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LOCALIZED COMPLEX SOLUTIONS FOR THE CHOQUARD EQUATION WITH MAGNETIC FIELD

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Abstract. We study the Choquard equation $(\frac{\varepsilon}{i}\nabla - A(x))^2 u + W(x)u = (|x|^{-\mu} * F(|u|^2))f(|u|^2)u$ in presence of a magnetic field A, $i = \sqrt{-1}$ and $0 < \mu < 2$. We prescribe the lowest number of complex solutions $u \in H^1(\mathbb{R}^N, \mathbb{C})$. The quantity is at least equals to the number of global minima of the potential W, when $\varepsilon > 0$ is sufficiently small. We prove a projection lemma of the Nehari manifold corresponding to the energy functional, and then we use it to estimate the energy of each solution and to localize them around each minima of the potential W. We analyze the energy levels of the solutions to distinguish them from each other. We do not use topological arguments, and no symmetric function spaces are used.

Keywords. Choquard equation; Multiple solutions; Magnetic field; Variational methods.

1. INTRODUCTION

We are concerned with the nonlinear Choquard complex equation

$$\left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + W(x)u = \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) f(|u|^2)u, \quad x \in \mathbb{R}^N, \ u \in H^1(\mathbb{R}^N, \mathbb{C}), \tag{1.1}$$

where $N \geq 3$, $F(s) = \int_0^s f(t)dt$, and $i = \sqrt{-1}$ is the imaginary unit. The function $A : \mathbb{R}^N \to \mathbb{R}^N$ denotes a continuous magnetic field associated with a magnetic field B by means of $curl(A) = B$. The symbol $*$ denotes the convolution and the fact that $0 < \mu < 2$ is linked to an integrability property that we will discuss later on. The operator is defined by

$$\left(\frac{\varepsilon}{i}\nabla - A\right)^2 \psi = -\varepsilon^2 \Delta\psi - \frac{2\varepsilon}{i} A.\nabla\psi + |A|^2\psi - \frac{\varepsilon}{i}\psi divA.$$

The function W is a real electric potential and the nonlinear term f is a superlinear function with properties that will be timely displayed. Introducing the change of variables $v(x) = u(\varepsilon x)$, we obtain that (1.1) is equivalent to

$$\left(\frac{1}{i}\nabla - A_\varepsilon(x)\right)^2 v + W_\varepsilon(x)v = \left(\frac{1}{|x|^\mu} * F(|v|^2)\right) f(|v|^2)v, \ x \in \mathbb{R}^N, \ v \in H^1(\mathbb{R}^N, \mathbb{C}), \tag{1.2}$$

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with $A_\varepsilon(x) = A(\varepsilon x)$ and $W_\varepsilon(x) = W(\varepsilon x)$. We denote by $H^1(\mathbb{R}^N, \mathbb{C})$ the Hilbert space obtained by the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ under the scalar product $\langle u, v\rangle_\varepsilon = \Re\left(\int \nabla_\varepsilon u \overline{\nabla_\varepsilon v} + W_\varepsilon(x) u\bar{v}\right)$, where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$, $\bar{z}$ is its conjugated, $\nabla_\varepsilon u = (D_1^\varepsilon u, D_2^\varepsilon u, \ldots, D_N^\varepsilon u)$, and

$$D_k^\varepsilon u(x) = \frac{\partial_k u(x)}{i} - A_k(\varepsilon x) u(x) \text{ for } k = 1, \ldots, N.$$

Notation. Everywhere in the paper we write simply $\int$ to denote the integrals $\int_{\mathbb{R}^N} \cdots dx$ and we use $\int_{set}$ to mean $\int_{set} \cdots dx$.

The norm induced by the inner product $\langle ., .\rangle_\varepsilon$ is defined by

$$\|u\|_\varepsilon = \left(\int |\nabla_\varepsilon u|^2 + W_\varepsilon(x)|u|^2\right)^{1/2}.$$

Hereafter, we say that a function $u \in H^1(\mathbb{R}^N, \mathbb{C})$ is a weak solution to (1.1) if

$$\Re\left(\int \nabla_\varepsilon u \overline{\nabla_\varepsilon v} + W_\varepsilon(x) u\bar{v} - \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) f(|u|^2) u\bar{v}\right) = 0, \text{ for each } v \in H^1(\mathbb{R}^N, \mathbb{C}).$$

We proceed to quote the conditions on the functions A, W and f.

$$\text{The magnetic potential } A : \mathbb{R}^N \to \mathbb{R}^N \text{ is continuous.} \tag{1.3}$$

The potential $W : \mathbb{R}^N \to \mathbb{R}$ is continuous, it has a minimum value

$$W_0 = \min_{x \in \mathbb{R}^N} W(x); \tag{1.4}$$

a limit for large $|x|$

$$W_\infty = \lim_{|x| \to \infty} W(x) > W_0; \tag{1.5}$$

and it has multiple wells

$$W(m_j) = W_0, \; j = 1, \ldots, l \text{ with } m_1 = 0 \text{ and } m_j \neq m_k \text{ if } j \neq k. \tag{1.6}$$

We suppose that $f : [0, +\infty) \to \mathbb{R}$ is C^1 and recall that $0 < \mu < 2$. We assume the following growth conditions

$$\lim_{t \to 0^+} f(t) = 0 \tag{1.7}$$

and there exists $s \in \left(2, \frac{2N - \mu}{N - 2}\right)$ verifying

$$\lim_{t \to \infty} \frac{f(t)}{t^{(s-2)/2}} = 0 \tag{1.8}$$

and

$$f'(t) > 0 \text{ for each } t > 0. \tag{1.9}$$

An example is $f(t) = t^{\frac{s-1}{2}}$ for $s \in (2, \frac{2N-\mu}{N-2})$.

We state our main result.

Theorem 1.1. *Suppose that assumptions (1.3)-(1.9) on A, W, and f hold. Then, there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, problem (1.2) has at least l nontrivial distinct solutions.*

The Riesz potential

$$I_\alpha(x) = R_\alpha \frac{1}{|x|^{N-\alpha}}, \quad \text{where } R_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha} \text{ and } 0 < \alpha < N.$$

appears in the Choquard equation due to quantum physics motivations presented in [19, 22, 28, 30, 31, 32, 36, 40].

The Choquard equation with a source term

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)\,|u|^{p-2}u + \mu g,\ \ x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N, \mathbb{R}),$$

whenever $N = 3$, $p = \alpha = 2$, $V = 1$ and $g \in H^{-1}(\mathbb{R}^3)$ was studied by Küpper, Zhang and Xia [29]. They affirm that there are $\mu_2 \geq \mu_1 > 0$ such that there are two positive solutions if $0 < \mu < \mu_1$ and that no solution exists for $\mu > \mu_2$. A result in the same way for a nonconstant V, $N \geq 3$, $0 < \alpha < N$, and $(N-2)/(N+\alpha) < 1/p < N/(N+\alpha)$ was discussed in [50]; see also [53].

The singularly perturbed Choquard equation

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{-\alpha}\,(I_\alpha * |u|^p)\,|u|^{p-2}u,\ \ x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N, \mathbb{R}),$$

and the so-called semiclassical limit of the solutions u_ε as $\varepsilon \to 0$, was addressed in [47] in dimension $N = 3$, $p = \alpha = 2$ with $V > 0$ by means of Lyapunov–Schmidt finite dimension reduction method. They proved that there are multibump positive solutions v_ε concentrating at k given nondegenerate critical points of V. Similar equations with periodic V were studied in [35]; see also [43] for a result with a power-like V at infinity. A penalization approach was employed in [12] to find multibump solutions concentrating around minima of V; see also [1, 4, 14, 52] for other results about multiplicity and concentration of solutions.

