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ACCELERATING CONTINUOUS-TIME OPTIMIZATION WITH FIXED-TIME TIME-VARYING SECOND-ORDER DYNAMICS

LIEN T. NGUYEN[1], ANDREW EBERHARD[1,*], XINGHUO YU[2], CHAOJIE LI[3]

[1]*School of Science, RMIT University, Melbourne, VIC 3000, Australia*
[2]*School of Engineering, RMIT University, Melbourne, VIC 3000, Australia*
[3]*Department of Electrical Engineering, City University of Hong Kong, Kowloon, Hong Kong*

Abstract. This paper proposes a new fixed-time second-order time-varying dynamics to accelerate continuous-time optimization. We derive a sufficient condition for the existence and uniqueness of solutions to a general ordinary differential equation (ODE) and present a rigorous proof establishing the well-posedness of our proposed algorithm, addressing a significant gap in the literature. We show that the time-varying coefficients in the proposed dynamics can be used to accelerate convergence speed to optimal solutions. The efficiency of the proposed method is demonstrated by numerical simulations on the ridge regression and training neural networks.

Keywords. Fixed-time convergence; Gradient flow; Lyapunov stability; Second-order dynamics; Time-varying systems.

1. INTRODUCTION

Continuous-time optimization methods have received increasing interest in recent years. This is due to the need for fast algorithms to solve optimization problems such as machine learning applications where large data sets and long computational time are challenging. For instance, Brown and Bartholomew-Biggs [4] employed continuous-time dynamical systems to derive fast discrete-time methods. Su, Boyd, and Candes [22] utilized ordinary differential equations (ODEs) to model Nesterov's scheme. Despite much progress, the majority of the algorithms are grounded on asymptotic convergence theories.

Recent years have seen the emergence of finite-time control systems due to the need to deliver fast convergence [25]. Early work [3, 10] established a theoretical foundation for achieving stability in autonomous systems within a finite time. This was subsequently extended to solve a broad range of control problems, including terminal sliding mode control [25] and the finite-time stabilization of nonlinear systems [7, 9, 23, 26]. Nevertheless, the settling time may be dependent on the initial conditions, possibly becoming very large when the system state is far away from the equilibrium. To overcome this limitation, fixed-time stability was introduced in [20], which guarantees uniform upper bounds on the settling time for all initial conditions.

*Corresponding author.
E-mail address: thuy.lien.nguyen@rmit.edu.au (L.T. Nguyen), andy.eberhard@rmit.edu.au (A. Eberhard), xinghuo.yu@rmit.edu.au (X. Yu), Chaojie.li@unsw.edu.au (C. Li)

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Addressing continuous-time optimization problems using the finite-time/fixed-time stability properties of dynamical systems has received increasing interest. For dealing with objective functions that are twice continuously differentiable and strongly convex, [6] introduced two modified gradient flow algorithms that exhibit finite-time convergence. Without requiring the strong convexity, building upon the Kurdyka–Łojasiewicz (KL) exponent concept for the objective function, [21] presented two dynamical systems which ensure finite-time convergence if the starting point is close enough to the optimal solution. In [14], a fixed-time convergent gradient flow was introduced, relying on the strongly convex assumption of the objective function. Under the Polyak–Łojasiewicz (PL) condition, [8] also proposed a fixed-time convergent algorithm for addressing continuous-time optimization problems. Still assuming the PL condition, in [17], a fixed-time first order dynamics with time-varying coefficients was proposed to address the unconstrained optimization problems. In the case when the objective function only satisfies the KL condition, [16] proved the global finite-time convergence of the algorithm in [17] instead of a local result as in [21].

Second-order dynamics have been shown to be able to further enhance optimization performance, for example, the continuous-time model of Nesterov's accelerated gradient algorithm [22]. Many studies have investigated second-order sliding modes to achieve finite-time or fixed-time convergence [2, 19, 24]. On the other hand, to address optimization problems, in [5], a fixed-time convergent dynamics with momentum was developed under the strong convexity condition. However, the existence and uniqueness of solutions to this dynamical system is not guaranteed and is non-trivial. Also, this algorithm is a variation of the heavy ball method [18] which only has a convergence rate of $O\left(\frac{1}{t}\right)$ for convex functions and this rate does not surpass that of the gradient descent method. In [1], a second-order fixed-time time-invariant dynamics was proposed where the objective function satisfies the PL condition and has a unique minimizer. However, the existence and uniqueness of a solution to this dynamical system remain unproven. If a solution does not exist, fixed-time convergence cannot be achieved. Furthermore, an interesting question arises whether the time-varying coefficients for the second-order dynamics can further enhance the optimization performance, similar to the results in [17] for the first-order dynamics.

This paper attempts to answer the question above. We propose a new class of second-order time-varying dynamics for finding the minimizer of strongly convex functions in fixed time. We establish a sufficient condition for the existence and uniqueness of solutions to the ODE and then provide a rigorous proof for the existence and uniqueness of solutions of our proposed algorithm, thereby addressing a gap in the literature. Moreover, we show that using time-varying coefficients can offer flexibility to vary convergence speed more precisely for different purposes. We also show that Nesterov's accelerated gradient method can be considered a "special case" of our algorithm. To the best of our knowledge, this is the first work proposing a fixed-time second-order dynamics with time-varying coefficients to further accelerate optimization convergence.

The remaining of the paper is structured as follows. Section 2 provides sufficient conditions for the existence and uniqueness of ODE solutions. In Section 3, we introduce new fixed-time second-order time-varying dynamical systems to solve continuous-time optimization problems. Section 4 provides some numerical simulations to illustrate the efficacy of the proposed algorithm.

2. Existence and Uniqueness of Solutions Revisited

In this section, we provide a sufficient condition for the existence and uniqueness of solutions of dynamical system

$$\begin{cases} \dot{z}(t) & = \mathscr{G}(t, z(t)), \\ z(t_0) & = z_0, \end{cases} \tag{2.1}$$

where $\mathscr{G} : D \times \mathbb{R}^d \to \mathbb{R}^d$, D is an open interval in $\mathbb{R}$, and $(t_0, z_0) \in D \times \mathbb{R}^d$. We then use this condition to prove the existence and uniqueness of the proposed algorithm in Section 3.

Proposition 2.1. *Let $\mathscr{G} : (\tau_0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous, where $\tau_0 \in (-\infty, t_0)$. Suppose that $\mathscr{G}(t, w)$ is locally Lipschitz continuous in $w \in \mathbb{R}^d \setminus S_0$, where S_0 is a closed subset of $\mathbb{R}^d$, and the following hold:*

(1) For all $\bar{w} \in S_0$, there exists a function $V_{\bar{w}} : [t_0, +\infty) \times \mathbb{R}^d \to \mathbb{R}_+$ such that, for all $t \in [t_0, +\infty)$, $V_{\bar{w}}(t, \bar{w}) = 0$, $V_{\bar{w}}(t, w) > 0$ for all $w \neq \bar{w}$, and for every solution $z : J \to \mathbb{R}^d$ of (2.1) with $J \subseteq [t_0, +\infty)$, $\frac{d}{dt} V_{\bar{w}}(t, z(t)) \leq 0$.

(2) There exists a function $V : [t_0, +\infty) \times \mathbb{R}^d \to \mathbb{R}_+$ such that for any solution $z : J \to \mathbb{R}^d$ of (2.1) with $J \subseteq [t_0, +\infty)$, $\frac{d}{dt} V(t, z(t)) \leq 0$. Moreover, for any interval $I \subseteq [t_0, +\infty)$, $\{w \in \mathbb{R}^d : V(t, w) \leq V(t_0, z_0) \ \forall t \in I\}$ is bounded, and for all $t \in [t_0, +\infty)$, $V(t, w) = 0 \iff w \in S_0$.

Then there exists a unique solution $z : [t_0, +\infty) \to \mathbb{R}^d$ of (2.1).

