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ON MATHEMATICAL PROGRAMS WITH SIGN CONSTRAINTS AND THEIR APPLICATIONS IN REGULARIZATION THEORY

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Abstract. We study a novel class of nonsmooth optimization problems, called mathematical programs with sign constraints (MPSiC). For MPSiC, the critical point theory is developed. It includes the introduction of topologically relevant T-stationary points along with their T-index. Two typical results within the scope of Morse theory are demonstrated. Outside the set of T-stationary points, the MPSiC lower level sets remain homotopy-equivalent. If surpassing a nondegenerate T-stationary point, a cell of dimension equal to its T-index has to be attached for this purpose. Further, we apply our findings on the MPSiC class to the sign-type regularization of mathematical programs with complementarity constraints (MPCC) from [A. Kadrani, J.P. Dussault, A. Benchakroun, A new regularization scheme for mathematical programs with complementarity constraints, SIAM J. Optim. 20 (2009), 78–103]. By doing so, the convergence analysis for the sign-type regularization of MPCC is refined. Among the other results, an index shift for limiting C-stationary points of MPCC can not be avoided in the generic sense. In particular, a sequence of saddle points of the sign-type regularization might converge to a minimizer of MPCC. This phenomenon turns out to be stable with respect to C^2-perturbations of the MPCC defining functions. Moreover, due to the stably occurring bifurcations, the global structure of the sign-type regularization becomes much more involved than that of the underlying MPCC.

Keywords. Complementarity constraints; Genericity; Morse theory; Nondegenerate T-stationarity; Sign constraints; Regularization.

1. INTRODUCTION

We introduce and study a novel class of nonsmooth optimization problems, called mathematical programs with sign constraints:

$$\text{MPSiC}: \quad \min_x f(x) \quad \text{s.t.} \quad x \in \mathscr{S}$$

with

$$\mathscr{S} = \left\{ x \in \mathbb{R}^n \;\middle|\; g_i(x) \geq 0, i \in I, H_m(x) \cdot G_m(x) \leq 0, m = 1, \ldots, k \right\},$$

where $f \in C^2(\mathbb{R}^n, \mathbb{R})$, $g \in C^2(\mathbb{R}^n, \mathbb{R}^{|I|})$, and $H, G \in C^2(\mathbb{R}^n, \mathbb{R}^k)$. Our motivation of considering MPSiCs is their use in regularization theory for MPCC. In [6], it has been suggested to substitute

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the basic two-dimensional complementarity constraint

$$x_1 \cdot x_2 = 0, \quad x_1 \geq 0, x_2 \geq 0$$

by the sign constraint

$$x_1 + t \geq 0, x_2 + t \geq 0, \quad (x_1 - t) \cdot (x_2 - t) \leq 0,$$

where $t > 0$ is taken sufficiently small. We refer to Figure 1 for the illustration of the corresponding basic feasible sets.

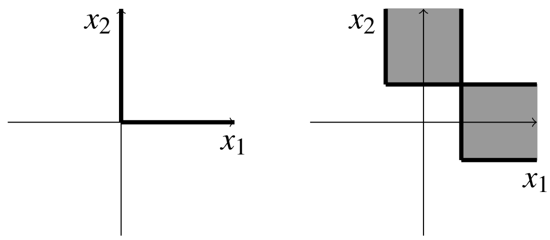

FIGURE 1. Complementarity constraint with its sign-type regularization

The contributions of this paper are twofold. First, we establish the critical point theory for the MPSiC class. This is to identify the topologically relevant notion of T-stationarity for MPSiC. The latter is meant to describe the changes of its lower level sets up to the homotopy-equivalence. By doing so, the local properties of T-stationary points and the global structure of MPSiC can be adequately captured in terms of the so-called Morse theory, cf. [3]. Crucial for the topological analysis performed is that a nondegenerate T-stationary point is uniquely characterized by its T-index. By means of the latter, minimizers, maximizers, and all kinds of saddle points can be successively classified in algebraic terms. We note that T-index consists of the quadratic part, which counts negative eigenvalues of the Lagrangian's Hessian restricted to an appropriate tangent space, and of the biactive part, which counts biactive sign constraints. For comparison, we refer to [1, 4, 5, 7, 15], where the critical point theory has been established for mathematical programs with complementarity, vanishing, orthogonality-type, switching, and disjunctive constraints, respectively. For example, we recall from [5] that for MPCC topologically relevant are C-stationary points along with their C-index. It also consists of the quadratic part, which again counts negative eigenvalues of the Lagrangian's Hessian restricted to an appropriate tangent space, and of the biactive part, which now counts pairs of negative biactive multipliers.

Second, we apply the established critical point theory for MPSiC to the sign-type regularization of MPCC from [6]. There, the sign-type regularization was dealt with as an instance of nonlinear programming (NLP). This typically leads to the violation of the usual linear independence constraint qualification (LICQ) at its feasible points with biactive sign constraints. Another issue is that the Karush-Kuhn-Tucker points of the sign-type regularization exclude its saddle points with nonzero biactive indices from consideration. Both obstacles make it impossible to study the global structure of the sign-type regularization within the NLP framework, and finally to understand its convergence properties to the full extent. Our paper tries to close this gap by dealing with the sign-type regularization as an MPSiC instance. In particular, we show that the MPSiC-tailored LICQ is inherited for the sign-type regularization under the MPCC-tailored LICQ. Moreover, T-stationary points of the sign-type regularization converge to a C-stationary point of MPCC. For this to hold, just MPCC-LICQ at the limit has to be assumed. Note that a similar result in [6] concerning the sequence of Karush-Kuhn-Tucker points of the sign-type regularization and its limiting S-stationary point of MPCC, additionally needs that the biactive multipliers of the latter do not vanish. Further, we trace indices of nondegenerate C-stationary points of MPCC and T-stationary points of its sign-type regularization while the latter approximate the former. As for our main findings, the C-index of the limiting C-stationary point of MPCC may shift if compared to the T-index of the approximating T-stationary points of its sign-type regularization. This change of the topological type may be caused by two different effects:

- The quadratic part of the C-index may shift by the number of vanishing multipliers taken with respect to the non-biactive complementarity constraints. The importance of non-biactive multipliers for index stability has been already observed in the previous paper [8], where we refined the convergence analysis for the Scholtes regularization of MPCC. There, it has been additionally shown that all non-biactive multipliers do not vanish at the nondegenerate C-stationary points of a generic MPCC.
- The biactive part of the C-index may shift by the number of pairs of positive biactive multipliers. In contrast to the quadratic, the biactive index shift turns out to be stable under any sufficiently small C^2-perturbations of the MPCC defining functions. In other words, it cannot be generically avoided that e.g. a sequence of saddle points of the sign-type regularization converges to a minimizer of MPCC. It comes even worse, since a minimizer of MPCC may then stably bifurcate into multiple T-stationary points of the corresponding sign-type regularization. Here, we conclude that the global structure of the sign-type regularization can be much more involved than that of the underlying MPCC. This concerns in particular the growing number of its T-stationary points.

The organization of our paper is as follows. Section 2 is devoted to the critical point theory for MPSiC. Preliminaries on the MPCC class can be found in Section 3. In Section 4, we present our results on the convergence behavior of the sign-type regularization.

Our notation is standard. We denote the n-dimensional Euclidean space by $\mathbb{R}^n$. Given a twice continuously differentiable function $f:\mathbb{R}^n \to \mathbb{R}$, we denote its differential by Df as a row vector and its Hessian matrix by $D^2 f$.

2. Mathematical Programs with Sign Constraints

Let us start with the presentation of the critical point theory for MPSiC. Here, we omit proofs of the corresponding results, since they are meanwhile standard within the framework of non-smooth optimization; see [14]. Readers, who are interested in the technicalities behind the proofs, we refer e.g. to [7], where the topological approach has been exemplarily elaborated for mathematical programs with orthogonality type constraints (MPOC). It is not hard, but tedious, to adjust the proofs from there for the case of MPSiC.

First, we associate with an MPSiC feasible point $x \in \mathscr{S}$ the following index sets:

$$I_g(x) = \{ i \in I \,|\, g_i(x) = 0 \},$$
$$b_{H,\neq}(x) = \{ m \,|\, H_m(x) = 0, G_m(x) \neq 0 \}, \quad b_{\neq,G}(x) = \{ m \,|\, H_m(x) \neq 0, G_m(x) = 0 \},$$
$$b_{H,G}(x) = \{ m \,|\, H_m(x) = 0, G_m(x) = 0 \}.$$

Finally, we need their index subsets:

$$b_{H,>}(x) = \{m \mid H_m(x) = 0, G_m(x) > 0\}, \quad b_{H,<}(x) = \{m \mid H_m(x) = 0, G_m(x) < 0\},$$
$$b_{>,G}(x) = \{m \mid H_m(x) > 0, G_m(x) = 0\}, \quad b_{<,G}(x) = \{m \mid H_m(x) < 0, G_m(x) = 0\}.$$

Note that $I_g(x)$ subsumes indices of the active inequality constraints, and $b_{H,G}(x)$ is the index set of biactive sign constraints at x. The index sets $b_{H,\neq}(x) = b_{H,>}(x) \cup b_{H,<}(x)$ and $b_{\neq,G}(x) = b_{>,G}(x) \cup b_{<,G}(x)$ stand for the other cases of sign constraints feasibility. Locally, they correspond to the inequality constraints.

Further, it is convenient to assume the following constraint qualification to be fulfilled on the MPSiC feasible set. This is a direct generalization of the standard linear independence constraint qualification, being well-known from the theory of nonlinear programming (NLP).

Definition 2.1 (MPSiC-LICQ)**.** The MPSiC-tailored linear independence constraint qualification (MPSiC-LICQ) is said to be fulfilled at $x \in \mathscr{S}$ if the following vectors are linearly independent:

$$Dg_i(x), i \in I_g(x), \quad DH_m(x), m \in b_{H,\neq}(x) \cup b_{H,G}(x), \quad DG_m(x), m \in b_{\neq,G}(x) \cup b_{H,G}(x).$$

Next result suggests that MPSiC-LICQ is not restrictive at all. To put it into mathematical language, we consider the strong C^2-topology on the space $C^2(\mathbb{R}^n, \mathbb{R})$ of twice differentiable functions. Denoted by C^2_s, it is sometimes referred to as the Whitney-topology; see [2]. The basis of C^2_s-topology is given by C^2-perturbations of a twice differentiable function which are controlled by means of a continuous positive function.