Infinitely many solutions for the equation

$$-\Delta u + u = (I_\alpha * |u|^p)\,|u|^{p-2}u,\ \ x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N, \mathbb{R}),$$

were proved in [33] whenever $p = 2$, by means of the Krasnoselskii genus theory. The equivariant critical point theory was used in [13] with $p > 2$ and in [16], the authors found sign changing solutions; see also [23, 41].

The Choquard equation

$$\left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + u = (I_\alpha * |u|^p)\,|u|^{p-2}u,\ x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N, \mathbb{C}),$$

with a skew-symmetric constant matrix A was studied in [24], where the authors proved existence of a groundstate when $N = 3$ and $p = \alpha = 2$. Infinitely many solutions were obtained in [13] under the assumptions $(N-2)/(N+\alpha) < 1/p < N/(N+\alpha)$ and $dim(KerA) \neq 1$; see also [21, 44]. An account for more details on groundstates appeared in [38] and [39].

The Choquard equation subject to a variable magnetic field $A : \mathbb{R}^N \to \mathbb{R}^N$, namely

$$\left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = \frac{1}{\varepsilon}\left(K * |u|^2 u\right),\ \ x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \tag{1.10}$$

was studied in [15]. There $N = 3$, $V \geq 0$, and $K > 0$ is an even homogeneous function. The solutions exist for $\varepsilon > 0$ sufficiently small and concentrate as $\varepsilon \to 0$ around a finite number of maximum points related to V. Whenever $V > 0$ and K is even, other types of solutions concentrating around a few points were found in [51] for a Schrödinger equation with magnetic field and Hartree type nonlinearity. Vortex type solutions for a magnetic nonlinear Choquard equation were found in [42]. Whenever K is the Riesz potential I_α, the equation

$$\left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = (I_\alpha * |u|^p)\,|u|^{p-2}u,\ x\in\mathbb{R}^N,\ u\in H^1(\mathbb{R}^N,\mathbb{C}), \tag{1.11}$$

was studied in [24] where the authors found groundstate solution when $N=3$ and $p=\alpha=2$ and V is bounded, see also [17, 45, 46] for other multiplicity and concentration behavior of solutions. For other recent results involving the diamagnetic operator and the Choquard term, we refer to [7, 8, 10, 27, 48].

In [11, 31, 33], the Choquard equation was studied with $\mu = 1$. Multibump solutions with $0<\mu<2$ were found in [3, 6, 26]; see also the quasilinear equation in [25]. Concluding this Introduction, it is worth to say that, in the literature, we find some papers where the authors studied the problems involving the nonlinearity of the Choquard type, such as [12, 18, 37] and the references therein.

Usually, the methods dealing with multiple solutions are of topological nature, like genus and category theory. Another way to obtain multiple solutions is by means of some periodic property or by exploiting a sort of symmetry of the underlying space of functions u and it is also related to a symmetric energy functional with f, V, and K symmetric; see (1.10) and (1.11). We do not use such techniques used in some above cited papers.

We find localized solutions for equation (1.1) with a potential W possessing multiple global minima (multiple well potential). The existence of multiple solutions correspond to number of these minima when the scaling parameter ε is sufficiently small. We prove a projection lemma of the Nehari manifold corresponding to the energy functional associated to (1.2), and then we use it to estimate the energy of each solution and to localize them around each minima of W. Part of the difficulties in the estimates we bypass is the fact that the solutions are complex valued. This reasoning is one of the main contribution of the present paper.

In Section 2, we define the function spaces, norms and functionals. We study the equations

$$-\Delta u + W_0 u = \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) f(|u|^2)u \ \text{ in } \ \mathbb{R}^N$$

and

$$-\Delta u + W_\infty u = \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) f(|u|^2)u \ \text{ in } \ \mathbb{R}^N.$$

We compare these two limiting cases to the situation with $W(x)$ of equation (1.1). In Section 3, we study more precisely Φ_ε to obtain suitable converging Palais-Smale sequences. In Section 4, we prove a projection lemma of the Nehari manifold corresponding to Φ_ε into $\mathbb{R}^N$. In Section 5, the last section, we prove Theorem 1.1. Lemmas 5.1 is used to estimate the energy of each solution and to localize them around each minima m_i of W, $i=1,\dots,l$.

2. Functional Setting

In the ongoing section, we define some functionals related to auxiliary equations which are useful to estimate critical levels. We bound equation (1.1), or more precisely (1.2) to bound the critical points of (2.6). By virtue of (1.7)-(1.8), one obtains, for every $\eta>0$, a constant $C_\eta>0$

such that $|f(t^2)| \le \eta|t| + C_\eta |t|^{s-1}$ for every $t \in \mathbb{R}$ and a subsequent integration implies

$$|F(t^2)| \le \frac{\eta}{2}t^2 + \frac{C_\eta}{s}|t|^s \quad \text{for every } t \in \mathbb{R}. \tag{2.1}$$

And (1.9) implies that

$$f(t^2)t^2 - F(t^2) \ge 0 \quad \text{for every } t \in \mathbb{R}. \tag{2.2}$$

The so-called diamagnetic inequality was proved by Esteban and Lions [20, Section II], and it states that, for any $u \in H^1(\mathbb{R}^N, \mathbb{C})$, there holds

$$|\nabla |u|(x)| = \left|\Re\left(\nabla u \frac{\overline{u}}{|u|}\right)\right| = \left|\Re\left((\nabla u - iA_\varepsilon u)\frac{\overline{u}}{|u|}\right)\right| \le |\nabla_\varepsilon u(x)|. \tag{2.3}$$

And it follows from (2.3) that if $u \in H^1(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$. Moreover, the embedding $H^1(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R})$ is continuous for each $2 \le q \le \frac{2N}{N-2}$. For each bounded set $\Omega \subset \mathbb{R}^N$ and $2 \le s < \frac{2N}{N-2}$, the embedding $H^1(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^s(\Omega, \mathbb{R})$. is compact. A function $u \in H^1(\mathbb{R}^N, \mathbb{C})$ is a weak solution to problem (1.1) if

$$\Re\left(\int \nabla_\varepsilon u \overline{\nabla_\varepsilon v} + W_\varepsilon(x) u\overline{v} - \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) f(|u|^2) u\overline{v}\right) = 0 \text{ for each } v \in H^1(\mathbb{R}^N, \mathbb{C}).$$

In order to use the variational method, we have

$$\left|\int \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) F(|u|^2)\right| < \infty \quad \text{for every } u \in H^1(\mathbb{R}^N, \mathbb{C}). \tag{2.4}$$

In the verification of (2.4), it is important to recall Hardy-Littlewood-Sobolev inequality [32], which asserts that if $s, r > 1$ and $0 < \mu < 2$ with $1/s + \mu/N + 1/r = 2$, then, for $f \in L^s(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, there exists a sharp constant $C(s, N, \mu, r) > 0$, independent of f, h, such that

$$\int\int \frac{f(x)h(y)}{|x-y|^\mu} dxdy \le C(s, N, \mu, r)\|f\|_s \|h\|_r. \tag{2.5}$$

Inequality (2.5) guarantees that (2.4) holds. Concluding this fact, recall (2.1). Then, by Hardy-Littlewood-Sobolev inequality and $\mu \in (0,2)$, $\int \left(\frac{1}{|x|^\mu} * |F(|u|^2)|\right)|F(|u|^2)|$ is finite if $F(|u|^2) \in L^t(\mathbb{R}^N, \mathbb{R})$ for $t > 1$ and $\frac{2}{t} + \frac{\mu}{N} = 2$. Therefore, $t = \frac{2N}{2N-\mu}$. Since $s \in \left(2, \frac{2N-\mu}{N-2}\right)$, we obtain $\int |F(|u|^2)|^t < \infty$ for all $u \in H^1(\mathbb{R}^N, \mathbb{C})$, showing (2.4).