Proof. Since $\mathscr{G}$ is continuous, according to [11, Theorems 1.1 and 2 .1], there exists a right maximally defined solution $z : [t_0, \bar{\tau}) \to \mathbb{R}^d$ of (2.1). By assumption (2), for all $t \in [t_0, \bar{\tau})$, $V(t, z(t)) \leq V(t_0, z(t_0)) = V(t_0, z_0)$, and so $(z(t))_{t \in [t_0, \bar{\tau})}$ is bounded. By using [3, Proposition 5.1], $\bar{\tau} = +\infty$. We now distinguish the following two cases.

Case 1: $z_0 \in S_0$. Let $z : [t_0, +\infty) \to \mathbb{R}^d$ be a solution of (2.1). It follows from (1) that for all $t \geq t_0$, $V_{z_0}(t, z(t)) \leq V_{z_0}(t_0, z_0) = 0$. By (1), for all $t \geq t_0$, $z(t) = z_0$, and hence (2.1) has the unique solution $z(\cdot) \equiv z_0$ on $[t_0, +\infty)$.

Case 2: $z_0 \notin S_0$. Suppose that there is more than one solution of dynamical system (2.1). Take any two arbitrary solutions z_1, z_2 of (2.1). Set $S := \{t \in [t_0, +\infty) : z_1(t) \in S_0 \text{ or } z_2(t) \in S_0\}$. If $S \neq \varnothing$, then it follows from the closedness of S_0 that S is closed, and we set $t_1 := \min_{t \in S} t$. If $S = \varnothing$, set $t_1 = +\infty$. Take an arbitrary $t_2 < t_1$. By the definition of t_1, for any $s \in [t_0, t_2]$, $z_1(s) \notin S_0$, $z_2(s) \notin S_0$, which yields $V(t_2, z_1(t_2)) > 0$ and $V(t_2, z_2(t_2)) > 0$ (by (2)). Moreover, $t \mapsto V(t, z_1(t))$ and $t \mapsto V(t, z_2(t))$ are nonincreasing on $[t_0, +\infty)$, and hence, for all $s \in [t_0, t_2]$,

$$0 < V(t_2, z_1(t_2)) \leq V(s, z_1(s)) \leq V(t_0, z_0),$$
$$0 < V(t_2, z_2(t_2)) \leq V(s, z_2(s)) \leq V(t_0, z_0).$$

Set $U := \{w \in \mathbb{R}^d : \min\{V(t_2, z_1(t_2)), V(t_2, z_2(t_2))\} \leq V(s, w) \leq V(t_0, z_0) \ \forall s \in [t_0, t_2]\}$. It follows from the continuity of V that U is closed. By (2), U is bounded, which implies that U is a compact set. Moreover, $S_0 \not\subseteq U$ since $V(t, w) = 0 \iff w \in S_0$. Therefore, for all $s \in [t_0, t_2]$, $z_1(s) \in U$, $z_2(s) \in U$ and $S_0 \not\subseteq U$.

By the locally Lipschitz property of $\mathscr{G}(t, w)$ in $w \in \mathbb{R}^d \setminus S_0$, there exists $C = C(U)$ such that for all $s \in [t_0, t_2]$,

$$\|\mathscr{G}(s, z_1(s)) - \mathscr{G}(s, z_2(s))\| \leq C \|z_1(s) - z_2(s)\|. \tag{2.2}$$

As z_1, z_2 are solutions of (2.1), for all $t \in [t_0, t_2]$, $z_i(t) = z_0 + \int_{t_0}^t \mathscr{G}(s, z_i(s))ds, \quad i = 1,2$. It follows from (2.2) that

$$\begin{aligned}\|z_1(t) - z_2(t)\| &= \left| \int_{t_0}^t (\mathscr{G}(s, z_1(s)) - \mathscr{G}(s, z_2(s)))\, ds \right| \\ &\leq C \int_{t_0}^t \|z_1(s) - z_2(s)\| ds.\end{aligned}$$

According to [11, Corollary 6.6], $\|z_1(t) - z_2(t)\| = 0$ for all $t \in [t_0, t_2]$, and so $z_1 = z_2$ on $[t_0, t_2]$. Since t_2 is an arbitrary point in $[t_0, t_1)$, it holds that $z_1 = z_2$ on $[t_0, t_1)$. If $t_1 = +\infty$, then $z_1 = z_2$ on $[t_0, +\infty)$. If $t_1 < +\infty$, then by the definition of t_1, we have $z_1(t_1) \in S_0$ or $z_2(t_1) \in S_0$. By the continuity of z_1, z_2 and the fact that $z_1 = z_2$ on $[t_0, t_1)$, we derive that $z_1(t_1) = z_2(t_1) := \bar{z}_1 \in S_0$. Using the same argument as in *Case 1*, it follows that $z_1(t) = z_2(t) = \bar{z}_1$ for all $t \in [t_1, +\infty)$, which implies that $z_1 = z_2$ on $[t_0, +\infty)$. Therefore, for any two arbitrary solutions z_1, z_2 of (2.1) on $[t_0, +\infty)$, we have $z_1 = z_2$ on $[t_0, +\infty)$, and hence there is a unique solution of (2.1) on $[t_0, +\infty)$. □

If S_0 contains a unique point, then the following holds.

Corollary 2.1. *Let $\mathscr{G} : (\tau_0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous, where $\tau_0 < t_0$. Suppose that $\mathscr{G}(t, w)$ is locally Lipschitz continuous in $w \in \mathbb{R}^d \setminus \{\bar{w}\}$ and there exists a function $V : [t_0, +\infty) \times \mathbb{R}^d \to \mathbb{R}_+$ such that for any solution $z : J \to \mathbb{R}^d$ of (2.1) with $J \subseteq [t_0, +\infty)$, $\frac{d}{dt}V(t, z(t)) \leq 0$, and for any interval $I \subseteq [t_0, +\infty)$, $\{w \in \mathbb{R}^d : V(t, w) \leq V(t_0, z_0)\ \forall t \in I\}$ is bounded, for all $t \in [t_0, +\infty)$, $V(t, w) = 0$ if and only if $w = \bar{w}$. Then there exists a unique solution $z : [t_0, +\infty) \to \mathbb{R}^d$ of (2.1).*

Proof. Set $V_{\bar{w}} := V$. The conclusion follows from Proposition 2.1. □

3. Second-order Fixed Time Dynamics

We consider the problem

$$\min_{x \in \mathbb{R}^d} f(x),$$

where $f \colon \mathbb{R}^d \to \mathbb{R}$ is differentiable, and propose second-order dynamics as accelerated gradient algorithms

$$\begin{cases} \dot{x}(t) = -u(t)\theta(t, x(t), u(t)) \\ \dot{u}(t) = \lambda(t)(\nabla f(x(t)) - \alpha(t)u(t))\omega(t, x(t), u(t)) - \ell(t)(\nabla f(x(t)) - \alpha(t)u(t)) \\ x(t_0) = x_0, u(t_0) = u_0. \end{cases} \tag{3.1}$$

where $x_0, u_0 \in \mathbb{R}^d$ and $t_0 \in [0, +\infty)$. Here $\theta : [t_0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is defined by $\theta(t, y, v) = \gamma(t)\psi(t, \max\{\|\nabla f(y)\|), \kappa\|\nabla f(y) - \alpha(t)v\|\} + k(t)$ and $\psi(t, s) = \begin{cases} \frac{\beta_1(t)}{s^p} + \frac{\beta_2(t)}{s^q} & \text{if } s > 0 \\ 0 & \text{if } s = 0, \end{cases}$ with $0 \leq p < 1$, $q \leq 0$, where $\beta_1, \beta_2 : [t_0, +\infty) \to (0, +\infty)$. The function $\omega : [t_0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is given by either $\omega(t, y, v) = \psi(t, \kappa\|\nabla f(y) - \alpha(t)v\|)$ or $\omega(t, y, v) = \theta(t, y, v)$ for all $(t, y, v) \in [t_0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

It can be observed that if $p=q=0$, and for all $t\in[t_0,+\infty)$, $\alpha(t)=\frac{\alpha}{t}$, $\beta_1(t)=\beta_2(t)=\lambda(t)=\gamma(t)=1$, and $l(t)=k(t)=0$, then (3.1) becomes

$$\begin{cases}\dot{x}(t)=-u(t)\\ \dot{u}(t)=\nabla f(x(t))-\frac{\alpha}{t}u(t),\end{cases}\tag{3.2}$$

which corresponds to Nesterov's accelerated gradient algorithm [15] when $\alpha=3$ and for $\alpha\geq 3$, and the trajectories of (3.2) satisfy $f(x(t))-\inf_{\mathscr{H}} f=O(\frac{1}{t^2})$. We will prove that the trajectories of (3.1) converge to a minimizer of f in fixed time if $p\in(0,1)$ and $q<0$.

