ and $\Re\left(\frac{wg'(w)}{g(w)}\right) > \eta$, $w \in \mathbb{U}$, where g is the analytic continuation of f^{-1} to $\mathbb{U}$.

Definition 1.2. (see [6]) A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{K}_\Sigma(\eta)$ $(0 \leqq \eta < 1)$ if $f \in \Sigma$ and $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \eta$, $z \in \mathbb{U}$, and $\Re\left(1 + \frac{wg''(w)}{g'(w)}\right) > \eta$, $w \in \mathbb{U}$, where g is the analytic continuation of f^{-1} to $\mathbb{U}$.

Definition 1.3. (see [14]) A function $f \in \mathscr{A}$ is said to be in the following class: $\mathscr{M}_\Sigma(\delta, \eta)$, $0 \leqq \eta < 1, 0 \leqq \delta \leqq 1)$, if

$$f \in \Sigma \text{ and } \Re\left((1-\delta)\left(\frac{zf'(z)}{f(z)}\right) + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \eta \quad (z \in \mathbb{U})$$

and

$$\Re\left((1-\delta)\left(\frac{wg'(w)}{g(w)}\right) + \delta\left(1 + \frac{wg''(w)}{g'(w)}\right)\right) > \eta \quad (w \in \mathbb{U}),$$

where g is the analytic continuation of f^{-1} to $\mathbb{U}$.

We also need the class $\mathscr{P}$ of analytic functions $\mathfrak{p}(z)$ of the form: $\mathfrak{p}(z)=1+\sum_{k=1}^{\infty}c_kz^k$, $z\in\mathbb{U}$ and satisfying $\Re(\mathfrak{p}(z))>0$. The class $\mathscr{P}$ is popularly named after Carathéodory.

Motivated by the earlier works of Srivastava *et al.* [29] and Kumar *et al.* [15], we, in this paper, obtain estimates on initial coefficients a_2, a_3, and a_4 for the functions of the class Σ when f is in some subclasses of univalent functions and f^{-1} is in some other subclasses of univalent functions.

Analytic and bi-univalent function classes are the main topic of this study. The fundamental concepts of some of these function classes are provided by our paper structure, which is displayed in Section 1. Section 2 presents the statements and proofs of Theorem 2.1 if $f\in\mathscr{S}^*_{\Sigma}(\eta)$ and $g\in\mathscr{M}_{\Sigma}(\delta,\eta)$ and Theorem 2.2 if $f\in\mathscr{K}_{\Sigma}(\eta)$ and $g\in\mathscr{M}_{\Sigma}(\delta,\eta)$. Applications of the theorems to classical subclasses of the class Σ are also found in Section 2 (see Corollaries 2.1, 2.2, and 2.3, and Remark 2.1). Section 3 presents the proofs of Theorem 3.1 if $f\in\mathscr{S}^{*,\gamma}_{\Sigma}$ and $g\in\mathscr{M}^{\gamma}_{\Sigma}(\delta)$ and Theorem 3.2 if $f\in\mathscr{K}^{\gamma}_{\Sigma}$ and $g\in\mathscr{M}^{\gamma}_{\Sigma}(\delta)$. Applications of the theorems to classical subclasses of the class Σ are also included in Section 3 (see Corollaries 3.1, 3.2, and 3.3, and Remark 3.1). Finally, in Section 4, we present the concluding remarks and observations pertaining to our investigation.

2. Coefficient Estimates for the Classes $\mathscr{S}^*_{\Sigma}(\eta)$ and $\mathscr{K}_{\Sigma}(\eta)$ with $\mathscr{M}_{\Sigma}(\delta,\eta)$

In this section, we first prove the results for function $f\in\mathscr{S}^*_{\Sigma}(\eta)$ and the inverse of function $g\in\mathscr{M}_{\Sigma}(\delta,\eta)$ $(0\leqq\eta<1; 0\leqq\delta\leqq 1)$. We also derive the consequences of the results of the following theorem for particular choices of the parameter δ.

Theorem 2.1. *Let the functions f and g, given by* (1.1) *and* (1.2), *respectively, be in the class Σ. If $f\in\mathscr{S}^*_{\Sigma}(\eta)$ and $g\in\mathscr{M}_{\Sigma}(\delta,\eta)$, $0\leqq\eta<1$ and $0\leqq\delta\leqq 1)$, then*

$$|a_2|\leqq\begin{cases}(1-\eta)\sqrt{\frac{4(1+\delta)}{(2+3\delta)(1-\eta)}} & \left(0\leqq\eta\leqq\frac{1}{2}\right),\\ (1-\eta)\sqrt{\frac{8(1+\delta)}{(2+3\delta)}} & \left(\frac{1}{2}\leqq\eta<1\right),\end{cases}\tag{2.1}$$

$$|a_3|\leqq\frac{(1-\eta)(4+5\delta)}{2+3\delta}\tag{2.2}$$

and

$$|a_4|\leqq\begin{cases}\frac{2(1-\eta)}{3}+\frac{(1-\eta)^2(8+14\delta)}{3(2+3\delta)}\sqrt{\frac{4(1+\delta)}{(1-\eta)(2+3\delta)}} & \left(0\leqq\eta\leqq\frac{1}{2}\right),\\ \frac{2(1-\eta)}{3}+\frac{(1-\eta)^2(8+14\delta)}{3(2+3\delta)}\sqrt{\frac{8(1+\delta)}{(2+3\delta)}} & \left(\frac{1}{2}\leqq\eta<1\right).\end{cases}\tag{2.3}$$

Proof. Let the function $f\in\Sigma$ be a member of the class $\mathscr{S}^*_{\Sigma}(\eta)$ and suppose that the function $g\in\Sigma$ is in the class $\mathscr{M}_{\Sigma}(\delta,\eta)$ $(0\leqq\eta<1; 0\leqq\delta\leqq 1)$. Then, by Definition 1.1 and 1.3, we have

$$\frac{zf'(z)}{f(z)}=\eta+(1-\eta)\mathfrak{p}(z)\quad(z\in\mathbb{U})\tag{2.4}$$

and

$$(1-\delta)\left(\frac{wg'(w)}{g(w)}\right)+\delta\left(1+\frac{wg''(w)}{g'(w)}\right)=\eta+(1-\eta)\mathfrak{q}(w)\quad(w\in\mathbb{U}),\tag{2.5}$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are members of the Carathéodory class $\mathscr{P}$ and have the following forms:

$$\mathfrak{p}(z)=1+c_1z+c_2z^2+c_3z^3+\cdots\quad(z\in\mathbb{U})\tag{2.6}$$

and

$$\mathfrak{q}(w) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \cdots \quad (w \in \mathbb{U}). \tag{2.7}$$

