A QUASI-NEWTON METHOD TO SOLVE UNCERTAIN MULTIOBJECTIVE OPTIMIZATION PROBLEMS WITH UNCERTAINTY SET OF FINITE CARDINALITY

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Abstract. In this paper, we derive an iterative scheme through quasi-Newton technique to capture robust weakly efficient points of uncertain multiobjective optimization problems under the upper set less relation. It is assumed that the set of uncertainty scenarios of the problems is of finite cardinality. We also assume that, corresponding to each given uncertain scenario from the uncertainty set, the objective function of the problem is twice continuously differentiable. In the proposed iterative scheme, at any iterate, by applying the *partition set* concept from set-valued optimization, we formulate an iterate-wise class of vector optimization problems to determine a descent direction. To evaluate this descent direction at the current iterate, we employ one iteration of the quasi-Newton scheme for vector optimization on the formulated class of vector optimization problems. As this class of vector optimization problems differs iterate-wise, the proposed quasi-Newton scheme is not a straight extension of the quasi-Newton method for vector optimization problems. Under commonly used assumptions, any limit point of a sequence generated by the proposed quasi-Newton technique is found to be a robust weakly efficient point of the problem. We analyze the well-definedness and global convergence of the proposed iterative scheme based on a regularity assumption on stationary points. Under the uniform continuity of the Hessian approximation function, we demonstrate a local superlinear convergence of the method. Finally, numerical examples are presented to demonstrate the effectiveness of the proposed method.

Keywords. Gerstewitz function; Partition set; Quasi-Newton method; Regular point; Uncertain optimization.

1. INTRODUCTION

Uncertain multiobjective optimization problems (UMOPs) encompass a category of optimization problems where multiple uncertain objectives of a conflicting nature must be addressed. In many practical optimization problems, the objectives are inherently uncertain due to incomplete information, imperfect data, unpredictable factors, or uncertain predictions [34].

Analysis of the effects of uncertain scenarios in optimization theory serves many crucial purposes, such as enhancing the robustness of solutions, improving decision-making under variability, trade-off analysis and managing risk in practical applications.

The area of uncertain (robust) optimization is growing quickly. For a detailed discussion on fundamental concepts and applications of robust optimization, we refer to exploring the research contributions of Ben-Tal, Nemirovski, and Ghaoui [1]. Ehrgott, Ide, and Schöbel [5], Kuroiwa and Lee [24] introduced the concept of minimax robustness for uncertain multiobjective optimization problems, replacing the objective of the uncertain problem with a multiobjective function that accounts for the worst-case outcomes of each objective component. This concept of robustness involves converting an uncertain multiobjective optimization problem into a deterministic one, known as the robust counterpart. Various types of robust counterparts (see [16] and [18, Section 15.4]) can be derived for uncertain multiobjective optimization problems. Approaches like the minimax strategy and the objective-wise worst-case robust counterpart [5] are used to address uncertainty by creating a deterministic formulation. These deterministic problems can be addressed using scalarization techniques, such as the popular normal-boundary intersection or the weighted sum method. For more information on these techniques, refer to Ehrgott [6]. In [14, 15], various definitions of solution concepts for UMOPs were proposed and reported based on several set less relations [7]. For additional insights, the references [20, 21, 28, 31, 32] offer detailed information on these developments, and for a new robust counterpart through aggregation operator, see recently developed "generalized ordered weighted aggregation robustness" [9, 19].

Regarding numerical methods for solving UMOPs, the authors in [22, 23] effectively tackled UMOPs with a finite uncertainty set. To convert an uncertain optimization problem into a deterministic one, the idea of an objective-wise robust counterpart has been utilized in [22, 23]. The set of efficient solutions derived from the objective-wise robust counterpart is a significantly smaller subset of the complete set of robust optimal solutions for UMOPs. In order to identify all robust weakly minimal solutions of UMOPs, recently Ghosh, Kishor, and Zhao [10] proposed a set-valued optimization view-point of UMOPs instead of just using objective-wise robust counterpart. In [10], a Newton method is proposed for solving UMOPs. However, in [10], the objective function for each uncertain scenario is assumed to be locally strong convex, which is restrictive. So, in this study, we focus on developing a quasi-Newton scheme to solve UMOPs without any convexity assumption of the objective. By transforming the considered UMOPs into their min-max counterparts, problems are observed as optimization problems with set-valued maps under the set-ordering relation given by upper less ordering. This reformulated optimization problem with set-valued objectives is solved using a quasi-Newton scheme, which helps to directly obtain robust weakly efficient solutions for the UMOPs under consideration. To achieve this, we construct a sequence of vector optimization problems (VOPs) for the given UMOP by employing the concept of a partition set, as defined in [2], to solve the pertaining set optimization problem. Based on the ordering cone of the UMOP we find descent direction at every iterate by applying quasi-Newton approach to the constructed sequence of VOPs. This approach aims to find sequences that converge to weakly robust efficient points of the considered UMOPs by systematically analyzing and partitioning the maximal elements within the uncertainty set of objective values across scenarios.

The entire study is organized as follows: In Section 2, we provide notations, definitions, and preliminary results that are used throughout the paper. Section 3 reports results on optimality conditions for set-valued optimization, which are later connected with robust weakly minimal solutions of UMOPs under consideration. With the help of these optimality conditions, Section 4 derives a quasi-Newton scheme for UMOPs. We establish the well-definedness of the proposed algorithm, supported by the existence of the Armijo step length and detailed convergence analysis. Section 5 provides numerical illustrations of the proposed method. Finally, the study is concluded in Section 6 by highlighting the key findings and suggesting potential directions for subsequent research.