In view of the above comments, the Euler-Lagrange functional $\Phi_\varepsilon : H^1(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ associated to (1.2) is well defined by

$$\Phi_\varepsilon(u) = \frac{1}{2}\int |\nabla_\varepsilon u|^2 + \frac{1}{2}\int W_\varepsilon(x)|u|^2 - \frac{1}{4}\int \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) F(|u|^2). \tag{2.6}$$

Furthermore, Φ_ε is C^1 and has the following derivative

$$\Phi_\varepsilon'(u)v = \Re\left(\int \nabla_\varepsilon u \overline{\nabla_\varepsilon v} + W_\varepsilon(x) u\overline{v} - \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) f(|u|^2) u\overline{v}\right). \tag{2.7}$$

Hence, the weak solutions of (1.2) are precisely the critical points of Φ_ε. By means of (1.4) and (1.5), we define in $H^1(\mathbb{R}^N, \mathbb{R})$ two equivalent norms, namely

$$\|u\|_0^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} W_0 u^2$$

and

$$\|u\|_\infty^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} W_\infty u^2.$$

Respectively, we define the functionals Φ_0, $\Phi_\infty : H^1(\mathbb{R}^N,\mathbb{R}) \to \mathbb{R}$ by

$$\Phi_0(u) = \frac{1}{2}\|u\|_0^2 - \frac{1}{4}\int \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) F(|u|^2) \tag{2.8}$$

and

$$\Phi_\infty(u) = \frac{1}{2}\|u\|_\infty^2 - \frac{1}{4}\int \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) F(|u|^2). \tag{2.9}$$

Our purpose now is to prove the existence of a solution for the following equations

$$-\Delta u + W_0 u = \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) f(|u|^2)u \ \text{ in } \ \mathbb{R}^N, \tag{2.10}$$

and

$$-\Delta u + W_\infty u = \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) f(|u|^2)u \ \text{ in } \ \mathbb{R}^N. \tag{2.11}$$

We are mostly concerned to (2.10), because the study related to (2.11) is similar.

The functional Φ_0 defined in (2.8) corresponds to the Euler-Lagrange equation (2.10). Moreover, Φ_0 is of class C^1 in $H^1(\mathbb{R}^N,\mathbb{R})$ with derivative

$$\Phi_0'(u)\phi = \int \nabla u \nabla \phi + \int W_0 u\phi - \left(\frac{1}{|x|^\mu} * F(|u|^2)\right) f(|u|^2)u\phi \text{ for every } \phi \in H^1(\mathbb{R}^N,\mathbb{R}).$$

Consequently, critical points of Φ_0 are precisely the weak solutions of (2.10). We denote by $\mathfrak{N}_0$ the Nehari manifold related to Φ_0, defined by

$$\mathfrak{N}_0 = \left\{u \in H^1(\mathbb{R}^N,\mathbb{R}) : u \neq 0,\ \ \Phi_0'(u)u = 0\right\}. \tag{2.12}$$

Using [5, Section 2], the hypotheses on f, Hardy-Littlewood-Sobolev inequality (2.5), a result due to Lions [34, Lemma I.1] and the invariance of $\mathbb{R}^N$ under translations, we can prove that problem (2.10) has a nontrivial positive solution. By means of [49, Theorem 1.15], we assert the existence of a Palais-Smale sequence (u_n) in $H^1(\mathbb{R}^N,\mathbb{R})$ at level β_0, that is, a sequence with the property

$$\Phi_0(u_n) \to \beta_0 \text{ and } \Phi_0'(u_n) \to 0,$$

where β_0 is the minimax level of the mountain pass theorem related to Φ_0, namely

$$\beta_0 = \inf_{\varsigma \in \mathfrak{J}} \max_{t\in[0,1]} \Phi_0(\varsigma(t)), \tag{2.13}$$

where $\mathfrak{J} = \left\{\varsigma \in C([0,1],H^1(\mathbb{R}^N,\mathbb{R})) : \varsigma(0) = 0 \text{ and } \Phi_0(\varsigma(1)) < 0\right\}$. More precisely, the solution is $U_0 > 0$ such that

$$U_0 \in H^1(\mathbb{R}^N,\mathbb{R}) \text{ such that } \Phi_0(U_0) = \beta_0 \text{ and } \Phi_0'(U_0) = 0. \tag{2.14}$$

Besides, for each $u \in H^1(\mathbb{R}^N,\mathbb{R})$ with $u \neq 0$, there exists a unique $t_0 = t_0(u) > 0$ such that $t_0 u \in \mathfrak{N}_0$ and $\Phi_0(t_0 u) = \max_{t\geq 0} \Phi_0(tu)$. Moreover $\beta_0 = \widetilde{\beta}_0 = \widehat{\beta}_0$, where

$$\widetilde{\beta}_0 = \inf_{u\in H^1(\mathbb{R}^N,\mathbb{R}), u\neq 0} \max_{t\geq 0} \Phi_0(tu) \tag{2.15}$$

and $\widehat{\beta}_0 = \inf_{\mathfrak{N}_0} \Phi_0$. We remark that the results for (2.10) are also true with respect to (2.11). Recall (2.9) and define the mountain pass level $\beta_\infty = \inf_{\zeta\in\mathfrak{T}} \max_{t\in[0,1]} \Phi_\infty(\zeta(t))$, where

$$\mathfrak{T} = \left\{ \zeta \in C([0,1], H^1(\mathbb{R}^N, \mathbb{R})) : \zeta(0) = 0 \text{ and } \Phi_\infty(\zeta(1)) < 0 \right\}$$

and the corresponding Nehari manifold

$$\mathfrak{N}_\infty = \left\{ u \in H^1(\mathbb{R}^N, \mathbb{R}) : u \neq 0, \;\; \Phi'_\infty(u)u = 0 \right\}. \tag{2.16}$$

By the above methods, equation (2.11) has a solution. In other words, there exists $U_\infty \in H^1(\mathbb{R}^N, \mathbb{R})$ such that $\Phi_\infty(U_\infty) = \beta_\infty$ and $\Phi'_\infty(U_\infty) = 0$. We define the next constant for later use

$$\sigma^* = \frac{\beta_\infty - \beta_0}{2} > 0. \tag{2.17}$$

3. Estimates of Critical Levels

In the present section, we prove the Palais-Smale property for Φ_ε and we estimate is critical levels. This is useful to localize its critical points as well as the solutions of (1.1) or (1.2) later.