Now, by equating the coefficients of (2.4), we have

$$a_2 = (1-\eta)c_1, \tag{2.8}$$

$$2a_3 - a_2^2 = (1-\eta)c_2, \tag{2.9}$$

and

$$3a_4 - 3a_2a_3 + a_2^3 = (1-\eta)c_3. \tag{2.10}$$

Similarly, a comparison of the coefficients of both sides of (2.5) yields

$$-(1+\delta)a_2 = (1-\eta)d_1, \tag{2.11}$$

$$(3+5\delta)a_2^2 - (2+4\delta)a_3 = (1-\eta)d_2, \tag{2.12}$$

and

$$-(10+22\delta)a_2^3 + (12+30\delta)a_2a_3 - (3+9\delta)a_4 = (1-\eta)d_3. \tag{2.13}$$

From (2.8) and (2.11), it is clear that $c_1 = -\frac{d_1}{1+\delta}$.

We first obtain refined estimates on $|c_1|$ by using the above relations. For this purpose, we add (2.9) with (2.12) to see that

$$(2+3\delta)a_2^2 = (1-\eta)[(1+2\delta)c_2 + d_2]. \tag{2.14}$$

On substituting $a_2 = (1-\eta)c_1$ in the above relation, we see after simplification:

$$c_1^2 = \frac{(1+2\delta)c_2 + d_2}{(2+3\delta)(1-\eta)}. \tag{2.15}$$

By applying the inequalities $|c_2| \leqq 2$ and $|d_2| \leqq 2$, the relation (2.15) gives the following refined estimates:

$$|c_1| \leqq \begin{cases} \sqrt{\frac{4(1+\delta)}{(2+3\delta)(1-\eta)}} & \left(0 \leqq \eta \leqq \frac{1}{2}\right), \\ \sqrt{\frac{8(1+\delta)}{(2+3\delta)}} & \left(\frac{1}{2} \leqq \eta < 1\right). \end{cases} \tag{2.16}$$

Using the estimate (2.16) in (2.8), we find that

$$|a_2| \leqq \begin{cases} (1-\eta)\sqrt{\frac{4(1+\delta)}{(2+3\delta)(1-\eta)}} & \left(0 \leqq \eta \leqq \frac{1}{2}\right) \\ (1-\eta)\sqrt{\frac{8(1+\delta)}{(2+3\delta)}} & \left(\frac{1}{2} \leqq \eta < 1\right) \end{cases},$$

which is precisely our estimate in (2.1). In order to find bounds on $|a_3|$, we subtract (2.12) from (2.9) and obtain

$$(4+8\delta)a_3 = (4+7\delta)a_2^2 + (1-\eta)[(1+2\delta)c_2 - d_2].$$

Using a_2^2 from (2.14) in the above equation, we find after simplification that

$$a_3 = \frac{1-\eta}{2(2+3\delta)}[(3+5\delta)c_2 + d_2]. \tag{2.17}$$

By applying the triangle inequality and the well-known estimates $|c_2| \leqq 2$ and $|d_2| \leqq 2$ in (2.17), we have

$$|a_3| \leqq \frac{(1-\eta)(4+5\delta)}{2+3\delta},$$

which is precisely our estimate in (2.2).

Next, adding (2.13) and (2.10), we have

$$(9+21\delta)a_2a_3-(9+19\delta)a_2^3=(1-\eta)[(1+3\delta)c_3+d_3]. \tag{2.18}$$

We now find estimates on $|a_4|$. We express a_4 in terms of the first three coefficients of the functions $\mathfrak{p}$ and $\mathfrak{q}$. Subtracting (2.13) from (2.10), we arrive at

$$\begin{aligned}6(1+3\delta)a_4&=(15+39\delta)a_2a_3-(11+25\delta)a_2^3+(1-\eta)[(1+3\delta)c_3-d_3]\\&=(9+21\delta)a_2a_3-(9+19\delta)a_2^3+(6+18\delta)a_2a_3-(2+6\delta)a_2^3\\&\quad+(1-\eta)[(1+3\delta)c_3-d_3].\end{aligned}$$

By using (2.8), (2.17) and, (2.18), we obtain

$$a_4=\frac{1-\eta}{3}c_3-\frac{(1-\eta)^3}{3}c_1^3+\frac{(1-\eta)^2c_1}{2(2+3\delta)}[(3+5\delta)c_2+d_2].$$

On replacing c_1^2, the above relation reduces to

$$a_4=\frac{1-\eta}{3}c_3+\frac{(1-\eta)^2(7+11\delta)}{6(2+3\delta)}c_1c_2+\frac{(1-\eta)^2}{6(2+3\delta)}c_1d_2. \tag{2.19}$$

By applying the usual estimates $|c_2|\leqq 2$, $|c_3|\leqq 2$, $|d_2|\leqq 2$ and the refined estimate (2.16) for $|c_1|$ in (2.19), conclude

$$\begin{aligned}|a_4|&\leqq\frac{1-\eta}{3}|c_3|+\frac{(1-\eta)^2(7+11\delta)}{6(2+3\delta)}|c_1|\cdot|c_2|+\frac{(1-\eta)^2}{6(2+3\delta)}|c_1|\cdot|d_2|\\&\leqq\begin{cases}\frac{2(1-\eta)}{3}+\frac{(1-\eta)^2(8+11\delta)}{3(2+3\delta)}\sqrt{\frac{4(1+\delta)}{(1-\eta)(2+3\delta)}} & \left(0\leqq\eta\leqq\frac{1}{2}\right)\\ \frac{2(1-\eta)}{3}+\frac{(1-\eta)^2(8+11\delta)}{3(2+3\delta)}\sqrt{\frac{8(1+\delta)}{(2+3\delta)}} & \left(\frac{1}{2}\leqq\eta<1\right),\end{cases}\end{aligned}$$

which is precisely our assertion in (2.3). Thus the proof of Theorem 2.1 is complete. □

Taking $\delta=1$ in Theorem 2.1, we have the following results.

