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PROFIT MAXIMIZATION AND PATIENT DISTRIBUTION IN STRATEGIC PHARMACY NETWORK MODELS

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Abstract. This paper addresses the problem of optimizing the spatial distribution of pharmacies within a regional healthcare system, with the aim of improving accessibility and equity in pharmaceutical services. We propose a nonlinear programming model based on the theory of variational inequalities, which captures the equilibrium behavior of users in choosing pharmacies, considering travel costs, service attractiveness, and territorial constraints. Unlike classical discrete location models, our approach allows for a realistic representation of users' decentralized decisions. A numerical example is presented to illustrate the applicability of the proposed framework and its potential to guide strategic decisions in pharmaceutical supply chain management. It is an illustrative setting designed to validate the internal consistency of the model. Its goal is to highlight how the theoretical formulation can be effectively implemented and to show the model's ability to capture the complex interactions among stakeholders in the pharmaceutical distribution network.

Keywords. Equilibrium problems; Nonlinear programming; Pharmaceutical supply chain; Variational inequality.

1. INTRODUCTION

The spatial organization of pharmaceutical services plays a central role in ensuring equitable access to healthcare, particularly in regions where demographic decline, rural dispersion, or economic hardship limit the effectiveness of centralized healthcare models. Pharmacies are not only providers of medicines but also operate as territorial healthcare facilities, particularly in Italy, where recent legislation (e.g., D.lgs. n. 153/2009 and successive decrees) has institutionalized their role as decentralized access points for primary health services, such as diagnostic testing, vaccination, and telemedicine consultations. In this evolving context, traditional administrative criteria for locating pharmacies, often based on rigid population thresholds and municipal boundaries, have shown limitations. These rules, although essential for regulatory compliance, may not reflect actual population needs, travel behaviors, or territorial inequalities. As a result, there is a growing interest in adopting quantitative, optimization-based approaches to support data-driven decisions in the planning of pharmacy networks. The organization of pharmaceutical distribution systems has been widely studied in the literature, with particular attention given to service accessibility, demand heterogeneity, and cost minimization. Early contributions have addressed healthcare

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facility location problems and emergency service planning, laying the foundation for current quantitative methods (see [1] and [2]). This paper introduces a novel mathematical model for the reorganization of pharmacy networks under decentralized decision-making and demand-side constraints. It proposes a nonlinear programming framework, based on the theory of variational inequalities (VIs), to model the equilibrium behavior of users in selecting pharmacies, taking into account travel costs, service attractiveness, and territorial constraints. Variational inequalities are a powerful mathematical tool to model equilibrium situations where decision-makers interact in a shared environment with possibly conflicting objectives. In this setting, users (patients) seek to minimize their individual disutility, while the planner seeks to configure the network in such a way that coverage and accessibility are guaranteed. Unlike classical location models focused on territorial service distribution, the present approach addresses a strategic supply-demand equilibrium problem where the decisions of each stakeholder are interdependent. Another key innovation of this model is that it includes the average delivery time delays for online drug purchases, where demand price functions are modeled as decreasing functions of these delay times, reflecting real-world market behavior (see [3]). Recent literature has emphasized the importance of equilibrium models in managing sustainable and decentralized supply chains, even in complex and fragmented contexts such as the pharmaceutical distribution sector (see [4, 5, 6, 7]). Our methodology is based on the theory of variational inequalities (VIs), a powerful framework for modeling competitive interactions among agents in network systems (see [8] and [9]). In terms of solution methodologies, accelerated first-order methods for variational inequality problems have gained significant attention in optimization and machine learning applications. A prominent approach in this domain is Nesterov's accelerated method (see [10]), which has demonstrated convergence and effectiveness for convex optimization and monotone inclusion problems. Within this research area, dual extrapolation techniques that generalize the extragradient principle to broader classes of problems have been developed (see [11]), and enhanced convergence analysis for strongly monotone cases has been obtained, establishing improved convergence rates (see [12]). The adoption of VIs allows for the representation of user equilibrium, congestion phenomena, and decentralized decision-making processes (see [13] and [14]). Variational inequality theory provides a robust framework to capture the interdependencies under behavioral and market constraints (see [15] and [16]). In particular, the structure of the decentralized interactions between pharmacies and distributors and the inclusion of capacity constraints and environmental considerations are inspired by recent developments in healthcare supply chain modeling (see [17]).

Traditional approaches to spatial accessibility in healthcare systems often rely on geometric or gravity-based metrics. However, these models can fail to capture equity concerns, especially in rural or underserved areas. Recent studies have proposed refined methods to address these gaps (see [18] and [19]), but the integration of demand-side responsiveness remains limited. The present model incorporates key market features such as pricing policies, purchasing preferences, and legal limitations on product availability to overcome such limitations. The purpose of the model is to identify optimality or equilibrium conditions at each decision-making level of the healthcare distribution network. The model aims to maximize profits for pharmaceutical companies, wholesalers, and retailers (including traditional pharmacies, mixed pharmacies, and parapharmacies); on the contrary, for patients, the focus is on identifying equilibrium conditions that determine the optimal quantity of prescription and non-prescription drugs to purchase. Patients' choices are modeled in terms of equilibrium behavior, taking into account available purchasing channels, whether through traditional physical access or online platforms, and through different types of retailers (traditional pharmacies, mixed pharmacies, or parapharmacies).

The remainder of this paper is organized as follows: Section 2 introduces the mathematical model and the structure of the network, which includes pharmaceutical companies, wholesalers, traditional and mixed pharmacies, para-pharmacies, and patients demanding both prescription and non-prescription drugs. The entire system is represented as a multi-tiered pharmaceutical supply chain, and all decision variables, functions, and parameters used in the model are formally defined. Section 3 analyzes the behavior of each decision-making level. Optimality conditions are derived for pharmaceutical companies, wholesalers, pharmacies, and para-pharmacies, while equilibrium conditions are established for patients. For each stakeholder group, a corresponding variational inequality is formulated to characterize either optimality or equilibrium. These are then integrated into a single unified variational inequality that describes the overall equilibrium of the pharmaceutical supply chain. Section 4 presents a numerical illustrative example to validate the proposed model. The problem is solved using the Korpelevich method, demonstrating the applicability and coherence of the theoretical framework. Section 5, the last section, concludes the paper with final remarks and discusses directions for future research.

2. The Mathematical Model

The healthcare pharmaceutical network can be described as a hierarchical supply chain that begins with the pharmaceutical companies and ends with the patients, involving multiple layers of distribution and retail.

In Figure 1, we represent a simplified model of the pharmaceutical distribution network within a healthcare system. The network distinguishes between traditional and electronic purchase channels and includes different types of retail outlets.

At the top of the network are the pharmaceutical manufacturers. These companies produce both prescription and non-prescription drugs and act as the primary source of supply. They distribute their products to:

- distributors, namely wholesalers, who act as intermediaries and manage large-scale logistics and redistribution.
- Traditional pharmacies, for direct retail to patients.
- Mixed pharmacies, which combine physical presence with online services.
- Parapharmacies, which sell mainly non-prescription drugs and health products.

Wholesalers act as intermediaries between producers and retailers. They handle the logistics and redistribution of drugs to various sales points, helping to optimize the flow and availability of products across the network.

As already mentioned, the retail level includes three types of distribution points that interact directly with patients:

- Traditional pharmacies: they have a physical location where patients can go in person to purchase both prescription and non-prescription drugs. They operate entirely through in-person service.
- Mixed pharmacies: they combine traditional physical service with online ordering capabilities. Patients can either visit the pharmacy or place their orders via the Internet, typically with home delivery or in-store pickup options.
- Parapharmacies: they are retail outlets that specialize in the sale of non-prescription drugs, also known as over-the-counter (OTC) medications, as well as medical devices, dietary supplements, and wellness products. These establishments operate through both physical and, in some cases,

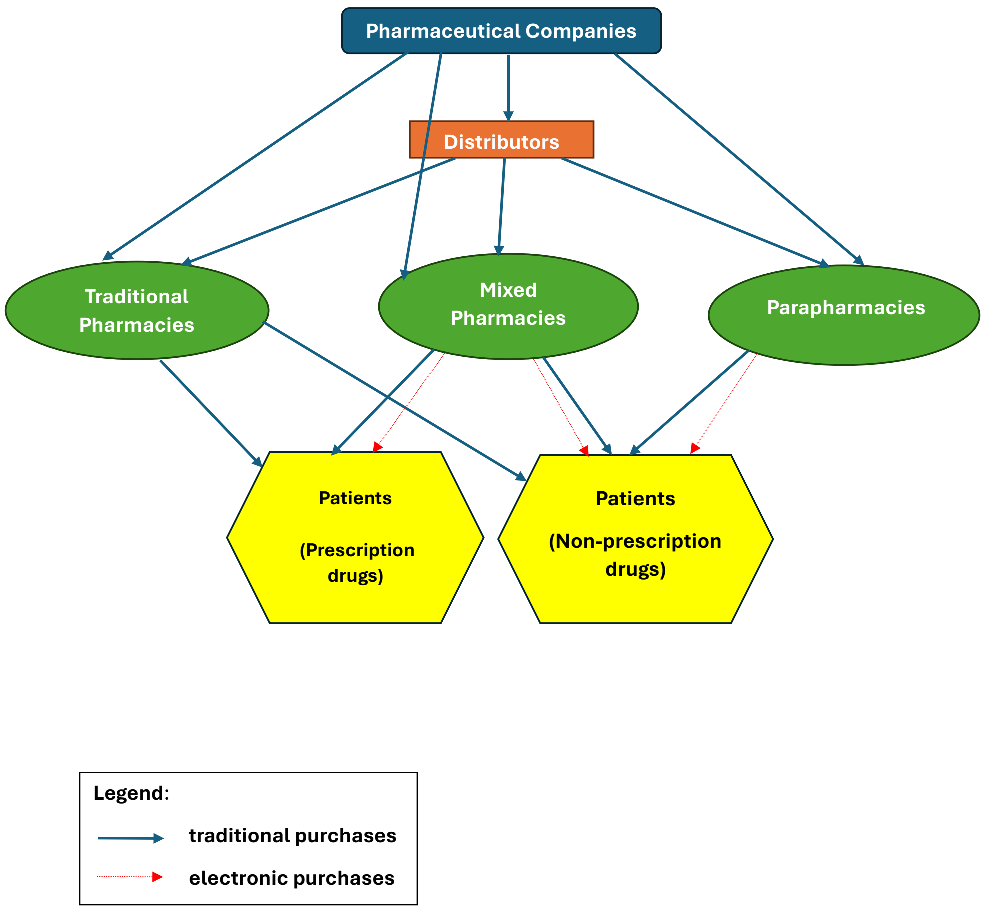

FIGURE 1. Pharmaceutical supply network

digital channels. Unlike certified pharmacies, parapharmacies are not authorized to dispense prescription drugs.

At the bottom of the network are the patients, who acquire medications through:

- Traditional channels, by visiting physical pharmacies or parapharmacies;
- Electronic channels, by placing orders online (primarily for OTC drugs and permitted categories).

This network reflects the diversification of pharmaceutical retail in the digital age and maintains regulatory distinctions between medication types and authorized sales channels. In Figure 1, solid lines indicate traditional purchases, while dashed lines indicate electronic purchases.

Now, assume we have

- I pharmaceutical companies, where the typical one is denoted by i;
- J distributors, where the typical one is denoted by j;

- K traditional pharmacies, where the typical one is denoted by k;
- H mixed pharmacies, where the typical one is denoted by h;
- M parapharmacies, where the typical one is denoted by m;
- N patients, where the typical one is denoted by n.

Each agent in the network performs decisions over production, pricing, procurement, or consumption, depending on their role. Specifically:

- Pharmaceutical company i decides on the optimal quantities of drugs to produce and the prices at which to sell them to distributors and retailers, aiming to maximize their profit under production and capacity constraints.
- Distributor j chooses the optimal quantities to purchase from companies and the optimal quantities to resell to pharmacies and parapharmacies, also seeking to maximize profit by managing procurement costs and sale revenues.
- Traditional pharmacy k, mixed pharmacy h, and parapharmacy m make decisions on purchasing from distributors or directly from producers, and set retail prices, aiming to maximize their respective profits from sales to final patients.
- Patient n selects optimal quantities of prescription and non-prescription drugs to buy from the available retail channels, namely traditional, mixed, or parapharmacies, either physically or online, by minimizing personal expenditure and satisfying their medical needs, under a set of equilibrium conditions. For modeling purposes, we further divide this set into two groups: N_1 representing patients who purchase prescription drugs (indexed as n_1) and N_2 representing those who purchase non-prescription drugs (indexed as n_2). This distinction allows us to accurately model purchasing behavior and constraints based on the type of medication involved.

We will use an additional index p to denote the type of drug, specifically:

$$p = \begin{cases} 1 & \text{refers to prescription drugs;} \\ 2 & \text{refers to non-prescription drugs.} \end{cases}$$

Finally, we use a superscript l to denote the mode of transportation, i.e., whether the purchase takes place physically or online. In particular:

$$l = \begin{cases} 1 & \text{denotes physical (in-person) purchases;} \\ 2 & \text{denotes online purchases.} \end{cases}$$

This notation allows us to distinguish both the nature of the drug and the channel through which it is acquired, which is essential to accurately represent the structure and constraints of the pharmaceutical distribution network.

So, a schematic network of our model is depicted in Figure 2, where the green and purple arrows indicate that the corresponding connections can be made both through traditional commerce and online ordering. These dual pathways reflect the flexibility of mixed and digital retail channels within the pharmaceutical distribution network.

The notation employed in our model is provided in Tables 1 and 2.