Proposition 2.1 (Genericity of MPSiC-LICQ)**.** *Let $\mathscr{F} \subset C^2\left(\mathbb{R}^n, \mathbb{R}^{1+|J|+2k}\right)$ denote the subset of MPSiC defining functions for which MPSiC-LICQ holds at all feasible points. Then, $\mathscr{F}$ is C^2_s-open and -dense.*

Now, we are ready to overview the stationarity notions for MPSiC.

Definition 2.2 (Stationarity for MPSiC)**.** A feasible point $x \in \mathscr{S}$ is called W-stationary for MPSiC if there exists multipliers

$$\mu_i, i \in I_g(x), \quad \eta_{H,m}, m \in b_{H,\neq}(x), \quad \eta_{G,m}, m \in b_{\neq,G}(x), \quad \zeta_{H,m}, \zeta_{G,m}, m \in b_{H,G}(x),$$

such that the following conditions hold:

$$\begin{aligned} Df(x) \;=\; & \sum_{i \in I_g(x)} \mu_i Dg_i(x) + \sum_{m \in b_{H,\neq}(x)} \eta_{H,m} DH_m(x) + \sum_{m \in b_{\neq,G}(x)} \eta_{G,m} DG_m(x) \\ & + \sum_{m \in b_{H,G}(x)} \left(\zeta_{H,m} DH_m(x) + \zeta_{G,m} DG_m(x)\right), \end{aligned}$$

$$\mu_i \geq 0 \text{ for all } i \in I_g(x),$$
$$\eta_{H,m} \leq 0 \text{ for all } m \in b_{H,>}(x), \quad \eta_{H,m} \geq 0 \text{ for all } m \in b_{H,<}(x),$$
$$\eta_{G,m} \leq 0 \text{ for all } m \in b_{>,G}(x), \quad \eta_{G,m} \geq 0 \text{ for all } m \in b_{<,G}(x).$$

Moreover, the W-stationary point x is called:

- T-stationary if $\zeta_{H,m} \cdot \zeta_{G,m} \geq 0$ for all $m \in b_{H,G}(x)$;
- M-stationary if $\zeta_{H,m} \cdot \zeta_{G,m} = 0$ for all $m \in b_{H,G}(x)$;
- S-stationary if $\zeta_{H,m} = 0, \zeta_{G,m} = 0$ for all $m \in b_{H,G}(x)$.

We point out that the notions of S-, M-, and W-stationarity are geometrically motivated. They can be equivalently stated in terms of the Fréchet, Mordukhovich, or Clarke normal cones to the MPSiC feasible set, cf. [9, 11], respectively:

$$0 \in D^T f(x) + \widehat{N}_{\mathscr{S}}(x), \quad 0 \in D^T f(x) + N_{\mathscr{S}}(x), \quad 0 \in D^T f(x) + \overline{N}_{\mathscr{S}}(x).$$

Under MPSiC-LICQ, the restrictions on biactive multipliers for S-, M-, and W-stationary points in Definition 2.2 correspond to the negatives of the Fréchet, Mordukhovich (limiting), or Clarke normal cones to the prototypical MPSiC feasible set

$$\mathbb{K} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \cdot x_2 \leq 0 \right\}.$$

We refer to Figure 2 for comparison of these concepts of stationarity. Obviously, S-, M-, T-, and W-stationarity gets less and less restrictive. From the variational analysis we know that S-stationary points generally include local minimizers; see, e.g., [11]. This is also the case for MPSiC.

Theorem 2.1 (Necessary optimality condition for MPSiC)**.** *Let $x \in \mathscr{S}$ be a local minimizer of MPSiC satisfying MPSiC-LICQ. Then x is an S-stationary point.*

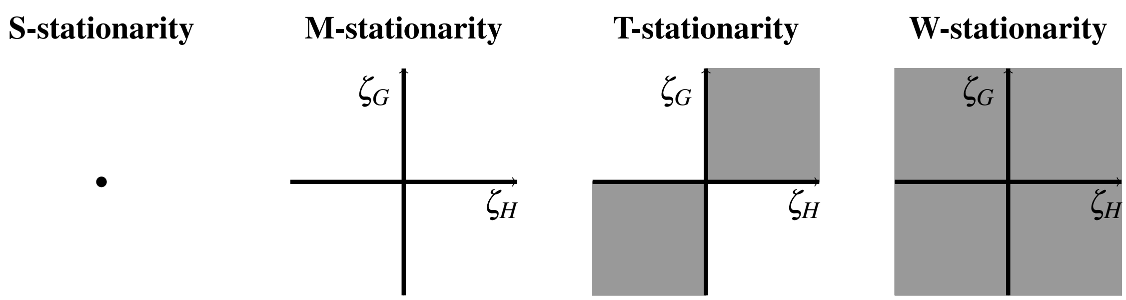

FIGURE 2. Biactive multipliers with S-, M-, T-, and W-stationarity for MPSiC

The aim of T-stationarity is to also incorporate saddle points into the study of MPSiC. For elaborating on this, given a T-stationary point $x \in \mathscr{S}$ with the multipliers (μ, η, ζ), we define the Lagrange function

$$\begin{aligned} L(\cdot) \;\; = \;\; & f(\cdot) - \sum_{i \in I_g(x)} \mu_i g_i(\cdot) - \sum_{m \in b_{H,\neq}(x)} \eta_{H,m} H_m(\cdot) - \sum_{m \in b_{\neq,G}(x)} \eta_{G,m} G_m(\cdot) \\ & - \sum_{m \in b_{H,G}(x)} \left(\zeta_{H,m} H_m(\cdot) + \zeta_{G,m} G_m(\cdot) \right). \end{aligned}$$

The corresponding tangent space is set as

$$\mathscr{T}_x = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \begin{array}{l} DH_m(x)\,\xi = 0, m \in b_{H,\neq}(x), DG_m(x)\,\xi = 0, m \in b_{\neq,G}(x), \\ Dg_i(x)\,\xi = 0, i \in I_g(x), DH_m(x)\,\xi = 0, DG_m(x)\,\xi = 0, m \in b_{H,G}(x) \end{array} \right\}.$$

Now, let us turn our attention to the so-called nondegenerate T-stationary points. Nondegeneracy is crucial for excluding singularities from our considerations and for being able to explicitely describe the local structure of MPSiC in the vicinity of a T-stationary point.

Definition 2.3 (Nondegeneracy for MPSiC)**.** A T-stationary point $x \in \mathscr{S}$ of MPSiC with multipliers (μ, η, ζ) is called nondegenerate if the following conditions are satisfied:

NDSi1: MPSiC-LICQ holds at x,
NDSi2: the strict complementarity (SC) holds for active inequality constraints, i. e. $\mu_i \neq 0$ for all $i \in I_g(x)$, $\eta_{H,m} \neq 0$ for all $m \in b_{H,\neq}(x)$ and $\eta_{G,m} \neq 0$ for all $m \in b_{\neq,G}(x)$,
NDSi3: the multipliers corresponding to biactive sign constraints do not vanish, i. e. $\zeta_{H,m} \neq 0$ and $\zeta_{G,m} \neq 0$ for all $m \in b_{H,G}(x)$,
NDSi4: the matrix $D^2L(x)$ restricted to $\mathscr{T}_x$ is nonsingular.

It turns out that for a generic MPSiC all T-stationary points are nondegenerate.

Proposition 2.2 (Genericity of nondegeneracy for MPSiC)**.** *Let $\mathscr{F} \subset C^2\left(\mathbb{R}^n, \mathbb{R}^{1+|J|+2k}\right)$ be the subset of MPSiC defining functions for which each T-stationary point is nondegenerate. Then, $\mathscr{F}$ is C^2_s-open and -dense.*

As the main invariant of a nondegenerate T-stationary point we introduce further its T-index. The T-index unanimously defines the topological type of a T-stationary point. We may thus distinguish local minimizers and different kinds of saddle points just by comparing their T-indices.

Definition 2.4 (T-index for MPSiC)**.** Let $x \in \mathscr{S}$ be a nondegenerate T-stationary point of MPSiC with the unique multipliers (μ, η, ζ). The number of negative eigenvalues of the matrix $D^2L(x) \restriction_{\mathscr{T}_x}$ is called its quadratic index (MPSiC-*QI*). The number of biactive sign constraints equals $\left|b_{H,G}(x)\right|$ and is called the biactive index (MPSiC-*BI*) of x. We define the T-index (MPSiC-*TI*) as the sum of both, i. e. MPSiC-$TI =$ MPSiC-$QI +$ MPSiC-BI.

Considering a nondegenerate local minimizer $x \in \mathscr{S}$ of MPSiC with multipliers (μ, η, ζ), we know that it fulfills MPSiC-LICQ, SC, and the second-order sufficient condition (SOSC), i. e. the matrix $D^2L(x) \restriction_{\mathscr{T}_x}$ is positive definite. Note that these properties are standard and have been studied in-depth already for the class of NLP. What is new here, NDSi3 requires the biactive sign constraints to be absent at a nondegenerate minimizer of MPSiC, i. e. $b_{H,G}(x) = \emptyset$. Otherwise, we would be dealing with a saddle point instead. Thus, the following result can be obtained.

Corollary 2.1 (Sufficient optimality condition)**.** *A nondegenerate T-stationary point of MPSiC is a local minimizer if and only if its T-index vanishes.*

In order to justify the introduction of T-stationary points, we study the topological properties of MPSiC lower level sets

$$\mathscr{S}_a = \{x \in N \mid f(x) \leq a\},$$

where $a \in \mathbb{R}$ is varying. For that, we define intermediate sets for $a < b$:

$$\mathscr{S}_a^b = \{x \in N \mid a \leq f(x) \leq b\}.$$

Two typical results from Morse theory can be shown for the class of MPSiC.