2. NOTATION AND PRELIMINARIES

The set of all nonempty subsets of $\mathbb{R}^m$ is represented as $P(\mathbb{R}^m)$. The notation $\|\cdot\|$ denotes either the Euclidean norm for vectors or the spectral norm for matrices, depending on the context. The cardinality of a finite set A is denoted by $|A|$. For any $p \in \mathbb{N}$, we denote $[p] := \{1, 2, \ldots, p\}$. The notations $\mathscr{N}[\bar{x}, \delta]$ and $\mathscr{N}(\bar{x}, \delta)$, respectively, represent the closed and open balls centred at $\bar{x} \in \mathbb{R}^n$ with radius δ. We denote

$$\mathbb{R}^m_{\geqq} := \{z \in \mathbb{R}^m : z \geqq 0\}, \text{ where } z \geqq 0 \text{ means no component of } z \text{ is negative,}$$

$$\mathbb{R}^m_{\geq} := \{z \in \mathbb{R}^m : z \geq 0\}, \text{ where } z \geq 0 \text{ means } z \geqq 0 \text{ but } z \neq 0,$$

$$\text{and } \mathbb{R}^m_{>} := \{z \in \mathbb{R}^m : z > 0\}, \text{ where } z > 0 \text{ means all components of } z \text{ are positive.}$$

A nonempty set $K \in P(\mathbb{R}^m)$ is called a cone if $\tau z \in K$ for every $z \in K$ and $\tau \geq 0$. A cone K is called pointed if $K \cap (-K) = \{0\}$, and solid if its interior, denoted by $\text{int}(K)$, is nonempty.

The dual cone of a cone K is the set

$$K^* := \{z \in \mathbb{R}^m : z^\top y \geq 0 \text{ for all } y \in K\}.$$

The following definitions are known in the literature; see [6, 12, 18] and the references therein.

Definition 2.1. If K is a pointed, closed, and convex cone, then it induces a partial order $\preceq$ on $\mathbb{R}^m$ defined as follows. For $y, z \in \mathbb{R}^m$, $z \preceq y \iff y - z \in K$. In addition, if K is a solid cone, then K induces a strict order $\prec$ on $\mathbb{R}^m$: $z \prec y \iff y - z \in \text{int}(K)$.

A deterministic vector optimization problem is given by

$$\min_{x \in S \subseteq \mathbb{R}^n} f(x), \tag{2.1}$$

where S is a nonempty subset of $\mathbb{R}^n$, $f : S \to \mathbb{R}^m$ is a function given by

$$f(x) := (f_1(x), f_2(x), \ldots, f_m(x))^\top,$$

and the optimality concept is given by the following definition with respect to a pointed, closed and convex cone K.

Definition 2.2. For the optimization problem (2.1), we categorize a point $\bar{x} \in S$ as

 (i) efficient if there is no $x \in S$ for which $f(x) \preceq f(\bar{x})$, and
 (ii) weakly efficient if, for no $x \in S$, $f(x) \prec f(\bar{x})$.

In this paper, we consider an UVOP as a family of parameterized optimization problems:

$$\mathscr{P}(U) := \{\mathscr{P}(\xi) : \xi \in U\},$$

where, for any $\xi \in U$, the parameterized problem is

$$\min_{x \in S \subseteq \mathbb{R}^n} F(x, \xi) := \big(f_1(x, \xi), f_2(x, \xi), \ldots, f_m(x, \xi)\big)^\top, \qquad (\mathscr{P}(\xi))$$

where U is a nonempty subset of $\mathbb{R}^r$ and $F : \mathbb{R}^n \times U \to \mathbb{R}^m$ is a vector-valued function. The minimization in $(\mathscr{P}(\xi))$ is to be understood in the sense of Definition 2.2 with respect to a partial order induced by a proper, closed, convex, pointed, solid cone K. The set U represents the collection of uncertain (parametric) scenarios. Importantly, note that corresponding to each $\xi \in U$, the problem $(\mathscr{P}(\xi))$ is a deterministic vector optimization problem.

Throughout this paper, we assume the following for the class of problems $\mathscr{P}(U)$.

(i) The uncertainties in the problem are given by a finite set of scenarios ξ, which collectively form the uncertainty set $U \subseteq \mathbb{R}^r$. Specifically, we denote $U := \{\xi_1, \xi_2, \ldots, \xi_p\}$.

(ii) The feasible set S is assumed to be independent of the uncertainties. The cone $K \subseteq \mathbb{R}^m$ is closed, convex, pointed and solid. Furthermore, let $e \in \text{int}(K)$ be a given element.

(iii) The vector-valued functions $F(\cdot, \xi_1), F(\cdot, \xi_2), \ldots, F(\cdot, \xi_p) : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable.

For any given $x \in S$, the notation $F_U(x)$ will be used to represent the set $\{F(x, \xi) : \xi \in U\}$. It is important to note that for a given $\mathscr{P}(U)$, the mapping $F_U : S \rightrightarrows \mathbb{R}^m$ defines a set-valued map.

In the next definition, we introduce (weakly) robust efficient elements as solutions of a robust counterpart problem to UVOP $\mathscr{P}(U)$ (see [5]).

Definition 2.3. For a given UVOP $\mathscr{P}(U)$, a feasible solution $\bar{x} \in S$ is named

(i) robust efficient if there is no $x \in S$ such that $F_U(x) \subseteq F_U(\bar{x}) - K$, and

(ii) weakly robust efficient if there is no $x \in S$ such that $F_U(x) \subset F_U(\bar{x}) - \text{int}(K)$.

Next, we present the concept of maximal and weakly maximal elements of a set with respect to $\preceq$.

Definition 2.4. Let $C \in P(\mathbb{R}^m)$ and K be a pointed, closed, convex and solid cone.

(i) The set of maximal elements of C with respect to K is defined as

$$\text{Max}(C, K) := \{z \in C : (z + K) \cap C = \{z\}\}.$$

(ii) The set of weakly maximal elements of C with respect to K is defined as

$$\text{WMax}(C, K) := \{z \in C : (z + \text{int}(K)) \cap C = \emptyset\}.$$

In [16], it was shown that the solution concept in Definition 2.3 can be formulated using the upper set less relation introduced by Kuroiwa in [25, 26, 27]. Employing other set less relations (see [18]), it is possible to obtain other concepts of robustness as discussed in [16].