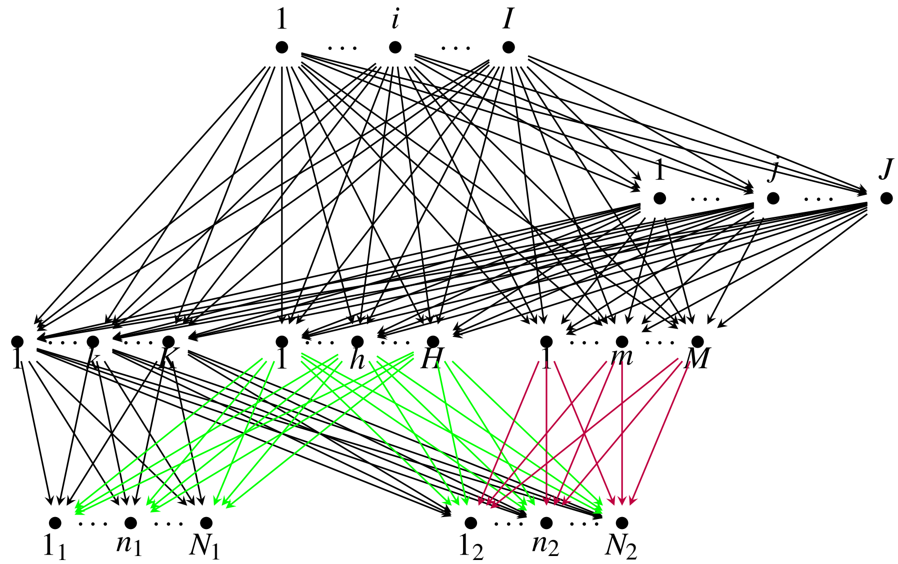

FIGURE 2. Schematic Representation of the Supply Chain

TABLE 1. Variables

Symbol	Definition
M_i	Amount of product produced by i
$Q^1 = (q^1_{ijp})_{ijp}$	Total amount of product sold by the companies to distributors
$Q^2 = (q^1_{ikp})_{ikp}$	Total amount of product sold by the companies to traditional pharmacies
$Q^3 = (q^1_{ihp})_{ihp}$	Total amount of product sold by the companies to mixed pharmacies
$Q^4 = (q^1_{im2})_{im}$	Total amount of product sold by the companies to parapharmacies
$Q^5 = (q^1_{jkp})_{jkp}$	Total amount of product sold by distributors to traditional pharmacies
$Q^6 = (q^1_{jhp})_{jhp}$	Total amount of product sold by distributors to mixed pharmacies
$Q^7 = (q^1_{jm2})_{jm}$	Total amount of product sold by distributors to parapharmacies
$Q^8 = (q^1_{kn_1 1})_{kn_1}$	Total amount of product with prescription sold by pharmacies to patients
$Q^9 = (q^1_{kn_2 2})_{kn_2}$	Total amount of product without prescription sold by pharmacies to patients
$Q^{10} = (q^l_{hn_1 1})_{hn_1 l}$	Total amount of product with prescription sold by mixed pharmacies to patients via both modes

Table 1 – continued from previous

Symbol	Definition
$Q^{11} = (q^l_{hn_2 2})_{hn_2 l}$	Total amount of product without prescription sold by mixed pharmacies to patients via both modes
$Q^{12} = (q^l_{mn_2 2})_{mn_2 l}$	Total amount of product without prescription sold by parapharmacies to patients via both modes

TABLE 2. Prices and costs

Symbol	Definition
ρ_{ijp}	Unit price charged by i to j for products of type p
ρ_{ikp}	Unit price charged by i to k for products of type p
ρ_{ihp}	Unit price charged by i to h for products of type p
ρ_{im2}	Unit price charged by i to m for products of type 2
ρ_{jkp}	Unit price charged by j to k for products of type p
ρ_{jhp}	Unit price charged by j to h for products of type p
ρ_{jm2}	Unit price charged by j to m for products of type 2
$\bar{\rho}^1_{kp}$	Unit price charged by k for products of type p
$\bar{\rho}^l_{hp}$	Unit price charged by h for products of type p
$\bar{\rho}_{m2}$	Unit price charged by m for products of type 2
$p_i(Q^1, Q^2, Q^3, Q^4)$	Production cost of i
$c^1_{ijp}(q^1_{ijp})$	Transportation cost related to the sold products of type p from i to j
$c^1_{ikp}(q^1_{ikp})$	Transportation cost related to the sold products of type p from i to k
$c^1_{ihp}(q^1_{ihp})$	Transportation cost related to the sold products of type p from i to h
$c^1_{im2}(q^1_{im2})$	Transportation cost related to the sold products of type p from i to m
$c^1_{jkp}(q^1_{jkp})$	Transportation cost related to the sold products of type p from j to k
$\hat{c}^1_{ijp}(q^1_{ijp})$	Transaction cost related to the purchased products of type p from i to j charged to the distributor j
$c^1_{jhp}(q^1_{jhp})$	Transportation cost related to the sold products of type p from j to h
$c^1_{jm2}(q^1_{jm2})$	Transportation cost related to the sold products of type p from j to m
$c_j(Q^1)$	Operating cost of j
$\bar{\rho}^{SSN}_{k1}$	Unit reimbursement to k for sold products of type p = 1
$\bar{\rho}^{SSN}_{h1}$	Unit reimbursement to h for sold products of type p = 1

$\bar{c}_k(Q^2, Q^5)$	Handling cost of k
$\bar{c}_h(Q^3, Q^6)$	Handling cost of h
$\bar{c}_m(Q^4, Q^7)$	Handling cost of m
$tr_{ikp}(q^1_{ikp})$	Transaction cost associated with pharmacy k transacting with company i
$tr_{ihp}(q^1_{ihp})$	Transaction cost associated with pharmacy h transacting with company i
$tr_{jkp}(q^1_{jkp})$	Transaction cost associated with pharmacy k transacting with distributor j
$tr_{jhp}(q^1_{jhp})$	Transaction cost associated with pharmacy h transacting with distributor j
$tr_{im2}(q^1_{im2})$	Transaction cost associated with parapharmacy m transacting with company i for products of type $p = 2$
$tr_{jm2}(q^1_{jm2})$	Transaction cost associated with parapharmacy m transacting with distributor j for products of type $p = 2$
$sub_h(Q^{10}_2, Q^{11}_2)$	Subscription cost of h associated with using the platform that manages the online sales
$sub_m(Q^{12}_2)$	Subscription cost of m associated with using the platform that manages the online sales
$\rho_{n_1}(d^1, T^1_{ave})$	Demand price function for patient n_1
$\rho_{n_2}(d^2, T^2_{ave})$	Demand price function for patient n_2
$\bar{\bar{c}}^1_{kn_1 1}(Q^8)$	Transaction cost associated with obtaining the products of type $p = 1$ for patient n_1 from pharmacy k with mode $l = 1$
$\bar{\bar{c}}^l_{hn_1 1}(Q^{10})$	Transaction cost associated with obtaining the products of type $p = 1$ for patient n_1 from pharmacy h with mode l
$\bar{\bar{c}}^1_{kn_2 2}(Q^9)$	Transaction cost associated with obtaining the products of type $p = 2$ for patient n_2 from pharmacy k with mode $l = 1$
$\bar{\bar{c}}^l_{hn_2 2}(Q^{11})$	Transaction cost associated with obtaining the products of type $p = 2$ for patient n_2 from pharmacy h with mode l
$\bar{\bar{c}}^l_{mn_2 2}(Q^{12})$	Transaction cost associated with obtaining the products of type $p = 2$ for patient n_2 from parapharmacy m with mode l

Our approach is conceptually similar to that in [20], where nonlinear constraints are incorporated into equilibrium problems, and it follows the methodological path used in pandemic-related distribution network models (see [9] and [17]).

In the next session, we describe the behavior of the pharmaceutical companies, the distributor, the pharmacies, and the patients. We then state the equilibrium conditions of the pharmaceutical supply chain network and provide the variational inequality formulation.

3. Optimality and Equilibrium Conditions

3.1. **Behavior of the Pharmaceutical Companies.** Each pharmaceutical company aims to determine the most profitable production and distribution strategy by optimizing its product portfolio and sales allocation to distributors. Let ρ_{ijp} represent the price that the pharmaceutical company i charges to distributor j, ρ_{ikp} the price that the pharmaceutical company i charges to pharmacy k, ρ_{ihp} the price that the pharmaceutical company i charges to mixed pharmacy h, and ρ_{im2} the price that the pharmaceutical company i charges to parapharmacy m. Let c^1_{ijp} be the transportation cost from the pharmaceutical company i to distributor j, c^1_{ikp} the transportation cost from the pharmaceutical company i to the traditional pharmacy k, c^1_{ihp} the transportation cost from the pharmaceutical company i to mixed pharmacy h, and ρ_{im2} the transportation cost from the pharmaceutical company i to parapharmacy m.

Finally, let $p_i = p_i(Q^1, Q^2, Q^3, Q^4)$ be the production cost of the pharmaceutical company i. Thus the criterion of profit maximization for the company i can be expressed as an optimization problem as follows

$$\max \Big(\sum_{j=1}^{J} \sum_{p=1}^{2} \rho_{ijp} q^1_{ijp} + \sum_{k=1}^{K} \sum_{p=1}^{2} \rho_{ikp} q^1_{ikp} + \sum_{h=1}^{H} \sum_{p=1}^{2} \rho_{ihp} q^1_{ihp} + \sum_{m=1}^{M} \rho_{im2} q^1_{im2} - p_i(Q^1, Q^2, Q^3, Q^4)$$

$$- \sum_{j=1}^{J} \sum_{p=1}^{2} c^1_{ijp}(q^1_{ijp}) - \sum_{k=1}^{K} \sum_{p=1}^{2} c^1_{ikp}(q^1_{ikp}) - \sum_{h=1}^{H} \sum_{p=1}^{2} c^1_{ihp}(q^1_{ikp}) - \sum_{m=1}^{M} c^1_{im2}(q^1_{im2}) \Big) \tag{3.1}$$

$$\sum_{j=1}^{J} \sum_{p=1}^{2} q^1_{ijp} + \sum_{k=1}^{K} \sum_{p=1}^{2} q^1_{ikp} + \sum_{h=1}^{H} \sum_{p=1}^{2} q^1_{ihp} + \sum_{m=1}^{M} q^1_{im2} \le M_i, \tag{3.2}$$

$$q^1_{ijp} \ge 0, \quad \forall j, \forall p, \quad q^1_{ikp} \ge 0, \forall k \,\forall p, \quad q^1_{ihp} \ge 0, \forall h, \forall p, \quad q^1_{im2} \ge 0, \forall m. \tag{3.3}$$

The objective function is given by the difference between revenues and costs. Specifically, the revenues are derived by selling the product to all distributors, pharmacies, and parapharmacies, whereas the costs include production and transportation costs. Constraint (3.2) ensures that the product flow from pharmaceutical company i to all distributors and retailers does not exceed the total amount of production. Lastly, constraints (3.3) guarantee the nonnegativity of the flow variables.

Under the assumptions on the convexity and regularity of the cost functions, we can state the following theorem.

Theorem 3.1. *A vector $(Q^{1*}, Q^{2*}, Q^{3*}, Q^{4*}, \lambda^{1*}) \in \mathbb{R}_+^{PI(J+K+H)+IM+I}$ is an optimal solution to problem (3.1)-(3.3) if and only if it is a solution to the Variational Inequality: Find $(Q^{1*}, Q^{2*}, Q^{3*}, Q^{4*}, \lambda^{1*}) \in$*

$\mathbb{R}_+^{PI(J+H+K)+IM+I}$ *such that, for all* $(Q^1,Q^2,Q^3,Q^4,\lambda^1) \in \mathbb{R}_+^{PI(J+H+K)+IM+I}$,

$$\begin{aligned}
&\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{p=1}^{2}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ijp}}+\frac{\partial c^1_{ijp}(q^{1*}_{ijp})}{\partial q^1_{ijp}}-\rho_{ijp}+\lambda_i^{1*}\right)(q^1_{ijp}-q^{1*}_{ijp})\\
&+\sum_{i=1}^{I}\sum_{k=1}^{K}\sum_{p=1}^{2}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ikp}}+\frac{\partial c^1_{ikp}(q^{1*}_{ikp})}{\partial q^1_{ikp}}-\rho_{ikp}+\lambda_i^{1*}\right)\left(q^1_{ikp}-q^{1*}_{ikp}\right)\\
&+\sum_{i=1}^{I}\sum_{h=1}^{H}\sum_{p=1}^{2}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ihp}}+\frac{\partial c^1_{ihp}(q^{1*}_{ihp})}{\partial q^1_{ihp}}-\rho_{ihp}+\lambda_i^{1*}\right)\left(q^1_{ihp}-q^{1*}_{ihp}\right)\\
&+\sum_{i=1}^{I}\sum_{m=1}^{2}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{im2}}+\frac{\partial c^1_{im2}(q^{1*}_{im2})}{\partial q^1_{im2}}-\rho_{im2}+\lambda_i^{1*}\right)(q^1_{im2}-q^{1*}_{im2}) \qquad (3.4)\\
&-\sum_{i=1}^{I}\left(\sum_{j=1}^{J}\sum_{p=1}^{2}q^{1*}_{ijp}+\sum_{k=1}^{K}\sum_{p=1}^{2}q^{1*}_{ikp}+\sum_{h=1}^{H}\sum_{p=1}^{2}q^{1*}_{ihp}+\sum_{m=1}^{M}q^{1*}_{im2}-M_i\right)(\lambda_i^1-\lambda_i^{1*})\geq 0,
\end{aligned}$$

where $\lambda^1=\left(\lambda_i^1\right)_i \in \mathbb{R}^I_+$ *is the vector of the Lagrange multipliers associated with constraints* (3.2).

3.2. **Behavior of the Distributors.** The distributors act as intermediaries, conducting business transactions with pharmaceutical companies on one side and with pharmacies and parapharmacies on the other side. Let ρ_{jkp} represent the price that the distributor j charges to traditional pharmacy k, ρ_{jhp} the price that the distributor j charges to mixed pharmacy h, and ρ_{jm2} the price that the distributor j charges to parapharmacy m. Let c^1_{jkp} be the transportation cost from distributor j to traditional pharmacy k, c^1_{jhp} the transportation cost from the distributor j to mixed pharmacy h, and c^1_{jm2} the transportation cost from distributor j to parapharmacy m. Let $\hat{c}^1_{ijp}$ be the transaction cost from distributor j to pharmaceutical company i. Finally, let $c_j=c_j(Q^1)$ be the operating cost of distributor j in imperfect competition. However, if the operating cost functions c_j depend solely on the product handled by the distributor j rather than also on the products handled by other distributors, then the dependence of these functions on Q^1 can be simplified accordingly.