By analogy with [2, Lemma 2.3], there exists a Palais-Smale sequence (u_n) in $H^1(\mathbb{R}^N, \mathbb{C})$ associated to Φ_ε at a level β_ε; see (2.6) and (2.7), that is, a sequence satisfies

$$\Phi_\varepsilon(u_n) \to \beta_\varepsilon \text{ and } \Phi'_\varepsilon(u_n) \to 0,$$

where β_ε is the minimax level of the mountain pass theorem related to Φ_ε, namely

$$\beta_\varepsilon = \inf_{\iota\in\mathfrak{I}} \max_{t\in[0,1]} \Phi_\varepsilon(\iota(t)),$$

where

$$\mathfrak{I} = \left\{ \iota \in C([0,1], H^1(\mathbb{R}^N, \mathbb{C})) : \iota(0) = 0 \text{ and } \Phi_\varepsilon(\iota(1)) < 0 \right\}.$$

By [2, Lemma 2.4], without loss of generality, we assume that (u_n) is bounded. Hence, for a subsequence, there exists $u \in H^1(\mathbb{R}^N, \mathbb{C})$ such that

$$u_n \rightharpoonup u \text{ weakly in } H^1(\mathbb{R}^N, \mathbb{C}) \text{ and } u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N.$$

By a similar reasoning as in [5, Section 2], it turns out that β_ε verifies

$$\beta_\varepsilon = \inf_{u\in H^1(\mathbb{R}^N,\mathbb{C}), u\neq 0} \sup_{t\geq 0} \Phi_\varepsilon(tu) = \inf_{u\in\mathfrak{N}_\varepsilon} \Phi_\varepsilon(u). \tag{3.1}$$

where $\mathfrak{N}_\varepsilon$ denotes the Nehari manifold related to Φ_ε, that is,

$$\mathfrak{N}_\varepsilon = \left\{ u \in H^1(\mathbb{R}^N, \mathbb{C}) : u \neq 0, \;\; \Phi'_\varepsilon(u)u = 0 \right\}. \tag{3.2}$$

Lemma 3.1. *Recall (2.15) and (3.1), then $\beta_\varepsilon \to \beta_0$ as $\varepsilon \to 0$.*

Proof. Let $u \in \mathfrak{N}_\varepsilon$, $v(x) = |u|(x)$, and $t > 0$ such that $tv \in \mathfrak{N}_0$. Then, from (1.4) and (2.3), we deduce

$$\beta_0 \leq \Phi_0(tv) \leq \Phi_\varepsilon(tu) \leq \max_{t\geq 0} \Phi_\varepsilon(tu) = \Phi_\varepsilon(u)$$

implying $\beta_0 \leq \beta_\varepsilon$ for every $\varepsilon > 0$. Let $t_\varepsilon > 0$ and

$$u_{\varepsilon,m_j}(x) = U_0\left(\frac{\varepsilon x - m_j}{\varepsilon}\right) \exp\left(i\gamma_{m_j}\left(\frac{\varepsilon x - m_j}{\varepsilon}\right)\right),$$

where

$$\gamma_{m_j}(y) = \sum_{k=1}^{N} A_k(m_j) y_k$$

is such that $t_\varepsilon u_{\varepsilon,m_j} \in \mathfrak{N}_\varepsilon$, where U_0 is the solution to (2.10); see (2.14). Then,

$$\|u_{\varepsilon,m_j}\|_\varepsilon^2 = \int \left(\frac{1}{|x|^\mu} * F(t_\varepsilon^2 |u_{\varepsilon,m_j}|^2) \right) f(t_\varepsilon^2 |u_{\varepsilon,m_j}|^2) |u_{\varepsilon,m_j}|^2. \tag{3.3}$$

From (2.2), we have $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} f(t) = \infty$. Thus, we conclude that (t_ε) is bounded, and for a subsequence, $t_\varepsilon \to t_0$ as $\varepsilon \to 0$ and $t_0 \geq 0$. Furthermore, by the change of variable

$$y = \frac{(\varepsilon x - m_j)}{\varepsilon},$$

we obtain

$$\int W_\varepsilon(x) |u_{\varepsilon,m_j}|^2 = \int W(\varepsilon y + m_1) |U_0|^2 = \int W_0 |U_0|^2 + o_\varepsilon(1), \tag{3.4}$$

$$\begin{aligned} \int |\nabla_\varepsilon u_{\varepsilon,m_j}|^2 &= \int \left| \frac{\nabla U_0 \left(\frac{\varepsilon x - m_j}{\varepsilon}\right) \exp\left(i\gamma_{m_j} \frac{\varepsilon x - m_j}{\varepsilon}\right)}{i} + \big(A(m_j) - A(\varepsilon x)\big) U_0 \left(i\gamma_{m_j} \frac{\varepsilon x - m_j}{\varepsilon}\right) \right|^2 \\ &= \int |\nabla U_0 + [A(m_j) - A(\varepsilon y + m_j)]\,|^2 = o_\varepsilon(1), \end{aligned}$$

By the mean value theorem and Lebesgue theorem, we conclude that

$$\int |\nabla_\varepsilon u_{\varepsilon,m_j}|^2 = \int |\nabla U_0|^2 + o_\varepsilon(1). \tag{3.5}$$

Arguing as in (3.4), we also obtain

$$\begin{aligned} &\int \left(\frac{1}{|x|^\mu} * F(t_\varepsilon^2 |u_\varepsilon, m_j|^2) \right) f(t_\varepsilon^2 |u_{\varepsilon,m_j}|^2) |u_{\varepsilon,m_j}|^2 \\ &= \int \left(\frac{1}{|x|^\mu} * F(t_0^2 |U_0|^2) \right) f(t_0^2 |U_0|^2) |U_0|^2 + o_\varepsilon(1). \end{aligned} \tag{3.6}$$

Letting $\varepsilon \to 0$ in (3.3) and using (3.4), (3.5) and (3.6), we obtain

$$\|U_0\|_0^2 = \int \left(\frac{1}{|x|^\mu} * F(t_0^2 |U_0|^2) \right) f(t_0^2 |U_0|^2) |U_0|^2.$$

Since $U_0 \in \mathfrak{N}_0$, we conclude from (2.12) that $t_0 = 1$ and

$$\beta_0 \leq \liminf_{\varepsilon\to 0} \beta_\varepsilon \leq \limsup_{\varepsilon\to 0} \beta_\varepsilon \leq \limsup_{\varepsilon\to 0} \Phi_\varepsilon(t_\varepsilon U_0) \leq \Phi_0(t_0 U_0) \leq \Phi_0(U_0) = \beta_0.$$

□

The next lemma will be used to show the convergence of some Palais-Smale sequences related to Φ_ε.