Definition 2.5. [25, 26, 27] For a given pointed, closed, convex and solid cone $K \subset \mathbb{R}^m$, the upper set less relation $\preceq^u$ is a binary relation defined on $P(\mathbb{R}^m)$ as, for all C, D in $P(\mathbb{R}^m)$, $C \preceq^u D \iff C \subseteq D - K$. Similarly, if K is solid, the strict upper set less relation $\prec^u$ is a binary relation defined on $P(\mathbb{R}^m)$ by, for all C, D in $P(\mathbb{R}^m)$, $C \prec^u D \iff C \subseteq D - \text{int}(K)$.

Obviously, the (strict) upper set less relation $\preceq^u$ is involved in the formulation of the solution concept in Definition 2.3.

Proposition 2.1. [10] *Let $C \in P(\mathbb{R}^m)$ be compact, and K be a closed, convex and pointed cone. Then, $C - K = \operatorname{Max}(C, K) - K$.*

The Gerstewitz scalarizing function will be key in deriving the main results of this study. This function is frequently used in vector optimization problems to convert them into scalar problems. In the following definition, we consider a special case of the Gerstewitz function (see [18, Chapter 5]).

Definition 2.6. Let K be a closed, convex, pointed, and solid cone. For a given element $e \in \operatorname{int}(K)$, the Gerstewitz function $\Theta_e : \mathbb{R}^m \to \mathbb{R}$ associated with e and K is defined by

$$\Theta_e(z) := \min\{t \in \mathbb{R} : te \in z + K\}. \tag{2.2}$$

A few useful properties of the Gerstewitz scalarizing function Θ_e are provided in the following proposition; see [18, Chapter 5] and the references therein.

Proposition 2.2. *Let $y, z \in \mathbb{R}^m$. The Gerstewitz function defined in (2.2) has the following properties.*

 (i) Θ_e *is sublinear, i.e., $\Theta_e(y + z) \le \Theta_e(y) + \Theta_e(z)$.*
 (ii) Θ_e *is Lipschitz continous on $\mathbb{R}^m$.*
 (iii) Θ_e *is monotone, i.e., $\forall y, z \in \mathbb{R}^m$, $z \preceq y \implies \Theta_e(z) \le \Theta_e(y)$ and $\forall y, z \in \mathbb{R}^m$, $z \prec y \implies \Theta_e(z) < \Theta_e(y)$.*
 (iv) Θ_e *has the representability property,* i.e., $-K = \{z \in \mathbb{R}^m : \Theta_e(z) \le 0\}$ *and* $-\operatorname{int}(K) = \{z \in \mathbb{R}^m : \Theta_e(z) < 0\}$.

Definition 2.7. Consider the UVOP $\mathscr{P}(U)$ and the associated set-valued mapping $F_U : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by $F_U(x) := \{F(x, \xi) : \xi \in U\}$ for all x in $\mathbb{R}^n$. Using the upper set less relation $\preceq^u$, we introduce the following robust counterpart problem (SOP) to the given UVOP $\mathscr{P}(U)$:

$$\begin{cases} \min & F_U(x) \\ \text{subject to} & x \in S \subseteq \mathbb{R}^n. \end{cases} \tag{SOP}$$

The robust counterpart problem (SOP) to $\mathscr{P}(U)$ is a deterministic set-valued optimization problem and the minimization in (SOP) is to be understood with respect to the set less relation $\preceq^u$, see Definitions 2.3 and 2.5. A point $\bar{x} \in \mathbb{R}^n$ is said to be a local weakly efficient of (SOP) if there exists a neighbourhood $\mathscr{N}(\bar{x}, \delta)$ such that $\nexists x \in \mathscr{N}(\bar{x}, \delta) : F_U(x) \prec^u F_U(\bar{x})$.

In this paper, we propose a quasi-Newton method aimed at finding robust weakly efficient solutions for $\mathscr{P}(U)$ through solving the robust counterpart problem (SOP).

3. OPTIMALITY CONDITIONS

In this section, we report optimality conditions for robust weakly efficient points of $\mathscr{P}(U)$. First, we start this section by mentioning some important index-related set-valued mapping sets. The following notions are introduced in [2].

Definition 3.1. Consider the UVOP $\mathscr{P}(U)$ and the associated set-valued mapping $F_U : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ as defined above. Let x be a given element from S.

(i) The active index for maximal elements of $F_U(x)$ is $\mathscr{I} : \mathbb{R}^n \rightrightarrows [p]$, defined by

$$\mathscr{I}(x) := \{i \in [p] : F(x, \xi_i) \in \text{Max}(F_U(x), K)\}.$$

(ii) The active index for weakly maximal elements of $F_U(x)$ is $\mathscr{I}_w : \mathbb{R}^n \rightrightarrows [p]$, given by

$$\mathscr{I}_w := \{i \in [p] : F(x, \xi_i) \in \text{WMax}(F_U(x), K)\}.$$

(iii) For a vector $v \in \mathbb{R}^m$, the index set $\mathscr{I}_v : \mathbb{R}^n \rightrightarrows [p]$ is defined as

$$\mathscr{I}_v(x) := \{i \in \mathscr{I}(x) : F(x, \xi_i) = v\}.$$

Definition 3.2. The mapping $\omega : \mathbb{R}^n \to \mathbb{N} \cup \{0\}$ represents the cardinality of the set of maximal elements of F_U with respect to K, i.e., $\omega(x) := |\text{Max}(F_U(x), K)|$. Additionally, for a given $\bar{x} \in \mathbb{R}^n$, we define $\bar{\omega} = \omega(\bar{x})$.

Next, we give the definition of the partition set of a point $x \in \mathbb{R}^n$ in order to systematically identify weakly maximal points of the (SOP).