Theorem 2.2 (Deformation for MPSiC)**.** *Let $\mathscr{S}_a^b$ be compact and MPSiC-LICQ be fulfilled at all points $x \in \mathscr{S}_a^b$. If $\mathscr{S}_a^b$ contains no T-stationary points for MPOC, then $\mathscr{S}_a$ is homeomorphic to $\mathscr{S}_b$.*

Theorem 2.3 (Cell-Attachment for MPSiC)**.** *Let $\mathscr{S}_a^b$ be compact and suppose that it contains exactly one T-stationary point x for MPSiC. Furthermore, let x be nondegenerate with T-index* *equal to t. If $a < f(x) < b$, then $\mathscr{S}_b$ is homotopy-equivalent to $\mathscr{S}_a$ with a t-dimensional cell attached along its boundary.*

Let us illustrate these results by means of the following examples.

Example 2.1 (Deformation)**.** We consider the following MPSiC:

$$\min_{x_1,x_2} f(x_1,x_2) = -x_1 + x_2 \quad \text{s. t.} \quad x_1 \cdot x_2 \le 0.$$

There are no T-stationary points for this MPSiC. Due to Theorem 2.2, the lower level sets will not topologically change as e.g. zero is surpassed. This process is illustrated in Figure 3. □

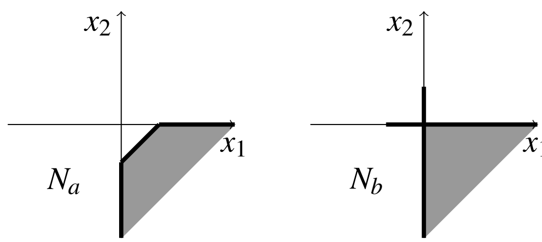

FIGURE 3. Deformation for $a < 0 < b$

Example 2.2 (Cell-attachement)**.** We consider the following MPSiC:

$$\min_{x_1,x_2} f(x_1,x_2) = x_1 + x_2 \quad \text{s. t.} \quad x_1 \cdot x_2 \le 0.$$

Its only T-stationary point is $(0,0)$ with $f(0,0) = 0$ as it is straightforward to see. It is also nondegenerate. Due to Theorem 2.3, the lower level sets will topologically change when zero is surpassed, namely by attaching a cell of proper dimension. This dimension equals the T-index of $(0,0)$, thus, a one dimensional cell has to be attached. This process is illustrated in Figure 4. □

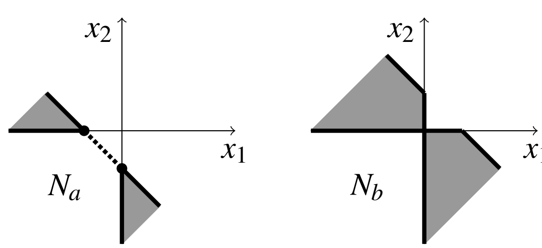

FIGURE 4. Cell-attachment for $a < 0 < b$

A global interpretation of our results is typical for the Morse theory, cf. [3]. For that, we assume that the MPSiC feasible set is compact and connected, that MPOC-LICQ holds at all feasible points, and that all T-stationary points are nondegenerate with pairwise different functional values. Then, a connected component of the lower level set is created as soon as we pass a level corresponding to a local minimizer. Different components can only be connected if a one-dimensional cell is attached. This implies that there are at least $(k-1)$ T-stationary points with T-index equal to one. Here, k denotes the number of local minimizers of MPSiC.

In the context of global optimization the latter result usually is referred to as mountain pass. For nonlinear programming it is well known that Karush-Kuhn-Tucker points with quadratic index equal to one naturally appear along with local minimizers. For MPSiC, however, not only T-stationary points with quadratic index equal to one, but also with biactive index equal to one may become relevant, see next Example 2.3.

Example 2.3 (Saddle point)**.** We consider the following MPSiC:

$$\min_{x_1,x_2} (x_1+1)^2+(x_2+1)^2 \quad \text{s.t.} \quad x_1\cdot x_2 \leq 0. \tag{2.1}$$

Obviously, $(-1,0)$ and $(0,-1)$ are nondegenerate minimizers for (2.1). Hence, there should exist an additional T-stationary point with T-index one. This nondegenerate T-stationary point is $(0,0)$. In fact, the sign constraint is biactive at $(0,0)$. Moreover, the corresponding multipliers $\zeta_1=\zeta_2=2$ are positive. Thus, its quadratic index vanishes and its biactive index equals one, i. e. MPSiC-$QI=0$, MPSiC-$BI=1$. Note that $(0,0)$ is neither M- nor S-stationary. □

3. Mathematical Programs with Complementarity Constraints

We consider mathematical programs with complementarity constraints:

$$\text{MPCC}: \quad \min_x f(x) \quad \text{s.t.} \quad x\in\mathcal{M}$$

with

$$\mathcal{M}=\left\{x\in\mathbb{R}^n \,\middle|\, F_{1,j}(x)\cdot F_{2,j}(x)=0, F_{1,j}(x)\geq 0, F_{2,j}(x)\geq 0, j=1,\ldots,\kappa\right\},$$

where $f\in C^2(\mathbb{R}^n,\mathbb{R})$, $F_1,F_2\in C^2(\mathbb{R}^n,\mathbb{R}^\kappa)$. In what follows, we briefly recall the basics from the theory of MPCC, see e.g. [14]. To this end, the index sets of active constraints associated with $\bar{x}\in\mathcal{M}$ will be helpful:

$$a_{01}(\bar{x})=\left\{j \,\middle|\, F_{1,j}(\bar{x})=0, F_{2,j}(\bar{x})>0\right\}, \quad a_{10}(\bar{x})=\left\{j \,\middle|\, F_{1,j}(\bar{x})>0, F_{2,j}(\bar{x})=0\right\},$$
$$a_{00}(\bar{x})=\left\{j \,\middle|\, F_{1,j}(\bar{x})=0, F_{2,j}(\bar{x})=0\right\}.$$

We start with the MPCC-tailored linear independence constraint qualification, which is known to be generic for MPCC.

Definition 3.1 (MPCC-LICQ)**.** We say that a feasible point $\bar{x}\in\mathcal{M}$ satisfies the MPCC-tailored linear independence constraint qualification (MPCC-LICQ) if the following vectors are linearly independent:

$$DF_{1,j}(\bar{x}), j\in a_{01}(\bar{x})\cup a_{00}(\bar{x}), \quad DF_{2,j}(\bar{x}), j\in a_{10}(\bar{x})\cup a_{00}(\bar{x}).$$

Next, we proceed with the different stationarity notions for MPCC used in the literature.

Definition 3.2 (Stationarity for MPCC)**.** A feasible point $\bar{x}\in\mathcal{M}$ is called W-stationary for MPCC if there exist multipliers

$$\bar{\sigma}_{1,j}, j\in a_{01}(\bar{x}), \quad \bar{\sigma}_{2,j}, j\in a_{10}(\bar{x}), \quad \bar{\rho}_{1,j},\bar{\rho}_{2,j}, j\in a_{00}(\bar{x}),$$

such that

$$\begin{aligned} Df(\bar{x}) &= \sum_{j\in a_{01}(\bar{x})}\bar{\sigma}_{1,j}DF_{1,j}(\bar{x})+\sum_{j\in a_{10}(\bar{x})}\bar{\sigma}_{2,j}DF_{2,j}(\bar{x}) \\ &\quad +\sum_{j\in a_{00}(\bar{x})}\left(\bar{\rho}_{1,j}DF_{1,j}(\bar{x})+\bar{\rho}_{2,j}DF_{2,j}(\bar{x})\right). \end{aligned} \tag{3.1}$$

Moreover, the W-stationary point $\bar{x}$ is called:

- C-stationary if

$$\bar{\rho}_{1,j} \cdot \bar{\rho}_{2,j} \geq 0 \text{ for all } j \in a_{00}(\bar{x});$$

- M-stationary if

$$\bar{\rho}_{1,j} > 0, \bar{\rho}_{2,j} > 0 \text{ or } \bar{\rho}_{1,j} \cdot \bar{\rho}_{2,j} = 0 \text{ for all } j \in a_{00}(\bar{x}); \tag{3.2}$$

- S-stationary if

$$\bar{\rho}_{1,j} \geq 0, \bar{\rho}_{2,j} \geq 0 \text{ for all } j \in a_{00}(\bar{x}). \tag{3.3}$$

Note that S-stationarity turns out to be necessary for optimality; see [12]. For a W-stationary point $\bar{x} \in \mathscr{M}$ with multipliers $(\bar{\sigma}, \bar{\rho})$ – which are unique under MPCC-LICQ – it is convenient to define the appropriate Lagrange function as

$$L(\cdot) \;=\; f(\cdot) - \sum_{j \in a_{01}(\bar{x})} \bar{\sigma}_{1,j} F_{1,j}(\cdot) - \sum_{j \in a_{10}(\bar{x})} \bar{\sigma}_{2,j} F_{2,j}(\cdot) - \sum_{j \in a_{00}(\bar{x})} \left(\bar{\rho}_{1,j} F_{1,j}(\cdot) + \bar{\rho}_{2,j} F_{2,j}(\cdot)\right).$$

The corresponding tangent space is given by

$$\mathscr{T}_{\bar{x}} = \left\{ \xi \in \mathbb{R}^n \;\middle|\; DF_{1,j}(\bar{x})\,\xi = 0, j \in a_{01}(\bar{x}) \cup a_{00}(\bar{x}), DF_{2,j}(\bar{x})\,\xi = 0, j \in a_{10}(\bar{x}) \cup a_{00}(\bar{x}) \right\}.$$

Now, we turn our attention to the also generic notion of nondegeneracy for C-stationary points. Here, we mention the additional condition NDC4, which has been proven useful in the context of Scholtes regularization of MPCC; see [8].

Definition 3.3 (Nondegenerate C-stationarity for MPCC)**.** A C-stationary point $\bar{x} \in \mathscr{M}$ of MPCC with multipliers $(\bar{\sigma}, \bar{\rho})$ is called nondegenerate if

NDC1: MPCC-LICQ holds at $\bar{x}$,

NDC2: the multipliers corresponding to biactive complementarity constraints do not vanish, i.e. $\bar{\rho}_{1,j} \neq 0$ and $\bar{\rho}_{2,j} \neq 0$ for all $j \in a_{00}(\bar{x})$,

NDC3: the matrix $D^2 L(\bar{x}) \restriction_{\mathscr{T}_{\bar{x}}}$ is nonsingular.