Corollary 2.1. *Let the functions f and g, given by* (1.1) *and* (1.2), *respectively, be in the class* Σ. *If $f\in\mathcal{S}_\Sigma^*(\eta)$ and $g\in\mathcal{M}_\Sigma(1,\eta):=\mathcal{K}_\Sigma(\eta)$, $0\leqq\eta<1$, then*

$$|a_2|\leqq\begin{cases}(1-\eta)\sqrt{\frac{8}{5(1-\eta)}} & \left(0\leqq\eta\leqq\frac{1}{2}\right)\\(1-\eta)\sqrt{\frac{16}{5}} & \left(\frac{1}{2}\leqq\eta<1\right)\end{cases},$$

$|a_3|\leqq\frac{9(1-\eta)}{5}$ *and*

$$|a_4|\leqq\begin{cases}\frac{2(1-\eta)}{3}+\frac{22(1-\eta)^2}{15}\sqrt{\frac{8}{5(1-\eta)}} & \left(0\leqq\eta\leqq\frac{1}{2}\right)\\ \frac{2(1-\eta)}{3}+\frac{22(1-\eta)^2}{15}\sqrt{\frac{16}{5}} & \left(\frac{1}{2}\leqq\eta<1\right).\end{cases}$$

Remark 2.1. By setting $\delta=0$ in Theorem 2.1, we have a known result given by Mishra and Soren [17, Theorem 2.2].

We next demonstrate the results for the function $f\in\mathcal{K}_\Sigma(\eta)$ and the inverse of the function $g\in\mathcal{M}_\Sigma(\delta,\eta)$, $0\leqq\eta<1$ and $0\leqq\delta\leqq1$.

Theorem 2.2. *Let the functions f and g, given by (1.1) and (1.2), respectively, be in the class Σ. If $f \in \mathscr{K}_\Sigma(\eta)$ and $g \in \mathscr{M}_\Sigma(\delta,\eta)$, $0 \leqq \eta < 1$ and $0 \leqq \delta \leqq 1$, then*

$$|a_2| \leqq \begin{cases} 2(1-\eta)\sqrt{\frac{2+\delta}{(1-\eta)(5+7\delta)}} & \left(0 \leqq \eta \leqq \frac{1}{2}\right), \\ 2(1-\eta)\sqrt{\frac{2(2+\delta)}{(5+7\delta)}} & \left(\frac{1}{2} \leqq \eta < 1\right), \end{cases} \tag{2.20}$$

$$|a_3| \leqq \frac{(1-\eta)(7+5\delta)}{5+7\delta} \tag{2.21}$$

and

$$|a_4| \leqq \begin{cases} \frac{1-\eta}{6} + \frac{(1-\eta)^2(31+29\delta)}{3(5+7\delta)}\sqrt{\frac{2+\delta}{(1-\eta)(5+7\delta)}} & \left(0 \leqq \eta \leqq \frac{1}{2}\right), \\ \frac{1-\eta}{6} + \frac{(1-\eta)^2(31+29\delta)}{3(5+7\delta)}\sqrt{\frac{2(2+\delta)}{(5+7\delta)}} & \left(\frac{1}{2} \leqq \eta < 1\right). \end{cases}$$

Proof. Let the function $f \in \Sigma$ be in the class $\mathscr{K}_\Sigma(\eta)$ and let the function $g \in \Sigma$ be in $\mathscr{M}_\Sigma(\delta,\eta)$, $0 \leqq \eta < 1$ and $0 \leqq \delta \leqq 1$. By Definitions 1.2 and 1.3, we have

$$1 + \frac{zf''(z)}{f'(z)} = \eta + (1-\eta)\mathfrak{p}(z) \quad (z \in \mathbb{U}) \tag{2.22}$$

and

$$(1-\delta)\left(\frac{wg'(w)}{g(w)}\right) + \delta\left(1 + \frac{wg''(w)}{g'(w)}\right) = \eta + (1-\eta)\mathfrak{q}(w) \quad (w \in \mathbb{U}),$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are members of the Carathéodory class $\mathscr{P}$, given in (2.6) and (2.7), respectively. Equating the coefficients of (2.22) yields

$$2a_2 = (1-\eta)c_1, \tag{2.23}$$

$$6a_3 - 4a_2^2 = (1-\eta)c_2, \tag{2.24}$$

and

$$12a_4 - 18a_2a_3 + 8a_2^3 = (1-\eta)c_3. \tag{2.25}$$

From (2.11) and (2.23), it is clear that $c_1 = -\frac{2d_1}{1+\delta}$. We first obtain refined estimates on $|c_1|$ by using the above relations. For this purpose, adding (2.12) to (2.23), we have

$$(5+7\delta)a_2^2 = (1-\eta)[(1+2\delta)c_2 + 3d_2]. \tag{2.26}$$

On substituting a_2 from (2.23) in the above relation, we find after simplification that

$$c_1^2 = \frac{4[(1+2\delta)c_2 + 3d_2]}{(1-\eta)(5+7\delta)}. \tag{2.27}$$

By applying the inequalities $|c_2| \leqq 2$ and $|d_2| \leqq 2$, relation (2.27) gives

$$|c_1| \leqq \begin{cases} \sqrt{\frac{16(2+\delta)}{(1-\eta)(5+7\delta)}} & \left(0 \leqq \eta \leqq \frac{1}{2}\right), \\ \sqrt{\frac{32(2+\delta)}{(5+7\delta)}} & \left(\frac{1}{2} \leqq \eta < 1\right). \end{cases} \tag{2.28}$$

Using estimate (2.28) in (2.23), we find

$$|a_2| \leqq \begin{cases} 2(1-\eta)\sqrt{\frac{2+\delta}{(1-\eta)(5+7\delta)}} & \left(0 \leqq \eta \leqq \frac{1}{2}\right), \\ 2(1-\eta)\sqrt{\frac{2(2+\delta)}{(5+7\delta)}} & \left(\frac{1}{2} \leqq \eta < 1\right), \end{cases}$$

which is precisely our estimate in (2.20).

In order to find a bounds on $|a_3|$, subtracting (2.12) from (2.24), we obtain

$$12(1+2\delta)a_3=(13+23\delta)a_2^2+(1-\eta)[(1+2\delta)c_2-3d_2].$$

Using a_2^2 from (2.26) in the above equation, we find after simplification that

$$a_3=(1-\eta)\left[\frac{3+5\delta}{2(5+7\delta)}c_2+\frac{2}{5+7\delta}d_2\right]. \tag{2.29}$$

By applying the triangle inequality and the well-known estimates $|c_2|\leqq 2$ and $|d_2|\leqq 2$ in (2.29), we obtain $|a_3|\leqq\frac{(1-\eta)(7+5\delta)}{5+7\delta}$, which is precisely our estimate in (2.21).