Definition 3.3. Let $\{v_1^x, v_2^x, \ldots, v_{\omega(x)}^x\}$ be an enumeration of the set $\text{Max}(F(x), K)$ at a given $x \in \mathbb{R}^n$. The *partition set* at x is $P_x := \prod_{j=1}^{\omega} \mathscr{I}_{v_j^x}(x)$.

Lemma 3.1. [10] *Consider problem* (SOP). *Let $\bar{x}$ be a given element from S. Denote the cone* $\prod_{j=1}^{\bar{\omega}} K$ *by $\tilde{K}$. Furthermore, for every $\beta := (\beta_1, \beta_2, \ldots, \beta_{\bar{\omega}}) \in P_{\bar{x}}$, define a function $\tilde{F}(\cdot, \xi_\beta) :$* $\mathbb{R}^n \times U \to \prod_{j=1}^{\bar{\omega}} \mathbb{R}^m$ *by* $\tilde{F}(x, \xi_\beta) := \left(F(x, \xi_{\beta_1}), F(x, \xi_{\beta_2}), \ldots, F(x, \xi_{\beta_{\bar{\omega}}})\right)^\top$. *Then, $\bar{x}$ is a local weakly efficient solution to* (SOP) *if and only if, for every $\beta \in P_{\bar{x}}$, $\bar{x}$ is a local weakly efficient solution to the vector optimization problem*

$$\preceq_{\tilde{K}} -(\text{VOP})(\xi_\beta) \begin{cases} \min & \tilde{F}(x, \xi_\beta) \\ \text{subject to} & x \in \mathbb{R}^n. \end{cases}$$

Next, to find a necessary optimality condition for the UVOP $\mathscr{P}(U)$, we recall the concept of stationary point for (SOP).

Definition 3.4. [2] We call $\bar{x}$ a stationary point for the UVOP $\mathscr{P}(U)$ if there exists a nonempty subset $A \subseteq P_{\bar{x}}$ such that

$$\forall \beta \in A, \; \exists \; \gamma_1, \gamma_2, \ldots, \gamma_{\bar{\omega}} \in K^* : \sum_{j=1}^{\bar{\omega}} \nabla F(\bar{x}, \xi_{\beta_j}) \gamma_j = 0, \; (\gamma_1, \gamma_2, \ldots, \gamma_{\bar{\omega}}) \neq 0.$$

Proposition 3.1. [2] *Let $A \subseteq P_{\bar{x}}$ be given. Then, $\bar{x}$ is a stationary point for* (SOP) *with respect to A if and only if*

$$\forall \beta \in A, p \in \mathbb{R}^n, \exists \; j \in [\bar{\omega}] \text{ for which } \nabla F(\bar{x}, \xi_{\beta_j})^\top p \notin -\text{int}(K).$$

Note that if $\bar{x}$ is a non-stationary point, then, for all $j \in [\bar{\omega}]$, there exist $\beta \in P_{\bar{x}}$ and $p \in \mathbb{R}^n$ such that $\nabla F(\bar{x}, \xi_{\beta_j})^\top p \in -\text{int}(K)$, i.e., $\Theta_e(\nabla F(\bar{x}, \xi_{\beta_j})^\top p) < 0$.

Definition 3.5. [2] If the following two conditions are satisfied at a point $\bar{x}$, then $\bar{x}$ is called a regular point of F_U of (SOP):

(i) $\text{Max}(F_U(\bar{x}), K) = \text{WMax}(F_U(\bar{x}), K)$, i.e., the set of maximal elements is equal to the set of weakly maximal elements.

(ii) There exists a $\delta > 0$ such that $\omega(\bar{x})$ is constant in $\mathscr{N}(\bar{x}, \delta)$.

Lemma 3.2. [2] *If $\bar{x} \in \mathbb{R}^n$ is a regular point of F_U, then there exists $\delta > 0$ such that for every $x \in \mathcal{N}(\bar{x}, \delta)$, we have $P_x \subseteq P_{\bar{x}}$ and $\omega(x) = \bar{\omega}$.*

4. QUASI-NEWTON METHOD AND ITS CONVERGENCE ANALYSIS

For a given initial point x_0, for each $i \in [p]$, we construct a sequence $\{B(x_k, \xi_i)\}$ of Hessian approximations of $F(\cdot, \xi_i)$ beginning with an initial matrix $B(x_0, \xi_i)$. For each function $F(\cdot, \xi_i) : \mathbb{R}^n \to \mathbb{R}^m$, we use the following quadratic approximation around x_k:

$$q(x, \xi_i) := F(x_k, \xi_i) + \nabla F(x_k, \xi_i)^\top (x - x_k) + \frac{1}{2}(x - x_k)^\top B(x_k, \xi_i)(x - x_k).$$

We choose $\{B(x_k, \xi_i)\}$ that satisfies $B(x_{k+1}, \xi_i)s_k = y_k^i$, $k = 0, 1, 2, \ldots$, where $s_k := x_{k+1} - x_k$ and $y_k^i := \nabla F(x_{k+1}, \xi_i) - \nabla F(x_k, \xi_i)$. To maintain the symmetric and positive definite properties of all terms in the sequence $B(x_k, \xi_i)$, we apply the BFGS update formula starting from a given symmetric and positive definite matrix $B(x_0, \xi_i)$:

$$B(x_{k+1}, \xi_i) := B(x_k, \xi_i) - \frac{B(x_k, \xi_i)s_k s_k^\top B(x_k, \xi_i)}{s_k^\top B(x_k, \xi_i)s_k} + \frac{y_k^i y_k^{i\top}}{s_k^\top y_k^i}. \tag{4.1}$$

From [33, Section 6.1], it can be noted that if $B(x_k, \xi_i)$ is K positive definite, i.e., $p^\top B(x_k, \xi_i)p \in \text{int}(K)$ for all p in $\mathbb{R}^n$, then $B(x_{k+1}, \xi_i)$ will also be K positive definite.