For a nondegenerate C-stationary point we eventually use the following additional condition:

NDC4: the multipliers corresponding to local equality constraints do not vanish, i.e. $\bar{\sigma}_{1,j} \neq 0$ for all $j \in a_{01}(\bar{x})$ and $\bar{\sigma}_{2,j} \neq 0$ for all $j \in a_{10}(\bar{x})$.

For a nondegenerate C-stationary point its C-index becomes a crucial invariant; see also [10].

Definition 3.4 (C-index for MPCC)**.** Let $\bar{x} \in \mathscr{M}$ be a nondegenerate C-stationary point of MPCC with unique multipliers $(\bar{\sigma}, \bar{\rho})$. The number of negative eigenvalues of the matrix $D^2 L(\bar{x}) \restriction_{\mathscr{T}_{\bar{x}}}$ is called its quadratic index (MPCC-*QI*). The number of negative pairs $\bar{\rho}_{1,j}, \bar{\rho}_{2,j}, j \in a_{00}(\bar{x})$ with $\bar{\rho}_{1,j}, \bar{\rho}_{2,j} < 0$ is called the biactive index (MPCC-*BI*) of $\bar{x}$. We define the C-index as the sum of both, i. e. MPCC-*CI* = MPCC-*QI* + MPCC-*BI*.

We emphasize that C-stationary points are topologically relevant in the sense of Morse theory. It is to say that they adequately describe the topological changes of lower level sets of MPCC while their levels rise. Moreover, nondegenerate C-stationary points with C-index equal to zero are local minimizers of MPCC. For nonvanishing C-indices we obtain all kinds of saddle points for MPCC. Overall, the C-index uniquely determines the topological type of a nondegenerate C-stationary point.

Finally, we define the following auxiliary index subsets, which depend on the signs of multipliers $(\bar{\sigma}, \bar{\rho})$ corresponding to a W-stationary point $\bar{x} \in \mathscr{M}$:

$$\begin{aligned} a_{01}^{-}(\bar{x}) &= \left\{ j \in a_{01}(\bar{x}) \,\middle|\, \bar{\sigma}_{1,j} < 0 \right\}, \quad a_{10}^{-}(\bar{x}) = \left\{ j \in a_{10}(\bar{x}) \,\middle|\, \bar{\sigma}_{2,j} < 0 \right\}, \\ a_{01}^{0}(\bar{x}) &= \left\{ j \in a_{01}(\bar{x}) \,\middle|\, \bar{\sigma}_{1,j} = 0 \right\}, \quad a_{10}^{0}(\bar{x}) = \left\{ j \in a_{10}(\bar{x}) \,\middle|\, \bar{\sigma}_{2,j} = 0 \right\}, \\ a_{01}^{+}(\bar{x}) &= \left\{ j \in a_{01}(\bar{x}) \,\middle|\, \bar{\sigma}_{1,j} > 0 \right\}, \quad a_{10}^{+}(\bar{x}) = \left\{ j \in a_{10}(\bar{x}) \,\middle|\, \bar{\sigma}_{2,j} > 0 \right\}, \end{aligned}$$

$$\begin{aligned} a_{00}^{-}(\bar{x}) &= \left\{ j \in a_{00}(\bar{x}) \,\middle|\, \bar{\rho}_{1,j}, \bar{\rho}_{2,j} < 0 \right\}, \\ a_{00}^{0}(\bar{x}) &= \left\{ j \in a_{00}(\bar{x}) \,\middle|\, \bar{\rho}_{1,j} \cdot \bar{\rho}_{2,j} = 0 \right\}, \\ a_{00}^{+}(\bar{x}) &= \left\{ j \in a_{00}(\bar{x}) \,\middle|\, \bar{\rho}_{1,j}, \bar{\rho}_{2,j} > 0 \right\}. \end{aligned}$$

4. SIGN-TYPE REGULARIZATION OF MPCC

With a given MPCC, the authors in [6] suggested to associate the following parametric MPSiC:

$$\mathrm{MPSiC}(t): \quad \min_{x} f(x) \quad \text{s. t.} \quad x \in \mathscr{M}^t$$

with

$$\mathscr{M}^t = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{l} F_{1,j}(x) + t \geq 0, F_{2,j}(x) + t \geq 0, \\ \left(F_{1,j}(x) - t\right) \cdot \left(F_{2,j}(x) - t\right) \leq 0, j = 1, \ldots, \kappa \end{array} \right\},$$

where the parameter $t > 0$ is positive. Note that for $\mathrm{MPSiC}(t)$ the relevant concepts from Section 2 can be applied. Thus, we can say whether MPSiC-LICQ holds at a feasible point of $\mathrm{MPSiC}(t)$, how S-, M-, T-, or W-stationarity is defined, which of the nondegeneracy conditions NDSi1-NDSi4 are fulfilled at a T-stationary point, and how its T-index has to be determined. Without providing too many details here, let us nevertheless fix the notation for active index sets and multipliers associated with a W-stationary point $x^t \in \mathscr{M}^t$. The index sets for the active inequality constraints $F_{1,j}(x^t) + t = 0$ and $F_{2,j}(x^t) + t = 0$ will be denoted by $J_1(x^t)$ and $J_2(x^t)$, respectively. For the corresponding multipliers we use the notation $\mu_{1,j}^t$ and $\mu_{2,j}^t$. By further borrowing the notation from Section 2, let $b_{1,2}(x^t)$ stand for the index set of biactive constraints $F_{1,j}(x^t) - t = 0$, $F_{2,j}(x^t) - t = 0$ with the corresponding biactive multipliers $\zeta_{1,j}^t$, $\zeta_{2,j}^t$. The active index set for the local inequality constraints $F_{1,j}(x^t) - t = 0, F_{2,j}(x^t) - t \neq 0$ is $b_{1,\neq}(x^t) = b_{1,>}(x^t) \cup b_{1,<}(x^t)$ and, vice versa, in case of $F_{1,j}(x^t) - t \neq 0, F_{2,j}(x^t) - t = 0$ we use the notation $b_{\neq,2}(x^t) = b_{>,2}(x^t) \cup b_{<,2}(x^t)$. The corresponding multipliers would be set as $\eta_{1,j}^t$ and $\eta_{2,j}^t$, respectively. Overall, a W-stationary point $x^t \in \mathscr{M}^t$ of $\mathrm{MPSiC}(t)$ has multipliers (μ^t, η^t, ζ^t).

Theorem 4.1 (MPCC-LICQ vs. MPSiC-LICQ). *Let a feasible point $\bar{x} \in M$ of MPCC fulfill MPCC-LICQ. Then, MPSiC-LICQ holds at all feasible points $x^t \in \mathscr{M}^t$ of $\mathrm{MPSiC}(t)$ for all sufficiently small t, whenever they are sufficiently close to $\bar{x}$.*

Proof. MPSiC-LICQ is satisfied at $x^t \in \mathscr{M}^t$ if the following vectors are linearly independent:

$$DF_{1,j}(x^t), j \in J_1(x^t) \cup b_{1,\neq}(x^t), \quad DF_{2,j}(x^t), j \in J_2(x^t) \cup b_{\neq,2}(x^t).$$

Suppose $j \in J_1(x^t) \cup b_{1,\neq}(x^t)$ along some sequence of $x^t \in \mathrm{MPSiC}(t)$ with $x^t \to \bar{x}$ for $t \to 0$. Then, $\left|F_{1,j}(x^t)\right| = t$. Taking the limit, we get $F_{1,j}(\bar{x}) = 0$. Hence, for all t sufficiently small it has to hold $j \in a_{01}(\bar{x}) \cup a_{00}(\bar{x})$. Analogously, for $j \in J_2(x^t) \cup b_{\neq,2}(x^t)$ we get $j \in a_{10}(\bar{x}) \cup a_{00}(\bar{x})$ for all t sufficiently small. Thus, MPCC-LICQ yields the linear independence of the vectors

$$DF_{1,j}(\bar{x}), j \in J_1(x^t) \cup b_{1,\neq}(x^t), \quad DF_{2,j}(\bar{x}), j \in J_2(x^t) \cup b_{\neq,2}(x^t),$$

for all t sufficiently small and for all x^t sufficiently close to $\bar{x}$. MPSiC-LICQ follows in view of the continuity of DF_1 and DF_2. □ □

Note that the sign-type regularization MPSiC(t) can be alternatively treated within the scope of NLP theory. In [6], the usual linear independence constraint qualification (LICQ) is shown to hold at those of its feasible points ($x^t \in \mathscr{M}^t$ – in a sufficiently small neighborhood of $\bar{x} \in M$ with MPCC-LICQ and for t sufficiently close to zero, – which do not have biactive sign constraints, i.e. $b_{1,2}(x^t) = \emptyset$. Otherwise, LICQ can fail at the corresponding feasible points of MPSiC(t), even if $\bar{x}$ fulfills MPCC-LICQ. Taking this into account, we would say that MPSiC(t) viewed as NLP instances turn out to be degenerate. In particular, if x^t happens to be T-stationary, $b_{1,2}(x^t) \neq \emptyset$ implies that its biactive index MPSiC-*BI* does not vanish, hence, x^t is a saddle point of MPSiC(t). We conclude that LICQ will normally be violated at the saddle points of MPSiC(t) with the nonvanishing biactive index. In other words, LICQ would be not inherited there from MPCC-LICQ at $\bar{x}$. Therefore, the study of the global structure of MPSiC(t) by means of the standard Morse theory for NLP, see [3], becomes hampered, if not even impossible. This observation motivated us to introduce the new MPSiC class, as well as to adjust the Morse theory for the latter.