Moreover, our algorithm provides us with more flexibility in choosing not only the parameters but also the function ω. We have two options for ω: either $\omega(t,y,v)=\psi(t,\kappa\|\nabla f(y)-\alpha(t)v\|)$ or $\omega(t,y,v)=\theta(t,y,v)$ for all $(t,y,v)\in[t_0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. Additionally, the terms $\ell(t)(\nabla f(x(t))-\alpha(t)u(t))$ in the equation related to $\dot{u}$ and the function $k(t)$ in the definition of θ also contribute to speeding up convergence. We will delve further into these aspects in the numerical simulation section. In order to reduce notational complexity, we drop explicitly stating the dependence of (x,u) on t and the dependence of θ and ω on $(t,x(t),u(t))$.

We now prove the existence and uniqueness of solutions to dynamical system (3.1). It can be seen that the dynamics (3.1) can be written as

$$\begin{cases}\dot{Z}(t)+\mathscr{G}(t,Z(t))=0\\ Z(t_0)=(x_0,u_0)\end{cases}\tag{3.3}$$

where $Z(t)=(x(t),u(t))$ and $\mathscr{G}:[t_0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^d$ is defined by $\mathscr{G}(t,y,v)=(\mathscr{G}_1(t,y,v),\mathscr{G}_2(t,y,v))$ with $\mathscr{G}_1(t,y,v)=v\theta(t,y,v)$ and

$$\mathscr{G}_2(t,y,v)=-\lambda(t)(\nabla f(y)-\alpha(t)v)\omega(t,y,v)+\ell(t)(\nabla f(y)-\alpha(t)v).$$

Lemma 3.1. *Let $f\colon\mathbb{R}^d\to\mathbb{R}$ be a differentiable function and let $\alpha,\beta_1,\beta_2,\lambda,\gamma:[t_0,+\infty)\to(0,+\infty)$, $\ell:[t_0,+\infty)\to\mathbb{R}$, and $k:[t_0,+\infty)\to[0,+\infty)$ be continuous functions. Then $\mathscr{G}$ in (3.3) is continuous on $[t_0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.*

Proof. Set $\phi(t,y,v)=\max\{\|\nabla f(y)\|,\kappa\|\nabla f(y)-\alpha(t)v\|\}$. Then $\phi(t,y,v)=0$ if and only if $\nabla f(y)=0$ and $v=0$. Take an arbitrary critical point $\bar{y}$ of f. It can be seen that $\mathscr{G}_1$ is continuous at any (t,y,v) where $y\neq\bar{y}$ or $v\neq 0$. So, let us now consider the continuity of $\mathscr{G}_1$ at $(\bar{t},\bar{y},0)$ with $\bar{t}\in[t_0,+\infty)$. By Cauchy–Schwarz inequality, for all (t,y,v) such that $\phi(t,y,v)\neq 0$, we have

$$\begin{aligned}\frac{\|v\|}{\phi^p(t,y,v)}&\leq\frac{1}{\alpha(t)}\frac{\|\nabla f(y)\|+\|\nabla f(y)-\alpha(t)v\|}{\phi^p(t,y,v)}\\&\leq\frac{1}{\alpha(t)}\left(1+\frac{1}{\kappa}\right)\phi^{1-p}(t,y,v).\end{aligned}$$

Since $\phi(t,\bar{y},0)=0$, $\lim_{\substack{(t,y,v)\to(\bar{t},\bar{y},0)\\ \phi(t,y,v)\neq 0}}\|\mathscr{G}_1(t,y,v)\|=0$. Moreover, $\lim_{\substack{(t,y,v)\to(\bar{t},\bar{y},0)\\ \phi(t,y,v)=0}}\|\mathscr{G}_1(t,y,v)\|=0$. Thus $\lim_{(t,y,v)\to(\bar{t},\bar{y},0)}\mathscr{G}_1(t,y,v)=0=\mathscr{G}_1(\bar{t},\bar{y},0)$, and so $\mathscr{G}_1$ is continuous at any $(t,y,v)\in[t_0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

Now, let us prove the continuity of $\mathscr{G}_2$. We distinguish the following two cases.

Case 1: $\omega(t,y,v)=\psi(t,\kappa\|\nabla f(y)-\alpha(t)v\|)$. It can be seen that $\mathscr{G}_2$ is continuous at any $(t,y,v)\in[t_0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ with $\|\nabla f(y)-\alpha(t)v\|\neq 0$. Moreover,

$$\begin{aligned}\|\mathscr{G}_2(t,y,v)\| \leq{}& \frac{1}{\kappa^p}\lambda(t)\beta_1(t)\|\nabla f(y)-\alpha(t)v\|^{1-p}+\frac{1}{\kappa^q}\lambda(t)\beta_2(t)\|\nabla f(y)-\alpha(t)v\|^{1-q}\\ &+|\ell(t)|\|\nabla f(y)-\alpha(t)v\|.\end{aligned}$$

Now take an arbitrary $(\tilde{t},\tilde{y},\tilde{v})$ such that $\|\nabla f(\tilde{y})-\alpha(\tilde{t})\tilde{v}\|=0$. Then, $\lim_{(t,y,v)\to(\tilde{t},\tilde{y},\tilde{v})}\mathscr{G}_2(t,y,v)=0=\mathscr{G}_2(\tilde{t},\tilde{y},\tilde{v})$. Thus, $\mathscr{G}_2$ is continuous at any $(t,y,v)\in[t_0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

Case 2: $\omega(t,y,v)=\theta(t,y,v)$. Take an arbitrary critical point $\bar{y}$ of f. It can be seen that $\mathscr{G}_2$ is continuous at any (t,y,v) where $y\neq\bar{y}$ or $v\neq 0$. We now prove the continuity of $\mathscr{G}_2$ at any $(\bar{t},\bar{y},0)$ with $\bar{t}\in[t_0,+\infty)$. Indeed, using Cauchy–Schwarz inequality, we derive that

$$\begin{aligned}\|\nabla f(y)-\alpha(t)v\| &\leq \|\nabla f(y)\|+\|\alpha(t)v\|\\ &\leq \|\nabla f(y)\|+(\|\nabla f(y)\|+\|\nabla f(y)-\alpha(t)v\|)\\ &\leq \left(2+\frac{1}{\kappa}\right)\phi(t,y,v),\end{aligned}$$

which implies that

$$\begin{aligned}\|\mathscr{G}_2(t,y,v)\| \leq{}& \left(2+\frac{1}{\kappa}\right)\lambda(t)\beta_1(t)\phi^{1-p}(t,y,v)+\|\nabla f(y)-\alpha(t)v\|\lambda(t)\beta_2(t)\phi^{-q}(t,y,v)\\ &+\|\nabla f(y)-\alpha(t)v\|\left(\lambda(t)k(t)-\ell(t)\right).\end{aligned}$$

Combining this with the fact that $p\in(0,1)$ and $q\in(-\infty,0)$, we have $\lim_{(t,y,v)\to(\bar{t},\bar{y},0)}\mathscr{G}_2(t,y,v)=0=\mathscr{G}_2(\bar{t},\bar{y},0)$ and hence we obtain the continuity of $\mathscr{G}_2$. Therefore, $\mathscr{G}$ is continuous on $[t_0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. □

Lemma 3.2. *Suppose that* ∇f *is L-Lipschitz continuous. Then* $\mathscr{G}(t,y,v)$ *is locally Lipschitz in* $(y,v)\in(\mathbb{R}^d\setminus\{x^*\})\times(\mathbb{R}^d\setminus\{0\})$.