We now present the process to generate a sequence of iterates $\{x_k\}$ starting from an x_0, which is built upon the result established in Lemma 3.1. To proceed from an iterate x_k to the next iterate, we use $x_{k+1} := x_k + \tau_k p_k$, where p_k is a descent direction of F_U at x_k, and $\tau_k > 0$. In the following, we formulate a quasi-Newton process to generate p_k. At the current iterate x_k, an element β_k from the partition set P_{x_k} is selected, and then, a descent direction for $(\text{VOP})(\xi_{\beta_k})$ is found by employing the quasi-Newton method for vector optimization. For a tactful choice of β_k, we ensure later (in Theorem 4.3) that a decent direction p_k of $(\text{VOP})(\xi_{\beta_k})$ is a descent direction of F_U at x_k. The selection of such β_k is detailed below. After determining a descent direction, we follow the Armijo-type line search to choose a suitable step size τ_k.

For an appropriate choice of β_k, define the following parametric family of functionals $\{\varphi_x\}_{x \in \mathbb{R}^n}$, whose elements $\varphi_x : P_x \times \mathbb{R}^n \to \mathbb{R}$ are defined as follows:

$$\varphi_x(\beta, p) := \max_{j \in [\omega(x)]} \left\{ \Theta_e \left(\nabla F(x, \xi_{\beta_j})^\top p + \frac{1}{2} p^\top B(x, \xi_{\beta_j}) p \right) \right\}, \ \beta \in P_x, p \in \mathbb{R}^n. \tag{4.2}$$

If for any given $x \in \mathbb{R}^n$ and $\beta \in P_x$, the matrix $B(x, \xi_{\beta_j})$ is positive definite, then we note that the functional $\varphi_x(\beta, \cdot)$ is strongly convex on $\mathbb{R}^n$ because Θ_e is sublinear. Therefore, $\varphi_x(\beta, \cdot)$ attains a unique minimum over $\mathbb{R}^n$. Note that P_x is finite for any $x \in \mathbb{R}^n$. So, φ_x attains its minimum over $P_x \times \mathbb{R}^n$. Hence, the functional $\phi : \mathbb{R}^n \to \mathbb{R}$ given by

$$\phi(x) := \min_{(\beta, p) \in P_x \times \mathbb{R}^n} \varphi_x(\beta, p) \tag{4.3}$$

is well-defined. In determining a descent direction for F_U at a given nonstationary iterate x_k, we choose to employ quasi-Newton method on $\text{VOP}(\xi_{\beta_k})$ corresponding to that β_k for which we get

$$(\beta_k, p_k) := \underset{(\beta, p) \in P_{x_k} \times \mathbb{R}^n}{\text{argmin}} \ \varphi_{x_k}(\beta, p). \tag{4.4}$$

This is how a β_k value is chosen. We show below in Theorem 4.3 that at a nonstationary x_k, the p_k determined in (4.4) is a descent direction of F_U at x_k. To arrive at this result, we require the following Proposition 4.1. Notice that the formulation of VOP(β_k) is x_k-dependent and the direction p_k is obtained from VOP(ξ_{β_k}). So, even if we aim to employ the existing quasi-Newton method for vector optimization [30] on VOP(ξ_{β_k}), the extension is not straightforward because VOP(ξ_{β_k}) is determined point-wise, i.e., VOP(ξ_{β_k}) differs once x_k gets differed.

Proposition 4.1. *Consider the functions φ_x and ϕ given in* (4.2) *and* (4.3)*, respectively. Suppose $(\bar{\beta}, \bar{p}) \in P_{\bar{x}} \times \mathbb{R}^n$ satisfies the condition $\phi(\bar{x}) = \varphi_{\bar{x}}(\bar{\beta}, \bar{p})$. Then, all of the following three statements are equivalent: (i) $\bar{x}$ is a nonstationary point of UVOP $\mathscr{P}(U)$; (ii) $\phi(\bar{x}) < 0$; and (iii) $\bar{p} \neq 0$.*

Proof. $(i) \implies (ii)$. Let $\bar{x}$ be a nonstationary point of the UVOP $\mathscr{P}(U)$. Then, we see from Proposition 3.1 that, for all $j \in [\bar{\omega}]$, there exists a $\beta := (\beta_1, \beta_2, \ldots, \beta_{\bar{\omega}}) \in P_{\bar{x}}$ and $\bar{p} \in \mathbb{R}^n$ with

$$\nabla F(\bar{x}, \xi_{\beta_j})^\top \bar{p} \in -\mathrm{int}(K), \text{ i.e., } \Theta_e\left(\nabla F(\bar{x}, \xi_{\beta_j})^\top \bar{p}\right) < 0.$$

Note that

$$
\begin{aligned}
\phi(\bar{x}) &= \min_{(\beta,p) \in P_{\bar{x}} \times \mathbb{R}^n} \varphi_{\bar{x}}(\beta, p) \\
&\leq \varphi_{\bar{x}}(\beta, t\bar{p}), \text{ for some } t > 0 \\
&= \max_{j \in [\omega(\bar{x})]} \left\{ \Theta_e\left(\nabla F(\bar{x}, \xi_{\beta_j})^\top t\bar{p} + \frac{t}{2}\bar{p}^\top B(\bar{x}, \xi_{\beta_j}) t\bar{p}\right) \right\} \\
&= t \max_{j \in [\omega(\bar{x})]} \left\{ \Theta_e\left(\nabla F(\bar{x}, \xi_{\beta_j})^\top \bar{p} + \frac{t}{2}\bar{p}^\top B(\bar{x}, \xi_{\beta_j}) \bar{p}\right) \right\},
\end{aligned}
$$

which is negative for sufficiently small $t > 0$. Consequently, (ii) holds.