Each distributor is assumed to pursue the maximization of the profit. Therefore, the profit-maximization problem for distributor j can be formulated as:

$$\begin{aligned}
&\max\Big(\sum_{k=1}^{K}\sum_{p=1}^{2}\rho_{jkp}q^1_{jkp}+\sum_{h=1}^{H}\sum_{p=1}^{2}\rho_{jhp}q^1_{jhp}+\sum_{m=1}^{M}\rho_{jm2}q^1_{jm2}-c_j(Q^1)-\sum_{i=1}^{I}\sum_{p=1}^{2}\rho_{ijp}q^1_{ijp}\\
&-\sum_{i=1}^{I}\sum_{p=1}^{2}\hat{c}^1_{ijp}(q^1_{ijp})-\sum_{k=1}^{K}\sum_{p=1}^{2}c^1_{jkp}(q^1_{jkp})-\sum_{h=1}^{H}\sum_{p=1}^{2}c^1_{jhp}(q^1_{ihp})-\sum_{m=1}^{M}c^1_{jm2}(q^1_{jm2})\Big) \qquad (3.5)
\end{aligned}$$

$$\sum_{k=1}^{K}q^1_{jk1}+\sum_{h=1}^{H}q^1_{jh1}\leq\sum_{i=1}^{I}q^1_{ij1}, \qquad (3.6)$$

$$\sum_{k=1}^{K}q^1_{jk2}+\sum_{h=1}^{H}q^1_{jh2}+\sum_{m=1}^{M}q^1_{jm2}\leq\sum_{i=1}^{I}q^1_{ij2}, \qquad (3.7)$$

$$q^1_{ijp}\geq 0,\quad \forall i,\,\forall p,\quad q^1_{jkp}\geq 0,\,\forall k,\,\forall p,\quad q^1_{jhp}\geq 0,\,\forall h,\,\forall p,\quad q^1_{jm2}\geq 0,\,\forall m. \qquad (3.8)$$

The objective function represents the difference between total revenues and total costs. More specifically, revenues are generated from product sales to all pharmacies and parapharmacies, while costs include purchasing, operating, and transportation expenses. Constraints (3.6)-(3.7) ensure that the total product flow from distributor j to all pharmacies and parapharmacies does not exceed the total amount of the products purchased from the pharmaceutical companies. Finally, constraints (3.8) enforce the non-negativity requirement for all flow variables.

We suppose that the transportation costs are all continuously differentiable and convex, and that the distributors compete in a noncooperative way. Then, we can state the following theorem.

Theorem 3.2. *A vector $(Q^{1*}, Q^{5*}, Q^{6*}, Q^{7*}, \lambda^{2*}, \mu^{2*}) \in \mathbb{R}_+^{IJP+JKP+JHP+JM+2J}$ is an optimal solution for the maximization problem (3.5)-(3.8) if and only if it is a solution to the Variational Inequality: Find $(Q^{1*}, Q^{5*}, Q^{6*}, Q^{7*}, \lambda^{2*}, \mu^{2*}) \in \mathbb{R}_+^{IJP+JKP+JHP+JM+2J}$ such that:*

$$
\begin{aligned}
&\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{\partial \hat{c}_{ij1}(q^{1*}_{ij1})}{\partial q^1_{ij1}}+\frac{\partial c_j(Q^{1*})}{\partial q^1_{ij1}}+\rho_{ij1}-\lambda^{2*}_j\right)\left(q^1_{ij1}-q^{1*}_{ij1}\right)\\
&+\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{\partial \hat{c}_{ij2}(q^{1*}_{ij2})}{\partial q^1_{ij2}}+\frac{\partial c_j(Q^{1*})}{\partial q^1_{ij2}}+\rho_{ij2}-\mu^{2*}_j\right)\left(q^1_{ij2}-q^{1*}_{ij2}\right)\\
&+\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{\partial c^1_{jk1}(q^{1*}_{jk1})}{\partial q^1_{jk1}}-\rho_{jk1}+\lambda^{2*}_j\right)\left(q^1_{jk1}-q^{1*}_{jk1}\right)\\
&+\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{\partial c^1_{jk2}(q^{1*}_{jk2})}{\partial q^1_{jk2}}-\rho_{jk2}+\mu^{2*}_j\right)\left(q^1_{jk2}-q^{1*}_{jk2}\right)\\
&+\sum_{j=1}^{J}\sum_{h=1}^{H}\left(\frac{\partial c^1_{jh1}(q^{1*}_{jh1})}{\partial q^1_{jh1}}-\rho_{jh1}+\lambda^{2*}_j\right)\left(q^1_{jh1}-q^{1*}_{jh1}\right)\\
&+\sum_{j=1}^{J}\sum_{h=1}^{H}\left(\frac{\partial c^1_{jh2}(q^{1*}_{jh2})}{\partial q^1_{jh2}}-\rho_{jh2}+\mu^{2*}_j\right)\left(q^1_{jh2}-q^{1*}_{jh2}\right)\\
&+\sum_{j=1}^{J}\sum_{m=1}^{M}\left(\frac{\partial c^1_{jm2}(q^{1*}_{jm2})}{\partial q^1_{jm2}}-\rho_{jm2}+\mu^{2*}_j\right)\left(q^1_{jm2}-q^{1*}_{jm2}\right)
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{J}\sum_{m=1}^{M}\left(\frac{\partial c^1_{jm2}(q^{1*}_{jm2})}{\partial q^1_{jm2}}-\rho_{jm2}+\mu^{2*}_j\right)\left(q^1_{jm2}-q^{1*}_{jm2}\right)\\
&-\sum_{j=1}^{J}\left(\sum_{k=1}^{K}q^{1*}_{jk1}+\sum_{h=1}^{H}q^{1*}_{jh1}-\sum_{i=1}^{I}q^{1*}_{ij1}\right)\left(\lambda^2_j-\lambda^{2*}_j\right)\\
&-\sum_{j=1}^{J}\left(\sum_{k=1}^{K}q^{1*}_{jk2}+\sum_{h=1}^{H}q^{1*}_{jh2}+\sum_{m=1}^{M}q^{1*}_{jm2}-\sum_{i=1}^{I}q^{1*}_{ij2}\right)\left(\mu^2_j-\mu^{2*}_j\right)\geq 0,\\
&\qquad\forall (Q^1, Q^5, Q^6, Q^7, \lambda^2, \mu^2) \in \mathbb{R}_+^{IJP+JKP+JHP+JM+2J},
\end{aligned}
$$

where $\lambda^2 = (\lambda^2_j)_j \in \mathbb{R}^J_+$ and $\mu^2 = (\mu^2_j)_j \in \mathbb{R}^J_+$ are the vectors of the Lagrange multipliers associated with constraints (3.6) and (3.7), respectively.

3.3. **Behavior of the Pharmacies and Parapharmacies.** Pharmacies and parapharmacies engage in transactions with pharmaceutical companies, distributors, and patients. They acquire products from pharmaceutical companies and distributors and sell them to patients. In this way, a pharmacy or a parapharmacy serves as a link between pharmaceutical companies and patients, as well as between distributors and patients.

3.3.1. *Traditional Pharmacies.* A traditional pharmacy k faces a handling cost, which may include, for example, the display and storage cost associated with the product. We denote this cost by $\bar{c}_k$ and, in the simplest case, $\bar{c}_k = \bar{c}_k(Q^2, Q^5)$, namely, it is a function of the amount of product the pharmacy has obtained from the various producers and distributors. Pharmacies, in turn, also incur transaction costs when dealing with pharmaceutical companies and distributors. We denote the transaction cost associated with pharmacy k transacting with company i by tr_{ikp} and assume that this function depends on the shipment of the company's product q^1_{ikp}. We denote the transaction cost associated with pharmacy k transacting with distributor j by tr_{jkp} and assume that this function depends on the shipment of the distributor's product q^1_{jkp}.

Let $q_{kn_1 1}$ denote the amount of the product with prescription ($p = 1$) purchased by patient n_1 from pharmacy k and $q_{kn_2 2}$ the amount of the product without prescription ($p = 2$) purchased by patient n_2 from pharmacy k. We group these consumption quantities into the column vector Q^8 and Q^9, respectively. The pharmacy k associates a price with the product p at their retail outlet, which is denoted by $\bar{\rho}^1_{kp}$. In the Italian healthcare system, pharmacies receive reimbursement from the National Health Service (SSN) for the dispensing of prescription medications to patients. Thus, we denote by $\bar{\rho}^{SSN}_{k1}$ the unit reimbursement for sold products of type $p = 1$.

Assuming, that the pharmacies are also profit maximizers, the optimization problem of a pharmacy k is given by:

$$\max\Big(\sum_{n_1=1}^{N_1} (\bar{\rho}^1_{k1} + \bar{\rho}^{SSN}_{k1}) q^1_{kn_1 1} + \sum_{n_2=1}^{N_2} \bar{\rho}^1_{k2} q^1_{kn_2 2} - \bar{c}_k(Q^2, Q^5) - \sum_{i=1}^{I}\sum_{p=1}^{2} tr_{ikp}(q^1_{ikp})$$

$$- \sum_{j=1}^{J}\sum_{p=1}^{2} tr_{jkp}(q^1_{jkp}) - \sum_{i=1}^{I}\sum_{p=1}^{2} \rho_{ikp} q^1_{ikp} - \sum_{j=1}^{J}\sum_{p=1}^{2} \rho_{jkp} q^1_{jkp}\Big) \tag{3.10}$$

$$\sum_{n_1=1}^{N_1} q^1_{kn_1 1} \le \sum_{i=1}^{I} q^1_{ik1} + \sum_{j=1}^{J} q^1_{jk1}, \tag{3.11}$$

$$\sum_{n_2=1}^{N_2} q^1_{kn_2 2} \le \sum_{i=1}^{I} q^1_{ik2} + \sum_{j=1}^{J} q^1_{jk2}, \tag{3.12}$$

$$q^1_{kn_1 1} \ge 0, \quad \forall n_1, \quad q^1_{kn_2 2} \ge 0, \quad \forall n_2, \quad q^1_{ikp} \ge 0, \quad \forall i, \forall p, \quad q^1_{jkp} \ge 0, \quad \forall j, \forall p. \tag{3.13}$$

Constraints (3.11)-(3.12) ensure that the quantity purchased by patients from the pharmacy k does not exceed the available stock held by that pharmacy. Finally, constraint (3.13) represents the non-negativity requirement for all flow variables.

We suppose that the handling and transaction costs are all continuously differentiable and convex, and that the traditional pharmacies compete in a noncooperative way.

Then, we can state the following theorem.

Theorem 3.3. *A vector* $(Q^{2*}, Q^{5*}, Q^{8*}, Q^{9*}, \lambda^{3*}, \mu^{3*}) \in \mathbb{R}_+^{IKP+JKP+KN_1+KN_2+2K}$ *is an optimal solution for the maximization problem* (3.10)-(3.13) *if and only if it is a solution to the Variational Inequality: Find* $(Q^{2*}, Q^{5*}, Q^{8*}, Q^{9*}, \lambda^{3*}, \mu^{3*}) \in \mathbb{R}_+^{IKP+JKP+KN_1+KN_2+2K}$ *such that:*

$$\begin{aligned}
&\sum_{i=1}^{I}\sum_{k=1}^{K}\left(\frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{ik1}}+\frac{\partial tr_{ik1}(q^{1*}_{ik1})}{\partial q^1_{ik1}}+\rho_{ik1}-\lambda^{3*}_k\right)(q^1_{ik1}-q^{1*}_{ik1})\\
&+\sum_{i=1}^{I}\sum_{k=1}^{K}\left(\frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{ik2}}+\frac{\partial tr_{ik2}(q^{1*}_{ik2})}{\partial q^1_{ik2}}+\rho_{ik2}-\mu^{3*}_k\right)(q^1_{ik2}-q^{1*}_{ik2})\\
&+\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{jk1}}+\frac{\partial tr_{jk1}(q^{1*}_{jk1})}{\partial q^1_{jk1}}+\rho_{jk1}-\lambda^{3*}_k\right)\left(q^1_{jk1}-q^{1*}_{jk1}\right)\\
&+\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{jk2}}+\frac{\partial tr_{jk2}(q^{1*}_{jk2})}{\partial q^1_{jk2}}+\rho_{jk2}-\mu^{3*}_k\right)\left(q^1_{jk2}-q^{1*}_{jk2}\right)\\
&+\sum_{k=1}^{K}\sum_{n_1=1}^{N_1}\left(-\left(\bar{\rho}^1_{k1}+\bar{\rho}^{SSN}_{k1}\right)+\lambda^{3*}_k\right)(q^1_{kn_11}-q^{1*}_{kn_11}) \qquad (3.14)\\
&+\sum_{k=1}^{K}\sum_{n_1=1}^{N_1}(-\bar{\rho}^1_{k2}+\mu^{3*}_k)(q^1_{kn_22}-q^{1*}_{kn_22})\\
&-\sum_{k=1}^{K}\left(\sum_{n_1=1}^{N_1}q^{1*}_{kn_11}-\sum_{i=1}^{I}q^{1*}_{ik1}-\sum_{j=1}^{J}q^{1*}_{jk1}\right)(\lambda^3_k-\lambda^{3*}_k)\\
&-\sum_{k=1}^{K}\left(\sum_{n_2=1}^{N_2}q^{1*}_{kn_22}-\sum_{i=1}^{I}q^{1*}_{ik2}-\sum_{j=1}^{J}q^{1*}_{jk2}\right)(\mu^3_k-\mu^{3*}_k)\geq 0,\\
&\qquad\qquad \forall (Q^2,Q^5,Q^8,Q^9,\lambda^3,\mu^3)\in\mathbb{R}_+^{IKP+JKP+KN_1+KN_2+2K},
\end{aligned}$$

where $\lambda^3 = (\lambda^3_k)_k \in \mathbb{R}^K_+$ *and* $\mu^3 = (\mu^3_k)_k \in \mathbb{R}^K_+$ *are the vectors of the Lagrange multipliers associated with constraints* (3.11) *and* (3.12)*, respectively.*

3.3.2. *Mixed Pharmacies.* A mixed pharmacy h is faced with a handling cost denoted by $\bar{c}_h$ and, in the simplest case, $\bar{c}_h = \bar{c}_h(Q^3, Q^6)$, namely it is a function of how much product the pharmacy has obtained from the various producers and distributors. The mixed pharmacies, in turn, also incur transaction costs when dealing with pharmaceutical companies and distributors. We denote by tr_{ihp} the transaction costs with pharmaceutical companies, and we assume that such a function can depend upon the company product shipment q^1_{ihp}. We denote the transaction cost associated with pharmacy h transacting with distributor j by tr_{jhp} and we assume that such a function can depend upon the distributor product shipment q^1_{jhp}.