Theorem 4.2 (Convergence for T- or M-stationarity)**.** *Suppose a sequence of T- or M-stationary points $x^t \in \mathscr{M}^t$ of MPSiC(t) with multipliers (μ^t, ζ^t, η^t) converges to $\bar{x}$ for $t \to 0$. Let MPCC-LICQ be fulfilled at $\bar{x} \in M$. Then, $\bar{x}$ is a C- or M-stationary point of MPCC, respectively.*

Proof. Suppose that $x^t \in \mathscr{M}^t$ are T-stationary points of MPSiC(t). By applying Definition 2.2 to MPSiC(t), we have

$$\begin{aligned} Df\left(x^t\right) \;=\;& \sum_{j\in J_1(x^t)} \mu^t_{1,j} DF_{1,j}\left(x^t\right) + \sum_{j\in J_2(x^t)} \mu^t_{2,j} DF_{2,j}\left(x^t\right) \\ &+ \sum_{j\in b_{1,\neq}(x^t)} \eta^t_{1,j} DF_{1,j}\left(x^t\right) + \sum_{j\in b_{\neq,2}(x^t)} \eta^t_{2,j} DF_{2,j}\left(x^t\right) \\ &+ \sum_{j\in b_{1,2}(x^t)} \left(\zeta^t_{1,j} DF_{1,j}\left(x^t\right) + \zeta^t_{2,j} DF_{2,j}\left(x^t\right)\right), \end{aligned}$$

with the multipliers satisfying

$$\mu^t_{1,j} \geq 0 \text{ for all } j \in J_1\left(x^t\right), \quad \mu^t_{2,j} \geq 0 \text{ for all } j \in J_2\left(x^t\right), \tag{4.1}$$

$$\eta^t_{1,j} \leq 0 \text{ for all } m \in b_{1,>}\left(x^t\right), \quad \eta^t_{1,j} \geq 0 \text{ for all } m \in b_{1,<}\left(x^t\right), \tag{4.2}$$

$$\eta^t_{2,j} \leq 0 \text{ for all } m \in b_{>,2}\left(x^t\right), \quad \eta^t_{2,j} \geq 0 \text{ for all } m \in b_{<,2}\left(x^t\right), \tag{4.3}$$

and

$$\zeta^t_{1,j} \cdot \zeta^t_{2,j} \geq 0 \text{ for all } j \in b_{1,2}\left(x^t\right). \tag{4.4}$$

From the proof of Theorem 4.1 we have for all t sufficiently small:

$$J_1\left(x^t\right) \cup b_{1,\neq}\left(x^t\right) \subset a_{01}(\bar{x}) \cup a_{00}(\bar{x}) \quad \text{and} \quad J_2\left(x^t\right) \cup b_{\neq,2}\left(x^t\right) \subset a_{10}(\bar{x}) \cup a_{00}(\bar{x}).$$

Using similar arguments as there, we also derive $b_{1,2}(x^t) \subset a_{00}(\bar{x})$. Further, the sets $J_1(x^t)$, $b_{1,\neq}(x^t)$, $b_{1,2}(x^t)$, and $J_2(x^t)$, $b_{\neq,2}(x^t)$, $b_{1,2}(x^t)$ are pairwise disjoint, respectively. Thus, the following multipliers (σ^t, ρ^t) are well-defined:

$$\sigma^t_{1,j} = \begin{cases} \mu^t_{1,j} & \text{for } j \in J_1(x^t) \cap a_{01}(\bar{x}), \\ \eta^t_{1,j} & \text{for } j \in b_{1,\neq}(x^t) \cap a_{01}(\bar{x}), \\ 0 & \text{else}, \end{cases} \qquad \sigma^t_{2,j} = \begin{cases} \mu^t_{2,j} & \text{for } j \in J_2(x^t) \cap a_{10}(\bar{x}), \\ \eta^t_{2,j} & \text{for } j \in b_{\neq,2}(x^t) \cap a_{10}(\bar{x}), \\ 0 & \text{else}, \end{cases}$$

$$\rho^t_{1,j} = \begin{cases} \mu^t_{1,j} & \text{for } j \in J_1(x^t) \cap a_{00}(\bar{x}), \\ \eta^t_{1,j} & \text{for } j \in b_{1,\neq}(x^t) \cap a_{00}(\bar{x}), \\ \zeta^t_{1,j} & \text{for } j \in b_{1,2}(x^t), \\ 0 & \text{else}, \end{cases} \qquad \rho^t_{2,j} = \begin{cases} \mu^t_{2,j} & \text{for } j \in J_2(x^t) \cap a_{00}(\bar{x}), \\ \eta^t_{2,j} & \text{for } j \in b_{\neq,2}(x^t) \cap a_{00}(\bar{x}), \\ \zeta^t_{2,j} & \text{for } j \in b_{1,2}(x^t), \\ 0 & \text{else}. \end{cases}$$

This yields

$$\begin{aligned} Df(x^t) &= \sum_{j \in a_{01}(\bar{x})} \sigma^t_{1,j} DF_{1,j}(x^t) + \sum_{j \in a_{10}(\bar{x})} \sigma^t_{2,j} DF_{2,j}(x^t) \\ &\quad + \sum_{j \in a_{00}(\bar{x})} \left(\rho^t_{1,j} DF_{1,j}(x^t) + \rho^t_{2,j} DF_{2,j}(x^t) \right). \end{aligned}$$

Due to MPCC-LICQ at $\bar{x}$, the multipliers (σ^t, ρ^t) converge to a vector $(\bar{\sigma}, \bar{\rho})$ for $t \to 0$. Thus, $\bar{x}$ together with $(\bar{\sigma}, \bar{\rho})$ fulfills (3.1). Next, let us assume there exists $\mathrm{j} \in a_{00}(\bar{x})$ with $\bar{\rho}_{1,\mathrm{j}} < 0$ and $\bar{\rho}_{2,\mathrm{j}} > 0$, hence, it holds $\rho^t_{1,\mathrm{j}} < 0$ and $\rho^t_{2,\mathrm{j}} > 0$ for all t sufficiently small. It follows from the definition of ρ^t and (4.1)-(4.4) that $\mathrm{j} \in b_{1,>}(x^t) \cup b_{1,2}(x^t)$. However, if $\mathrm{j} \in b_{1,>}(x^t)$, then $F_{2,j}(x^t) > t$. In particular, $\mathrm{j} \notin J_2(x^t) \cup b_{\neq,2}(x^t) \cup b_{1,2}(x^t)$. Hence, it must hold $\rho^t_{2,\mathrm{j}} = 0$, a contradiction. Instead, assume $\mathrm{j} \in b_{1,2}(x^t)$. Then, from (4.4) we get $\rho^t_{2,\mathrm{j}}(x^t) \leq 0$, a contradiction again. A similar argumentation shows that there is no $\mathrm{j} \in a_{00}(\bar{x})$ with $\bar{\rho}_{1,\mathrm{j}} > 0$ and $\bar{\rho}_{2,\mathrm{j}} < 0$. Overall, we conclude $\bar{\rho}_{1,\mathrm{j}} \cdot \bar{\rho}_{2,\mathrm{j}} \geq 0$ for all $j \in a_{00}(\bar{x})$. Thus, $\bar{x}$ is a C-stationary point for MPCC.

Let us now assume that x^t are M-stationary, i.e. $\zeta^t_{1,j} \cdot \zeta^t_{2,j} = 0$ for all $j \in b_{1,2}(x^t)$. Assume there exists $\mathrm{j} \in a_{00}(\bar{x})$ with $\bar{\rho}_{1,\mathrm{j}} \cdot \bar{\rho}_{2,\mathrm{j}} \neq 0$. Then, for all t sufficiently small it must hold $\rho^t_{1,\mathrm{j}} \cdot \rho^t_{2,\mathrm{j}} \neq 0$ and, thus, $\mathrm{j} \notin b_{1,2}(x^t)$. We use the latter as well as the definition of ρ^t to derive

$$\mathrm{j} \in \left(J_1(x^t) \cup b_{1,\neq}(x^t) \right) \cap \left(J_2(x^t) \cup b_{\neq,2}(x^t) \right).$$

Suppose $\mathrm{j} \in J_1(x^t)$. Then, the feasibility of x^t yields $\mathrm{j} \in b_{<,2}(x^t)$. We conclude from $\rho^t_{1,\mathrm{j}} \cdot \rho^t_{2,\mathrm{j}} \neq 0$, (4.1), and (4.3) that it holds $\rho^t_{1,\mathrm{j}} > 0$ and $\rho^t_{2,\mathrm{j}} > 0$. After taking the limit here and recalling $\bar{\rho}_{1,\mathrm{j}} \cdot \bar{\rho}_{2,\mathrm{j}} \neq 0$, we get $\bar{\rho}_{1,\mathrm{j}} > 0$ and $\bar{\rho}_{2,\mathrm{j}} > 0$. Now, suppose $\mathrm{j} \in b_{1,\neq}(x^t)$, instead. It follows $\mathrm{j} \in J_2(x^t)$ and, thus, $\mathrm{j} \in b_{1,<}(x^t)$ as above. Analogously, we come to the same conclusion $\bar{\rho}_{1,\mathrm{j}} > 0$ and $\bar{\rho}_{2,\mathrm{j}} > 0$. Overall, (3.2) holds. Thus, $\bar{x}$ is an M-stationary point of MPCC. □

Now, we turn our attention to the convergence towards S-stationary points of MPCC. Note that the notion of S-stationary points for MPSiC(t) coincides with that of its Karush-Kuhn-Tucker points. This is again if MPSiC(t) is viewed as an NLP instance. In [6], it has been investigated if Karush-Kuhn-Tucker points of MPSiC(t) provide in the limit an S-stationary point of MPCC. For this result to hold, MPCC-LICQ and NDC2 are assumed there. For the sake of completeness, we state a similar result in Corollary 4.1. Its proof easily follows from Theorem 4.2. Moreover, we would like to stress that for the convergence with respect to S-stationarity an additional condition NDC2 is needed. For, T- or M-stationary points the latter can be omitted.