Proof. Let U_0 be an compact set in $(\tau_0,+\infty)$, U_1 be an compact set in $\mathbb{R}^d\setminus\{x^*\}$, and U_2 be an compact set in $\mathbb{R}^d\setminus\{0\}$. For all $s\in U_0, y_1,y_2\in U_1$, $v_1,v_2\in U_2$,

$$\begin{aligned}\|\mathscr{G}_1(s,y_1,v_1)-\mathscr{G}_1(s,y_2,v_2)\| &= \|v_1\theta(s,y_1,v_1)-v_2\theta(s,y_2,v_2)\|\\ &\leq \|v_1\|\|\theta(s,y_1,v_1)-\theta(t,y_2,v_2)\|+\|v_1-v_2\|\theta(s,y_2,v_2).\end{aligned}\tag{3.4}$$

Set $\phi(s,y,v)=\max\{\|\nabla f(y)\|,\kappa\|\nabla f(y)-\alpha(t)v\|\}$. Since the function x^α with $\alpha\in(0,+\infty)$ is locally Lipschitz continuous everywhere except 0, for $p\in(0,1)$, there exists K_p such that, for all $s\in U_0$, $y_1,y_2\in U_1$, and $v_1,v_2\in U_2$,

$$\begin{aligned}|\phi^p(s,y_2,v_2)-\phi^p(s,y_1,v_1)| &\leq K_p|\phi(s,y_2,v_2)-\phi(s,y_1,v_1)|\\ &\leq K_p\|\nabla f(y_2)-\nabla f(y_1)\|+K_p\kappa\|\nabla f(y_2)-\nabla f(y_1)\|\\ &\quad+K_p\kappa\max_{s\in U_0}\alpha(s)\|v_2-v_1\|\\ &\leq K_pL(1+\kappa)\|y_2-y_1\|+K_p\kappa\max_{s\in U_0}\alpha(s)\|v_2-v_1\|.\end{aligned}\tag{3.5}$$

On the other hand, since ϕ is continuous, $U_1 \subseteq \mathbb{R}^d \setminus \{x^*\}$, $U_2 \subseteq \mathbb{R}^d \setminus \{0\}$, and U_i $(i = 0,1,2)$ are compact sets, there exists $m_\phi, M_\phi > 0$ such that $m_\phi \leq \phi(s,y_2,v_2) \leq M_\phi$. Therefore,

$$\left|\frac{1}{\phi^p(s,y_2,v_2)} - \frac{1}{\phi^p(s,y_1,v_1)}\right| \leq \frac{K_p L(1+\kappa)}{m_\phi^{2p}}\|y_2 - y_1\| + \frac{\kappa K_p}{m_\phi^{2p}} \max_{s\in U_0} \alpha(s)\|v_2 - v_1\|.$$

Similar to (3.5), for any $q < 0$,

$$\begin{aligned}&|\phi^{-q}(s,y_2,v_2) - \phi^{-q}(s,y_1,v_1)| \\ &\leq K_{-q}L(1+\kappa)\|y_2 - y_1\| + K_{-q}\kappa \max_{s\in U_0} \alpha(s)\|v_2 - v_1\|.\end{aligned}$$

This, together with the definition of θ, yields that there exists $C_1, C_2 > 0$ such that

$$\|\theta(s,y_1,v_1) - \theta(s,y_2,v_2)\| \leq C_1\|y_1 - y_2\| + C_2\|v_1 - v_2\|.$$

Combining this with (3.4), it follows that

$$\|\mathscr{G}_1(s,y_1,v_1) - \mathscr{G}_1(s,y_2,v_2)\| \leq \bar{C}_1\|y_1 - y_2\| + \bar{C}_2\|v_1 - v_2\|.$$

Similarly, $\mathscr{G}(t,y,v)$ is locally Lipschitz in $(y,v) \in (\mathbb{R}^d \setminus \{x^*\}) \times (\mathbb{R}^d \setminus \{0\})$. Therefore, there is k such that, for all $s \in U_0, y_1, y_2 \in U_1$ and $v_1, v_2 \in U_2$, $\|\mathscr{G}(s,y_1,v_1) - \mathscr{G}(s,y_2,v_2)\| \leq k\|(y_1,v_1) - (y_2,v_2)\|$. The proof is complete. □

The existence and uniqueness of the global solution of (3.1) will be established in Theorem 3.1. The analysis of the fixed-time convergence of the proposed algorithm relies on the following two lemmas.

Lemma 3.3. *Let $g : [t_0,+\infty) \to [0,+\infty)$ and, for $i \in \{1,2,3\}$, let $u_i : [t_0,+\infty) \to [0,+\infty)$ and $m_i \in (0,+\infty)$ with $m_1 \leq m_3 \leq m_2$. Then the following hold:*

(1) $g(t)^{m_3} \leq g(t)^{m_1} + g(t)^{m_2}$.
(2) $-u_1(t)g(t)^{m_1} - u_2(t)g(t)^{m_2} + u_3(t)g(t)^{m_3} \leq -(u_1(t) - u_3(t))g(t)^{m_1} - (u_2(t) - u_3(t))g(t)^{m_2}$.

Proof. (1): If $g(t) \leq 1$, then $g(t)^{m_3} \leq g(t)^{m_1}$ since $m_1 \leq m_3$. If $g(t) \geq 1$, then $g(t)^{m_3} \leq g(t)^{m_2}$ since $m_3 \leq m_2$. Therefore, $g(t)^{m_3} \leq g(t)^{m_1} + g(t)^{m_2}$.

(2): From (1), we have $u_3(t)g(t)^{m_3} \leq u_3(t)g(t)^{m_1} + u_3(t)g(t)^{m_2}$, which implies the conclusion. □

We now derive a critical inequality for our Lyapunov convergence analysis.

Lemma 3.4. *Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is twice differentiable μ-strongly convex function with L-Lipschitz continuous gradient and that $\dot{\alpha}(t) \leq 0$. Let $(x,u)\colon [t_0,\bar{\tau}) \to \mathbb{R}^d \times \mathbb{R}^d$ be a solution to (3.1), where $\bar{\tau} > 0$. For any $t \in [t_0,\bar{\tau})$, set $V(t) := \alpha(t)\|\nabla f(x)\|^2 + \alpha(t)\|\nabla f(x) - \alpha(t)u\|^2$ and set $\ell_+(t) = \max\{\ell(t), 0\}$. Then, for any $t \in [t_0,\bar{\tau})$,*

$$\begin{aligned}\dot{V}(t) \leq &-2\theta\mu\|\nabla f(x)\|^2 + 2\theta L\|\nabla f(x) - \alpha(t)u\|^2 \\ &- 2\lambda(t)\left(\alpha^2(t)\omega - \ell_+(t)\right)\|\nabla f(x) - \alpha(t)u\|^2. \end{aligned} \tag{3.6}$$

Proof. By the definition of V,

$$\begin{aligned}\dot V(t) &= 2\alpha(t)\langle \nabla f(x), \nabla^2 f(x)\dot x(t)\rangle + \dot\alpha(t)\|\nabla f(x)\|^2 + \dot\alpha(t)\|\nabla f(x)-\alpha(t)u\|^2\\ &\quad + 2\alpha(t)\langle \nabla f(x)-\alpha(t)u, \nabla^2 f(x)\dot x(t)\rangle - 2\alpha(t)\langle \nabla f(x)-\alpha(t)u, \dot\alpha(t)u+\alpha(t)\dot u(t)\rangle\\ &= -2\theta\langle \nabla f(x), \nabla^2 f(x)\nabla f(x)\rangle + 2\theta\langle \nabla f(x), \nabla^2 f(x)(\nabla f(x)-\alpha(t)u)\rangle\\ &\quad + \dot\alpha(t)\|\nabla f(x)\|^2 - 2\theta\langle \nabla f(x)-\alpha(t)u, \nabla^2 f(x)\alpha(t)u\rangle - 2\dot\alpha(t)\langle \nabla f(x)-\alpha(t)u, \alpha(t)u\rangle\\ &\quad + 2\alpha^2(t)\ell(t)\|\nabla f(x)-\alpha(t)u\|^2 - 2\lambda(t)\alpha^2(t)\omega\|\nabla f(x)-\alpha(t)u\|^2\\ &\quad + \dot\alpha(t)\|\nabla f(x)-\alpha(t)u\|^2. \end{aligned}\tag{3.7}$$