Lemma 3.2. *Let (v_n) be a Palais-Smale sequence in $H^1(\mathbb{R}^N, \mathbb{C})$ for Φ_ε in the level c such that $c \leq \beta_0 + \sigma^*$. If $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N, \mathbb{C})$, then $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathscr{B}_1(y)} |v_n|^2 = 0$.*

Proof. Assume on the contrary that there exist a sequence (y_n) in $\mathbb{R}^N$ and $\Xi > 0$ such that $\int_{\mathscr{B}_1(y_n)} |v_n|^2 \geq \Xi$ for every $n \in \mathbb{N}$. Since $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N, \mathbb{C})$, then the sequence (y_n) is unbounded. Thus, for a subsequence, $|y_n| \to \infty$ as $n \to \infty$. Define $w_n(x) = |v_n|(x + y_n)$ and observe from (1.5) that, for a $\xi > 0$, there exists $C > 0$ such that $W_\infty - \xi \leq W(x)$ for every $|x| \geq C$. Then, using (2.3), for a subsequence, we deduce

$$\begin{aligned}
\int |\nabla w_n|^2 + (W_\infty - \xi) \int |w_n|^2 &\leq \int |\nabla_\varepsilon v_n|^2 + (W_\infty - \xi) \int |v_n|^2 \\
&\leq \int |\nabla_\varepsilon v_n|^2 + \int W(x) |v_n|^2 \\
&= \int \left(\frac{1}{|x|^\mu} * F(|v_n|^2) \right) f(|v_n|^2) |v_n|^2 \\
&= \int \left(\frac{1}{|x|^\mu} * F(|w_n|^2) \right) f(|w_n|^2) |w_n|^2.
\end{aligned}$$

Since $\xi > 0$ is arbitrary and $\int |w_n|^2$ is bounded, we have that $\Phi'_\infty(w_n) w_n \leq 0$. Then there exists $0 < t_n \leq 1$ such that $t_n w_n \in \mathfrak{N}_\infty$, recall (2.16). From the fact that $\Phi_\varepsilon(v_n) \to c$ and $\Phi'_\varepsilon(v_n) \to 0$ as $n \to \infty$, (1.9), and (2.2), we obtain

$$\begin{aligned}
\beta_\infty &\leq \liminf_{n\to\infty} \left[\Phi_\infty(t_n w_n) - \frac{1}{2} \Phi'_\infty(t_n w_n) t_n w_n \right] \\
&= \liminf_{n\to\infty} \int \left(\frac{1}{|x|^\mu} * F(|t_n w_n|^2) \right) \left[\frac{1}{2} f(|t_n w_n|^2) |t_n w_n|^2 - \frac{1}{4} F(|t_n w_n|^2) \right] \\
&\leq \liminf_{n\to\infty} \int \left(\frac{1}{|x|^\mu} * F(|w_n|^2) \right) \left[\frac{1}{2} f(|w_n|^2) |w_n|^2 - \frac{1}{4} F(|w_n|^2) \right] \\
&= \liminf_{n\to\infty} \int \left(\frac{1}{|x|^\mu} * F(|v_n|^2) \right) \left[\frac{1}{2} f(|v_n|^2) |v_n|^2 - \frac{1}{4} F(|v_n|^2) \right] \\
&= \liminf_{n\to\infty} \left[\Phi_\varepsilon(v_n) - \frac{1}{2} \Phi'_\varepsilon(v_n) v_n \right] = \liminf_{n\to\infty} \Phi_\varepsilon(v_n) = c \leq \beta_0 + \sigma^*,
\end{aligned}$$

which is an absurd. □

Proposition 3.1. *Recall* (2.6)*,* (2.14)*, and* (2.17)*. The functional Φ_ε satisfies the Palais-Smale property at any level $c \leq \beta_0 + \sigma^*$.*

Proof. Let (u_n) be a sequence in $H^1(\mathbb{R}^N, \mathbb{C})$ such that $\Phi_\varepsilon(u_n) \to c$ and $\Phi'_\varepsilon(u_n) \to 0$. By a similar reasoning developed in [2, Lemma 2.4], we conclude that (u_n) is bounded in $H^1(\mathbb{R}^N, \mathbb{C})$ and that there exists $u \in H^1(\mathbb{R}^N, \mathbb{C})$ such that, for a subsequence, $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N, \mathbb{C})$. A density argument yields $\Phi'_\varepsilon(u) = 0$. Setting $v_n = u_n - u$, by Lemma 3.2, we obtain

$$v_n \to 0 \text{ in } L^s(\mathbb{R}^N) \text{ for } s \in \left(2, \frac{2N - \mu}{N - 2}\right),$$

which implies

$$\liminf_{n\to\infty} \int \left(\frac{1}{|x|^\mu} * F(|v_n|^2) \right) f(|v_n|^2) |v_n|^2 dx \to 0.$$

The fact that $\Phi'_\varepsilon(v_n) v_n = o_n(1)$ leads to $\|v_n\|_\varepsilon^2 = o_n(1)$. □

4. Localization of Solutions

In this section, we prove the existence of multiple solutions for (1.2), as enounced in Theorem 1.1. Lemma 4.2 turns out to be a sort of localization of the solutions, this will be fully understood in Section 5.

Recalling assumption (1.6), we take ρ_0, $r_0 > 0$ satisfying

$$\overline{\mathscr{B}_{\rho_0(m_i)}} \cap \overline{\mathscr{B}_{\rho_0(m_j)}} = \emptyset \text{ for } i \neq j \text{ and } i, j \in \{1,2,...,l\},$$

$$\mathscr{B}_{\rho_0(m_1)} \cup \ldots \cup \mathscr{B}_{\rho_0(m_l)} \subset \mathscr{B}_{r_0}(0),$$

and define

$$\Gamma_{\frac{\rho_0}{2}} = \overline{\mathscr{B}_{\rho_0(m_1)}} \cup \ldots \cup \overline{\mathscr{B}_{\rho_0(m_l)}}.$$

Let $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ be defined as

$$\Upsilon(x) = \begin{cases} x & \text{for } \quad |x| \leq r_0 \\ \dfrac{r_0 x}{|x|} & \text{for } \quad |x| \geq r_0. \end{cases}$$

We also define $\mathscr{T}_\varepsilon : \mathfrak{N}_\varepsilon \to \mathbb{R}^N$ by

$$\mathscr{T}_\varepsilon(u) = \frac{\displaystyle\int \Upsilon(\varepsilon x)|u(x)|^s}{\displaystyle\int |u(x)|^s}, \tag{4.1}$$

where $s \in \left(2, \dfrac{2N-\mu}{N-2}\right)$ appeared in the hypothesis (1.8).

Lemma 4.1. *There are sequences $u_n, v_n \in H^1(\mathbb{R}^N, \mathbb{C})$ such that $\|u_n - v_n\|_{\varepsilon_n} = o_n(1)$, $\Phi_{\varepsilon_n}(v_n) = \beta_0 + o_n(1)$ and $\|\Phi'_{\varepsilon_n}(v_n)\|_{\varepsilon_n} = o_n(1)$.*

Proof. By Lemma 3.1, we see that there exists $u_n \in \mathfrak{N}_{\varepsilon_n}$ and $\Phi_{\varepsilon_n}(u_n) \to \beta_0$. Moreover, arguing as in [2, Lemma 2.4], we conclude that (u_n) is bounded in $H^1(\mathbb{R}^N, \mathbb{C})$. Using the Ekeland variational principle as in [9, Lemma 7], we find that there exists $v_n \in H^1(\mathbb{R}^N, \mathbb{C})$ as stated. □

Lemma 4.2. *There exist $\varepsilon^* > 0$ and $\delta_0 > 0$ such that if $u \in \mathfrak{N}_\varepsilon$ and $\Phi_\varepsilon(u) \leq \beta_0 + \delta_0$, then $\mathscr{T}_\varepsilon(u) \in \Gamma_{\frac{\rho_0}{2}}$ for every $\varepsilon \in (0, \varepsilon^*)$.*

Proof. By contradiction, we suppose that there exist $\varepsilon_n \to 0$ and $u_n \in \mathfrak{N}_{\varepsilon_n}$ such that

$$\mathscr{T}_{\varepsilon_n}(u_n) \notin \Gamma_{\frac{\rho_0}{2}} \text{ and } \beta_{\varepsilon_n} \leq \Phi_{\varepsilon_n}(u_n) \leq \beta_0 + \frac{1}{n} \text{ for every } n \in \mathbb{N}.$$

By Lemma 3.1, we have that $u_n \in \mathfrak{N}_{\varepsilon_n}$ and $\Phi_{\varepsilon_n}(u_n) \to \beta_0$. Moreover, arguing as in [2, Lemma 2.4], we conclude that (u_n) is bounded in $H^1(\mathbb{R}^N, \mathbb{C})$. By the existence of a sequence $v_n \in H^1(\mathbb{R}^N, \mathbb{C})$ as in Lemma 4.1. Hence, for a subsequence, one has $\mathscr{T}_{\varepsilon_n}(v_n) \notin \Gamma_{\frac{\rho_0}{4}}$.