Next, we add (2.25) and (2.13) to get

$$(30+66\delta)a_2a_3-(32+64\delta)a_2^3=(1-\eta)[(1+3\delta)c_3+4d_3]. \tag{2.30}$$

We now find estimates on $|a_4|$ by first expressing a_4 in terms of the first three coefficients of $\mathfrak{p}$ and $\mathfrak{q}$. Subtracting (2.13) from (2.25), we obtain

$$\begin{aligned}24(1+3\delta)a_4&=(66+174\delta)a_2a_3-(48+112\delta)a_2^3+(1-\eta)[(1+3\delta)c_3-4d_3]\\&=(30+66\delta)a_2a_3-(32+64\delta)a_2^3+(36+108\delta)a_2a_3-(16+48\delta)a_2^3\\&\qquad+(1-\eta)[(1+3\delta)c_3-4d_3].\end{aligned}$$

Using (2.23), (2.29), and (2.30), we thus have

$$a_4=\frac{1-\eta}{12}c_3-\frac{(1-\eta)^3c_1^3}{12}+\frac{3(1-\eta)^2c_1}{8(5+7\delta)}\left[(3+5\delta)c_2+4d_2\right].$$

On replacing c_1^2 from (2.27), the above relation reduces to

$$a_4=\frac{1-\eta}{12}c_3+\frac{(1-\eta)^2(19+29\delta)}{24(5+7\delta)}c_1c_2+\frac{(1-\eta)^2}{2(5+7\delta)}c_1d_2. \tag{2.31}$$

By applying the usual estimates $|c_2|\leqq 2$, $|c_3|\leqq 2$ and $|d_2|\leqq 2$, and the refined estimate (2.28) for $|c_1|$ in (2.31), we get

$$\begin{aligned}|a_4|&\leqq\frac{1-\eta}{12}|c_3|+\frac{(1-\eta)^2(19+29\delta)}{24(5+7\delta)}|c_1|\cdot|c_2|+\frac{(1-\eta)^2}{2(5+7\delta)}|c_1|\cdot|d_2|\\&\leqq\begin{cases}\frac{1-\eta}{6}+\frac{(1-\eta)^2(31+29\delta)}{3(5+7\delta)}\sqrt{\frac{2+\delta}{(1-\eta)(5+7\delta)}} & \left(0\leqq\eta\leqq\frac{1}{2}\right),\\ \frac{1-\eta}{6}+\frac{(1-\eta)^2(31+29\delta)}{3(5+7\delta)}\sqrt{\frac{2(2+\delta)}{(5+7\delta)}} & \left(\frac{1}{2}\leqq\eta<1\right),\end{cases}\end{aligned}$$

which is precisely our assertion in (2.3). Thus, the proof of the Theorem 2.2 is complete. □

The implications of Theorem 2.1, with specifically chosen the values of δ, yield the following results. For example, by taking $\delta=1$ in Theorem 2.2, we have Corollary 2.2.

Corollary 2.2. *Let the functions f and g given by* (1.1) *and* (1.2), *respectively, be in the class* Σ. *If $f\in\mathscr{K}_\Sigma(\eta)$ and $g\in\mathscr{M}_\Sigma(1,\eta):=\mathscr{K}_\Sigma(\eta)$, $0\leqq\eta<1$, then*

$$|a_2|\leqq\begin{cases}2(1-\eta)\sqrt{\frac{1}{3(1-\eta)}} & \left(0\leqq\eta\leqq\frac{1}{2}\right)\\ 2(1-\eta)\sqrt{\frac{2}{3}} & \left(\frac{1}{2}\leqq\eta<1\right),\end{cases}$$

$|a_3| \leqq 1-\eta$ *and*

$$|a_4| \leqq \begin{cases} \frac{1-\eta}{6}+\frac{5(1-\eta)^2}{3}\sqrt{\frac{1}{3(1-\eta)}} & \left(0 \leqq \eta \leqq \frac{1}{2}\right) \\ \frac{1-\eta}{6}+\frac{5(1-\eta)^2}{3}\sqrt{\frac{2}{3}} & \left(\frac{1}{2} \leqq \eta < 1\right). \end{cases}$$

Upon setting $\delta = 0$ in Theorem 2.2, we have the following results.

Corollary 2.3. *Let the functions f and g given by* (1.1) *and* (1.2), *respectively, be in the class* Σ. *If $f \in \mathscr{K}_\Sigma(\eta)$ and $g \in \mathscr{M}_\Sigma(0,\eta) := \mathscr{S}^*_\Sigma(\eta)$, $0 \leqq \eta < 1$, then*

$$|a_2| \leqq \begin{cases} 2(1-\eta)\sqrt{\frac{2}{5(1-\eta)}} & \left(0 \leqq \eta \leqq \frac{1}{2}\right), \\ 2(1-\eta)\sqrt{\frac{4}{5}} & \left(\frac{1}{2} \leqq \eta < 1\right), \end{cases}$$

$|a_3| \leqq \frac{7(1-\eta)}{5}$ *and*

$$|a_4| \leqq \begin{cases} \frac{1-\eta}{6}+\frac{31(1-\eta)^2}{15}\sqrt{\frac{2}{5(1-\eta)}} & \left(0 \leqq \eta \leqq \frac{1}{2}\right), \\ \frac{1-\eta}{6}+\frac{31(1-\eta)^2}{15}\sqrt{\frac{4}{5}} & \left(\frac{1}{2} \leqq \eta < 1\right). \end{cases}$$

3. Coefficient Estimates for the Classes $\mathscr{S}^{*,\gamma}_\Sigma$ and $\mathscr{K}^{\gamma}_\Sigma$ with $\mathscr{M}^{\gamma}_\Sigma(\delta)$

This section demonstrates our findings by using each of the following definitions.

Definition 3.1. (see [6]) A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{S}^{*,\gamma}_\Sigma$ $(0 < \gamma \leqq 1)$ if $f \in \Sigma$ and $\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\gamma\pi}{2}$, $z \in \mathbb{U}$ and $\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\gamma\pi}{2}$, $w \in \mathbb{U}$, where g is the analytic continuation of f^{-1} to $\mathbb{U}$.

Definition 3.2. (see [6]) A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{K}^{\gamma}_\Sigma$ $(0 < \gamma \leqq 1)$ if $f \in \Sigma$ and $\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\gamma\pi}{2}$, $z \in \mathbb{U}$ and $\left|\arg\left(1+\frac{wg''(w)}{g'(w)}\right)\right| < \frac{\gamma\pi}{2}$, $w \in \mathbb{U}$, where g is the analytic continuation of f^{-1} to $\mathbb{U}$.