Let $q^l_{hn_11}$ denote the amount of the product with prescription ($p = 1$) purchased by patient n_1 from pharmacy h with mode l and $q^l_{hn_22}$ the amount of the product without prescription ($p = 2$) purchased by patient n_2 from pharmacy h with mode l. We aggregate these quantities into the column vectors Q^{10} and Q^{11}, which represent the total amounts of prescription and non-prescription drugs delivered from all mixed pharmacies to the appropriate groups of patients via both physical ($l = 1$) and electronic ($l = 2$) channels, respectively. Additionally, the vectors Q^{10}_2 and Q^{11}_2 specifically refer to the quantities of prescription and non-prescription drugs, respectively, delivered exclusively through electronic commerce (i.e., for $l=2$). The pharmacy h associates a price with the product p, which is denoted by $\bar{\rho}^l_{hp}$. In the Italian healthcare system, pharmacies receive reimbursement from the National Health Service (SSN) for the dispensing of prescription medications to patients. Thus, we denote by $\bar{\rho}^{SSN}_{h1}$ the unit reimbursement for products sold of type $p=1$. Lastly, mixed pharmacies are subject to a subscription cost associated with using the platform that manages the online sales. This cost is denoted sub_h and depends on the total volume of product sold online Q^{10}_2 and Q^{11}_2.

Assuming, that the pharmacies are also profit maximizers, the optimization problem of a pharmacy h is given by:

$$\max\Big(\sum_{n_1=1}^{N_1}\sum_{l=1}^{2}(\bar{\rho}^l_{h1}+\bar{\rho}^{SSN}_{h1})q^l_{hn_1 1}+\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}\bar{\rho}^l_{h2}q^l_{hn_2 2}-\bar{c}_h(Q^3,Q^6)-\sum_{i=1}^{I}\sum_{p=1}^{2}tr_{ihp}(q^1_{ihp})$$
$$-\sum_{j=1}^{J}\sum_{p=1}^{2}tr_{jhp}(q^1_{jhp})-sub_h(Q^{10}_2,Q^{11}_2)-\sum_{i=1}^{I}\sum_{p=1}^{2}\rho_{ihp}q^1_{ihp}-\sum_{j=1}^{J}\sum_{p=1}^{2}\rho_{jhp}q^1_{jhp}\Big) \tag{3.15}$$
$$\sum_{n_1=1}^{N_1}\sum_{l=1}^{2}q^l_{hn_1 1}\le\sum_{i=1}^{I}q^1_{ih1}+\sum_{j=1}^{J}q^1_{jh1}, \tag{3.16}$$
$$\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}q^l_{hn_2 2}\le\sum_{i=1}^{I}q^1_{ih2}+\sum_{j=1}^{J}q^1_{jh2}, \tag{3.17}$$
$$q^l_{hn_1 1}\ge 0,\quad \forall n_1,\forall l,\quad q^l_{hn_2 2}\ge 0,\quad \forall n_2,\forall l,\quad q^1_{ihp}\ge 0,\quad \forall i,\forall p,\quad q^1_{jhp}\ge 0,\quad \forall j,\forall p. \tag{3.18}$$

Constraints (3.16)-(3.17) ensure that the quantity purchased by patients from the pharmacy h does not exceed the available stock held by that pharmacy. Finally, constraint (3.18) represents the non-negativity requirement for all flow variables. We suppose that the handling, transaction and subscription costs are all continuously differentiable and convex, and that the mixed pharmacies compete in a noncooperative way.

Then, we can state the following theorem.

Theorem 3.4. *A vector* $(Q^{3*},Q^{6*},Q^{10*},Q^{11*},\lambda^{4*},\mu^{4*})\in\mathbb{R}_+^{IHP+JHP+HN_1L+HN_2L+2H}$ *is an optimal solution for the maximization problem* (3.15)-(3.18) *if and only if it is a solution to the Variational Inequality: Find* $(Q^{3*},Q^{6*},Q^{10*},Q^{11*},\lambda^{4*},\mu^{4*})\in\mathbb{R}_+^{IHP+JHP+HN_1L+HN_2L+2H}$ *such that:*

$$\sum_{i=1}^{I}\sum_{h=1}^{H}\left(\frac{\partial\bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{ih1}}+\frac{\partial tr_{ih1}(q^{1*}_{ih1})}{\partial q^1_{ih1}}+\rho_{ih1}-\lambda^{4*}_h\right)(q^1_{ih1}-q^{1*}_{ih1})$$
$$+\sum_{i=1}^{I}\sum_{h=1}^{H}\left(\frac{\partial\bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{ih2}}+\frac{\partial tr_{ih2}(q^{1*}_{ih2})}{\partial q^1_{ih2}}+\rho_{ih2}-\mu^{4*}_h\right)(q^1_{ih2}-q^{1*}_{ih2})$$
$$+\sum_{j=1}^{J}\sum_{h=1}^{H}\left(\frac{\partial\bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{jh1}}+\frac{\partial tr_{jh1}(q^{1*}_{jh1})}{\partial q^1_{jh1}}+\rho_{jh1}-\lambda^{4*}_h\right)\left(q^1_{jh1}-q^{1*}_{jh1}\right)$$
$$+\sum_{j=1}^{J}\sum_{h=1}^{H}\left(\frac{\partial\bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{jh2}}+\frac{\partial tr_{jh2}(q^{1*}_{jh2})}{\partial q^1_{jh2}}+\rho_{jh2}-\mu^{4*}_h\right)\left(q^1_{jh2}-q^{1*}_{jh2}\right)$$

$$
\begin{aligned}
&+\sum_{h=1}^{H}\sum_{n_1=1}^{N_1}\sum_{l=1}^{2}\left(\frac{\partial sub_h(Q_2^{10*},Q_2^{11*})}{\partial q_{hn_11}^l}-\left(\bar{\rho}_{h1}^l+\bar{\rho}_{h1}^{SSN}\right)+\lambda_h^{4*}\right)\left(q_{hn_11}^l-q_{hn_11}^{l*}\right)\\
&+\sum_{h=1}^{H}\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}\left(\frac{\partial sub_h(Q_2^{10*},Q_2^{11*})}{\partial q_{hn_22}^l}-\bar{\rho}_{h2}^l+\mu_h^{4*}\right)\left(q_{hn_22}^l-q_{hn_22}^{l*}\right)\\
&-\sum_{h=1}^{H}\left(\sum_{n_1=1}^{N_1}\sum_{l=1}^{2}q_{hn_11}^{l*}-\sum_{i=1}^{I}q_{ih1}^{1*}-\sum_{j=1}^{J}q_{jh1}^{1*}\right)(\lambda_h^4-\lambda_h^{4*})\\
&-\sum_{h=1}^{H}\left(\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}q_{hn_22}^{l*}-\sum_{i=1}^{I}q_{ih2}^{1*}-\sum_{j=1}^{J}q_{jh2}^{1*}\right)(\mu_h^4-\mu_h^{4*})\geq 0,\\
&\qquad\qquad\forall(Q^3,Q^6,Q^{10},Q^{11},\lambda^4,\mu^4)\in\mathbb{R}_+^{IHP+JHP+HN_1L+HN_2L+2H},
\end{aligned}
\tag{3.19}
$$

where $\lambda^4=(\lambda_h^4)_h\in\mathbb{R}_+^H$ *and* $\mu^4=(\mu_h^4)_h\in\mathbb{R}_+^H$ *are the vectors of the Lagrange multipliers associated with constraints* (3.13) *and* (3.16)*, respectively.*

3.3.3. *Parapharmacies.* A parapharmacy m is faced with a handling cost denoted by $\bar{c}_m$ and, in the simplest case, $\bar{c}_m=\bar{c}_m(Q^4,Q^7)$, namely, it is a function of the amount of product the parapharmacy has obtained from various producers and distributors. The parapharmacies, in turn, also incur transaction costs when dealing with pharmaceutical companies and distributors. We denote by tr_{im2} the transaction cost with the pharmaceutical company i and assume that such a function can depend on the company product shipment q_{im2}^1. We denote the transaction cost associated with pharmacy m transacting with distributor j by tr_{jm2} and assume that such a function can depend on the distributor product shipment q_{jm2}^1.

Let $q_{mn_22}^l$ the amount of the product without prescription ($p=2$) purchased by patient n_2 from parapharmacy m with mode l. We group these consumption quantities into the column vector Q^{12}. Additionally, the vector Q_2^{12} specifically refers to the quantities of non-prescription drugs delivered exclusively through electronic commerce (i.e., for l = 2). The parapharmacy m associates a price with the product of type $p=2$, which is denoted by $\bar{\rho}_{m2}^l$. Lastly, parapharmacies are subject to a subscription cost associated with using the platform that manages the online sales. This cost is denoted by sub_m and depends on the total volume of product sold online Q_2^{12}.

Assuming, that the parapharmacies are also profit maximizers, the optimization problem of a pharmacy m is given by:

$$
\begin{aligned}
&\max\Big(\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}\bar{\rho}_{m2}^l q_{mn_22}^l-\bar{c}_m(Q^4,Q^7)-\sum_{i=1}^{I}tr_{im2}(q_{im2}^1)-\sum_{j=1}^{J}tr_{jm2}(q_{jm2}^1)\\
&-sub_m(Q_2^{12})-\sum_{i=1}^{I}\rho_{im2}q_{im2}^1-\sum_{j=1}^{J}\rho_{jm2}q_{jm2}^1\Big)
\end{aligned}
\tag{3.20}
$$

$$
\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}q_{mn_22}^l\leq\sum_{i=1}^{I}q_{im2}^1+\sum_{j=1}^{J}q_{jm2}^1, \tag{3.21}
$$

$$
q_{mn_22}^l\geq 0,\quad\forall n_2,\,\forall l,\quad q_{im2}^1\geq 0,\quad\forall i,\quad q_{jm2}^1\geq 0,\quad\forall j. \tag{3.22}
$$

We suppose that the handling, transaction and subscription costs are all continuously differentiable and convex, and that the mixed pharmacies compete in a noncooperative way.

Then, we can state the following theorem.

Theorem 3.5. *A vector* $(Q^{4*}, Q^{7*}, Q^{12*}, \mu^{5*}) \in \mathbb{R}_+^{IM+JM+MN_2+M}$ *is an optimal solution for the maximization problem* (3.20)-(3.22) *if and only if it is a solution to the Variational Inequality:*
Find $(Q^{4*}, Q^{7*}, Q^{12*}, \mu^{5*}) \in \mathbb{R}_+^{IM+JM+MN_2+M}$ *such that:*

$$\begin{aligned}
&\sum_{i=1}^{I}\sum_{m=1}^{M}\left(\frac{\partial \bar{c}_m(Q^{4*},Q^{7*})}{\partial q^1_{im2}}+\frac{\partial tr_{im2}(q^{1*}_{im2})}{\partial q^1_{im2}}+\rho_{im2}-\mu^{5*}_m\right)\left(q^1_{im2}-q^{1*}_{im2}\right)\\
&+\sum_{j=1}^{J}\sum_{m=1}^{M}\left(\frac{\partial \bar{c}_m(Q^{4*},Q^{7*})}{\partial q^1_{jm2}}+\frac{\partial tr_{jm2}(q^{1*}_{jm2})}{\partial q^1_{jm2}}+\rho_{jm2}-\mu^{5*}_m\right)\left(q^1_{jm2}-q^{1*}_{jm2}\right)\\
&+\sum_{m=1}^{M}\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}\left(\frac{\partial sub_m(Q^{12*}_2)}{\partial q^l_{mn_22}}-\bar{\rho}^l_{m2}+\mu^{5*}_m\right)\left(q^l_{mn_22}-q^{l*}_{mn_22}\right)\\
&-\sum_{m=1}^{M}\left(\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}q^{l*}_{mn_22}-\sum_{i=1}^{I}q^{1*}_{im2}-\sum_{j=1}^{J}q^{1*}_{jm2}\right)\left(\mu^5_m-\mu^{5*}_m\right)\geq 0,\\
&\qquad\qquad\forall (Q^4,Q^7,Q^{12},\mu^5)\in\mathbb{R}_+^{IM+JM+MN_2+M},
\end{aligned} \tag{3.23}$$

where $\mu^5 = (\mu^5_m)_m \in \mathbb{R}^M_+$ *is the vector of the Lagrange multipliers associated with constraint* (3.18).

3.4. **Behavior of the Patients.** We now describe the behavior of the patients. Patients make purchasing decisions based on both the prices that pharmacies charge for products and the transaction costs they incur when acquiring those products. On the other hand, firms compete not only on price but also on time, with guaranteed delivery times serving as strategic variables. Given that each product requires both order and delivery, we can consider the time consumed by these supply chain network activities.

3.4.1. *Patients with prescription.* Let $\bar{\bar{c}}^1_{kn_11}$ denote the transaction cost associated with obtaining the products of type $p = 1$ for patient n_1 from pharmacy k with mode $l = 1$. We assume that the transaction cost is continuous and of the general form $\bar{\bar{c}}^1_{kn_11} = \bar{\bar{c}}^1_{kn_11}(Q^8)$. Hence, we allow for the transaction cost to depend not only upon the flow of the product transacted from a pharmacy to a patient n_1 but also on other product flows between pharmacies and patients. The patients take the price charged by the pharmacy $\bar{\rho}_{k1}$ plus the transaction cost, in making their consumption decisions.