Claim. We affirm that there exists a sequence (x_n) in $\mathbb{R}^N$ such that

$$\varepsilon_n x_n \to m_j \in \{m_1, m_2, ..., m_l\} \text{ for some } i = 1, ..., l, \tag{4.2}$$

where $W(m_i) = W_0$, and define $\widetilde{v}_n(x) = v_n(x + x_n)$,

$$\text{there exits } \widetilde{v} \in X \text{ such that } \widetilde{v}_n \to \widetilde{v} \text{ in } L^s(\mathbb{R}^N) \text{ for } s \in \left(2, \frac{2N-\mu}{N-2}\right). \tag{4.3}$$

We postpone the verification of (4.2) and (4.3), but if they are true, we obtain

$$\begin{aligned} \mathscr{T}_{\varepsilon_n}(v_n) &= \frac{\displaystyle\int \Upsilon(\varepsilon_n x)|v_n(x)|^s}{\displaystyle\int |v_n(x)|^s} = \frac{\displaystyle\int \Upsilon(\varepsilon_n x + \varepsilon_n x_n)|\widetilde{v}_n(x)|^s}{\displaystyle\int |\widetilde{v}_n(x)|^s} \\ &= \frac{\displaystyle\int m_j |\widetilde{v}(x)|^s}{\displaystyle\int |\widetilde{v}(x)|^s} + o_n(1) = m_j + o_n(1) \in \Gamma_{\frac{\rho_0}{2}}. \end{aligned}$$

Then $\mathscr{T}_{\varepsilon_n}(v_n) \in \Gamma_{\frac{\rho_0}{4}}$ for n large, which is a contradiction and the proof of the lemma is finished.

We proceed to show (4.2) and (4.3). The sequence (u_n) is bounded in $H^1(\mathbb{R}^N, \mathbb{C})$. Hence, from Lemma 4.1, one sees that (v_n) is also bounded in $H^1(\mathbb{R}^N, \mathbb{C})$. Then, for a subsequence, there exists $v \in H^1(\mathbb{R}^N, \mathbb{C})$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N, \mathbb{C})$. Since

$$o_n(1) + \|v_n\|_{\varepsilon_n}^2 = \int \left(\frac{1}{|x|^\mu} * F(|v_n|^2)\right) f(|v_n|^2)|v_n|^2,$$

we conclude by Lemma 4.1 that v_n does not converges 0 in $L^s(\mathbb{R}^N)$. Define

$$\rho_n(x) = \frac{|v_n(x)|}{\displaystyle\int |v_n|^s},$$

where $s \in \left(2, \frac{2N-\mu}{N-2}\right)$ was given in (1.8). The sequence ρ_n is bounded in $L^1(\mathbb{R}^N)$. According to the concentration compactness principle [34], after a careful analysis, we achieve that only one of the next three items is correct.

Vanishing.

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathscr{B}_R(y)} \rho_n = 0 \text{ for every } R > 0.$$

Dichotomy. We use several times the characteristic function χ_K on a set K. There are a sequence (x_n) in $\mathbb{R}^N$, $0 < \rho < 1$, $\widehat{R} > 0$ and another sequence $R_n \to \infty$ such that mass functions $\rho_{n,1}(x) = \chi_{\mathscr{B}_{\widehat{R}}(x_n)}(x)\rho_n(x)$ and $\rho_{n,2}(x) = \chi_{(\mathbb{R}^N - \mathscr{B}_{\widehat{R}}(x_n))}(x)\rho_n(x)$ verify

$$\int_{\mathbb{R}^N} \rho_{1,n} \to \rho \tag{4.4}$$

and

$$\int_{\mathbb{R}^N} \rho_{2,n} \to 1 - \rho.$$

Compactness. There exists (x_n) in $\mathbb{R}^N$ such that, for every $\varpi > 0$, there exists $R > 0$ such that $\int_{\mathscr{B}_R(x_n)} \rho_n \geq 1 - \varpi$ for every $n \in \mathbb{N}$.

Vanishing is excluded. Assume on the contrary if it is true, by means of [34], we obtain $\rho_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < \frac{2N-\mu}{N-2}$. This Implies that $v_n \to 0$ in $L^s(\mathbb{R}^N)$, which is an absurd.

Dichotomy is ruled out. In fact, we analyze two cases with respect to the sequence (x_n).

First case. Suppose that (x_n) is bounded. By (4.4), there exists $\delta > 0$ such that $\int_{\mathscr{B}_R(x_n)} |v_n|^s \geq \delta$ for sufficiently large n. From the fact that (x_n) is bounded, one sees that there exists $R_0 > R$ such that $\int_{\mathscr{B}_{R_0}(0)} |v|^s = \int_{\mathscr{B}_{R_0}(0)} |v_n|^s \geq \delta$, which yields $v \neq 0$. Otherwise, we define a cutoff $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$0 \leq \varphi(x) \leq 1,\ \varphi \equiv 1 \text{ in } \mathscr{B}_R(0) \text{ and } 0 \leq \varphi(x) \leq 1,\ \varphi \equiv 0 \text{ in } \mathbb{R}^N - \mathscr{B}_{2R}(0). \tag{4.5}$$

For $\varphi_R(x) = \varphi(x/R_0)$, it follows that $\Phi'_{\varepsilon_n}(v_n)(\varphi_R v_n) = o_n(1)$. Using (2.3), (1.4), and Fatou's Lemma and letting $n \to \infty$ and $R \to \infty$, we obtain $\Phi_0(v)v \leq 0$. Then, there exists $t_0 \in (0,1]$ such that $t_0 v \in \mathfrak{N}_0$. By (1.9), we conclude that

$$\begin{aligned}
\beta_0 &\leq \Phi_0(t_0v) - \frac{1}{2}\Phi_0'(t_0v)t_0v = \int \left(\frac{1}{|x|^\mu} * F(|tv|^2)\right)\left[\frac{1}{2}f(|tv|^2)|tv|^2 - \frac{1}{4}F(|tv|^2)\right] \\
&\leq \int \left(\frac{1}{|x|^\mu} * F(|v|^2)\right)\left[\frac{1}{2}f(|v|^2)|v|^2 - \frac{1}{4}F(v|^2)\right].
\end{aligned}$$