Definition 3.3. (see [14]) A function $f \in \mathscr{A}$ is said to be in $\mathscr{M}^{\gamma}_\Sigma(\delta)$, $0 < \gamma \leqq 1$ and $0 \leqq \delta \leqq 1$, if

$$f \in \Sigma \text{ and } \left|\arg\left((1-\delta)\left(\frac{zf'(z)}{f(z)}\right)+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\right| < \frac{\gamma\pi}{2} \quad (z \in \mathbb{U})$$

and

$$\left|\arg\left((1-\delta)\left(\frac{wg'(w)}{g(w)}\right)+\delta\left(1+\frac{wg''(w)}{g'(w)}\right)\right)\right| < \frac{\gamma\pi}{2} \quad (w \in \mathbb{U}),$$

where g is the analytic continuation of f^{-1} to $\mathbb{U}$.

We will now prove the result for the function $f \in \mathscr{S}^{*,\gamma}_\Sigma$ and the inverse of the function $g \in \mathscr{M}^{\gamma}_\Sigma(\delta)$ $(0 < \gamma \leqq 1; 0 \leqq \delta \leqq 1)$. We then derived the consequences of the result of the following theorem, particularly values of δ.

Theorem 3.1. *Let the functions f and g given by* (2.1) *and* (2.2), *respectively, be in the class* Σ. *If* $f \in \mathscr{S}_{\Sigma}^{*,\gamma}$ *and* $g \in \mathscr{M}_{\Sigma}^{\gamma}(\delta)$ $(0 < \gamma \leqq 1; 0 \leqq \delta \leqq 1)$, *then*

$$|a_2| \leqq 2\gamma\sqrt{\frac{2+2\delta}{\gamma(2+2\delta-\delta^2)+(2+4\delta+\delta^2)}}, \tag{3.1}$$

$$|a_3| \leqq \frac{(8+8\delta-\delta^2)\gamma^2+(2\delta+\delta^2)\gamma}{(2+2\delta-\delta^2)\gamma+(2+4\delta+\delta^2)} \tag{3.2}$$

and

$$|a_4| \leqq \frac{2\gamma}{3}+2\frac{(64+64\delta-15\delta^2)\gamma^3+(-12+18\delta+21\delta^2)\gamma^2-(4+16\delta+6\delta^2)\gamma}{9[(2+2\delta-\delta^2)\gamma+(2+4\delta+\delta^2)]} \cdot\sqrt{\frac{2+2\delta}{\gamma(2+2\delta-\delta^2)+(2+4\delta+\delta^2)}}. \tag{3.3}$$

Proof. Let the functions $f \in \Sigma$ and $g \in \Sigma$ be in the classes $\mathscr{S}_{\Sigma}^{*,\gamma}$ $(0 < \gamma \leqq 1)$ and $\mathscr{M}_{\Sigma}^{\gamma}(\delta)$ $(0 < \gamma \leqq 1; 0 \leqq \delta \leqq 1)$, respectively. Then, by Definition 3.1 and 3.3, we have

$$\frac{zf'(z)}{f(z)} = [\mathfrak{p}(z)]^{\gamma} \quad (z \in \mathbb{U}) \tag{3.4}$$

and

$$(1-\delta)\left(\frac{wg'(w)}{g(w)}\right)+\delta\left(1+\frac{wg''(w)}{g'(w)}\right) = [\mathfrak{q}(z)]^{\gamma} \quad (w \in \mathbb{U}), \tag{3.5}$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are members of the Carathéodory class $\mathscr{P}$ and are given by (2.6) and (2.7), respectively. Equating the coefficients of (3.4), we have

$$a_2 = \gamma c_1, \tag{3.6}$$

$$-a_2^2+2a_3 = \gamma c_2+\frac{\gamma(\gamma-1)}{2}c_1^2 \tag{3.7}$$

and

$$a_2^3-3a_2a_3+3a_4 = \gamma c_3+\gamma(\gamma-1)c_1c_2+\frac{\gamma(\gamma-1)(\gamma-2)}{6}c_1^3. \tag{3.8}$$

Similarly, a comparison of the coefficients of both sides of (3.5) yields

$$-(1+\delta)a_2 = \gamma d_1, \tag{3.9}$$

$$(3+5\delta)a_2^2-(2+4\delta)a_3 = \gamma d_2+\frac{\gamma(\gamma-1)}{2}d_1^2, \tag{3.10}$$

and

$$-(10+22\delta)a_2^3+(12+30\delta)a_2a_3-(3+9\delta)a_4 = \gamma d_3+\gamma(\gamma-1)d_1d_2+\frac{\gamma(\gamma-1)(\gamma-2)}{6}d_1^3. \tag{3.11}$$

From (3.6) and (3.9), it is clearly seen that

$$c_1 = -\frac{d_1}{1+\delta}. \tag{3.12}$$

We first obtain refined estimates on $|c_1|$ by using the above relations. For this purpose, we solve the equation (3.7) and (3.10) to see

$$(2+3\delta)a_2^2 = \gamma[(1+2\delta)c_2 + d_2] + \frac{\gamma(\gamma-1)}{2}[(1+2\delta)c_1^2 + d_1^2]. \tag{3.13}$$

Using $a_2 = \gamma c_1$ and $d_1^2 = (1+\delta)^2 c_1^2$ in (3.13), we have the following relation:

$$c_1^2 = \frac{2[(1+2\delta)c_2 + d_2]}{\gamma(2+2\delta-\delta^2)+(2+4\delta+\delta^2)}. \tag{3.14}$$

Relation (3.14) also gives the following refined estimates:

$$|c_1| \leqq 2\sqrt{\frac{2+2\delta}{\gamma(2+2\delta-\delta^2)+(2+4\delta+\delta^2)}}. \tag{3.15}$$

Using the estimate (3.15) in (3.6), we obtain

$$|a_2| \leqq 2\gamma\sqrt{\frac{2+2\delta}{\gamma(2+2\delta-\delta^2)+(2+4\delta+\delta^2)}},$$

which is precisely our estimate in (3.1). In order to find bounds on $|a_3|$, solving (3.7) and (3.10) yields

$$4(1+2\delta)a_3 = (4+7\delta)a_2^2 + \gamma[(1+2\delta)c_2 - d_2] + \frac{\gamma(\gamma-1)}{2}[(1+2\delta)c_1^2 - d_1^2].$$