Corollary 4.1 (Convergence for S-stationarity, cf. [6]). *Suppose a sequence of M-stationary points $x^t \in \mathscr{M}^t$ of MPSiC(t) with multipliers (μ^t, η^t, ζ^t) converges to $\bar{x}$ for $t \to 0$. Let MPCC-LICQ and NDC2 be fulfilled at $\bar{x} \in M$. Then, $\bar{x}$ is an S-stationary point of MPCC.*

Proof. By virtue of Theorem 4.2, $\bar{x}$ is an M-stationary point of MPCC. In particular (3.2) holds. Additionally, we have $\bar{\rho}_{1,\mathrm{j}} \cdot \bar{\rho}_{2,\mathrm{j}} \neq 0$ due to NDC2. Thus, (3.3) holds and $\bar{x}$ is an S-stationary point of MPCC. $\square$

In order to relate the active index sets of MPCC and MPSiC(t) in the context of Theorem 4.2, we define for a W-stationary point $x^t \in \mathscr{M}^t$:

$$b^-_{1,2}\left(x^t\right) = \left\{j \in b_{1,2}\left(x^t\right) \mid \zeta^t_{1,j}, \zeta^t_{2,j} < 0\right\}, \quad b^+_{1,2}\left(x^t\right) = \left\{j \in b_{1,2}\left(x^t\right) \mid \zeta^t_{1,j}, \zeta^t_{2,j} > 0\right\},$$
$$b^0_{1,2}\left(x^t\right) = \left\{j \in b_{1,2}\left(x^t\right) \mid \zeta^t_{1,j} \cdot \zeta^t_{2,j} = 0\right\}.$$

Corollary 4.2 (Active index sets). *Under the assumptions of Theorem 4.2 for C-stationarity, we have for the active index subsets of the multipliers $(\bar{\sigma}, \bar{\rho})$ of $\bar{x} \in M$ for all sufficiently small t:*

a) $a^-_{01}(\bar{x}) \subset b_{1,>}(x^t)$,
b) $a^+_{01}(\bar{x}) \subset J_1(x^t) \backslash b_{<,2}(x^t) \cup b_{1,<}(x^t) \backslash J_2(x^t)$,
c) $a^-_{10}(\bar{x}) \subset b_{>,2}(x^t)$,
d) $a^+_{10}(\bar{x}) \subset J_2(x^t) \backslash b_{1,<}(x^t) \cup b_{<,2}(x^t) \backslash J_1(x^t)$,
e) $a^-_{00}(\bar{x}) \subset b^-_{1,2}(x^t)$,
f) $a^+_{00}(\bar{x}) \subset b^+_{1,2}(x^t) \cup (J_1(x^t) \cap b_{<,2}(x^t)) \cup (b_{1,<}(x^t) \cap J_2(x^t))$.

If additionally $\bar{x}$ fulfills NDC2, then inclusions e) and f) become equalities. If additionally $\bar{x}$ fulfills NDC2 and NDC4, then also inclusions a)–d) become equalities.

Proof. Inclusions a) and c) follow directly by comparing the signs of multipliers in the proof of Theorem 4.2 and the definition of σ^t therein. Following the proof of Theorem 4.2, we also obtain $a^+_{01}(\bar{x}) \subset J_1(x^t) \cup b_{1,<}(x^t)$ for all t sufficiently small. Now, assume $j \in J_1(x^{t_k}) \cap b_{<,2}(x^{t_k})$ along some subsequence. Thus, $F_{1,j}(x^{t_k}) = -t_k, F_{2,j}(x^{t_k}) = t_k$. Taking the limit $t_k \to 0$, we get $F_{1,j}(\bar{x}) = F_{2,j}(\bar{x}) = 0$. Thus, $j \notin a^+_{01}(\bar{x})$. If instead $j \in b_{1,<}(x^{t_k}) \cap J_2(x^{t_k})$, we follow a similar reasoning to conclude $j \notin a^+_{01}(\bar{x})$. Overall, inclusion b) follows. Inclusion d) is proven analogously to b). Further, it follows again from the proof of Theorem 4.2 that

$$a^-_{00}(\bar{x}) \subset \left(b_{1,>}\left(x^t\right) \cup b^-_{1,2}\left(x^t\right)\right) \cap \left(b_{>,2}\left(x^t\right) \cup b^-_{1,2}\left(x^t\right)\right).$$

Since $b_{1,>}(x^t) \cap \left(b_{>,2}(x^t) \cup b^-_{1,2}(x^t)\right) = \emptyset$ and $b_{>,2}(x^t) \cap \left(b_{1,>}(x^t) \cup b^-_{1,2}(x^t)\right) = \emptyset$, inclusion e) follows. Finally, using the proof of Theorem 4.2 once more, we derive

$$a^+_{00}(\bar{x}) \subset \left(J_1\left(x^t\right) \cup b_{1,>}\left(x^t\right) \cup b^+_{1,2}\left(x^t\right)\right) \cap \left(J_2\left(x^t\right) \cup b_{>,2}\left(x^t\right) \cup b^+_{1,2}\left(x^t\right)\right).$$

It is straightforward to conclude

$$a^-_{00}(\bar{x}) \subset b^+_{1,2}\left(x^t\right) \cup \left(\left(J_1\left(x^t\right) \cup b_{1,>}\left(x^t\right)\right) \cap \left(J_2\left(x^t\right) \cup b_{>,2}\left(x^t\right)\right)\right).$$

We notice $J_1(x^t) \cap J_2(x^t) = b_{1,>}(x^t) \cap b_{>,2}(x^t) = \emptyset$. Hence, inclusion f) follows.

Let us assume that NDC2 and NDC4 hold at $\bar{x}$. Then,

$$a^-_{01}(\bar{x}) \cup a^+_{01}(\bar{x}) \cup a^-_{10}(\bar{x}) \cup a^+_{10}(\bar{x}) \cup a^-_{00}(\bar{x}) \cup a^+_{00}(\bar{x}) = \{1, \ldots, \kappa\}.$$

Note that all sets here are pairwise disjoint. Furthermore, it is straightforward, but laborious to see that the sets $b_{1,>}(x^t)$, $J_1(x^t)\backslash b_{<,2}(x^t)\cup b_{1,<}(x^t)\backslash J_2(x^t)$, $b_{>,2}(x^t)$, $J_2(x^t)\backslash b_{1,<}(x^t)\cup b_{<,2}(x^t)\backslash J_1(x^t)$, $b^-_{1,2}(x^t)$, $b^+_{1,2}(x^t)\cup(J_1(x^t)\cap b_{<,2}(x^t))\cup(b_{1,<}(x^t)\cap J_2(x^t))$ are pairwise disjoint. Consequently, the inclusions given in a)–f) become equalities.

We finish by showing that NDC4 can be omitted to attain equalities in e) and f). For this, we start with f). Suppose $j\in b^+_{1,2}(x^t)$, i.e., $F_{1,j}(x^t)=F_{2,j}(x^t)=t$. We obtain $j\in a_{00}(\bar{x})$ by taking the limit $t\to 0$. From the proof of Theorem 4.2 we see that $\rho^t_1,\rho^t_2\geq 0$. In view of NDC2, it must, thus, hold $\bar{\rho}_1,\bar{\rho}_2>0$, i.e. $j\in a^+_{00}(\bar{x})$. If instead $j\in J_1(x^t)\cap b_{<,2}(x^t)$ we have $F_{1,j}(x^t)=-t, F_{2,j}(x^t)=t$. From here we similarly deduce $j\in a^+_{00}(\bar{x})$. Lastly, it follows from $j\in J_1(x^t)\cap b_{<,2}(x^t)$ that $j\in a^+_{00}(\bar{x})$ in an analogous way. Therefore, equality in f) follows under NDC2. The proof of the equality in e) follows a similar argumentation. □

Corollary 4.3 (Multipliers). *If additionally $\bar{x}$ fulfills NDC2 in Corollary 4.2, then for the multipliers it holds:*

$$\lim_{t\to 0}\mu^t_{1,j}=\begin{cases}\bar{\sigma}_{1,j} & \textit{for } j\in a^+_{01}(\bar{x}),\\ \bar{\rho}_{1,j} & \textit{for } j\in a^+_{00}(\bar{x}),\\ 0 & \textit{else},\end{cases}\qquad \lim_{t\to 0}\mu^t_{2,j}=\begin{cases}\bar{\sigma}_{2,j} & \textit{for } j\in a^+_{10}(\bar{x}),\\ \bar{\rho}_{2,j} & \textit{for } j\in a^+_{00}(\bar{x}),\\ 0 & \textit{else},\end{cases}$$

$$\lim_{t\to 0}\eta^t_{1,j}=\begin{cases}\bar{\sigma}_{1,j} & \textit{for } j\in a^-_{01}(\bar{x})\cup a^+_{01}(\bar{x}),\\ \bar{\rho}_{1,j} & \textit{for } j\in a^+_{00}(\bar{x}),\\ 0 & \textit{else},\end{cases}\qquad \lim_{t\to 0}\eta^t_{2,j}=\begin{cases}\bar{\sigma}_{2,j} & \textit{for } j\in a^-_{10}(\bar{x})\cup a^+_{01}(\bar{x}),\\ \bar{\rho}_{2,j} & \textit{for } j\in a^+_{00}(\bar{x}),\\ 0 & \textit{else},\end{cases}$$

$$\lim_{t\to 0}\zeta^t_{1,j}=\begin{cases}\bar{\rho}_{1,j} & \textit{for } j\in a_{00}(\bar{x}),\\ 0 & \textit{else},\end{cases}\qquad \lim_{t\to 0}\zeta^t_{2,j}=\begin{cases}\bar{\rho}_{2,j} & \textit{for } j\in a_{00}(\bar{x}),\\ 0 & \textit{else}.\end{cases}$$

Now, we are ready to state the main result of this section. Theorem 4.3 describes possible changes of the topological type of the corresponding stationary points while applying the sign-type regularization MPSiC(t) to MPCC.

Theorem 4.3 (Convergence and index tracing). *Suppose a sequence of nondegenerate T-stationary points $x^t\in\mathscr{M}^t$ of MPSiC(t) with the T-index MPSiC-TI, consisiting of quadratic and biactive indices MPSiC-QI and MPSiC-BI, respectively, converges to $\bar{x}$ for $t\to 0$. If $\bar{x}\in M$ is a nondegenerate C-stationary point of MPCC with multipliers $(\bar{\sigma},\bar{\rho})$, then we have for its quadratic and biactive indices MPCC-QI and MPCC-BI, respectively:*

$$\max\left\{\textit{MPSiC-QI}-\left|a^0_{01}(\bar{x})\right|-\left|a^0_{10}(\bar{x})\right|,0\right\}\leq \textit{MPCC-QI}\leq \textit{MPSiC-QI}$$

and

$$\max\left\{\textit{MPSiC-BI}-\left|a^+_{00}(\bar{x})\right|,0\right\}\leq \textit{MPCC-BI}\leq \textit{MPSiC-BI}.$$

Thus, for its C-index we have:

$$\max\left\{\textit{MPSiC-TI}-\left|a^0_{01}(\bar{x})\right|-\left|a^0_{10}(\bar{x})\right|-\left|a^+_{00}(\bar{x})\right|,0\right\}\leq \textit{MPCC-CI}\leq \textit{MPSiC-TI}.$$

Proof. We prove the assertions in four steps.