Since f is a twice differentiable μ-strongly convex function and ∇f is L-Lipschitz continuous, we obtain

$$\begin{aligned}&\langle \nabla f(x), \nabla^2 f(x)\nabla f(x)\rangle \geq \mu\|\nabla f(x)\|^2 \quad \text{and}\\ &\langle \nabla f(x), \nabla^2 f(x)(\nabla f(x)-\alpha(t)u)\rangle - \langle \nabla f(x)-\alpha(t)u, \nabla^2 f(x)\alpha(t)u\rangle\\ &= \langle \nabla f(x)-\alpha(t)u, \nabla^2 f(x)(\nabla f(x)-\alpha(t)u)\rangle\\ &\leq L\|\nabla f(x)-\alpha(t)u\|^2. \end{aligned}\tag{3.8}$$

Moreover, since $\dot\alpha(t)\leq 0$ for any $t\in[t_0,+\infty)$, we have

$$\begin{aligned}&-2\dot\alpha(t)\langle \nabla f(x)-\alpha(t)u, \alpha(t)u\rangle\\ &= 2\dot\alpha(t)\|\nabla f(x)-\alpha(t)u\|^2 - 2\dot\alpha(t)\langle \nabla f(x), \nabla f(x)-\alpha(t)u\rangle\\ &\leq (2\dot\alpha(t)+|\dot\alpha(t)|)\|\nabla f(x)-\alpha(t)u\|^2 + |\dot\alpha(t)|\|\nabla f(x)\|^2\\ &\leq \dot\alpha(t)\|\nabla f(x)-\alpha(t)u\|^2 + |\dot\alpha(t)|\|\nabla f(x)\|^2. \end{aligned}$$

This, together with (3.7) and (3.8) yields that

$$\begin{aligned}\dot V(t) &\leq -(2\theta\mu-\dot\alpha(t))\|\nabla f(x)\|^2 + 2\theta L\|\nabla f(x)-\alpha(t)u\|^2\\ &\quad + \dot\alpha(t)\|\nabla f(x)-\alpha(t)u\|^2 + |\dot\alpha(t)|\|\nabla f(x)\|^2 + 2\alpha^2(t)\ell_+(t)\|\nabla f(x)-\alpha(t)u\|^2\\ &\quad - 2\lambda(t)\alpha^2(t)\omega\|\nabla f(x)-\alpha(t)u\|^2 + \dot\alpha(t)\|\nabla f(x)-\alpha(t)u\|^2\\ &\leq -2\theta\mu\|\nabla f(x)\|^2 + 2\theta L\|\nabla f(x)-\alpha(t)u\|^2 - 2\lambda(t)\left(\alpha^2(t)\omega-\ell_+(t)\right)\|\nabla f(x)-\alpha(t)u\|^2, \end{aligned}$$

where the last inequality is obtained by using the assumption that $\ell_+(t)=\max\{\ell(t),0\}$ and $\dot\alpha(t)\leq 0$. □

We now provide the fixed-time convergence of our proposed algorithm.

Theorem 3.1 (Fixed-time convergence to a minimizer)**.** *Suppose that $f:\mathbb{R}^d\to\mathbb{R}$ is a twice differentiable μ-strongly convex function whose gradient is L-Lipschitz continuous, $p\in(0,1)$, $q<0$, $\dot\alpha(t)\leq 0$, $\inf_{t\in[t_0,\tau)}\alpha(t)>0$ for all $\tau<+\infty$, $\int_{t_0}^{+\infty}\gamma(t)\beta_1(t)\left(\frac{1}{\alpha(t)}\right)^{\frac{2-p}{2}}dt=+\infty$ and $\int_{t_0}^{+\infty}\gamma(t)\beta_2(t)\left(\frac{1}{\alpha(t)}\right)^{\frac{2-q}{2}}dt=+\infty$, and one of the following holds:*

(1) $\omega(t,x,u)=\psi(t,\kappa\|\nabla f(x)-\alpha(t)u\|)$, $\kappa>\sqrt{\frac{L}{\mu}}$, $u_1(t)>0$, $\int_{t_0}^{+\infty}u_1(t)dt=+\infty$, $u_2(t)>0$, and $\int_{t_0}^{+\infty}u_2(t)dt=+\infty$, where $u_1(t)=\left(\kappa^{-p}v_1(t)\beta_1(t)-v_2(t)\right)\alpha(t)^{\frac{p-2}{2}}$,

$u_2(t) = (\kappa^{-q} v_1(t)\beta_2(t) - v_2(t))\,\alpha(t)^{\frac{q-2}{2}}$, $v_1(t) = \lambda(t)\alpha^2(t) - L\gamma(t)$ and $v_2(t) = \alpha^2(t)\ell_+(t)$ $-Lk(t)$.

(2) $\omega(t,x,u) = \theta(t,x,u)$, $\ell(t) \le 0$ and $\lambda(t)\alpha^2(t) > L$.

Then for any initial value (x_0,u_0), there exists a unique solution $(x,u)\colon [t_0,+\infty) \to \mathbb{R}^d \times \mathbb{R}^d$ of (3.1). Moreover $x(t)$ converges to the minimizer x^* of f in fixed time and $(x^*,0)$ is fixed-time stable.

Proof. By Lemma 3.1 and [11, Theorem 2.1], there exists a right maximally defined solution $(x,u)\colon [t_0,\bar{\tau}) \to \mathbb{R}^d \times \mathbb{R}^d$ of dynamical system (3.1), where $\bar{\tau} > t_0$. Let $V(t) := \alpha(t)\|\nabla f(x(t))\|^2 + \alpha(t)\|\nabla f(x(t)) - \alpha(t)u(t)\|^2$, for $t \in [t_0,\bar{\tau})$.

(1): $\omega(t,x,u) = \psi(t,\kappa\|\nabla f(x) - \alpha(t)u\|)$. Applying Lemma 3.4, we obtain that, for all $t \in [t_0,\bar{\tau})$,

$$\begin{aligned}\dot{V}(t) \le& -2\theta\mu\|\nabla f(x)\|^2 + 2\theta L\|\nabla f(x) - \alpha(t)u\|^2\\&-2\lambda(t)\alpha^2(t)\psi(t,\kappa\|\nabla f(x) - \alpha(t)u\|)\|\nabla f(x) - \alpha(t)u\|^2\\&+2\alpha^2(t)\ell_+(t)\|\nabla f(x) - \alpha(t)u\|^2.\end{aligned}\tag{3.9}$$

By the definition of ψ and Lemma 3.3,

$$\begin{aligned}&-\lambda(t)\alpha^2(t)\psi(t,\kappa\|\nabla f(x) - \alpha(t)u\|)\|\nabla f(x) - \alpha(t)u\|^2\\&+\alpha^2(t)\ell_+(t)\|\nabla f(x) - \alpha(t)u\|^2\\&\le -\left(\kappa^{-p}\lambda(t)\alpha^2(t)\beta_1(t) - \alpha^2(t)\ell_+(t)\right)\|\nabla f(x) - \alpha(t)u\|^{2-p}\\&\quad -\left(\kappa^{-q}\lambda(t)\alpha^2(t)\beta_2(t) - \alpha^2(t)\ell_+(t)\right)\|\nabla f(x) - \alpha(t)u\|^{2-q}.\end{aligned}\tag{3.10}$$

Now let us consider the following two cases.