Using $d_1^2 = (1+\delta)^2 c_1^2$ from (3.12), $a_2 = \gamma c_1$ from (3.6) and

$$c_1^2 = \frac{2[(1+\delta)c_2 + d_2]}{\gamma(2+2\delta-\delta^2)+(2+4\delta+\delta^2)}$$

in the above equation, we find after simplification that

$$a_3 = \frac{(5+8\delta-\delta^2)\gamma^2 + (1+2\delta+\delta^2)\gamma}{(2+2\delta-\delta^2)\gamma+(2+4\delta+\delta^2)}\frac{c_2}{2} + \frac{3\gamma^2-\gamma}{(2+2\delta-\delta^2)\gamma+(2+4\delta+\delta^2)}\frac{d_2}{2}. \tag{3.16}$$

Now, by applying the triangle inequality and the usual estimates $|c_2| \leqq 2$ and $|d_2| \leqq 2$ in (3.16), we obtain

$$|a_3| \leqq \frac{(8+8\delta-\delta^2)\gamma^2 + (2\delta+\delta^2)\gamma}{(2+2\delta-\delta^2)\gamma+(2+4\delta+\delta^2)},$$

which is precisely our estimate in (3.2).

Next, adding (3.8) and (3.11), we obtain

$$(9+21\delta)a_2a_3 - (9+19)a_2^3 = \gamma[(1+3\delta)c_3 + d_3] + \gamma(\gamma-1)[(1+3\delta)c_1c_2 + d_1d_2] + \frac{\gamma(\gamma-1)(\gamma-2)}{6}((1+3\delta)c_1^3 + d_1^3). \tag{3.17}$$

We now find estimates on $|a_4|$. We express a_4 in terms of the first three coefficients of $\mathfrak{p}$ and $\mathfrak{q}$. For this, we subtract (3.11) from (3.8) to see

$$\begin{aligned}6(1+3\delta)a_4 &= (15+39\delta)a_2a_3-(11+25\delta)a_2^3+\gamma[(1+3\delta)c_3-d_3]\\&\quad+\gamma(\gamma-1)[(1+3\delta)c_1c_2-d_1d_2]+\frac{\gamma(\gamma-1)(\gamma-2)}{6}[(1+3\delta)c_1^3-d_1^3)\\&=(9+21\delta)a_2a_3-(9+19\delta)a_2^3+(6+18\delta)a_2a_3-(2+6\delta)a_2^3+\gamma[(1+3\delta)c_3-d_3]\\&\quad+\gamma(\gamma-1)[(1+3\delta)c_1c_2-d_1d_2]+\frac{\gamma(\gamma-1)(\gamma-2)}{6}[(1+3\delta)c_1^3-d_1^3].\end{aligned}$$

Using (3.16) and (3.17), we obtain

$$\begin{aligned}3a_4 &= 3\gamma c_1\left(\frac{(5+8\delta-\delta^2)\gamma^2+(1+2\delta+\delta^2)\gamma}{(2+2\delta-\delta^2)\gamma+(2+4\delta+\delta^2)}\frac{c_2}{2}+\frac{3\gamma^2-\gamma}{(2+2\delta-\delta^2)\gamma+(2+4\delta+\delta^2)}\frac{d_2}{2}\right)\\&\quad+\gamma c_3+\gamma(\gamma-1)c_1c_2+\frac{\gamma(\gamma-1)(\gamma-2)}{6}c_1^3-\gamma^3c_1^3.\end{aligned}$$

Again, by substituting c_1^2 from (3.14), we have

$$\begin{aligned}3a_4 &= \gamma c_3+\frac{(47+64\delta-15\delta^2)\gamma^3+(3+18\delta+21\delta)\gamma^2-(8+16\delta+6\delta^2)\gamma}{(2+2\delta-\delta^2)\gamma+(2+4\delta+\delta^2)}\frac{c_1c_2}{6}\\&\quad+\frac{17\gamma^3-15\gamma^2+4\gamma}{(2+2\delta-\delta^2)\gamma+(2+4\delta+\delta^2)}\frac{c_1d_2}{6}.\end{aligned}$$

We apply the usual estimates $|c_3|\leqq 2$, $|c_2|\leqq 2$, and $|d_2|\leqq 2$, and the refined estimate (3.15) for the $|c_1|$. This yields

$$\begin{aligned}|a_4| &\leqq \frac{2\gamma}{3}+2\frac{(64+64\delta-15\delta^2)\gamma^3+(-12+18\delta+21\delta^2)\gamma^2-(4+16\delta+6\delta^2)\gamma}{9[(2+2\delta-\delta^2)\gamma+(2+4\delta+\delta^2)]}\\&\quad\cdot\sqrt{\frac{2+2\delta}{\gamma(2+2\delta-\delta^2)+(2+4\delta+\delta^2)}},\end{aligned}$$

which is precisely our estimate in (3.3). □

Taking $\delta=1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let the functions f and g given by* (2.1) *and* (2.2), *respectively, be in the class* Σ. *If $f\in\mathscr{S}_\Sigma^{*,\gamma}$ and $g\in\mathscr{M}_\Sigma^{\gamma}(1)=\mathscr{K}_\Sigma^{\gamma}$, $0<\gamma\leqq 1$), then* $|a_2|\leqq 2\gamma\sqrt{\frac{4}{3\gamma+7}}$, $|a_3|\leqq\frac{15\gamma^2+3\gamma}{3\gamma+7}$, *and* $|a_4|\leqq\frac{2\gamma}{3}+2\frac{113\gamma^3+27\gamma^2-26\gamma}{9[3\gamma+7]}\sqrt{\frac{4}{3\gamma+7}}$.

Remark 3.1. By setting $\delta=0$ in Theorem 3.1, we readily derive a result of Mishra and Soren [17, Theorem 2.1].

For the function $f\in\mathscr{S}_\Sigma^{*,\gamma}$ and its inverse, given by $g\in\mathscr{M}_\Sigma^{\gamma}(\delta)$, $0<\gamma\leqq 1; 0\leqq\delta\leqq 1$, we state the following theorem.