Including the time component in transactions is essential because products acquired through e-commerce have inherently time-sensitive value. There exists only a narrow time window during which the product's value remains positive, and this value decreases continuously over time; see [3]. Therefore, we introduce the following expression to capture the time-dependent delay associated with online order and delivery processes:

$$t^2_{hn_11}q_{hn_11} + \tau^2_{hn_11} = T^2_{hn_11}, \quad h = 1,\ldots,H; n_1 = 1,\ldots,N_1, \tag{3.24}$$

where $t^2_{hn_11}$ and $\tau^2_{hn_11}$ are nonnegative and reflect the actual time consumption associated with delivering products of type $p = 1$ to patient n_1 via mode $l = 2$. $T^2_{hn_11}$ represents the time between the order of product from pharmacy h and the delivery to n_1. We denote by T_{ave,n_1} the average time for delivery the products to consumer n_1

$$T_{ave,n_1} = \frac{\sum_{h=1}^{H} T_{hn_1 1}^2 q_{hn_1 1}^2}{d_{n_1}},$$

where

$$d_{n_1} = \sum_{k=1}^{K} q_{kn_1 1}^1 + \sum_{h=1}^{H} q_{hn_1 1}^1 + \sum_{h=1}^{H} q_{hn_1 1}^2.$$

We group the demands into the vector $d^1 \in \mathbb{R}_+^{N_1}$. We group the average times for all the patients into the vector $T_{ave}^1 \in \mathbb{R}_+^{N_1}$. Then, we can define the demand price functions as follows:

$$\rho_{n_1}(d^1, T_{ave}^1) \equiv \hat{\rho}_{n_1}(Q^8, Q^{10}), \quad n_1 = 1, \dots, N_1. \tag{3.25}$$

Incorporating time into consumer behavior is crucial because time directly affects the perceived value and utility of a product, especially in the context of pharmaceutical supply chains. Patients do not make decisions based solely on price and availability, since they are also sensitive to delivery delays and waiting times. This is particularly true in healthcare markets, where the need for medication can be urgent and time-sensitive.

Since pharmaceutical products, especially over-the-counter drugs, have a value that often diminishes over time (e.g., a cold remedy is less useful if it arrives after the symptoms have subsided), the time of delivery becomes a strategic variable for firms and a cost component for patients.

This justifies modeling consumer behavior using equilibrium conditions that depend not only on price and transaction costs but also on delivery time. By doing so, the model captures more realistic decision-making and enables suppliers to compete on both price and time, reflecting the modern dynamics of e-commerce.

The equilibrium conditions for the consumer n_1, thus, take the following form. For all pharmacies $k = 1, \dots, K$, we have

$$\bar{\rho}_{k1} + \bar{\bar{c}}_{kn_1 1}^1(Q^{8*}) \begin{cases} = \hat{\rho}_{n1}(Q^{8*}, Q^{10*}) & \text{if } q_{kn_1 1}^{1*} > 0 \\ \geq \hat{\rho}_{n1}(Q^{8*}, Q^{10*}) & \text{if } q_{kn_1 1}^{1*} = 0. \end{cases} \tag{3.26}$$

Conditions (3.26) state that patients will purchase the product from pharmacy k, if the price charged by the pharmacy for the product plus the transaction cost does not exceed the price that the patients are willing to pay for the product.

For all mixed pharmacies $h = 1, \dots, H$ and for all model $l = 1, 2$, we have the equilibrium conditions

$$\bar{\rho}_{h1} + \bar{\bar{c}}_{hn_1 1}^l(Q^{10*}) \begin{cases} = \hat{\rho}_{n1}(Q^{8*}, Q^{10*}) & \text{if } q_{hn_1 1}^{l*} > 0 \\ \geq \hat{\rho}_{n1}(Q^{8*}, Q^{10*}) & \text{if } q_{hn_1 1}^{l*} = 0. \end{cases} \tag{3.27}$$

Now, we derive the variational formulation of the equilibrium conditions of patients with prescription (3.26)-(3.27), simultaneously. Then, we can state the following theorem (see [16] and [21]).

Theorem 3.6. *A vector* $(Q^{8*}, Q^{10*}) \in \mathbb{R}_+^{KN_1 + HN_1 L}$ *satisfies equilibrium conditions* (3.26)-(3.27) *if and only if such a vector is a solution to the variational inequality:*

Find $(Q^{8*}, Q^{10*}) \in \mathbb{R}_+^{KN_1+HN_1L}$ *such that:*

$$\begin{aligned} & \sum_{k=1}^{K}\sum_{n_1=1}^{N_1}\left[\bar{\rho}_{k1}+\bar{\bar{c}}^1_{kn_11}(Q^{8*})-\hat{\rho}_{n1}(Q^{8*},Q^{10*})\right]\times(q^1_{kn_11}-q^{1*}_{kn_11}) \\ + \quad & \sum_{h=1}^{H}\sum_{n_1=1}^{N_1}\sum_{l=1}^{2}\left[\bar{\rho}_{h1}+\bar{\bar{c}}^l_{hn_11}(Q^{10*})-\hat{\rho}_{n1}(Q^{8*},Q^{10*})\right]\times(q^l_{hn_11}-q^{l*}_{hn_11})\geq 0, \\ & \forall (Q^8,Q^{10})\in\mathbb{R}_+^{KN_1+HN_1L}. \end{aligned} \tag{3.28}$$

Moreover, variational inequality (3.28) *admits at least one solution.*

Variational inequality (3.28) represents the equilibrium for all patients with a prescription simultaneously. The solution gives the optimal amount (of medicines) in volumetric weight that patients receive from traditional and mixed pharmacies through physical links and the two types of purchases, respectively.

3.4.2. *Patients without prescription.* Similarly to the previous case, we denote by $\bar{\bar{c}}^1_{kn_22}$ the transaction cost associated with obtaining the products of type $p=2$ for a patient n_2 from pharmacy k with mode $l=1$. We assume that the transaction cost is continuous and of the general form $\bar{\bar{c}}^1_{kn_22}=\bar{\bar{c}}^1_{kn_22}(Q^9)$. The patients take the price charged by the pharmacy $\bar{\rho}_{k2}$ plus the transaction cost, in making their consumption decisions. We express the time-dependent delay associated with online order and delivery processes as follows:

$$t^2_{hn_22}q_{hn_22}+\tau^2_{hn_22}=T^2_{hn_22}, \quad h=1,\ldots,H; n_2=1,\ldots,N_2, \tag{3.29}$$

where $t^2_{hn_22}$ and $\tau^2_{hn_22}$ are nonnegative and reflect the actual time consumption associated with delivering products of type $p=2$ to patient n_2 via mode $l=2$. $T^2_{hn_22}$ represents the time between the order of product from pharmacy h and the delivery to n_2. We denote by T_{ave,n_2} the average time for delivery the products to consumer n_2 $T_{ave,n_2}=\dfrac{\sum_{h=1}^{H}T^2_{hn_22}q^2_{hn_22}}{d_{n_2}}$, where $d_{n_2}=\sum_{k=1}^{K}q^1_{kn_22}+\sum_{m=1}^{M}q^2_{mn_22}+\sum_{h=1}^{H}q^2_{hn_22}$. We group the demands into the vector $d^2\in\mathbb{R}_+^{N_2}$. We group the average times for all the patients into the vector $T^2_{ave}\in\mathbb{R}_+^{N_1}$. Then, we can define the demand price functions as follows:

$$\rho_{n_2}(d^2,T^2_{ave})=\hat{\rho}_{n_2}(Q^9,Q^{11}), \quad n_2=1,\ldots,N_2. \tag{3.30}$$

The equilibrium conditions for the consumer n_2, thus, take the following form. For all pharmacies $k=1,\ldots,K$, we have

$$\bar{\rho}_{k2}+\bar{\bar{c}}^1_{kn_22}(Q^{9*})\begin{cases} =\hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*})=0 & \text{if } q^{1*}_{kn_22}>0 \\ \geq\hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*})=0 & \text{if } q^{1*}_{kn_22}=0. \end{cases} \tag{3.31}$$

For all mixed pharmacies $h=1,\ldots,H$ and for all model $l=1,2$, we have the equilibrium conditions

$$\bar{\rho}_{h2}+\bar{\bar{c}}^l_{hn_22}(Q^{11*})\begin{cases} =\hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*})=0 & \text{if } q^{l*}_{hn_22}>0 \\ \geq\hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*})=0 & \text{if } q^{l*}_{hn_22}=0. \end{cases} \tag{3.32}$$

For all parapharmacies $m = 1, \dots, M$ and for all models $l = 1, 2$, we have the equilibrium conditions

$$\bar{\rho}_{m2} + \bar{\bar{c}}^{l}_{mn_2 2}(Q^{12*}) \begin{cases} = \hat{\rho}_{n2}(Q^{9*}, Q^{11*}, Q^{12*}) = 0 & \text{if } q^{l*}_{mn_2 2} > 0 \\ \geq \hat{\rho}_{n2}(Q^{9*}, Q^{11*}, Q^{12*}) = 0 & \text{if } q^{l*}_{mn_2 2} = 0. \end{cases} \tag{3.33}$$

Now, we derive the variational formulation of the equilibrium conditions of patients with prescription (3.31)-(3.33), simultaneously. Then, we can state the following theorem.

Theorem 3.7. *A vector* $(Q^{9*}, Q^{11*}, Q^{12*}) \in \mathbb{R}^{KN_2+HN_2L+MN_2L}_{+}$ *satisfies equilibrium conditions* (3.31)-(3.33) *if and only if such a vector is a solution to the variational inequality:*
Find $(Q^{9*}, Q^{11*}, Q^{12*}) \in \mathbb{R}^{KN_2+HN_2L+MN_2L}_{+}$ *such that:*

$$\begin{aligned} & \sum_{k=1}^{K} \sum_{n_2=1}^{N_2} \left[\bar{\rho}_{k2} + \bar{\bar{c}}^{1}_{kn_2 2}(Q^{9*}) - \hat{\rho}_{n2}(Q^{9*}, Q^{11*}, Q^{12*})\right] \times (q^{1}_{kn_2 2} - q^{1*}_{kn_2 2}) \\ + \quad & \sum_{h=1}^{H} \sum_{n_2=1}^{N_2} \sum_{l=1}^{2} \left[\bar{\rho}_{h2} + \bar{\bar{c}}^{l}_{hn_2 2}(Q^{11*}) - \hat{\rho}_{n2}(Q^{9*}, Q^{11*}, Q^{12*})\right] \times (q^{l}_{hn_2 2} - q^{l*}_{hn_2 2}) \\ + \quad & \sum_{m=1}^{M} \sum_{n_2=1}^{N_2} \sum_{l=1}^{2} \left[\bar{\rho}_{m2} + \bar{\bar{c}}^{l}_{mn_2 2}(Q^{12*}) - \hat{\rho}_{n2}(Q^{9*}, Q^{11*}, Q^{12*})\right] \times (q^{l}_{mn_2 2} - q^{l*}_{mn_2 2}) \geq 0, \\ & \qquad \forall (Q^{9}, Q^{11}, Q^{12}) \in \mathbb{R}^{KN_2+HN_2L+MN_2L}_{+}. \end{aligned} \tag{3.34}$$

Moreover, variational inequality (3.34) *admits at least one solution.*

Variational inequality (3.34) represents the equilibrium for all patients without prescription simultaneously. The solution gives the optimal amount (of medicines) in volumetric weight that patients receive from traditional and mixed pharmacies as well as parapharmacies, through physical links and the two types of purchases, respectively.

3.5. **Comprehensive Variational Inequality Formulation for the Entire Supply Chain Network.** In this Subsection we provide a unique variational formulation for the entire network. This general formulation is supported by the fact that the quantity of medicines, expressed in volumetric weight, transported from the pharmaceutical companies to every patient via the l shipping method must match the quantity that the patient receives from the pharmaceutical companies and is willing to pay for. Consequently, the optimal solution, for the entire network, by solving the variational inequality given by the sum of the seven variational inequalities (3.4), (3.9), (3.14), (3.19), (3.23), (3.28), and (3.34), in order to formalize the agreements between the tiers. We state formally this concept in the following definition.

Definition 3.1. An equilibrium state of the presented network is reached if the flows between the tiers of the network coincide and the product shipments and prices satisfy the sum of the optimality conditions (3.4), (3.9), (3.14), (3.19), (3.23), (3.28), and (3.34).

The following result represents a variational inequality formulation of the governing equilibrium conditions according to Definition 3.1 (see [22] for the proof).

Theorem 3.8. *A vector*

$$(Q^{1*}, Q^{2*}, Q^{3*}, Q^{4*}, Q^{5*}, Q^{6*}, Q^{7*}, Q^{8*}, Q^{9*}, Q^{10*}, Q^{11*}, Q^{12*}, \lambda^{1*}, \lambda^{2*}, \mu^{2*}, \lambda^{3*}, \mu^{3*}, \lambda^{4*}, \mu^{4*}, \mu^{5*})$$
$$\in \mathbb{R}^{IJP+IKP+IHP+IM+JPK+JHP+JM+KN_1+KN_2+HN_1L+HN_2L+MN_2L+I+2J+2K+2H+M}_{+}$$

is an equilibrium state of the supply chain network according to Definition 3.1 if and only if such a vector is a solution to the variational inequality:
Find