Case 1: $\|\nabla f(x)\| \ge \kappa\|\nabla f(x) - \alpha(t)u\|$. Then $V(t) \le (1+\kappa^{-2})\alpha(t)\|\nabla f(x)\|^2$. By the definition of θ and Lemma 3.31,

$$\begin{aligned}2\theta L\|\nabla f(x) - \alpha(t)u\|^2 &= 2L\gamma(t)\left(\frac{\beta_1(t)}{\|\nabla f(x)\|^p} + \frac{\beta_2(t)}{\|\nabla f(x)\|^q}\right)\|\nabla f(x) - \alpha(t)u\|^2\\&\quad + 2Lk(t)\|\nabla f(x) - \alpha(t)u\|^2\\&\le \frac{2L}{\kappa^p}\gamma(t)\beta_1(t)\|\nabla f(x) - \alpha(t)u\|^{2-p} + \frac{2L}{\kappa^2}\gamma(t)\beta_2(t)\|\nabla f(x)\|^{2-q}\\&\quad + 2Lk(t)\|\nabla f(x) - \alpha(t)u\|^{2-p} + 2Lk(t)\|\nabla f(x) - \alpha(t)u\|^{2-q}.\end{aligned}$$

Combining this with (3.9) and (3.10), we have

$$\begin{aligned}\dot{V}(t) \le& -2\mu\gamma(t)\beta_1(t)\|\nabla f(x)\|^{2-p} - 2(\mu - \frac{L}{\kappa^2})\gamma(t)\beta_2(t)\|\nabla f(x)\|^{2-q}\\&-2\left(\kappa^{-p}v_1(t)\beta_1(t) - v_2(t)\right)\|\nabla f(x) - \alpha(t)u\|^{2-p}\\&-2\left(\kappa^{-q}v_1(t)\beta_2(t) - v_2(t)\right)\|\nabla f(x) - \alpha(t)u\|^{2-q}.\end{aligned}$$

It follows from $V(t) \le (1+\kappa^{-2})\alpha(t)\|\nabla f(x)\|^2$ that

$$\begin{aligned}\dot{V}(t) \le& -2\mu\gamma(t)\beta_1(t)\left((1+\kappa^{-2})\alpha(t)\right)^{\frac{p-2}{2}}V^{\frac{2-p}{2}}(t)\\&-2(\mu - \frac{L}{\kappa^2})\gamma(t)\beta_2(t)\left((1+\kappa^{-2})\alpha(t)\right)^{\frac{q-2}{2}}V^{\frac{2-q}{2}}(t).\end{aligned}\tag{3.11}$$

Since $\kappa > \sqrt{\frac{L}{\mu}}$, we have $\mu - \frac{L}{\kappa^2} > 0$, and so $\dot{V}(t) \le 0$, which implies that $\alpha(t)\|\nabla f(x)\|^2 \le V(t_0)$. This, together with strong convexity of f, yields that $\frac{\mu^2}{2}\alpha(t)\|x-x^*\|^2 \le V(t_0)$. If $\bar{\tau} < +\infty$, then by the assumption, there exists α_0 such that $\alpha(t) \ge \alpha_0 > 0$ for all $t \in [t_0, \bar{\tau})$. Thus, $(x(t))_{t\in[t_0,\bar{\tau})}$ is bounded. Let $\bar{w} = (x^*, 0)$, $S_0 = \{\bar{w}\}$. By Corollary 2.1, Lemma 3.1, and Lemma 3.2, for any initial point (x_0, u_0), there exists a unique solution $(x,u)\colon [t_0, +\infty) \to \mathbb{R}^d \times \mathbb{R}^d$ of (3.1). Moreover, by [17, Theorem 2.2], there is $T \ge t_0$ such that $V(t) = 0$ for all $t \ge T$, which yields that $\nabla f(x(t)) = 0$ and $u(t) = 0$, and hence $\dot{x}(t) = 0$. Therefore $x(t)$ is constant for all $t \ge T$, and so $\forall t \ge T$, $x(t) = x^*$, where x^* is the minimizer of f.

Case 2: $\|\nabla f(x)\| < \kappa\|\nabla f(x) - \alpha(t)u\|$. Then $V(t) \le (1+\kappa^2)\alpha(t)\|\nabla f(x) - \alpha(t)u\|^2$ and by Lemma 3.32,

$$
\begin{aligned}
2\theta L\|\nabla f(x) - \alpha(t)u\|^2 \le\, & 2L\kappa^{-p}\gamma(t)\beta_1(t)\|\nabla f(x) - \alpha(t)u\|^{2-p} \\
& + 2L\kappa^{-q}\gamma(t)\beta_2(t)\|\nabla f(x) - \alpha(t)u\|^{2-q} \\
& + 2Lk(t)\|\nabla f(x) - \alpha(t)u\|^{2-p} \\
& + 2Lk(t)\|\nabla f(x) - \alpha(t)u\|^{2-q}.
\end{aligned}
$$

Combining this with (3.9) and (3.10), we obtain that

$$
\begin{aligned}
\dot{V}(t) &\le -2u_1(t)\alpha(t)^{\frac{2-p}{2}}\|\nabla f(x) - \alpha(t)u\|^{2-p} - 2u_2(t)\alpha(t)^{\frac{2-q}{2}}\|\nabla f(x) - \alpha(t)u\|^{2-q} \\
&\le -2\left(1+\kappa^2\right)^{\frac{p-2}{2}} u_1 V^{\frac{2-p}{2}}(t) - 2\left(1+\kappa^2\right)^{\frac{q-2}{2}} u_2 V^{\frac{2-q}{2}}(t).
\end{aligned}
$$

The conclusion follows from using the same arguments as in Case 1.

(2): $\omega = \theta$. Since $\ell(t) \le 0$ for all $t \ge t_0$, we have $\ell_+(t) = \max\{\ell(t), 0\} = 0$. Using Lemma 3.4 with $\omega = \theta$,

$$
\dot{V}(t) \le -2\theta\mu\|\nabla f(x)\|^2 - 2\theta(\lambda(t)\alpha^2(t) - L)\|\nabla f(x) - \alpha(t)u(t)\|^2. \tag{3.12}
$$

We now distinguish into the following two cases.

Case 1: $\|\nabla f(x)\| \ge \kappa\|\nabla f(x) - \alpha(t)u\|$. Then $V(t) \le \alpha(t)(1+\kappa^{-2})\|\nabla f(x)\|^2$. In view of $\lambda(t)\alpha^2(t) > L$, by (3.12), one has

$$
\begin{aligned}
\dot{V}(t) &\le -2\mu\gamma(t)\beta_1(t)\|\nabla f(x)\|^{2-p} - 2\mu\gamma(t)\beta_2(t)\|\nabla f(x)\|^{2-q} - 2\mu k(t)\|\nabla f(x)\|^2 \\
&\le -2\mu\gamma(t)\beta_1(t)\left(\alpha(t)(1+\kappa^{-2})\right)^{\frac{p-2}{2}} V^{\frac{2-p}{2}}(t) - 2\mu\gamma(t)\beta_2(t)\left(\alpha(t)(1+\kappa^{-2})\right)^{\frac{q-2}{2}} V^{\frac{2-q}{2}}(t).
\end{aligned} \tag{3.13}
$$

Case 2: $\|\nabla f(x)\| < \kappa\|\nabla f(x) - \alpha(t)u\|$. Then $V(t) \le \alpha(t)(1+\kappa^2)\|\nabla f(x) - \alpha(t)u\|^2$ and

$$
\begin{aligned}
\dot{V}(t) &\le -2\theta(\lambda(t)\alpha^2(t) - L)\|\nabla f(x) - \alpha(t)u\|^2 \\
&\le -2\gamma(t)\beta_1(t)(\lambda(t)\alpha^2(t) - L)\left(\alpha(t)(1+\kappa^2)\right)^{\frac{p-2}{2}} V^{\frac{2-p}{2}}(t) \\
&\quad - 2\gamma(t)\beta_2(t)(\lambda(t)\alpha^2(t) - L)\left(\alpha(t)(1+\kappa^2)\right)^{\frac{q-2}{2}} V^{\frac{2-q}{2}}(t).
\end{aligned} \tag{3.14}
$$

The conclusion follows from Corollary 2.1 and [17, Theorem 2.2]. □

Regarding Theorem 3.1, the conditions for κ, functions k and γ in Theorem 3.12 are weaker than those in Theorem 3.11.

Remark 3.1 (Selection of time-varying coefficients)**.** There are many choices of coefficients such that the assumptions in Theorem 3.11 are fulfilled. Here, we present several of these notable scenarios, some of which are investigated in our numerical work in Section 4:

(1) $\lambda(t) \equiv \lambda$, $\alpha(t) \equiv \alpha$, $\beta_1(t) \equiv \beta_1$, $\beta_2(t) \equiv \beta_2$, $\gamma(t) \equiv \gamma$, $\ell(t) \leq 0$, $k(t) = k$, and $\lambda > \max\{\frac{1}{\alpha^2}(\frac{Lk\kappa^p}{\beta_1} + L\gamma), \frac{1}{\alpha^2}(\frac{Lk\kappa^q}{\beta_2} + L\gamma)\}$.