Theorem 3.2. *Let the functions f and g given by* (1.1) *and* (1.2), *respectively, be in the class* Σ. *If $f \in \mathscr{K}_\Sigma^\gamma$ and $g \in \mathscr{M}_\Sigma^\gamma(\delta)$, $0 < \gamma \leqq 1$ and $0 \leqq \delta \leqq 1$, then*

$$|a_2| \leqq 2\gamma\sqrt{\frac{4+2\delta}{(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)}}, \tag{3.18}$$

$$|a_3| \leqq \frac{\gamma^2(85+159\delta+51\delta^2-7\delta^3)-\gamma(15+11\delta-19\delta^2-7\delta^3)}{(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]} \tag{3.19}$$

and

$$\begin{aligned}|a_4| \leqq \frac{\gamma}{6}+\Bigg(&\frac{\gamma^3(1585+2899\delta+727\delta^2-315\delta^3)+\gamma^2(-765-621\delta+945\delta^2+441\delta^3)}{9(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\\ &-\frac{\gamma(-110+106\delta+454\delta^2+126\delta^3)}{9(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\Bigg)\\ &\cdot\sqrt{\frac{4+2\delta}{(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)}}.\end{aligned} \tag{3.20}$$

Proof. Let the functions $f \in \Sigma$ and $g \in \Sigma$ belong to the class $\mathscr{K}_\Sigma^\gamma$ and $\mathscr{M}_\Sigma^\gamma(\delta)$, respectively. Then, by Definitions 3.2 and 3.3, we have

$$1+\frac{zf''(z)}{f'(z)} = [\mathfrak{p}(z)]^\gamma \quad (z \in \mathbb{U}) \tag{3.21}$$

and

$$(1-\delta)\left(\frac{wg'(w)}{g(w)}\right)+\delta\left(1+\frac{wg''(w)}{g'(w)}\right) = [\mathfrak{q}(z)]^\gamma \quad (w \in \mathbb{U}),$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are members of the Carathéodory class $\mathscr{P}$ and are given by (2.6) and (2.7), respectively. From (3.21), we find that

$$2a_2 = \gamma c_1, \tag{3.22}$$

$$6a_3 - 4a_2^2 = \gamma c_2 + \frac{\gamma(\gamma-1)}{2}c_1^2 \tag{3.23}$$

and

$$12a_4 - 18a_2a_3 + 8a_2^3 = \gamma c_3 + \gamma(\gamma-1)c_1c_2 + \frac{\gamma(\gamma-1)(\gamma-2)}{6}c_1^3. \tag{3.24}$$

Moreover, from the equations (3.22) and (3.9), it is clear that

$$c_1 = -\frac{2d_1}{(1+\delta)}. \tag{3.25}$$

We shall first obtain refined estimates on $|c_1|$ by using the above relations. For this purpose, we solve the equations (3.23) and (3.10), and we get

$$(5+7\delta)a_2^2 = \gamma[(1+2\delta)c_2+3d_2] + \frac{\gamma(\gamma-1)}{8}[(1+2\delta)c_1^2+3d_1^2]. \tag{3.26}$$

Thus, by using $d_1^2 = \frac{(1+\delta)^2}{4}c_1^2$ from (3.25) and $a_2 = \frac{\gamma c_1}{2}$ from (3.22) in (3.26), we find after simplification that

$$c_1^2 = \frac{8[(1+2\delta)c_2+3d_2]}{(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)}. \tag{3.27}$$

The relation (3.27) also gives the following refined estimates:

$$|c_1| \leqq 4\sqrt{\frac{4+2\delta}{(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)}} \tag{3.28}$$

Now, using the estimate (3.28) in (3.22), we have

$$|a_2| \leqq 2\gamma\sqrt{\frac{4+2\delta}{(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)}},$$

which proves (3.18). In order to find bounds on $|a_3|$, we solve the equations (3.10) and (3.23) to get

$$12(1+\delta)a_3 = (13+23\delta)a_2^2+\gamma[(1+2\delta)c_2-3d_2]+\frac{\gamma(\gamma-1)}{2}[(1+2\delta)c_1^2-3d_1^2]. \tag{3.29}$$

Using $d_1^2 = \frac{(1+\delta)^2}{4}c_1^2$ from (3.25) and a_2^2 from (3.26) in the equation (3.29), we find after simplification that

$$a_3 = \frac{\gamma(3+5\delta)}{2(5+7\delta)}c_2+\frac{2\gamma}{5+7\delta}d_2+\frac{\gamma(\gamma-1)(4+7\delta+\delta^2)}{4(5+7\delta)}c_1^2,$$

which, on replacing c_1^2 from (3.28), reduces to

$$\begin{aligned} a_3 = &\left(\frac{\gamma^2(25+75\delta+51\delta^2-7\delta^3)+\gamma(5+17\delta+19\delta^2+7\delta^3)}{2(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\right)c_2 \\ &+\left(\frac{\gamma^2(60+84\delta)-\gamma(20+28\delta)}{2(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\right)d_2. \end{aligned} \tag{3.30}$$

By applying the triangle inequality with the usual estimates $|c_2| \leqq 2$ and $|d_2| \leqq 2$ in (3.30), we get

$$|a_3| \leqq \frac{\gamma^2(85+159\delta+51\delta^2-7\delta^3)-\gamma(15+11\delta-19\delta^2-7\delta^3)}{(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]},$$

which is precisely our estimate in (3.19).

Next, we add the equations (3.11) and (3.24) to find that

$$\begin{aligned} (30+66\delta)a_2a_3-(32+64\delta)a_2^3 = &\gamma[(1+3\delta)c_3+4d_3]+\gamma(\gamma-1)[(1+3\delta)c_1c_2+4d_1d_2] \\ &+\frac{\gamma(\gamma-1)(\gamma-2)}{6}[(1+3\delta)c_1^2+4d_1^2]. \end{aligned} \tag{3.31}$$

We now find the estimates on $|a_4|$. For this purpose, we express a_4 in terms of the first three coefficients of the functions $\mathfrak{p}$ and $\mathfrak{q}$. Then, making use of (3.11) to (3.24), we find by subtraction that

$$\begin{aligned}24(1+3\delta)a_4 &= (66+174\delta)a_2a_3-(48+112\delta)a_2^3+\gamma[(1+3\delta)c_3-4d_3]\\&\quad+\gamma(\gamma-1)[(1+3\delta)c_1c_2-4d_1d_2]\\&\quad+\frac{\gamma(\gamma-1)(\gamma-2)}{6}[(1+3\delta)c_1^3-4d_1^3]\\&=(30+66\delta)a_2a_3-(32+64\delta)a_2^3+(36+108\delta)a_2a_3-(16+48\delta)a_2^3\\&\quad+\gamma[(1+3\delta)c_3-4d_3]+\gamma(\gamma-1)[(1+3\delta)c_1c_2-4d_1d_2]\\&\quad+\frac{\gamma(\gamma-1)(\gamma-2)}{6}[(1+3\delta)c_1^3-4d_1^3]. \qquad (3.32)\end{aligned}$$