Step 1: We show

$$\lim_{t\to 0}D^2L^{\mathscr{M}^t}\left(x^t\right)=D^2L(\bar{x}), \tag{4.5}$$

where $L^{\mathscr{M}^t}(\cdot)$ is the Lagrange function for MPSiC(t) given x^t and $L(\cdot)$ is the Lagrange function for MPCC given $\bar{x}$. For the former, we have

$$\begin{aligned} D^2L^{\mathscr{M}^t}\left(x^t\right) &= D^2f\left(x^t\right) - \sum_{j\in J_1(x^t)} \mu^t_{1,j}D^2F_{1,j}\left(x^t\right) - \sum_{j\in J_2(x^t)} \mu^t_{2,j}D^2F_{2,j}\left(x^t\right) \\ &\quad - \sum_{j\in b_{1,\neq}(x^t)} \eta^t_{1,j}D^2F_{1,j}\left(x^t\right) - \sum_{j\in b_{\neq,2}(x^t)} \eta^t_{2,j}D^2F_{2,j}\left(x^t\right) \\ &\quad - \sum_{j\in b_{1,2}(x^t)} \left(\zeta^t_{1,j}D^2F_{1,j}\left(x^t\right) + \zeta^t_{2,j}D^2F_{2,j}\left(x^t\right)\right). \end{aligned}$$

We rewrite this as follows:

$$\begin{aligned} D^2L^{\mathscr{M}^t}\left(x^t\right) &= D^2f\left(x^t\right) - \sum_{j\in J_1(x^t)\cap a^+_{01}(\bar{x})} \mu^t_{1,j}D^2F_{1,j}\left(x^t\right) - \sum_{j\in J_1(x^t)\cap a^+_{00}(\bar{x})} \mu^t_{1,j}D^2F_{1,j}\left(x^t\right) \\ &\quad - \sum_{j\in J_2(x^t)\cap a^+_{10}(\bar{x})} \mu^t_{2,j}D^2F_{2,j}\left(x^t\right) - \sum_{j\in J_2(x^t)\cap a^+_{00}(\bar{x})} \mu^t_{2,j}D^2F_{2,j}\left(x^t\right) \\ &\quad - \sum_{j\in b_{1,\neq}(x^t)\cap a_{01}(\bar{x})} \eta^t_{1,j}D^2F_{1,j}\left(x^t\right) - \sum_{j\in b_{\neq,2}(x^t)\cap a_{10}(\bar{x})} \eta^t_{2,j}D^2F_{2,j}\left(x^t\right) \\ &\quad - \sum_{j\in b_{1,\neq}(x^t)\cap a^+_{00}(\bar{x})} \eta^t_{1,j}D^2F_{1,j}\left(x^t\right) - \sum_{j\in b_{\neq,2}(x^t)\cap a^+_{00}(\bar{x})} \eta^t_{2,j}D^2F_{2,j}\left(x^t\right) \\ &\quad - \sum_{j\in b_{1,2}(x^t)\cap a_{00}(\bar{x})} \left(\zeta^t_{1,j}D^2F_{1,j}\left(x^t\right) + \zeta^t_{2,j}D^2F_{2,j}\left(x^t\right)\right) - \varepsilon\left(x^t\right). \end{aligned}$$

We use Corollary 4.3 to deduce that $\lim\limits_{t\to 0}\varepsilon\left(x^t\right) = 0$. Then, (4.5) follows by means of Corollaries 4.2 and 4.3 if taking the limit $t \to 0$ in the last formula.

Step 2: We show

$$\text{MPCC-}QI \leq \text{MPSiC-}QI. \tag{4.6}$$

Let us consider the tangent spaces $\mathscr{T}^{\mathscr{M}^t}_{x^t}$ corresponding to $x^t \in \mathscr{M}^t$ and $\mathscr{T}_{\bar{x}}$ corresponding to $\bar{x} \in M$. For the former, we have

$$\mathscr{T}^{\mathscr{M}^t}_{x^t} = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \begin{array}{l} DF_{1,j}\left(x^t\right)\xi = 0, j \in J_1\left(x^t\right) \cup b_{1,\neq}\left(x^t\right) \cup b_{1,2}\left(x^t\right), \\ DF_{2,j}\left(x^t\right)\xi = 0, j \in J_2\left(x^t\right) \cup b_{\neq,2}\left(x^t\right) \cup b_{1,2}\left(x^t\right) \end{array} \right\}.$$

From the proof of Theorem 4.1 we know that for all t sufficiently small it holds:

$$J_1\left(x^t\right) \cup b_{1,\neq}\left(x^t\right) \subset a_{01}\left(\bar{x}\right) \cup a_{00}\left(\bar{x}\right).$$

In addition, Corollary 4.2 yields in view of NDC2:

$$a_{00}\left(\bar{x}\right) = b_{1,2}\left(x^t\right) \cup \left(J_1\left(x^t\right) \cap b_{<,2}\left(x^t\right)\right) \cup \left(b_{1,<}\left(x^t\right) \cap J_2\left(x^t\right)\right).$$

Combining this with the former, we obtain

$$a_{00}\left(\bar{x}\right) \subset J_1\left(x^t\right) \cup b_{1,\neq}\left(x^t\right) \cup b_{1,2}\left(x^t\right) \subset a_{01}\left(\bar{x}\right) \cup a_{00}\left(\bar{x}\right).$$

Similarly,

$$a_{00}\left(\bar{x}\right) \subset J_2\left(x^t\right) \cup b_{\neq,2}\left(x^t\right) \cup b_{1,2}\left(x^t\right) \subset a_{10}\left(\bar{x}\right) \cup a_{00}\left(\bar{x}\right).$$

In particular, it follows $\mathscr{T}_{\bar{x}} \subset \mathscr{T}^{\mathscr{M}^t}_{x^t}$.

Next, we note that due Step 1 together with NDC3, it holds for all $\xi \in \mathbb{R}^n$:

$$\xi^T D^2L^{\mathscr{M}^t}\left(x^t\right)\xi < 0 \text{ if and only if } \xi^T D^2L(\bar{x})\xi < 0.$$

Thus, (4.6) follows immediately.

Step 3: We show

$$\text{MPCC-}QI \geq \text{MPSiC-}QI + \left|a_{01}^{0}(\bar{x})\right| + \left|a_{10}^{0}(\bar{x})\right|. \tag{4.7}$$

From Corollary 4.2 we see that, for all t sufficiently small,

$$a_{01}^{-}(\bar{x}) \cup a_{01}^{+}(\bar{x}) \subset J_1\left(x^t\right) \cup b_{1,\neq}\left(x^t\right) \quad \text{and} \quad a_{10}^{-}(\bar{x}) \cup a_{10}^{+}(\bar{x}) \subset J_2\left(x^t\right) \cup b_{\neq,2}\left(x^t\right).$$

Recalling what we have seen in Step 2, we thus have

$$a_{01}^{-}(\bar{x}) \cup a_{01}^{+}(\bar{x}) \cup a_{00}(\bar{x}) \subset J_1\left(x^t\right) \cup b_{1,\neq}\left(x^t\right) \cup b_{1,2}\left(x^t\right)$$

and

$$a_{10}^{-}(\bar{x}) \cup a_{10}^{+}(\bar{x}) \cup a_{00}(\bar{x}) \subset J_2\left(x^t\right) \cup b_{\neq,2}\left(x^t\right) \cup b_{1,2}\left(x^t\right).$$

Therefore, we have

$$\mathscr{T}_{\bar{x}} = \mathscr{T}_{x^t}^{\mathscr{M}^t} \cap \left\{ \xi \in \mathbb{R}^n \;\middle|\; \begin{array}{l} DF_{1,j}(x)\,\xi = 0, j \in a_{01}^{0}(\bar{x}), \\ DF_{2,j}(x)\,\xi = 0, j \in a_{10}^{0}(\bar{x}) \end{array} \right\}.$$

From this we get (4.7) due to the same argument based on NDC3 as in Step 2.

Step 4: We show

$$\text{MPSiC-}BI - \left|a_{00}^{+}(\bar{x})\right| \leq \text{MPCC-}BI \leq \text{MPSiC-}BI. \tag{4.8}$$

In view of NDC2, we obtain the left-hand side of (4.8):

$$\text{MPSiC-}BI = \left|b_{1,2}\left(x^t\right)\right| \leq \left|a_{00}^{-}(\bar{x})\right| + \left|a_{00}^{+}(\bar{x})\right| = \text{MPCC-}BI + \left|a_{00}^{+}(\bar{x})\right|.$$

Further, due to Corollary 4.2, we get also the right-hand side of (4.8):

$$\text{MPCC-}BI = \left|a_{00}^{-}(\bar{x})\right| \leq \left|b_{1,2}\left(x^t\right)\right| = \text{MPSiC-}BI.$$

□

In Theorem 4.3, let us consider the inequality, which relates the quadratic parts of the T- and C-index of the approximating T-stationary points $x^t \in \mathscr{M}^t$ of MPSiC(t) and the limiting C-stationary point $\bar{x} \in M$ of MPCC. It surely holds MPCC-QI = MPSiC-QI if $a_{01}^{0}(\bar{x}) = \emptyset$ and $a_{10}^{0}(\bar{x}) = \emptyset$, i.e. the multipliers for non-biactive constraints do not vanish. This condition, called NDC4, turned out to be crucial for the stability analysis of the Scholtes regularization for MPCC, see [8]. There, it has been shown to guarantee that the topological type of the corresponding stationary points prevails. Moreover, NDC4 has been proven to hold generically for MPCC. Therefore, we conclude that under NDC4 also in the context of sign-type regularization the quadratic indices MPCC-QI and MPSiC-QI are equal. In absence of NDC4, however, the lower bound for MPCC-QI from Theorem 4.3 can be attained, see Example 4.1. We refer to this phenomenon as the quadratic index shift.