(2) $\lambda(t) \equiv \lambda$, $\alpha(t) \equiv \frac{\alpha}{t}$, $\gamma(t) \equiv \frac{\gamma}{t^2}$, $\sigma(t) \equiv \frac{\sigma}{t^2}$, $k(t) \equiv 0$, $\ell(t) \leq 0$, $\beta_1(t) \equiv \beta_1 t^n$, $\beta_2(t) \equiv \beta_2 t^m$, with $n \in \{\frac{2+p}{2}, \frac{p}{2}\}$, $m \in \{\frac{2+q}{2}, \frac{q}{2}\}$, $\lambda > \frac{L\gamma}{\alpha^2}$, and $t_0 > 0$. Then $n - 1 - \frac{p}{2} \in \{0, -1\}$. It can be seen that all assumptions in Theorem 3.1 are satisfied.

We also list here some special cases of time-varying coefficients satisfying conditions in Theorem 3.12:

(1) For any $t \geq t_0$, $\ell(t) \leq 0$, $k(t) \geq 0$, $\lambda(t) = \lambda$, $\alpha(t) = \alpha$, and $\lambda\alpha^2 > L$.

(2) For any $t \geq t_0 > 0$, $\ell(t) \leq 0$, $k(t) \geq 0$, $\gamma(t) = \gamma$, $\beta_1(t) = \beta_1 t^{\frac{p}{2}-1}$, $\beta_2(t) = \beta_2 t^{\frac{q}{2}}$, $\lambda(t) = \lambda t^2$, $\alpha(t) = \frac{\alpha}{t}$ such that $\lambda\alpha^2 > L$.

In dynamical system (3.1), we can choose $\alpha(t) = \frac{\alpha}{t}$ which is the same as the choice of $\alpha(t)$ in the continuous-time Nesterov's accelerated gradient algorithm (3.2).

4. Numerical simulation

This section presents several numerical experiments to demonstrate the effectiveness of the proposed method.

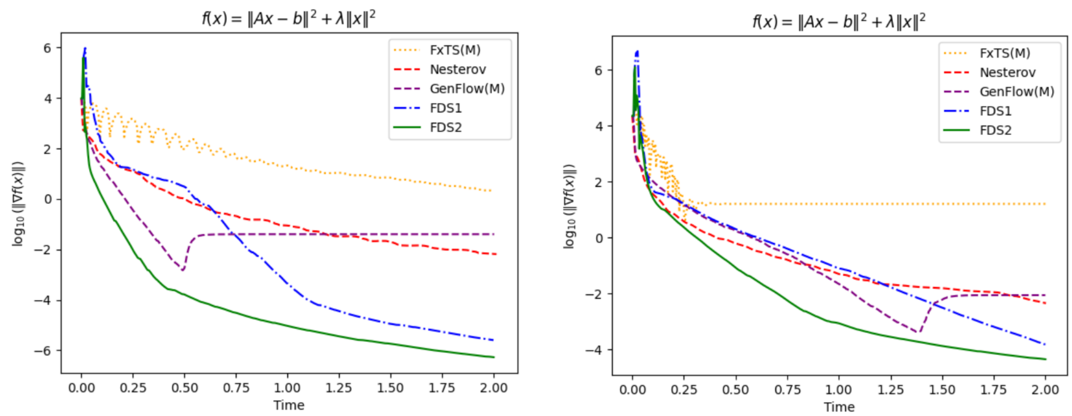
Example 4.1 (Ridge regression problem)**.** Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and consider the problem

$$\min_{x\in\mathbb{R}^n} f(x) := \tfrac{1}{2}\|Ax - b\|^2 + \lambda\|x\|^2.$$

We will show the effectiveness of the proposed approach by comparing it with FxTS(M) algorithm [5], GenFlow(M) algorithm [1], and Nesterov's accelerated gradient method [15]. Here we randomly choose a sparse matrix $A \in \mathbb{R}^{7000\times 1500}$, $b \in \mathbb{R}^{7000}$ in Figure 1(a); $A \in \mathbb{R}^{10000\times 2000}$, $b \in \mathbb{R}^{10000}$ in Figure 1(b) and $\lambda = 0.0001$. We see that f is β-strongly convex differentiable and ∇f is L-Lipschitz continuous. In this numerical simulation, for FDS1 algorithm (dynamical system (3.1) with $\omega = \psi(t, \kappa\|\nabla f(x) - \alpha(t)u\|)$), for all t, we choose $\lambda(t) = 0.01$, $\kappa = \sqrt{\frac{L}{\beta}} + 1$, $\gamma(t) = 1$, $\alpha(t) = \left(\frac{L\gamma+1}{\lambda}\right)^{1/2}$, $\beta_1(t) = \frac{50}{t}$, $\beta_2(t) = 5$, $p = \frac{1}{4}$, $q = \frac{-1}{3}$, $k(t) = 0$ and $\ell(t) = -80$. For FDS2 algorithm (dynamical system (3.1) with $\omega = \theta$), we choose $\lambda = 0.01$, $\kappa = 20$, $\alpha = \sqrt{\frac{L}{\lambda}} + 1$, $\beta_1(t) = \frac{50}{t}$, $\beta_2(t) = 5$, $p = \frac{1}{4}$, $q = \frac{-1}{3}$, $k(t) = 100$, and $\ell(t) = 0$. With these choices, all assumptions in Theorem 3.1 are satisfied. Figure 1 shows that our proposed algorithms FDS1 and FDS2 have better performance than other algorithms.

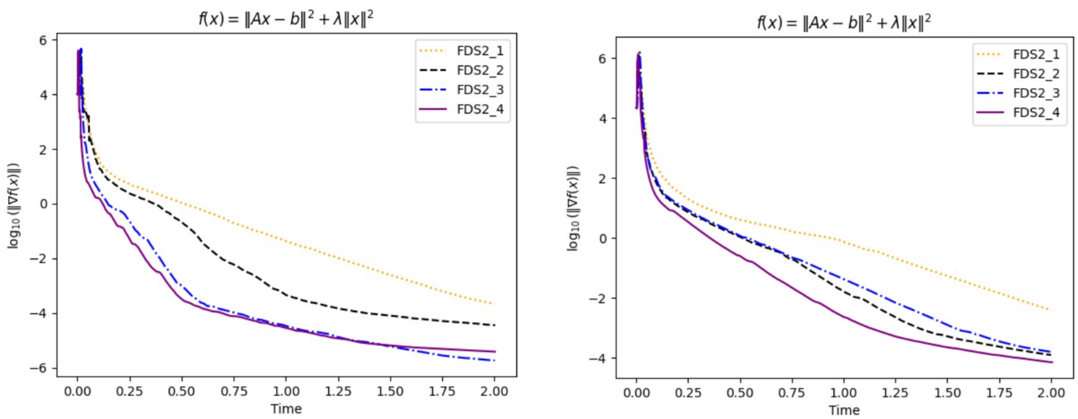
Still, considering this problem, Figure 2 shows the role of function ℓ and function k in dynamical system (3.1). Here, the coefficients λ, κ, α, $\beta_1(t)$, $\beta_2(t)$, p and q for FDS2-1, FDS2-2, FDS2-3, and FDS2-4 are the same as those for FDS2 algorithm. Moreover, we choose $\ell(t) = -100$, $k(t) = 0$ for FDS2-1; $\ell(t) = 0$, $k(t) = 0$ for FDS-2; $\ell(t) = -100$, $k(t) = 100$ for FDS-3, and $\ell(t) = 0$, $k(t) = 100$ for FDS2-4. It can be seen that FDS2-4 is more stable and converges faster than other algorithms.