By Fatou's lemma,

$$\begin{aligned}
&\int \left(\frac{1}{|x|^\mu} * F(|v|^2)\right)\left[\frac{1}{2}f(|v|^2)|v|^2 - \frac{1}{4}F(|v|^2)\right] \\
\leq\ &\int \left(\frac{1}{|x|^\mu} * F(|v_n|^2)\right)\left[\frac{1}{2}f(|v_n|^2)|v_n|^2 - \frac{1}{4}F(|v_n|^2)\right] \\
=\ &\liminf_{n\to\infty}\left[\Phi_{\varepsilon_n}(v_n) - \frac{1}{2}\Phi'_{\varepsilon_n}(v_n)v_n\right] = \beta_0.
\end{aligned}$$

Hence, $t_0 = 1$, $\Phi_0(v) = \beta_0$, and

$$\begin{aligned}
&\lim_{n\to\infty}\int \left(\frac{1}{|x|^\mu} * F(|v_n|^2)\right)\left[\frac{1}{2}f(|v_n|^2)|v_n|^2 - \frac{1}{4}F(|v_n|^2)\right] \\
&\quad = \int \left(\frac{1}{|x|^\mu} * F(|v|^2)\right)\left[\frac{1}{2}f(|v|^2)|v|^2 - \frac{1}{4}F(|v|^2)\right],
\end{aligned}$$

which yields

$$\lim_{n\to\infty}\int \left(\frac{1}{|x|^\mu} * F(|v_n|^2)\right) f(|v_n|^2)|v_n|^2 = \int \left(\frac{1}{|x|^\mu} * F(|v|^2)\right) f(|v|^2)|v|^2$$

and

$$\lim_{n\to\infty}\int \left(\frac{1}{|x|^\mu} * F(|v_n|^2)\right) F(|v_n|^2) = \int \left(\frac{1}{|x|^\mu} * F(|v|^2)\right) F(|v|^2).$$

Therefore $\|v_n - v\|_0^2 = o_n(1)$. For a subsequence, $v_n \to v$ in $L^s(\mathbb{R}^N)$. The boundedness of (x_n) and (4.4) imply $\int_{\mathbb{R}^N} \rho_{1,n} \to 0$, and then $\rho = 0$, which is an absurd in view of (4.4).

Second case. Suppose that (x_n) is unbounded. Hence, for a subsequence, we obtain $|x_n| \to \infty$ as $n \to \infty$. Define

$$w_n(x) = |v_n|(x + x_n). \tag{4.6}$$

The boundedness of (v_n) in $H^1(\mathbb{R}^N,\mathbb{C})$, implies the boundedness of (w_n) in $H^1(\mathbb{R}^N,\mathbb{C})$. Thus, for a subsequence one has $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^N,\mathbb{C})$. If $w = 0$, we obtain

$$\rho + o_n(1) = \int_{\mathbb{R}^N} \rho_{1,n} = \int_{\mathbb{R}^N} \chi_{\mathscr{B}_{r_1}(x_n)} \frac{|v_n|(x)}{\|v_n\|_{L^s}^s} = \int_{\mathbb{R}^N} \chi_{\mathscr{B}_{r_1}(0)} \frac{w_n(x - x_n))}{\|w_n\|_{L^s}^s} = o_n(1),$$

leading to an absurd, then $w \neq 0$.

Next we use the fact that $w_n \rightharpoonup w \neq 0$ weakly in $H^1(\mathbb{R}^N, \mathbb{C})$. We discus two possibilities related to the sequence $(\varepsilon_n x_n)$.

First possibility. If $(\varepsilon_n x_n)$ is unbounded, then, for a subsequence, $|\varepsilon_n x_n| \to \infty$ as $n \to \infty$. From (1.5) and for $\xi > 0$, there exists $C > 0$ such that $W_\infty - \xi \leq W(x)$ for every $|x| \geq C$. Thus, by the diamagnetic inequality (2.3), for a subsequence of $w_n(x) = |v_n|(x + \varepsilon_n x_n)$ defined in (4.6), we deduce

$$
\begin{aligned}
\int |\nabla w_n|^2 + (W_\infty - \xi)\int |w_n|^2 &\leq \int |\nabla_\varepsilon v_n|^2 + (W_\infty - \xi)\int |v_n|^2 \\
&\leq \int |\nabla_\varepsilon v_n|^2 + \int W(x)|v_n|^2 \\
&= \int \left(\frac{1}{|x|^\mu} * F(|v_n|^2)\right) f(|v_n|^2)|v_n|^2 \\
&= \int \left(\frac{1}{|x|^\mu} * F(|w_n|^2)\right) f(|w_n|^2)|w_n|^2 dx.
\end{aligned}
$$

Since $\xi > 0$ is arbitrary and $\int |w_n|^2$ is bounded, we conclude that $\Phi'_\infty(w_n)w_n \leq 0$. Then, there exists $0 < t_n \leq 1$ such that $t_n w_n \in \mathfrak{N}_\infty$. By (1.9) and (2.2), we have

$$
\begin{aligned}
\beta_\infty &\leq \liminf_{n\to\infty} \left[\Phi_\infty(t_n w_n) - \frac{1}{2}\Phi'_\infty(t_n w_n)t_n w_n\right] \\
&= \liminf_{n\to\infty} \int \left(\frac{1}{|x|^\mu} * F(|t_n w_n|^2)\right)\left[\frac{1}{2}f(|t_n w_n|^2)|t_n w_n|^2 - \frac{1}{4}F(|t_n w_n|^2)\right] \\
&\leq \liminf_{n\to\infty} \int \left(\frac{1}{|x|^\mu} * F(|w_n|^2)\right)\left[\frac{1}{2}f(|w_n|^2)|w_n|^2 - \frac{1}{4}F(|w_n|^2)\right] \\
&= \liminf_{n\to\infty} \int \left(\frac{1}{|x|^\mu} * F(|v_n|^2)\right)\left[\frac{1}{2}f(|v_n|^2)|v_n|^2 - \frac{1}{4}F(|v_n|^2)\right] \\
&= \liminf_{n\to\infty} \left[\Phi_\varepsilon(v_n) - \frac{1}{2}\Phi'_\varepsilon(v_n)v_n\right] = \liminf_{n\to\infty} \Phi_\varepsilon(v_n) = c \leq \beta_0 + \sigma^*,
\end{aligned}
$$

which is an absurd, compare with (2.17).

Second possibility. Consider the case that $(\varepsilon_n x_n)$ is bounded. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be as in (4.5). Then, for $\varphi_R(x) = \varphi(x/R_0)$, it implies that $\Phi'_{\varepsilon_n}(w_n)(\varphi_R w_n) = o_n(1)$. Using (2.3) again, (1.4), letting $n \to \infty$, Fatou's Lemma and $R \to \infty$, we obtain $\Phi_0(v)v \leq 0$. Then, there exists $t_0 \in (0,1]$ such that $t_0 v \in \mathfrak{N}_0$. By (1.9), we obtain

$$
\begin{aligned}
\beta_0 &\leq \Phi_0(t_0 w) - \frac{1}{2}\Phi'_0(t_0 w)t_0 w \\
&= \int \left(\frac{1}{|x|^\mu} * F(|tw|^2)\right)\left[\frac{1}{2}f(|tw|^2)|tw|^2 - \frac{1}{4}F(|tw|^2)\right] \\
&\leq \int \left(\frac{1}{|x|^\mu} * F(|w|^2)\right)\left[\frac{1}{2}f(|w|^2)|v|^2 - \frac{1}{4}F(|w|^2)\right].
\end{aligned}
$$