$(ii) \implies (iii)$. By (4.2), $\varphi_{\bar{x}}(\beta, 0) = 0$ for any β in $P_{\bar{x}}$. Since $\phi(\bar{x})$ is negative, $\bar{p} \neq 0$.

$(iii) \implies (i)$. Assume that $\bar{x}$ is a stationary point. Then, for any $\beta \in P_{\bar{x}}$ and $p \in \mathbb{R}^n$, there exists $j \in [\bar{\omega}]$ such that $\nabla F(\bar{x}, \xi_{\beta_j})^\top p \notin -\mathrm{int}(K)$, i.e., $\Theta_e\left(\nabla F(\bar{x}, \xi_{\beta_j})^\top p\right) \geq 0$. Since, for all $j \in [\omega(\bar{x})]$, $B(\bar{x}, \xi_{\beta_j})$'s are K positive definite matrices, we have

$$
\begin{aligned}
0 &\leq \Theta_e\left(\nabla F(\bar{x}, \xi_{\beta_j})^\top p\right) \\
&< \Theta_e\left(\nabla F(\bar{x}, \xi_{\beta_j})^\top p + \frac{1}{2}p^\top B(\bar{x}, \xi_{\beta_j}) p\right) \\
&\leq \max_{j \in [\omega(\bar{x})]} \left\{ \Theta_e\left(\nabla F(\bar{x}, \xi_{\beta_j})^\top p + \frac{1}{2}p^\top B(\bar{x}, \xi_{\beta_j}) p\right) \right\} \\
&= \varphi_{\bar{x}}(\beta, p), \text{ by}(4.2).
\end{aligned}
$$

That is,

$$0 \leq \varphi_{\bar{x}}(\beta, p) \leq \min_{(\beta,p) \in P_{\bar{x}} \times \mathbb{R}^n} \varphi_{\bar{x}}(\beta, p).$$

From (4.3), one sees that $0 \leq \phi(\bar{x}) < 0$. This leads to a contradiction, that is, $\bar{x}$ is a nonstationary point for the UVOP $\mathscr{P}(U)$. $\square$

Theorem 4.1. *Let S be a nonempty open subset of $\mathbb{R}^n$. Then, the function $\phi : S \to \mathbb{R}$, introduced in* (4.3)*, is continuous.*

Proof. The proof is similar to [10, Theorem 4.1]. Here, we omit the proof. $\quad\square$

Next, we propose a quasi-Newton scheme to solve the robust counterpart problem to $\mathscr{P}(U)$.

Algorithm 1 Quasi-Newton method to solve the robust counterpart problem to UMOP $\mathscr{P}(U)$

Step 0: Choose $x_0 \in \mathbb{R}^n$, $\gamma \in (0,1)$, and symmetric positive definite matrix $B(x_0, \xi_i) \in \mathbb{R}^{n \times n}$, for each $i \in [p]$. Set $k := 0$. Define $N := \{\frac{1}{2^n} : n = 0, 1, 2, \ldots\}$.
Step 1: Compute $M_k := \text{Max}(F_U(x_k), K)$, $P_k := P_{x_k}$, and $\omega_k := |\text{Max}(F_U(x_k), K)|$.
Step 2: Find

$$(\beta^k, p_k) := \underset{(\beta, p) \in P_k \times \mathbb{R}^n}{\text{argmin}} \max_{j \in [\omega_k]} \left\{ \Theta_e \left(\nabla F(x_k, \xi_{\beta_j})^\top p + \frac{1}{2} p^\top B(x_k, \xi_{\beta_j}) p \right) \right\}.$$

Step 3: If $p_k = 0$, terminate the process. Otherwise, proceed to Step 4.
Step 4: Choose τ_k as the largest $\tau \in N$ such that

$$F(x_k + \tau p_k, \xi_{\beta_j^k}) \preceq F(x_k, \xi_{\beta_j^k}) + \gamma \tau \left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k \right) \text{ for all } j \in [\omega_k].$$

Step 5: Set $x_{k+1} := x_k + \tau_k p_k$.
  Update $B(x_{k+1}, \xi_i)$ for $i \in [p]$ as in (4.1). Set $k = k + 1$, and go to **Step 1.**

With regard to the well-definedness of the steps of Algorithm 1, we note that if Step 2 and Step 4 are well-defined, then all steps are well-defined. Here, we note the following two points:

(i) For the existence of (β^k, p_k) in Step 2, notice that, at any point $x_k \in \mathbb{R}^n$, the map $p \mapsto (\nabla F(x_k, \xi_{\beta_j})^\top p + \frac{1}{2} p^\top B(x_k, \xi_{\beta_j}) p)$ is a convex function and $[\omega_k]$ is finite. Therefore, a minimum of $\varphi_{x_k}(\beta, p)$ defined in (4.2) always exists.

(ii) Existence of a step length τ_k in Step 4 is assured by Theorem 4.3.

Therefore, Algorithm 1 is well-defined.

Theorem 4.2. *Let $\{x_k\}$ be a sequence of nonstationary points generated by Algorithm 1, $\{p_k\}$ be the corresponding sequence of directions, and $\{x_k\}$ be convergent. Then, $\{p_k\}$ is bounded.*

Proof. The proof is similar to [11, Theorem 4.4]. We omit its proof here. $\quad\square$

Theorem 4.3. *Let γ be a fixed value in the interval $(0,1)$, and $\bar{x}$ be not a stationary point of* (SOP)*. Let $(\bar{\beta}, \bar{p}) \in P_{\bar{x}} \times \mathbb{R}^n$ be such that $\phi(\bar{x}) = \varphi_{\bar{x}}(\bar{\beta}, \bar{p})$. Under these conditions, the following statements are true.*

(i) There exists $\tilde{\tau} > 0$ such that

$$\forall \tau \in (0, \tilde{\tau}], j \in [\bar{\omega}] : F(\bar{x} + \tau \bar{p}, \xi_{\bar{\beta}_j}) \preceq F(\bar{x}, \xi_{\bar{\beta}_j}) + \gamma \tau \left(\nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} + \frac{1}{2} \bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j}) \bar{p} \right).$$