The choice of $\alpha(t) \equiv \frac{\alpha}{t}$, which might accelerate the convergence of the proposed method, is shown in Figure 3. For the FDS2-5, FDS2-6, and FDS2-7 algorithms, the coefficients are chosen



(A) $A \in \mathbb{R}^{7000\times1500}$ (B) $A \in \mathbb{R}^{10000\times2000}$

FIGURE 1. Ridge regression problem I.

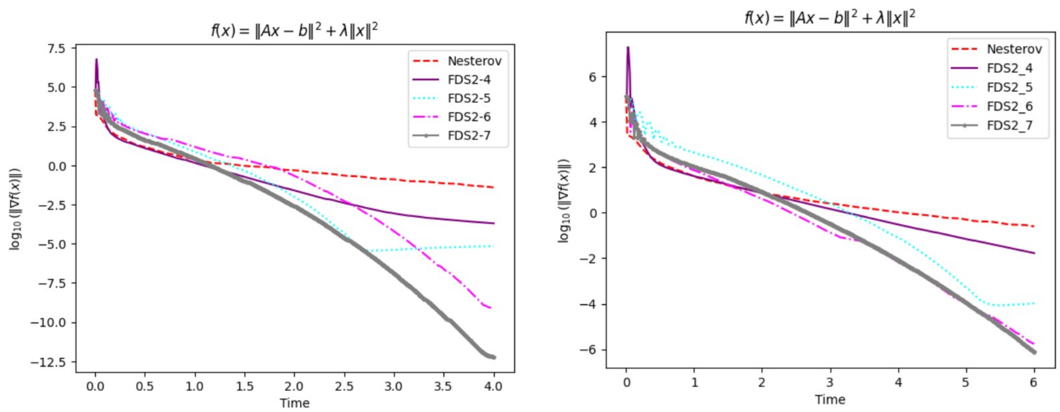


(A) $A \in \mathbb{R}^{7000\times1500}$ (B) $A \in \mathbb{R}^{10000\times2000}$

FIGURE 2. Ridge regression problem II.

as follows: $\lambda(t) = 0.01t^2$, $\alpha(t) = \frac{1}{t}\left(\sqrt{\frac{L}{0.01}} + 1\right)$, $\kappa = 20$, $\gamma = 5$, $q = \frac{-1}{3}$, $\beta_1(t) = 50t^{\frac{p}{2}-1}$, $\beta_2(t) = 5t^{\frac{q}{2}}$, $k(t) = 100$, and $\ell(t) = 0$. These coefficients satisfy the conditions in Theorem 3.12. We set $p = \frac{1}{10}$ for FDS2-5, $p = \frac{1}{20}$ for FDS2-6, and $p = \frac{1}{30}$ for FDS2-7. Figure 3 shows that the FDS2-5, FDS2-6, and FDS2-7 algorithms outperform FDS2-4 and Nesterov's accelerated gradient algorithm. Moreover, smaller values of p lead to better convergence for FDS2.

Example 4.2 (Training neural networks)**.** We now use our proposed algorithm FDS to train deep neural networks on MNIST dataset, where the training set contains 60000 examples and the test set contains 10000 examples [13], and CIFAR10, which comprises 50,000 training samples and 10,000 test samples [12]. We compare the proposed method FDS with GenFlow(M) algorithm and SGD algorithm. For classification on the MNIST dataset, we utilized a neural network comprising a single convolutional layer with ReLU activation, containing 32 filters of size 3×3. This is followed by a dense layer with ReLU activation and an output size of 128. The final linear layer converts the 128-dimensional input into a 10-dimensional output, corresponding to the 10 classes, using SoftMax activation. The neural network for CIFAR10



(A) $A \in \mathbb{R}^{15000\times3000}$ (B) $A \in \mathbb{R}^{20000\times4000}$

FIGURE 3. Ridge regression problem III

includes two convolutional layers, each paired with a max-pooling layer with a 2×2 window. The convolutional layers have 6 and 16 filters, each measuring 5×5 and using ReLU activation. Following these are two fully connected layers with ReLU activation, sized 120 and 84, respectively, and a final output layer with SoftMax activation corresponding to the 10 classes. For classification tasks, we apply L2-regularized cross-entropy loss. For FDS, we set $\alpha = 15$, $\kappa = 0.05$, $\gamma = 1$, $\lambda = 0.1$, $\beta_1 = 100/t$, $\beta_2 = 20$ for MNIST, $\beta_2 = 5$ for CIFAR10, $\ell = 0$, $k = 0$, $p = 1/3$, $q = -2/3$ and learning rate is 0.01. For SGD, the learning rate is 0.01. For GenFlow(M), we choose $(p,q) = (10, 1.98)$. It is illustrated from Figure 4 that FDS has better performance than SGD and GenFlow(M) in both training loss and testing accuracy on the MNIST dataset and on the CIFAR10 dataset as well.

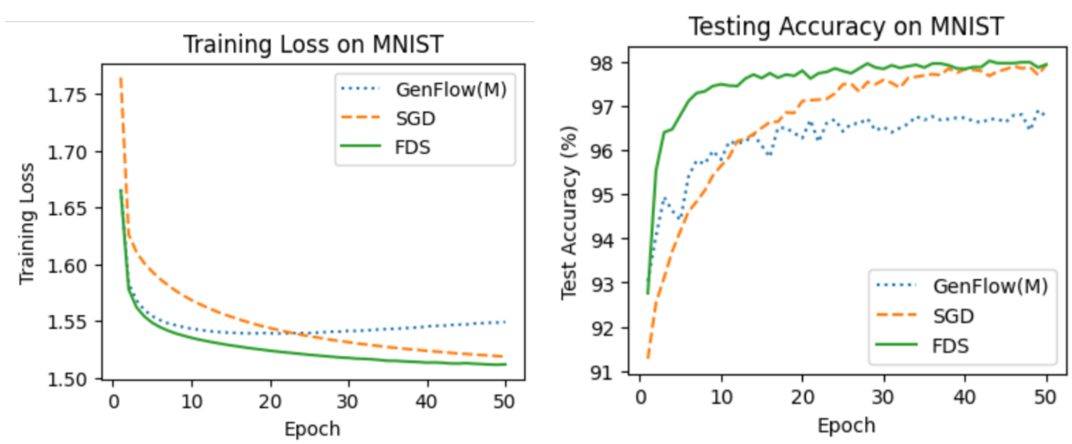


FIGURE 4. MNIST dataset

5. CONCLUSION

We proposed a class of novel fixed-time second-order time-varying dynamical systems to accelerate the convergence of continuous-time optimization. A rigorous proof was provided for the existence and uniqueness of solutions to the proposed dynamical system and its fixed-time

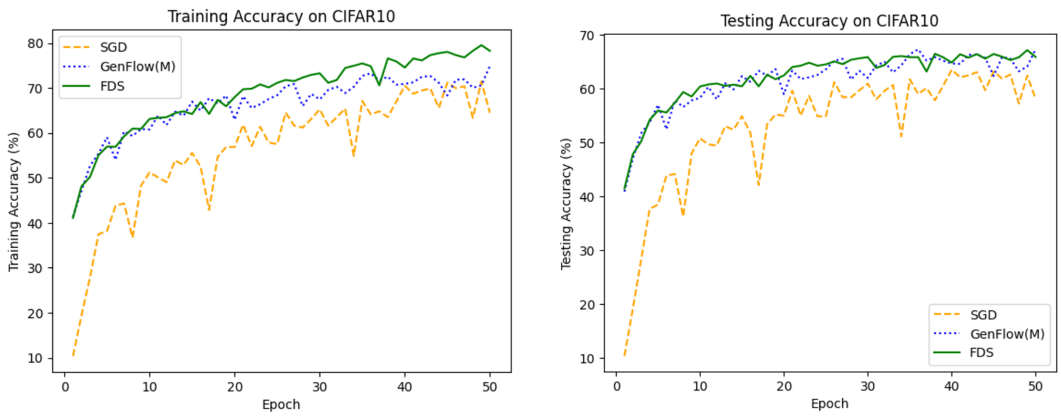


FIGURE 5. CIFAR10 dataset

convergence. By utilizing time-varying coefficients, our method presents flexibility in tuning the convergence transients. The outperformance of the proposed method was demonstrated through numerical simulations.

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