By Fatou's lemma, one obtains

$$\begin{aligned}
&\int \left(\frac{1}{|x|^{\mu}} * F(|w|^2)\right)\left[\frac{1}{2} f(|w|^2)|w|^2 - \frac{1}{4}F(|w|^2)\right] \\
&\leq \int \left(\frac{1}{|x|^{\mu}} * F(|w_n|^2)\right)\left[\frac{1}{2} f(|w_n|^2)|w_n|^2 - \frac{1}{4}F(|w_n|^2)\right] \\
&= \liminf_{n\to\infty}\left[\Phi_{\varepsilon_n}(w_n) - \frac{1}{2}\Phi'_{\varepsilon_n}(w_n)w_n\right] = \beta_0.
\end{aligned}$$

Hence, $t_0 = 1$, $\Phi_0(v) = \beta_0$, and

$$\begin{aligned}
&\lim_{n\to\infty}\int \left(\frac{1}{|x|^{\mu}} * F(|w_n|^2)\right)\left[\frac{1}{2} f(|w_n|^2)|w_n|^2 - \frac{1}{4}F(|w_n|^2)\right] \\
&= \int \left(\frac{1}{|x|^{\mu}} * F(|w|^2)\right)\left[\frac{1}{2} f(|w|^2)|w|^2 - \frac{1}{4}F(|w|^2)\right],
\end{aligned}$$

which implies

$$\lim_{n\to\infty}\int \left(\frac{1}{|x|^{\mu}} * F(|w_n|^2)\right) f(|w_n|^2)|v_n|^2 = \int \left(\frac{1}{|x|^{\mu}} * F(|w|^2)\right) f(|w|^2)|w|^2 \tag{4.7}$$

and

$$\lim_{n\to\infty}\int \left(\frac{1}{|x|^{\mu}} * F(|w_n|^2)\right) F(|w_n|^2) = \int \left(\frac{1}{|x|^{\mu}} * F(|w|^2)\right) F(|w|^2). \tag{4.8}$$

Thus $\|w_n - w\|_0^2 = o_n(1)$. In particular, for a subsequence, $w_n \to w$ in $L^s(\mathbb{R}^N)$. Since (x_n) is bounded, we find by (4.4) that $\int_{\mathbb{R}^N} \rho_{1,n} \to 0$ with $\rho = 0$, and we arrive at a contradiction. In synthesis, since Vanishing and Dichotomy do not occur, we conclude that Compactness must happen. Therefore, by the same reasoning of steps (4.7) and (4.8), one has $\|v_n - v\|_0^2 = o_n(1)$. Furthermore, for a subsequence, one obtains $\varepsilon_n x_n \to m_j$ for some $j = 1,\ldots,l$ (recall (1.6)) and $v_n \to v$ in $L^s(\mathbb{R}^N)$. Hence, the Claim that we formulated before at (4.2) and (4.3) is true and the proof of the lemma is accomplished. □

5. Proof of Theorem 1.1

We show that we can prescribe the lowest number of solutions for equation (1.2). The solutions are not identical since they have different energy levels. And each solution is located around a global minima of the potential function W. Recall (2.6), (3.1), (3.2), and (4.1). We define the following elements

$$\mathscr{K}_\varepsilon^j = \{u \in \mathfrak{N}_\varepsilon : |\mathscr{T}_\varepsilon(u) - m_j| < \rho_0\},$$

$$\partial\mathscr{K}_\varepsilon^j = \{u \in \mathfrak{N}_\varepsilon : |\mathscr{T}_\varepsilon(u) - m_j| = \rho_0\},$$

$$\alpha_\varepsilon^j = \inf_{u\in\mathscr{K}_\varepsilon^j}\Phi_\varepsilon(u), \quad \text{and} \quad \widetilde{\alpha}_\varepsilon^j = \inf_{u\in\partial\mathscr{K}_\varepsilon^j}\Phi_\varepsilon(u).$$

Lemma 5.1. *Recall (2.13) and (2.14). There exist $\varepsilon^* > 0$ and $\delta > 0$ such that*

$$\alpha_\varepsilon^j < \beta_0 + \frac{\delta}{2} \ \textit{ and } \ \alpha_\varepsilon^j < \widetilde{\alpha}_\varepsilon^j \textit{ for every } \varepsilon \in (0,\varepsilon^*).$$

Proof. Let $U_0 \in H^1(\mathbb{R}^N)$ be the solution of (2.10) settled in (2.14). Then $\Phi_0(U_0) = \beta_0$ and $\Phi_0'(U_0)U_0 = 0$. Recall (1.6) where we assumed that $W(m_j) = W_0$ for $j = 1, \dots, l$ and define for each $j = 1, 2, \dots, l$ the localized function

$$U_n^j = U_0\left(x - \frac{m_j}{\varepsilon_n}\right) \exp\left(i\gamma_{m_j}\left(\frac{\varepsilon x - m_j}{\varepsilon}\right)\right).$$

Let $t_{\varepsilon_n} > 0$ such that $t_{\varepsilon_n} U_n^j \in \mathfrak{N}_{\varepsilon_n}$. Then, by a change of variable and arguing as in the proof Lemma 3.1, we conclude that for a subsequence $t_{\varepsilon_n} \to t_0$ and

$$\limsup_{n\to\infty} \Phi_{\varepsilon_n}(t_{\varepsilon_n} U_n^j) \le \Phi_0(t_0 U_0) \le \Phi_0(U_0) = \beta_0.$$

The sequence ε_n is arbitrary and by the fact that $U_n^j \in \mathscr{K}_\varepsilon^i$, we obtain $\alpha_\varepsilon^j < \beta_0 + \dfrac{\delta}{2}$. The second inequality follows from Lemma 4.2. □

We end up with a result on the existence of solutions.

Proof of Theorem 1.1. By an application of the Ekeland variational principle as in [9, Lemma 7], there exist a Palais-Smale sequence (u_n^j) in $\mathscr{K}_\varepsilon^i$ at level α_ε^j for the functional Φ_ε for each $1 \le j \le l$. Since $\alpha_\varepsilon^j < \beta_0 + \delta < \beta_0 + \sigma^*$, from Proposition 3.1, there exists $u_\varepsilon^j \in H^1(\mathbb{R}^N, \mathbb{C})$ such that, for a subsequence, $u_n^j \to u_\varepsilon^j$ in $H^1(\mathbb{R}^N, \mathbb{C})$. Then,

$$u_\varepsilon^j \in \mathscr{K}_\varepsilon^j, \ \ \Phi_\varepsilon(u_\varepsilon^j) = \alpha_\varepsilon^j \ \text{ and } \ \Phi_\varepsilon'(u_\varepsilon^j)u_\varepsilon^j = 0.$$

Hence u_ε^j is a nontrivial solution of (1.2). Since

$$\mathscr{T}_\varepsilon(u_\varepsilon^j) \in \overline{\mathscr{B}_{\rho_0}(m_j)}, \ \mathscr{T}_\varepsilon(u_\varepsilon^k) \in \overline{\mathscr{B}_{\rho_0}(m_j)} \text{ and } \overline{\mathscr{B}_{\rho_0}(m_j)} \cap \overline{\mathscr{B}_{\rho_0}(m_k)} = \emptyset \text{ for } j \neq k,$$

we conclude that $u_\varepsilon^j \neq u_\varepsilon^k$. Hence, for $\varepsilon \in (0, \varepsilon^*)$,the functional Φ_ε has at least l nontrivial critical points, giving rise to l solutions. The solutions are not identical, since they have different energy levels. □

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