$$
\begin{aligned}
&(Q^{1*},Q^{2*},Q^{3*},Q^{4*},Q^{5*},Q^{6*},Q^{7*},Q^{8*},Q^{9*},Q^{10*},Q^{11*},Q^{12*},\lambda^{1*},\lambda^{2*},\mu^{2*},\lambda^{3*},\mu^{3*},\lambda^{4*},\mu^{4*},\mu^{5*})\\
&\quad \in \mathbb{R}_+^{IJP+IKP+IHP+IM+JPK+JHP+JM+KN_1+KN_2+HN_1L+HN_2L+MN_2L+I+2J+2K+2H+M} \text{ such that:}\\
&\quad \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ij1}}+\frac{\partial c^1_{ij1}(q^{1*}_{ij1})}{\partial q^1_{ij1}}+\frac{\partial \hat{c}_{ij1}(q^{1*}_{ij1})}{\partial q^1_{ij1}}+\frac{\partial c_j(Q^{1*})}{\partial q^1_{ij1}}+\lambda_i^{1*}-\lambda_j^{2*}\right)(q^1_{ij1}-q^{1*}_{ij1})\\
&+\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ij2}}+\frac{\partial c^1_{ij2}(q^{1*}_{ij2})}{\partial q^1_{ij2}}+\frac{\partial \hat{c}_{ij2}(q^{1*}_{ij2})}{\partial q^1_{ij2}}+\frac{\partial c_j(Q^{1*})}{\partial q^1_{ij2}}+\lambda_i^{1*}-\mu_j^{2*}\right)(q^1_{ij2}-q^{1*}_{ij2})\\
&+\sum_{i=1}^{I}\sum_{k=1}^{K}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ik1}}+\frac{\partial c^1_{ik1}(q^{1*}_{ik1})}{\partial q^1_{ik1}}+\frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{ik1}}+\frac{\partial tr_{ik1}(q^{1*}_{ik1})}{\partial q^1_{ik1}}+\lambda_i^{1*}-\lambda_k^{3*}\right)(q^1_{ik1}-q^{1*}_{ik1})\\
&+\sum_{i=1}^{I}\sum_{k=1}^{K}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ik12}}+\frac{\partial c^1_{ik2}(q^{1*}_{ik2})}{\partial q^1_{ik2}}+\frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{ik2}}+\frac{\partial tr_{ik2}(q^{1*}_{ik2})}{\partial q^1_{ik2}}+\lambda_i^{1*}-\mu_k^{3*}\right)(q^1_{ik2}-q^{1*}_{ik2})\\
&+\sum_{i=1}^{I}\sum_{h=1}^{H}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ih1}}+\frac{\partial c^1_{ih1}(q^{1*}_{ih1})}{\partial q^1_{ih1}}+\frac{\partial \bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{ih1}}+\frac{\partial tr_{ih1}(q^{1*}_{ih1})}{\partial q^1_{ih1}}+\lambda_i^{1*}-\lambda_h^{4*}\right)(q^1_{ih1}-q^{1*}_{ih1})\\
&+\sum_{i=1}^{I}\sum_{h=1}^{H}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ih2}}+\frac{\partial c^1_{ih2}(q^{1*}_{ih2})}{\partial q^1_{ih2}}+\frac{\partial \bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{ih2}}+\frac{\partial tr_{ih2}(q^{1*}_{ih2})}{\partial q^1_{ih2}}+\lambda_i^{1*}-\mu_h^{4*}\right)(q^1_{ih2}-q^{1*}_{ih2})\\
&+\sum_{i=1}^{I}\sum_{m=1}^{2}\left(\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{im2}}+\frac{\partial c^1_{im2}(q^{1*}_{im2})}{\partial q^1_{im2}}+\frac{\partial \bar{c}_m(Q^{4*},Q^{7*})}{\partial q^1_{im2}}+\frac{\partial tr_{im2}(q^{1*}_{im2})}{\partial q^1_{im2}}+\lambda_i^{1*}-\mu_m^{5*}\right)(q^1_{im2}-q^{1*}_{im2})\\
&+\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{\partial c^1_{jk1}(q^{1*}_{jk1})}{\partial q^1_{jk1}}+\frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{jk1}}+\frac{\partial tr_{jk1}(q^{1*}_{jk1})}{\partial q^1_{jk1}}+\lambda_j^{2*}-\lambda_k^{3*}\right)(q^1_{jk1}-q^{1*}_{jk1})\\
&+\sum_{j=1}^{J}\sum_{k=1}^{K}\left(\frac{\partial c^1_{jk2}(q^{1*}_{jk2})}{\partial q^1_{jk2}}+\frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{jk2}}+\frac{\partial tr_{jk2}(q^{1*}_{jk2})}{\partial q^1_{jk2}}+\lambda_j^{2*}-\mu_k^{3*}\right)(q^1_{jk2}-q^{1*}_{jk2})\\
&+\sum_{j=1}^{J}\sum_{h=1}^{H}\left(\frac{\partial c^1_{jh1}(q^{1*}_{jh1})}{\partial q^1_{jh1}}+\frac{\partial \bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{jh1}}+\frac{\partial tr_{jh1}(q^{1*}_{jh1})}{\partial q^1_{jh1}}+\lambda_j^{2*}-\lambda_h^{4*}\right)(q^1_{jh1}-q^{1*}_{jh1})\\
&+\sum_{j=1}^{J}\sum_{h=1}^{H}\left(\frac{\partial c^1_{jh2}(q^{1*}_{jh2})}{\partial q^1_{jh2}}+\frac{\partial \bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{jh2}}+\frac{\partial tr_{jh2}(q^{1*}_{jh1})}{\partial q^1_{jh2}}+\lambda_j^{2*}-\lambda_h^{4*}\right)(q^1_{jh2}-q^{1*}_{jh2})\\
&+\sum_{j=1}^{J}\sum_{m=1}^{M}\left(\frac{\partial c^1_{jm2}(q^{1*}_{jm2})}{\partial q^1_{jm2}}+\frac{\partial \bar{c}_m(Q^{4*},Q^{7*})}{\partial q^1_{jm2}}+\frac{\partial tr_{jm2}(q^{1*}_{jm2})}{\partial q^1_{jm2}}+\mu_j^{2*}-\mu_m^{5*}\right)(q^1_{jm2}-q^{1*}_{jm2})\\
&+\sum_{k=1}^{K}\sum_{n_1=1}^{N_1}\left(-\bar{\rho}^{SSN}_{k1}+\bar{\bar{c}}^1_{kn_11}(Q^{8*})-\hat{\rho}_{n1}(Q^{8*},Q^{10*})+\lambda_k^{3*}\right)(q^1_{kn_11}-q^{1*}_{kn_11})\\
&+\sum_{k=1}^{K}\sum_{n_2=1}^{N_2}\left(\bar{\bar{c}}^2_{kn_22}(Q^{9*})-\hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*})+\mu_k^{3*}\right)(q^1_{kn_2}-q^{1*}_{kn_22})\\
&+\sum_{h=1}^{H}\sum_{n_1=1}^{N_1}\sum_{l=1}^{2}\left(\frac{\partial sub_h(Q_2^{10*},Q_2^{11*})}{\partial q^l_{hn_11}}-\bar{\rho}^{SSN}_{h1}+\bar{\bar{c}}^1_{hn_11}(Q^{10*})-\hat{\rho}_{n1}(Q^{8*},Q^{10*})+\lambda_h^{4*}\right)\left(q^l_{hn_11}-q^{l*}_{hn_11}\right)
\end{aligned}
$$

$$+\sum_{h=1}^{H}\sum_{n_1=1}^{N_1}\sum_{l=1}^{2}\left(\frac{\partial sub_h(Q_2^{10*},Q_2^{11*})}{\partial q_{hn_22}^l}+\bar{\bar{c}}_{hn_22}^1(Q^{11*})-\hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*})+\mu_h^{4*}\right)\left(q_{hn_22}^l-q_{hn_22}^{l*}\right)$$

$$+\sum_{m=1}^{M}\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}\left[\frac{\partial sub_m(Q_2^{12*})}{\partial q_{mn_22}^l}+\bar{\bar{c}}_{mn_22}^l(Q^{12*})-\hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*})+\mu_m^{5*}\right]\times(q_{mn_22}^l-q_{mn_22}^{l*})$$

$$-\sum_{i=1}^{I}\left(\sum_{j=1}^{J}\sum_{p=1}^{2}q_{ijp}^{1*}+\sum_{k=1}^{K}\sum_{p=1}^{2}q_{ikp}^{1*}+\sum_{h=1}^{H}\sum_{p=1}^{2}q_{ihp}^{1*}+\sum_{m=1}^{M}q_{im2}^{1*}-M_i\right)(\lambda_i^1-\lambda_i^{1*})$$

$$-\sum_{j=1}^{J}\left(\sum_{k=1}^{K}q_{jk1}^{1*}+\sum_{h=1}^{H}q_{jh1}^{1*}-\sum_{i=1}^{I}q_{ij1}^{1*}\right)(\lambda_j^2-\lambda_j^{2*})$$

$$-\sum_{j=1}^{J}\left(\sum_{k=1}^{K}q_{jk2}^{1*}+\sum_{h=1}^{H}q_{jh2}^{1*}+\sum_{m=1}^{M}q_{jm2}^{1*}-\sum_{i=1}^{I}q_{ij2}^{1*}\right)(\mu_j^2-\mu_j^{2*})$$

$$-\sum_{k=1}^{K}\left(\sum_{n_1=1}^{N_1}q_{kn_11}^{1*}-\sum_{i=1}^{I}q_{ik1}^{1*}-\sum_{j=1}^{J}q_{jk1}^{1*}\right)(\lambda_k^3-\lambda_k^{3*})$$

$$-\sum_{k=1}^{K}\left(\sum_{n_2=1}^{N_2}q_{kn_22}^{1*}-\sum_{i=1}^{I}q_{ik2}^{1*}-\sum_{j=1}^{J}q_{jk2}^{1*}\right)(\mu_k^3-\mu_k^{3*})$$

$$-\sum_{h=1}^{H}\left(\sum_{n_1=1}^{N_1}\sum_{l=1}^{2}q_{hn_11}^{l*}-\sum_{i=1}^{I}q_{ih1}^{1*}-\sum_{j=1}^{J}q_{jh1}^{1*}\right)(\lambda_h^4-\lambda_h^{4*})$$

$$-\sum_{h=1}^{H}\left(\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}q_{hn_22}^{l*}-\sum_{i=1}^{I}q_{ih2}^{1*}-\sum_{j=1}^{J}q_{jh2}^{1*}\right)(\mu_h^4-\mu_h^{4*})$$

$$-\sum_{m=1}^{M}\left(\sum_{n_2=1}^{N_2}\sum_{l=1}^{2}q_{mn_22}^{l*}-\sum_{i=1}^{I}q_{im2}^{1*}-\sum_{j=1}^{J}q_{jm2}^{1*}\right)(\mu_m^5-\mu_m^{5*})\geq 0, \tag{3.35}$$

$$\forall(Q^1,Q^2,Q^3,Q^4,Q^5,Q^6,Q^7,Q^8,Q^9,Q^{10},Q^{11},Q^{12},\lambda^1,\lambda^2,\mu^2,\lambda^3,\mu^3,\lambda^4,\mu^4,\mu^5)\in \mathbb{R}_+^{IJP+IKP+IHP+IM+JPK+JHP+JM+KN_1+KN_2+HN_1L+HN_2L+MN_2L+I+2J+2K+2H+M}.$$

Note that in the unique VI (3.35) the terms ρ_{ijp}, ρ_{ikp}, ρ_{ihp}, ρ_{im2}, ρ_{jkp}, ρ_{jhp}, ρ_{jm2}, $\bar{\rho}_{kp}$, $\bar{\rho}_{hp}$, $\bar{\rho}_{m2}$ do not appear. Therefore, the optimal solutions are independent of the revenues generated from deliveries to patients using both types of purchases. Variational inequality (3.35) can be put in standard form, as follows:

Find $X^*\in K$ such that $F(X)^T(X-X^*)\geq 0$ for all $X\in K$, where $F(X^*)=(F_i(X^*))_{i=1,\dots,28}$ is a vector function with

$$F_1(X^*) = \left[\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q_{ij1}^1}+\frac{\partial c_{ij1}^1(q_{ij1}^{1*})}{\partial q_{ij1}^1}+\frac{\partial \hat{c}_{ij1}(q_{ij1}^{1*})}{\partial q_{ij1}^1}+\frac{\partial c_j(Q^{1*})}{\partial q_{ij1}^1}+\lambda_i^{1*}-\lambda_j^{2*}\right],\forall i,j,$$

$$F_2(X^*) = \left[\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q_{ij2}^1}+\frac{\partial c_{ij2}^1(q_{ij2}^{1*})}{\partial q_{ij2}^1}+\frac{\partial \hat{c}_{ij2}(q_{ij2}^{1*})}{\partial q_{ij2}^1}+\frac{\partial c_j(Q^{1*})}{\partial q_{ij2}^1}+\lambda_i^{1*}-\mu_j^{2*}\right],\forall i,j,$$