Example 4.1 (Quadratic index shift)**.** We consider the following sign-type regularization $\mathscr{M}^t$ with $n=3$ and $\kappa=3$:

$$\begin{array}{lll} \mathscr{M}^t: & \min\limits_{x} & -x_1 - x_2 + 2x_1x_2 + x_3^2 \\ & \text{s.t.} & x_1 + t \geq 0, x_3 - 2x_2 + t \geq 0, (x_1 - t)\cdot(x_3 - 2x_2 - t) \leq 0, \\ & & x_1 + x_2 + t \geq 0, 2 - x_3 + t \geq 0, (x_1 + x_2 - t)\cdot(2 - x_3 - t) \leq 0, \\ & & x_3 - 1 + t \geq 0, x_1 - x_2 + x_3 + t \geq 0, (x_3 - 1 - t)\cdot(x_1 - x_2 + x_3 - t)\} \leq 0. \end{array}$$

Suppose $0 < t < \frac{1}{3}$. We claim that $x^t = \left(\frac{t}{2}, \frac{t}{2}, 1-t\right)$ is a nondegenerate T-stationary point of $\mathscr{M}^t$. It is straightforward to check that, it is feasible with $J_1\left(x^t\right) = \{3\}$ and $b_{1,>}\left(x^t\right) = \{2\}$. Further, it holds

$$\begin{pmatrix} -1+t \\ -1+t \\ 2-2t \end{pmatrix} = \mu^t_{1,3} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \eta^t_{1,2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

with the unique multipliers $\mu^t_{1,3} = 2-2t$ and $\eta^t_{1,2} = t-1$. Clearly, NDSi1 and NDSi2 are fulfilled. For the tangent space at x^t we obtain

$$\mathscr{T}^{\mathscr{M}^t}_{x^t} = \left\{ \xi \in \mathbb{R}^3 \,|\, \xi_1 = -\xi_2, \xi_3 = 0 \right\}.$$

The Hessian matrix of the Lagrange function at x^t is

$$D^2 L^{\mathscr{M}^t}\left(x^t\right) = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Hence, for $\xi \in \mathscr{T}^{\mathscr{M}^t}_{x^t}$ with $\xi \neq 0$ it holds

$$\xi^T D^2 L^{\mathscr{M}^t}\left(x^t\right) \xi = -4\xi_2^2 < 0.$$

Thus, NDSi3 is fulfilled. Altogether, we have that x^t is nondegenerate. For its indices, we have MPSiC-$QI = 1$ and MPSiC-$BI = 0$.

Note, that the sequence x^t converges to $\bar{x} = (0,0,1)$. In [8] was shown, that $\bar{x}$ is a nondegenerate minimizer of the underlying MPCC. In particular, for the indices of $\bar{x}$ it holds MPCC-$QI = 0$ and MPCC-$BI = 0$. We conclude that the sequence x^t of saddle points of $\mathscr{M}^t$ converges to the minimizer $\bar{x}$ of MPCC. This index shift is caused by the violation of NDC4 at $\bar{x}$. It is not hard to see that this phenomenon can be avoided by almost any C^2-perturbation of the MPCC defining functions.

As a newly observed phenomenon an index shift might also be caused by the biactive constraints with corresponding positive multipliers, i.e. with $a^+_{00}(\bar{x}) \neq \emptyset$. More precisely, the biactive index of a C-stationary point $\bar{x} \in M$ of MPCC is possibly strictly lower than that of the approximating Karush-Kuhn-Tucker points $x^t \in \mathscr{M}^t$ of MPSiC(t). This is demonstrated in Example 4.2. It is noteworthy that the biactive index shift there is stable, i.e. it persists under any sufficiently small C^2-perturbation of the MPCC defining functions. This is the main difference between the quadratic and biactive index shifts in Theorem 4.3. The former is not stable, whereas the latter is.

Example 4.2 (Biactive index shift)**.** We consider the following MPCC with $n = 2$ and $\kappa = 1$:

$$\begin{array}{ll} \min\limits_{x} & (x_1+1)^2 + (x_2+1)^2 \\ \text{s.t.} & x_1 \cdot x_2 = 0, x_1 \geq 0, x_2 \geq 0. \end{array}$$

Its only C-stationary point is $\bar{x} = (0,0)$. Obviously, it is the unique minimizer of MPCC, fulfilling NDC1, NDC2, NDC3, and even NDC4. The C-index of $\bar{x}$ vanishes, i.e. MPCC-$CI = 0$. Let us focus on the corresponding sign-type regularization

$$\begin{array}{lll} \text{MPSiC}(t): & \min\limits_{x} & (x_1+1)^2 + (x_2+1)^2 \\ & \text{s.t.} & x_1 + t \geq 0, x_2 + t \geq 0, (x_1 - t)(x_2 - t) \leq 0, \end{array}$$

where $0 < t < 1$ is taken. Note that $x^t = (t,t)$ is a T-stationary points of MPSiC(t) with multipliers $\zeta^t_{1,1} = 1+t, \zeta^t_{2,1} = 1+t$. It is not hard to see that it is a nondegenerate saddle point with MPSiC-$TI = 1$. Thus, we observe an index shift with a saddle point x^t converging to the minimizer $\bar{x}$ for $t \to 0$. This is due to $a^+_{00}(\bar{x}) = \{1\} \neq \emptyset$. As Theorem 4.3 suggests, the biactive index of $\bar{x}$ might therefore shift. Indeed, we observe for its biactive index MPCC-$BI = 0$, while for the biactive index of x^t we have MPSiC-$BI = 1$. We note that the latter phenomenon persists under any sufficiently small C^2-perturbations of the MPCC defining functions.

□

Let us now focus on the well-posedness of MPSiC(t). In order to state a respective result of [6] the following index sets for MPCC are helpful:

$$a^1_{00}(\bar{x}) = \left\{ j \in a_{00}(\bar{x}) \,\middle|\, \bar{\rho}_{1,j} > 0 \right\}, \quad a^2_{00}(\bar{x}) = \left\{ j \in a_{00}(\bar{x}) \,\middle|\, \bar{\rho}_{2,j} > 0 \right\}.$$

Further, let us define the set of critical directions:

$$\mathscr{C}_{\bar{x}} = \left\{ \xi \in \mathbb{R}^n \,\middle|\, \begin{array}{l} DF_{1,j}(\bar{x})\,\xi = 0, j \in a^-_{01}(\bar{x}) \cup a^+_{01}(\bar{x}) \cup a^1_{00}(\bar{x}), \\ DF_{2,j}(\bar{x})\,\xi = 0, j \in a^-_{10}(\bar{x}) \cup a^+_{10}(\bar{x}) \cup a^2_{00}(\bar{x}) \end{array} \right\}.$$

Using the latter, the following MPCC-tailored second-order sufficient condition was defined in [13].

Definition 4.1 (MPCC-SSOSC, [13])**.** Let MPCC-LICQ hold at an S-stationary point $\bar{x} \in M$ of MPCC with the unique multipliers $(\bar{\sigma}, \bar{\rho})$. We say that it fulfills the strong second order sufficiency condition (MPCC-SSOSC) if for every $\xi \in \mathscr{C}_{\bar{x}} \backslash \{0\}$ it holds:

$$\xi^T D^2 L(\bar{x})\,\xi > 0.$$

With this we are now able to cite Theorem 5.2 from [6] on the well-posedness of the sign-type regularization. We do this under slightly stronger assumptions for the sake of further comparison.

Theorem 4.4 (Well-posedness, [6])**.** *Let $\bar{x}$ be an S-stationary point of MPCC fulfilling MPCC-LICQ, NDC2, and MPCC-SSOSC. Then, for all sufficiently small t there exists an S-stationary point $x^t \in \mathscr{M}^t$ of MPSiC(t) within a neighborhood of $\bar{x}$.*

Note that an S-stationary point $\bar{x}$ satisfying the assumptions of Theorem 4.4 is a strict local minimizer, see [6]. More specifically, under MPCC-LICQ, NDC2, and MPCC-SSOSC it is a nondegenerate local minimizer, since its C-index vanishes, i.e. MPCC-$CI = 0$. Additionally, it follows from the proof of Theorem 5.2, [6] that x^t is also a local minimizer of the corresponding MPSiC(t). However, while the existence of an approximating sequence of S-stationary points of MPSiC(t) can be ensured, the latter do not have to be locally unique, see Example 4.3. Moreover, this bifurcation phenomenon turns out to be stable with respect to sufficiently small C^2-perturbations of the MPCC defining functions.

Example 4.3 (Bifurcation)**.** We consider the MPCC instance given in Example 4.2. We recall that its only C-stationary point is $\bar{x} = (0,0)$. In particular it was the unique minimizer of MPCC, fulfilling NDC1, NDC2, NDC3, and even NDC4. The C-index of $\bar{x}$ vanishes, i.e. MPCC-$CI = 0$. Let us again focus on the corresponding sign-type regularization. The T-stationary points of MPSiC(t) include $\widetilde{x}^t = (-t,t)$ with the multipliers $\widetilde{\mu}^t_{1,1} = 1-t, \widetilde{\eta}^t_{2,1} = 1+t$; and $\widehat{x}^t = (t,-t)$ with the multipliers $\widehat{\mu}^t_{2,1} = 1-t, \widehat{\eta}^t_{1,1} = 1+t$, respectively. It is not hard to see that these T-stationary points are nondegenerate. Moreover, $\widetilde{x}^t$ and $\widehat{x}^t$ are minimizers of MPSiC(t). We observe here the bifurcation phenomenon with two different minimizers $\widetilde{x}^t$ and $\widehat{x}^t$ – both converging to $\bar{x}$ for $t \to 0$. As before, we note that the latter phenomenon persists under any sufficiently small C^2-perturbations of the MPCC defining functions.

As Examples 4.2 and 4.3 demonstrate, bifurcation in the sign-type regularization might even occur when considering a nondegenerate minimizer $\bar{x} \in M$ of MPCC. Particularly, there might exist multiple T-stationary points of MPSiC(t) in a neighborhood of $\bar{x}$. Thus, it becomes apparent from Section 2 that the global structure of the sign-type regularization MPSiC(t) might be much more involved than that of the underlying MPCC. Note that this stable bifurcation also hampers a generalization of Theorem 4.4 for C-stationary points of MPCC and T-stationary points of MPSiC(t), respectively. Without further restrictive assumptions, the local uniqueness of T-stationary points of the sign-type regularization cannot be ensured. Therefore, the implicit function theorem is not applicable here.

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