(ii) Additionally, for all $\tau \in (0, \tilde{\tau}]$ and $j \in [\bar{\omega}]$,

$$F_U(\bar{x} + \tau \bar{p}) \preceq^u \left\{ F(\bar{x}, \xi_{\bar{\beta}_j}) + \gamma \tau \left(\nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} + \frac{1}{2} \bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j}) \bar{p} \right) \right\}_{j \in [\bar{\omega}]}$$
$$\prec^u F_U(\bar{x}).$$

Proof. (i) Suppose that (i) is not satisfied. Therefore, there exists a sequence $\{\tau_k\}$ and $j \in [\bar{\omega}]$ such that $\tau_k \to 0$ and

$$\forall k \in \mathbb{N} : F(\bar{x} + \tau_k \bar{p}, \xi_{\bar{\beta}_j}) - F(\bar{x}, \xi_{\bar{\beta}_j}) - \gamma \tau_k \left(\nabla F(\bar{x}, \xi_{\bar{\beta}})^\top \bar{p} + \frac{1}{2} \bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j}) \bar{p} \right) \notin -K. \quad (4.5)$$

Multiply by $\frac{1}{\tau_k}$ in (4.5) for each $k \in \mathbb{N}$ obtains

$$\forall\, k \in \mathbb{N}: \frac{F(\bar{x} + \tau_k \bar{p}, \xi_{\bar{\beta}_j}) - F(\bar{x}, \xi_{\bar{\beta}_j})}{\tau_k} - \gamma\big(\nabla F(\bar{x}, \xi_{\bar{\beta}})^\top \bar{p} + \frac{1}{2}\bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j})\bar{p}\big) \notin -K. \qquad (4.6)$$

Now taking the limit $k \to \infty$ in (4.6), we have

$$\nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} - \gamma\big(\nabla F(\bar{x}, \xi_{\bar{\beta}})^\top \bar{p} + \frac{1}{2}\bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j})\bar{p}\big) \notin -\mathrm{int}(K). \qquad (4.7)$$

Note that $\bar{x}$ is not a stationary point of (SOP) and $(\bar{\beta}, \bar{p}) \in P_{\bar{x}} \times \mathbb{R}^n$ is such that $\phi(\bar{x}) = \varphi_{\bar{x}}(\bar{\beta}, \bar{p})$. Therefore, by Proposition 4.1, we obtain $\bar{p} \neq 0$ and $\phi(\bar{x}) < 0$. So, by the continuity of ϕ, (see Theorem 4.1) there exists $\delta > 0$ such that

$$\phi(x) \leq \frac{1}{2}\phi(\bar{x}) < 0 \text{ for all } x \in \mathcal{N}[\bar{x}, \delta]. \qquad (4.8)$$

Let us consider a sequence $\{p_k\}$ that lies in $\mathcal{N}[\bar{x}, \delta]$, i.e., $\|p_k - \bar{x}\| \leq \delta$, and converges to $\bar{p}$. By Taylor expansion, we obtain

$$F(\bar{x} + \tau_k p_k, \xi_{\beta_j}) = F(\bar{x}, \xi_{\beta_j}) + \tau_k \nabla F(\bar{x}, \xi_{\beta_j})^\top p_k + \circ_j(\tau_k p_k, x)e \text{ for all } j \in [\omega(\bar{x})],$$

where $\lim\limits_{k \to \infty} \frac{\circ_j(\tau_k p_k, \bar{x})e}{\tau_k \|p_k\|} = (0, 0, \ldots, 0)^\top$. Since $\{p_k\}$ is bounded, we have

$$\lim\limits_{k \to \infty} \frac{\circ_j(\tau_k p_k, \bar{x})e}{\tau_k} = (0, 0, \ldots, 0)^\top.$$

Now, we see $\nabla F(\bar{x}, \xi_{\beta_j})^\top p_k \preceq \nabla F(\bar{x}, \xi_{\beta_j})^\top p_k + \frac{1}{2} p_k^\top B(\bar{x}, \xi_{\beta_j}) p_k$ since $p_k^\top B(\bar{x}, \xi_{\beta_j}) p_k$ is positive definite for all $j \in [\omega(\bar{x})]$. If $\{x_k\}$ is a sequence generated by Algorithm 1 and converges to $\bar{x}$, then, for all $j \in [\omega(\bar{x})]$,

$$F(x_k + \tau_k p_k, \xi_{\beta_j^k}) \preceq F(x_k, \xi_{\beta_j^k}) + \tau_k\big(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k\big) + \circ_j(\tau_k p_k, x_k)e.$$

Therefore,

$$\frac{F(x_k + \tau_k p_k, \xi_{\beta_j^k}) - F(x_k, \xi_{\beta_j^k})}{\tau_k}$$

$$\preceq \gamma\big(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k\big)$$

$$+ \big[(1 - \gamma)\big(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k\big) + \frac{\circ_j(\tau_k p_k, x_k)e}{\tau_k}\big].$$

Taking limit $k \to \infty$ in the above inequality, we have from (4.8) that

$$\nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} \preceq \gamma\big(\nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} + \frac{1}{2}\bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j})\bar{p}\big),$$

i.e., $\nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} - \gamma\big(\nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} + \frac{1}{2}\bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j})\bar{p}\big) \in -\mathrm{int}(K)$ for all $j \in [\omega(\bar{x})]$,

which contradicts (4.7). Hence, (i) is proved.