$$
\begin{aligned}
F_3(X^*) &= \left[\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ik1}} + \frac{\partial c^1_{ik1}(q^{1*}_{ik1})}{\partial q^1_{ik1}} + \frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{ik1}} + \frac{\partial tr_{ik1}(q^{1*}_{ik1})}{\partial q^1_{ik1}} + \lambda_i^{1*} - \lambda_k^{3*}\right], \forall i,k,\\
F_4(X^*) &= \left[\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ik12}} + \frac{\partial c^1_{ik2}(q^{1*}_{ik2})}{\partial q^1_{ik2}} + \frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{ik2}} + \frac{\partial tr_{ik2}(q^{1*}_{ik2})}{\partial q^1_{ik2}} + \lambda_i^{1*} - \mu_k^{3*}\right], \forall i,k,\\
F_5(X^*) &= \left[\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ih1}} + \frac{\partial c^1_{ih1}(q^{1*}_{ih1})}{\partial q^1_{ih1}} + \frac{\partial \bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{ih1}} + \frac{\partial tr_{ih1}(q^{1*}_{ih1})}{\partial q^1_{ih1}} + \lambda_i^{1*} - \lambda_h^{4*}\right], \forall i,h,\\
F_6(X^*) &= \left[\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{ih2}} + \frac{\partial c^1_{ih2}(q^{1*}_{ih2})}{\partial q^1_{ih2}} + \frac{\partial \bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{ih2}} + \frac{\partial tr_{ih2}(q^{1*}_{ih2})}{\partial q^1_{ih2}} + \lambda_i^{1*} - \mu_h^{4*}\right], \forall i,h,\\
F_7(X^*) &= \left[\frac{\partial p_i(Q^{1*},Q^{2*},Q^{3*},Q^{4*})}{\partial q^1_{im2}} + \frac{\partial c^1_{im2}(q^{1*}_{im2})}{\partial q^1_{im2}} + \frac{\partial \bar{c}_m(Q^{4*},Q^{7*})}{\partial q^1_{im2}} + \frac{\partial tr_{im2}(q^{1*}_{im2})}{\partial q^1_{im2}} + \lambda_i^{1*} - \mu_m^{5*}\right], \forall i,m,\\
F_8(X^*) &= \left[\frac{\partial c^1_{jk1}(q^{1*}_{jk1})}{\partial q^1_{jk1}} + \frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{jk1}} + \frac{\partial tr_{jk1}(q^{1*}_{jk1})}{\partial q^1_{jk1}} + \lambda_j^{2*} - \lambda_k^{3*}\right], \ \forall j,k,\\
F_9(X^*) &= \left[\frac{\partial c^1_{jk2}(q^{1*}_{jk2})}{\partial q^1_{jk2}} + \frac{\partial \bar{c}_k(Q^{2*},Q^{5*})}{\partial q^1_{jk2}} + \frac{\partial tr_{jk2}(q^{1*}_{jk2})}{\partial q^1_{jk2}} + \lambda_j^{2*} - \mu_k^{3*}\right], \ \forall j,k,\\
F_{10}(X^*) &= \left[\frac{\partial c^1_{jh1}(q^{1*}_{jh1})}{\partial q^1_{jh1}} + \frac{\partial \bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{jh1}} + \frac{\partial tr_{jh1}(q^{1*}_{jh1})}{\partial q^1_{jh1}} + \lambda_j^{2*} - \lambda_h^{4*}\right], \ \forall j,h,\\
F_{11}(X^*) &= \left[\frac{\partial c^1_{jh2}(q^{1*}_{jh2})}{\partial q^1_{jh2}} + \frac{\partial \bar{c}_h(Q^{3*},Q^{6*})}{\partial q^1_{jh2}} + \frac{\partial tr_{jh2}(q^{1*}_{jh1})}{\partial q^1_{jh2}} + \lambda_j^{2*} - \lambda_h^{4*}\right], \ \forall j,h,\\
F_{12}(X^*) &= \left[\frac{\partial c^1_{jm2}(q^{1*}_{jm2})}{\partial q^1_{jm2}} + \frac{\partial \bar{c}_m(Q^{4*},Q^{7*})}{\partial q^1_{jm2}} + \frac{\partial tr_{jm2}(q^{1*}_{jm2})}{\partial q^1_{jm2}} + \mu_j^{2*} - \mu_m^{5*}\right], \ \forall j,m,\\
F_{13}(X^*) &= \left[-\bar{\rho}^{SSN}_{k1} + \bar{\bar{c}}^1_{kn_11}(Q^{8*}) - \hat{\rho}_{n1}(Q^{8*},Q^{10*}) + \lambda_k^{3*}\right], \ \forall k,n_1,\\
F_{14}(X^*) &= \left[\bar{\bar{c}}^2_{n_2k2}(Q^{9*}) - \hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*}) + \mu_k^{3*}\right], \quad \forall k,n_2,\\
F_{15}(X^*) &= \left[\frac{\partial sub_h(Q_2^{10*},Q_2^{11*})}{\partial q^1_{hn_11}} - \bar{\rho}^{SSN}_{h1} + \bar{\bar{c}}^1_{hn_11}(Q^{10*}) - \hat{\rho}_{n1}(Q^{8*},Q^{10*}) + \lambda_h^{4*}\right], \ \forall h,n_1,\\
F_{16}(X^*) &= \left[\frac{\partial sub_h(Q_2^{10*},Q_2^{11*})}{\partial q^2_{hn_11}} - \bar{\rho}^{SSN}_{h1} + \bar{\bar{c}}^1_{hn_11}(Q^{10*}) - \hat{\rho}_{n1}(Q^{8*},Q^{10*}) + \lambda_h^{4*}\right], \ \forall h,n_1,\\
F_{17}(X^*) &= \left[\frac{\partial sub_h(Q_2^{10*},Q_2^{11*})}{\partial q^1_{hn_22}} + \bar{\bar{c}}^1_{hn_22}(Q^{11*}) - \hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*}) + \mu_h^{4*}\right], \ \forall h,n_2,\\
F_{18}(X^*) &= \left[\frac{\partial sub_h(Q_2^{10*},Q_2^{11*})}{\partial q^2_{hn_22}} + \bar{\bar{c}}^1_{hn_22}(Q^{11*}) - \hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*}) + \mu_h^{4*}\right], \ \forall h,n_2,\\
F_{19}(X^*) &= \left[\frac{\partial sub_m(Q_2^{12*})}{\partial q^1_{mn_22}} + \bar{\bar{c}}^l_{mn_22}(Q^{12*}) - \hat{\rho}_{n2}(Q^{9*},Q^{11*},Q^{12*}) + \mu_m^{5*}\right], \ \forall m,n_1,
\end{aligned}
$$

$$
\begin{aligned}
F_{20}(X^*) &= \left[\frac{\partial sub_m(Q_2^{12*})}{\partial q^2_{mn_22}} + \bar{\bar{c}}^l_{mn_22}(Q^{12*}) - \hat{\rho}_{n2}(Q^{9*}, Q^{11*}, Q^{12*}) + \mu_m^{5*}\right], \quad \forall m, n_2, \\
F_{21}(X^*) &= -\left[\sum_{j=1}^{J}\sum_{p=1}^{2} q^{1*}_{ijp} + \sum_{k=1}^{K}\sum_{p=1}^{2} q^{1*}_{ikp} + \sum_{h=1}^{H}\sum_{p=1}^{2} q^{1*}_{ihp} + \sum_{m=1}^{M} q^{1*}_{im2} - M_i\right], \quad \forall i, \\
F_{22}(X^*) &= -\left[\sum_{k=1}^{K} q^{1*}_{jk1} + \sum_{h=1}^{H} q^{1*}_{jh1} - \sum_{i=1}^{I} q^{1*}_{ij1}\right], \quad \forall j, \\
F_{23}(X^*) &= -\left[\sum_{k=1}^{K} q^{1*}_{jk2} + \sum_{h=1}^{H} q^{1*}_{jh2} + \sum_{m=1}^{M} q^{1*}_{jm2} - \sum_{i=1}^{I} q^{1*}_{ij2}\right], \quad \forall j, \\
F_{24}(X^*) &= -\left[\sum_{n_1=1}^{N_1} q^{1*}_{kn_11} - \sum_{i=1}^{I} q^{1*}_{ik1} - \sum_{j=1}^{J} q^{1*}_{jk1}\right], \quad \forall k, \\
F_{25}(X^*) &= -\left[\sum_{n_2=1}^{N_2} q^{1*}_{kn_22} - \sum_{i=1}^{I} q^{1*}_{ik2} - \sum_{j=1}^{J} q^{1*}_{jk2}\right], \quad \forall k, \\
F_{26}(X^*) &= -\left[\sum_{n_1=1}^{N_1}\sum_{l=1}^{2} q^{l*}_{hn_11} - \sum_{i=1}^{I} q^{1*}_{ih1} - \sum_{j=1}^{J} q^{1*}_{jh1}\right], \quad \forall h, \\
F_{27}(X^*) &= -\left[\sum_{n_2=1}^{N_2}\sum_{l=1}^{2} q^{l*}_{hn_22} - \sum_{i=1}^{I} q^{1*}_{ih2} - \sum_{j=1}^{J} q^{1*}_{jh2}\right], \quad \forall h, \\
F_{28}(X^*) &= -\left[\sum_{n_2=1}^{N_2}\sum_{l=1}^{2} q^{l*}_{mn_22} - \sum_{i=1}^{I} q^{1*}_{im2} - \sum_{j=1}^{J} q^{1*}_{jm2}\right], \quad \forall m, \\
K &= \mathbb{R}_+^{IJP+IKP+IHP+IM+JPK+JHP+JM+KN_1+KN_2+HN_1L+HN_2L+MN_2L+I+2J+2K+2H+M}.
\end{aligned}
$$

Under the imposed assumptions on the costs and demand price functions, the operator of the variational inequality (3.35) is continuous. Moreover, if a coercivity condition on vector function F is assumed, then the existence of a solution to (3.35) is guaranteed by the classical theory of variational inequalities (see [23]).

4. Numerical Applications

In this section, we present an illustrative example designed to evaluate the validity of our model. In future work, more accurate simulations could be conducted using real data obtained from official databases. However, obtaining information for certain cost functions remains a challenge. The model demonstrates a high degree of flexibility and can be extended to more complex problems involving a larger number of variables. For the solution of the numerical example, we applied the classical extragradient method, see [24], implemented in Python.

In this example, we consider a problem consisting of one pharmaceutical company i, one distributor j, one traditional pharmacy k, one mixed pharmacy h, one parapharmacy m, one consumer of type n_1 and one of type n_2. The network topology for the example is depicted in Figure 3.

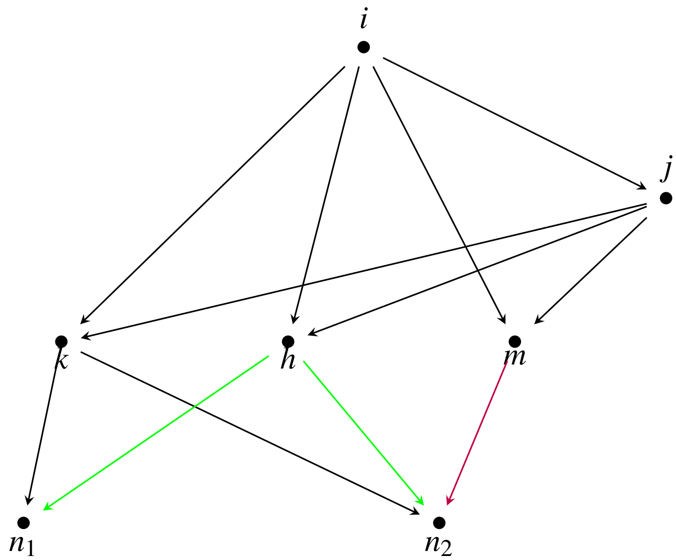

FIGURE 3. Topology of the example

The problem results in 28 variables: 20 variables represent the flows and 8 the Lagrange multipliers. We assume that all the flow balance conditions are satisfied as equality constraints. The input data for the example are as follows.

The production cost of company $i = 1$ is:

$$\begin{aligned} p_1(Q^1, Q^2, Q^3, Q^4) &= 2.5(x^1_{ij1} + x^1_{ij2} + x^1_{ik1} + x^1_{ik2} + x^1_{ih1} + x^1_{ih2} + x^1_{im2})^2 \\ &\quad +2(x^1_{ij1} + x^1_{ij2} + x^1_{ik1} + x^1_{ik2} + x^1_{ih1} + x^1_{ih2} + x^1_{im2}). \end{aligned}$$

The transportation costs of company i are:

$$\begin{aligned} c^1_{ij1}(x^1_{ij1}) &= 0.3\left(x^1_{ij1}\right)^2 + 3x^1_{ij1}, \quad c^1_{ij2}(x^1_{ij2}) = 0.2\left(x^1_{ij2}\right)^2 + 2x^1_{ij2}, \\ c^1_{ik1}(x^1_{ik1}) &= 0.4\left(x^1_{ik1}\right)^2 + 3x^1_{ik1}, \quad c^1_{ik2}(x^1_{ik2}) = 0.3\left(x^1_{ik2}\right)^2 + 2x^1_{ik2}, \\ c^1_{ih1}(x^1_{ih1}) &= 0.3\left(x^1_{ih1}\right)^2 + 3x^1_{ih1}, \quad c^1_{ih2}(x^1_{ih2}) = 0.2\left(x^1_{ih2}\right)^2 + 2x^1_{ih2}, \\ c^1_{im2}(x^1_{im2}) &= 0.3\left(x^1_{im2}\right)^2 + 3x^1_{im2}. \end{aligned}$$

The operating cost of the distributor j is:

$$c_j(Q^1) = 0.1(x^1_{ij1} + x^1_{ij2})^2.$$

The transportation costs of the distributor j are:

$$\begin{aligned} c^1_{jk1}(x^1_{jk1}) &= 0.4\left(x^1_{jk1}\right)^2 + 3x^1_{jk1}, \quad c^1_{jk2}(x^1_{jk2}) = 0.3\left(x^1_{jk2}\right)^2 + 2x^1_{jk2}, \\ c^1_{jh1}(x^1_{jh1}) &= 0.3\left(x^1_{jh1}\right)^2 + 3x^1_{jh1}, \quad c^1_{jh2}(x^1_{jh2}) = 0.2\cdot\left(x^1_{jh2}\right)^2 + 4x^1_{jh2}, \\ c^1_{jm2}(x^1_{jm2}) &= 0.3\left(x^1_{jm2}\right)^2 + 5x^1_{jm2}. \end{aligned}$$

The transaction costs of the distributor j are:

$$\hat{c}^1_{ij1}(x^1_{ij1}) = 0.2\left(x^1_{ij1}\right)^2 + 3x^1_{ij1}, \quad \hat{c}^1_{ij2}(x^1_{ij2}) = 0.2\left(x^1_{ij2}\right)^2 + 2x^1_{ij2}.$$

The handling cost of the pharmacy k is:

$$\bar{c}_k(Q^2, Q^5) = 0.3\left(x^1_{ik1} + x^1_{ik2} + x^1_{jk1} + x^1_{jk2}\right)^2.$$

The transaction costs of the pharmacy k are:

$$\begin{aligned} tr^1_{ik1}(x^1_{ik1}) &= 0.2\left(x^1_{ik1}\right)^2 + x^1_{ik1}, \quad tr^1_{ik2}(x^1_{ik2}) = 0.1\left(x^1_{ik2}\right)^2 + 2x^1_{ik2}, \\ tr^1_{jk1}(x^1_{jk1}) &= 0.2\left(x^1_{jk1}\right)^2 + 2x^1_{jk1}, \quad tr^1_{jk2}(x^1_{jk2}) = 0.1\left(x^1_{jk2}\right)^2 + 2x^1_{jk2}. \end{aligned}$$

The handling cost of the pharmacy h is:

$$\bar{c}_h(Q^3, Q^6) = 0.3\left(x^1_{ih1} + x^1_{ih2} + x^1_{jh1} + x^1_{jh2}\right)^2.$$

The transaction costs of the pharmacy h are:

$$\begin{aligned} tr^1_{ih1}(x^1_{ih1}) &= 0.2\left(x^1_{ih1}\right)^2 + 2x^1_{ih1}, \quad tr^1_{ih2}(x^1_{ih2}) = 0.1\left(x^1_{ih2}\right)^2 + 2x^1_{ih2}, \\ tr^1_{jh1}(x^1_{jh1}) &= 0.2\left(x^1_{jh1}\right)^2 + 2x^1_{jh1}, \quad tr^1_{jh2}(x^1_{jh2}) = 0.1\left(x^1_{jh2}\right)^2 + 2x^1_{jh2}. \end{aligned}$$