We now use (3.31) in the equation (3.32) and, upon simplifying the resulting equation, we obtain

$$a_4=\frac{\gamma}{12}c_3+\frac{3}{2}a_2a_3-\frac{2}{3}a_2^3+\frac{\gamma(\gamma-1)}{12}c_1c_2+\frac{\gamma(\gamma-1)(\gamma-2)}{72}c_1^3.$$

Next, we substitute a_3 from (3.30) and a_2 by $\frac{\gamma}{2}c_1$ form (3.22), and we get

$$\begin{aligned}a_4=\frac{\gamma}{12}c_3&+\left(\frac{\gamma^3(255+717\delta+429\delta^2-105\delta^3)+\gamma^2(85+349\delta+427\delta^2+147\delta^3)}{24(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\right.\\&\left.-\frac{\gamma(70+238\delta+226\delta^2+42\delta^3)}{24(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\right)c_1c_2\\&+\left(\frac{3\gamma^3(15+21\delta)-\gamma^2(5+7\delta)}{2(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\right)c_1d_2-\frac{5\gamma^3+3\gamma^2-2\gamma)}{72}c_1^3, \qquad (3.33)\end{aligned}$$

which, on using c_1^2 from (3.28), reduces to

$$\begin{aligned}a_4=\frac{\gamma}{12}c_3&+\left(\frac{\gamma^3(565+1471\delta+727\delta^2-315\delta^3)+\gamma^2(135+639\delta+945\delta^2+441\delta^3)}{72(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\right.\\&\left.-\frac{\gamma(130+442\delta+454\delta^2+126\delta^3)}{72(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\right)c_1c_2\\&+\left(\frac{\gamma^3(85+119\delta)-\gamma^2(75+105\delta)+\gamma(20+28\delta)}{6(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\right)c_1d_2. \qquad (3.34)\end{aligned}$$

Thus, clearly, we can apply the usual estimates $|c_3|\leqq 2$, $|c_2|\leqq 2$ and $|d_2|\leqq 2$, and the refined estimate (3.28) for $|c_1|$ in the inequality (3.34). This yields

$$\begin{aligned}|a_4|\leqq\frac{\gamma}{6}&+\left(\frac{\gamma^3(1585+2899\delta+727\delta^2-315\delta^3)+\gamma^2(-765-621\delta+945\delta^2+441\delta^3)}{9(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\right.\\&\left.-\frac{\gamma(-110+106\delta+454\delta^2+126\delta^3)}{9(5+7\delta)[(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)]}\right)\sqrt{\frac{4+2\delta}{(3-3\delta^2)\gamma+(7+14\delta+3\delta^2)}},\end{aligned}$$

which is precisely our estimate in (3.20). □

Taking $\delta=1$ in Theorem 3.2, we get the following corollary.

Corollary 3.2. *Let the functions f and g given by* (1.1) *and* (1.2), *respectively, be in the class* Σ. *If* $f \in \mathscr{K}_{\Sigma}^{\gamma}$ *and* $g \in \mathscr{M}_{\Sigma}^{\gamma}(1) := \mathscr{K}_{\Sigma}^{\gamma}$, $0 < \gamma \leqq 1$, *then* $|a_2| \leqq \gamma$, $|a_3| \leqq \gamma^2$, *and* $|a_4| \leqq \frac{\gamma}{6} + \frac{1}{2}\left(\frac{17\gamma^3-2\gamma}{9}\right) = \frac{17\gamma^3+\gamma}{18}$.

Taking $\delta = 0$ in Theorem 3.2, we deduce the following corollary.

Corollary 3.3. *Let the functions f and g given by* (1.1) *and* (1.2), *respectively, be in the class* Σ. *If* $f \in \mathscr{K}_{\Sigma}^{\gamma}$ *and* $g \in \mathscr{M}_{\Sigma}^{\gamma}(0) := \mathscr{S}_{\Sigma}^{*,\gamma}$, $0 < \gamma \leqq 1$, *then* $|a_2| \leqq 2\gamma\sqrt{\frac{4}{3\gamma+7}}$, $|a_3| \leqq \frac{17\gamma^2-3\gamma}{3\gamma+7}$, *and* $|a_4| \leqq \frac{\gamma}{6} + \left(\frac{317\gamma^3-153\gamma^2+22\gamma}{9[3\gamma+7]}\right)\sqrt{\frac{4}{3\gamma+7}}$.

4. Concluding Remarks

Each bi-univalent function $f(z)$ in $\mathbb{U}$ is, by definition, linked to a function $\mathfrak{p}(z) \in \mathscr{P}$, and its inverse function $g(w)$ is related to another function $\mathfrak{q}(w) \in \mathscr{P}$. The third and fourth coefficients of the functions in the classes $\mathscr{S}_{\Sigma}^{*}(\eta)$, $\mathscr{K}_{\Sigma}(\eta)$, $\mathscr{S}_{\Sigma}^{*,\gamma}$,and $\mathscr{K}_{\Sigma}^{\gamma}$, as well as those in the classes $\mathscr{M}_{\Sigma}(\delta,\eta)$ and $\mathscr{M}_{\Sigma}^{\gamma}(\delta)$, are obtained from these relationships and the refined estimates. Our study focus on deriving bounds for the initial Taylor-Maclaurin coefficients a_2, a_3 and a_4 for functions belonging to each of these subclasses. In Theorem 2.1, bounds for these coefficients are examined if $f \in \mathscr{S}_{\Sigma}^{*}(\eta)$ and $g \in \mathscr{M}_{\Sigma}(\delta,\eta)$ and are then investigated in Theorem 2.2 if $f \in \mathscr{K}_{\Sigma}(\eta)$ and $g \in \mathscr{M}_{\Sigma}(\delta,\eta)$. In Theorem 3.1, bounds for these coefficients are examined again if $f \in \mathscr{S}_{\Sigma}^{*,\gamma}$ and $g \in \mathscr{M}_{\Sigma}^{\gamma}(\delta)$ and are then investigated in Theorem 3.2 if $f \in \mathscr{K}_{\Sigma}^{\gamma}$ and $g \in \mathscr{M}_{\Sigma}^{\gamma}(\delta)$. All of the estimates, which we have proved in this study, are new. Presumably, therefore, this article will motivate and encourage future researches on coefficient bounds and coefficient estimates, as well as related developments on problems concerning, for example, the Hankel and Toeplitz determinants as well as the Fekete-Szegö functional for many different subclasses on the analytic and bi-univalent function class Σ; see, e.g., [1, 2, 3, 4] and the references therein.

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