The subscription cost of pharmacy h is:

$$sub_h(Q^{10}_2, Q^{11}_2) = 0.3\left(x^{11}_{hn1} + x^{12}_{hn1} + x^{21}_{hn2} + x^{22}_{hn2}\right)^2.$$

The handling cost of the parapharmacy m is:

$$\bar{c}_m(Q^4, Q^7) = 0.1\left(x^1_{im2} + x^1_{jm2}\right)^2.$$

The transaction costs of the parapharmacy m are:

$$tr^1_{im2} = 0.2\left(x^1_{im2}\right)^2 + 2x^1_{im2}, \quad tr^1_{jm2}(x^1_{jm2}) = 0.2\left(x^1_{jm2}\right)^2 + 4x^1_{jm2}.$$

The subscription cost of the parapharmacy m is:

$$sub_m(Q^{12}_2) = 0.3 \cdot \left(x^{21}_{mn2} + x^{22}_{mn2}\right)^2.$$

The transaction costs of consumers are:

$$\begin{aligned} tr^1_{kn_11}(x^1_{kn_11}) &= 0.1\left(x^1_{kn_11}\right)^2, \quad tr^1_{kn_22}(x^1_{kn_22}) = 0.1\left(x^1_{kn_22}\right)^2, \\ tr^1_{hn_11}(x^1_{hn_11}) &= 0.1\left(x^{11}_{hn_1}\right)^2, \quad tr^2_{hn_11}(x^1_{hn_11}) = 0.2\left(x^2_{hn_11}\right)^2, \\ tr^1_{hn_22}(x^1_{hn_22}) &= 0.1\left(x^1_{hn_22}\right)^2, \quad tr^2_{hn_22}(x^2_{hn_22}) = 0.2\left(x^2_{hn_22}\right)^2, \\ tr^1_{mn_22}(x^1_{mn_22}) &= 0.1\left(x^1_{mn_22}\right)^2, \quad tr^2_{mn_22}(x^2_{mn_22}) = 0.2\left(x^2_{mn_22}\right)^2. \end{aligned}$$

The demand price functions are:

$$\hat{\rho}_{n_1}(Q^8, Q^{10}) = 400 - \left(x^1_{hn_11} + x^2_{hn_11} + x^1_{kn_11}\right) - T_1,$$
$$\hat{\rho}_{n_1}(Q^9, Q^{11}) = 500 - \left(x^1_{hn_22} + x^2_{hn_22} + x^1_{kn_22} + x^1_{mn_22} + x^2_{mn_22}\right) - T_2,$$
$$T_1 = \frac{(0.1x^2_{hn_11} + 5)x^{12}_{hn_1}}{d_{n_1}}, \quad T_2 = \frac{(0.3x^2_{hn_22} + 8)x^2_{hn_22}}{d_{n_2}},$$
$$d_{n_1} = x^1_{kn_11} + x^1_{hn_11} + x^2_{hn_11}, \quad d_{n_2} = x^1_{kn_22} + x^1_{hn_22} + x^2_{hn_22} + x^1_{mn_22} + x^2_{mn_22}.$$

Finally, $M = 1000$, $\bar{\rho}^{SSN}_{k_1} = \bar{\rho}^{SSN}_{h_1} = 8$.
The Korpelevich method converged to the solutions reported in Table 3 in 353 iterations.

Variables	**Company** i	**Distributor** j	**Pharmacy** k	**Mixed Pharmacy** h	**Parapharmacy** m
Flows	$x^1_{ij1} = 76.1962$ $x^1_{ij2} = 115.2959$ $x^1_{ik1} = 150.9385$ $x^1_{ik2} = 178.5694$ $x^1_{ih1} = 122.2685$ $x^1_{ih2} = 156.3554$ $x^1_{im2} = 200.3761$	$x^1_{jk1} = 57.5007$ $x^1_{jk2} = 36.7848$ $x^1_{jh1} = 18.6955$ $x^1_{jh2} = 0$ $x^1_{jm2} = 78.5112$	$x^1_{kn_11} = 208.4392$ $x^1_{kn_22} = 215.3542$	$x^1_{hn_11} = 78.9762$ $x^2_{hn_11} = 61.9879$ $x^1_{hn_22} = 85.4103$ $x^2_{hn_22} = 70.9451$	$x^1_{mn_22} = 153.9224$ $x^2_{mn_22} = 124.9649$
Multipliers	$\lambda^1_i = 3.0593$	$\lambda^2_j = 0.6975$ $\mu^2_j = 2.2803$	$\lambda^3_k = 1.4607$ $\mu^3_k = 3.9259$	$\lambda^4_h = 1.8318$ $\mu^4_h = 0.9984$	$\mu^5_m = 2.5712$

TABLE 3. Equilibrium solutions

Based on the bar charts in Figure 4, the flow variables indicate that the company plays a central role in supplying drugs across the network, with significant volumes directed primarily to pharmacies and parapharmacies. Distributors handle smaller quantities, suggesting a more limited logistical role in a small, well-served area such as the one represented in this example. Parapharmacies receive and redistribute moderate volumes, highlighting their function as intermediaries with limited customer targets. This feature reflects their strategic positioning within the pharmaceutical distribution network, and define their market role. Their success depends on leveraging the advantages of specialization, personalized service (in-store or home delivery), and the focus on their specific customer segments within the broader distribution system.

The total sales data in Figure 5 shows pharmacy k leading with 42.4% of total sales, followed by mixed pharmacy h at 29.7% and parapharmacy m at 27.9%. This distribution suggests that traditional pharmacy models maintain competitive advantages in generating sales volume. Traditional pharmacy have clear advantages in building trust through face-to-face interactions. This allows pharmacists to develop deeper relationships with patients, understanding their health patterns, observing physical symptoms, and providing personalized counseling that goes beyond basic medication instructions. The physical location also provides immediate accessibility for urgent medication needs.

Panel (a) in Figure 6 shows that pharmacy k significantly outperforms mixed pharmacy h in terms of sales of prescription drugs, and pharmacy k achieves 59.7% compared to mixed pharmacy h's 40.3%. This suggests that traditional pharmacy models may have advantages in establishing a physical presence or meeting certain operational criteria. Panel (b) reveals a different pattern in the market share for non-prescription drugs. Parapharmacy m dominates with 42.9%, while pharmacy k holds 33.1% and mixed

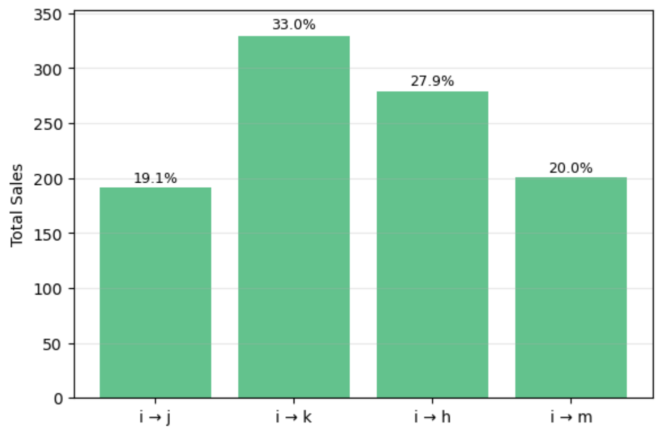

FIGURE 4. Sale distribution from the pharmaceutical company

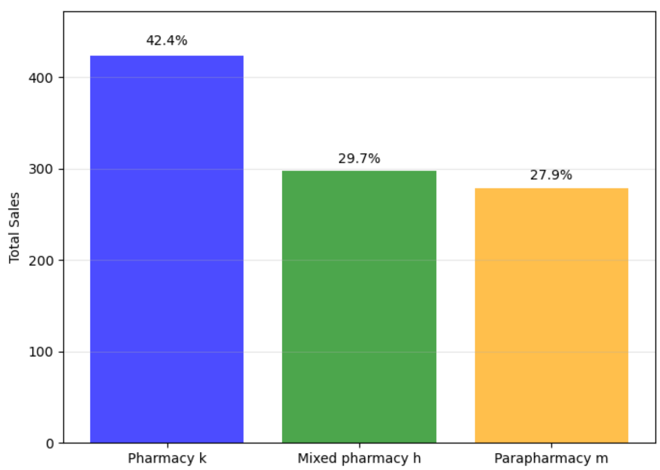

FIGURE 5. Sale distribution among pharmacies and parapharmacies

pharmacy h accounts for 24%. This indicates that parapharmacy models, despite potentially having smaller customer targets, may achieve better sales performance.

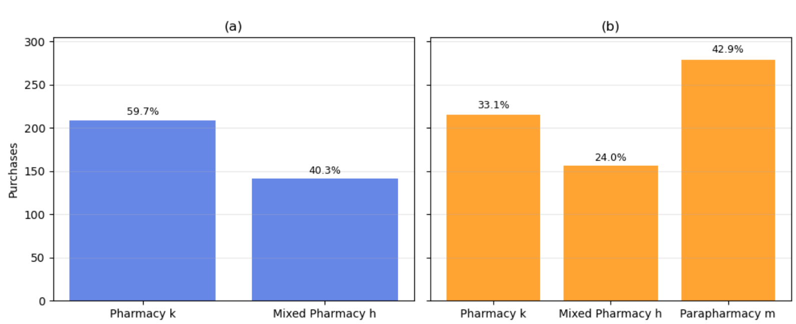

FIGURE 6. Purchase distribution for the consumer n_1 and n_2

Figure 7 illustrates the consumer purchasing behavior patterns across two distinct modes. In panel (a), traditional in-store shopping of prescription drugs demonstrates dominance at 82.3%, with online purchases accounting for only 17.7% of total purchases. Conversely, panel (b) reveals a more balanced distribution for non-prescription drugs, where in-store shopping maintains majority preference at 69.9%, while online purchasing represents a higher proportion at 30.1%.

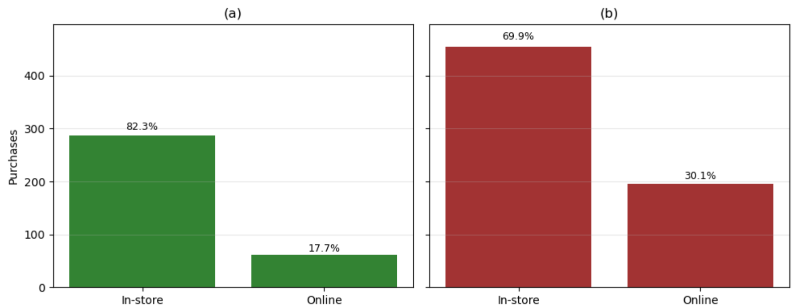

FIGURE 7. Distribution across the purchasing channels

We now set $\rho_{ij1} = 10$, $\rho_{ij2} = 12$, $\rho_{ik1} = \rho_{ik2} = 11$, $\rho_{ik2} = \rho_{ih2} = \rho_{im2} = 13$, and in Figure 8 illustrate the percentage sensitivity of the profit and total cost of company i to variations in a parameter M, ranging from -30% to $+30\%$. Total cost and profit show perfectly inverse relationships to M variations, with identical absolute sensitivity values but opposite signs. This suggests that the parameter M directly impacts the cost-profit balance. The symmetrical nature of the sensitivity suggests that both positive and negative deviations from the base value carry equal but opposite risks, making accurate parameter estimation and control essential for maintaining desired cost-profit ratios. Such information is crucial for strategic planning, helping to identify where additional investment or resource allocation could yield the greatest improvement.

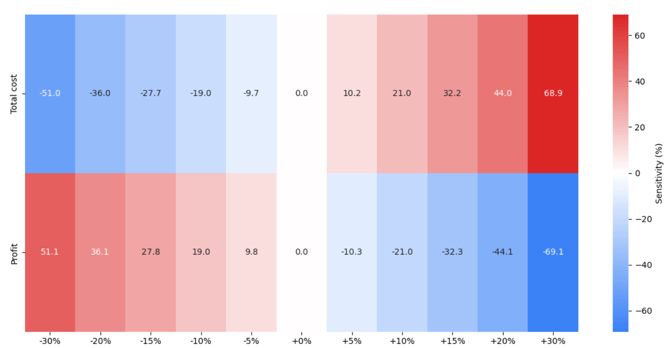

FIGURE 8. Sensitivity analysis under variation of parameter M

The data reveals a complex competitive network where different pharmacy models excel in different market segments. Traditional pharmacies appear to maintain advantages in total sales volume, while parapharmacies demonstrate efficiency in their specific performance area. Mixed pharmacy models occupy an intermediate position, suggesting they may represent a balanced approach but without dominant advantages in any single area. These findings have important implications for pharmacy business strategy and market positioning, suggesting to align the strategic focus with the operational model's strengths to optimize market performance.

5. Conclusion and Future Research

The proposed framework offers a rigorous and flexible modeling tool for supporting the strategic reorganization of pharmacy networks, with particular attention to issues of territorial coverage, user accessibility, and service equity. By adopting a formulation based on variational inequalities, the model successfully incorporates user behavior in a decentralized context, taking into account the role of congestion, distance-related costs, and heterogeneous attractiveness of service points. This represents a significant step beyond traditional discrete location models, as it allows for a more realistic depiction of interactions between supply structures and demand patterns. Our model incorporates the time factor that plays a fundamental role in consumer behavior because it directly influences how consumers perceive the value and usefulness of products, particularly within pharmaceutical distribution networks. When making purchasing decisions, patients consider more than just cost and product availability; they are also highly sensitive to delivery delays and waiting periods. This temporal sensitivity becomes especially critical in healthcare markets, where medication needs are often urgent and delays can have serious consequences for patient health outcomes.

The illustrative numerical example presented in this work, while not based on empirical data, has proven useful in highlighting the internal consistency of the model and its computational tractability in solving nontrivial allocation problems. The results underscore the model's ability to support the evaluation of alternative planning scenarios and to inform decision-making processes where multiple objectives, such as efficiency, equity, and territorial balance, must be jointly considered.

Future research directions include the integration of real-world data from regional healthcare systems to validate the model empirically and refine its calibration, and could build on the variational formulations proposed in [25]. Furthermore, future directions may explore the stability of coalitions in healthcare logistics systems (see [7]). Finally, extending the framework to multi-period settings or to dynamic environments, where demand evolves over time, would increase its relevance in operational planning. Stochastic components related to demand uncertainty, travel times, or service availability could also be incorporated to enhance robustness.

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