(ii) As $\bar{x}$ is not a stationary point, we see by Proposition 4.1 that

$$\phi(\bar{x}) < 0 \implies \varphi_{\bar{x}}(\bar{\beta}, \bar{p}) < 0$$

$$\implies \max_{j \in [\bar{\omega}]}\big\{\Theta_e\big(\nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} + \frac{1}{2}\bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j})\bar{p}\big)\big\} < 0$$

$$\implies \text{ for any } j \in [\bar{\omega}], \Theta_e\left(\nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} + \frac{1}{2}\bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j})\bar{p}\right) < 0$$

$$\implies \nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} + \frac{1}{2}\bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j})\bar{p} \in -\text{int}(K).$$

Using this in inequality (i), we have

$$\forall \tau \in (0, \tilde{\tau}], j \in [\bar{\omega}] : F(\bar{x} + \tau\bar{p}, \xi_{\bar{\beta}_j}) \prec F(\bar{x}, \xi_{\bar{\beta}_j}). \tag{4.9}$$

As $\text{Max}(F_U(\bar{x} + \tau\bar{p}), K) = \{F(\bar{x} + \tau\bar{p}, \xi_{\bar{\beta}_j}) : j \in [\bar{\omega}]\}$, it follows from Proposition 2.1 that

$$\begin{aligned}
F_U(\bar{x} + \tau\bar{p}) &\subseteq \left\{F\left(\bar{x} + \tau\bar{p}, \xi_{\bar{\beta}_j}\right)\right\}_{j \in [\bar{\omega}]} - K \\
&\overset{(4.9)}{\subseteq} \left\{F(\bar{x}, \xi_{\bar{\beta}_j}) + \gamma\tau\left(\nabla F(\bar{x}, \xi_{\bar{\beta}_j})^\top \bar{p} + \frac{1}{2}\bar{p}^\top B(\bar{x}, \xi_{\bar{\beta}_j})\bar{p}\right)\right\}_{j \in [\bar{\omega}]} - K \\
&\subseteq \left\{F(\bar{x}, \xi_{\bar{\beta}_j})\right\}_{j \in [\bar{\omega}]} - \text{int}(K) - K \text{ by Statement (i)} \\
&\subseteq \left\{F(\bar{x}, \xi_{\bar{\beta}_1}), F(\bar{x}, \xi_{\bar{\beta}_2}), \dots, F(\bar{x}, \xi_{\bar{\beta}_{\bar{\omega}}})\right\} - \text{int}(K) \\
&\subseteq F_U(\bar{x}) - \text{int}(K).
\end{aligned}$$

Hence, statement (ii) is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 4.4. *Let $\{x_k\}$ be a sequence produced by Algorithm 1 and $\bar{x}$ be an accumulation point for $\{x_k\}$. Additionally, let $\bar{x}$ be a regular point for F_U. Then, $\bar{x}$ is a stationary point of* (SOP).

Proof. Let $\Upsilon : P(\mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$ be the functional defined by $\Upsilon(A) := \sup_{z \in A} \Theta_e(z)$, $A \in P(\mathbb{R}^m)$. First, we show that, for all $k \in \mathbb{N} \cup \{0\}$, $(\Upsilon \circ F_U)(x_{k+1}) \leq (\Upsilon \circ F_U)(x_k) + \gamma\tau_k\phi(x_k)$. Due to the monotonicity property of Θ_e in Proposition 2.2 (iii), one sees that Υ is also monotone with respect to the $\preceq^u$, i.e., for all A and B in $P(\mathbb{R}^m)$, $A \preceq^u B \implies \Upsilon(A) \leq \Upsilon(B)$. In view of (ii) of Theorem 4.3, for all $k \in \mathbb{N} \cup \{0\}$, one has

$$F_U(x_k + \tau_k p_k) \preceq^u \left\{F(x_k, \xi_{\beta_j^k}) + \gamma\tau_k\left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2}p_k^\top B(x_k, \xi_{\beta_j^k})p_k\right)\right\}_{j \in [\omega_k]}.$$

Therefore, by applying the monotonicity of Υ and the sublinearity of Θ_e, we derive that, for any $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
(\Upsilon \circ F_U)(x_{k+1}) &\leq \max_{j \in [\omega_k]}\left\{\Theta_e\left(F(x_k, \xi_{\beta_j^k}) + \gamma\tau_k\left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2}p_k^\top B(x_k, \xi_{\beta_j^k})p_k\right)\right)\right\} \\
&\leq \max_{j \in [\omega_k]}\left\{\Theta_e\left(F(x_k, \xi_{\beta_j^k})\right) + \gamma\tau_k\Theta_e\left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2}p_k^\top B(x_k, \xi_{\beta_j^k})p_k\right)\right\} \\
&\leq \max_{j \in [\omega_k]}\Theta_e\left(F(x_k, \xi_{\beta_j^k})\right) + \gamma\tau_k\max_{j \in [\omega_k]}\Theta_e\left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2}p_k^\top B(x_k, \xi_{\beta_j^k})p_k\right) \\
&= (\Upsilon \circ F_U)(x_k) + \gamma\tau_k\max_{j \in [\omega_k]}\Theta_e\left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2}p_k^\top B(x_k, \xi_{\beta_j^k})p_k\right)
\end{aligned}$$

$$\implies -\gamma\tau_k\max_{j \in [\omega_k]}\Theta_e\left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2}p_k^\top B(x_k, \xi_{\beta_j^k})p_k\right) \leq (\Upsilon \circ F_U)(x_k) - (\Upsilon \circ F_U)(x_{k+1}).$$

By summing the above inequality for $k = 0, 1, 2, \ldots, \bar{k}$, we have

$$-\sum_{k=0}^{\bar{k}} \tau_k \max_{j \in [\omega_k]} \Theta_e \left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k \right) \leq \frac{(\Upsilon \circ F_U)(x_0) - (\Upsilon \circ F_U)(x_{\bar{k}+1})}{\gamma}.$$

As $\{x_k\}$ is a sequence of nonstationary points, one sees that

$$\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k \in -\mathrm{int}(K). \tag{4.10}$$

Specifically, by applying Proposition 2.2 in (4.10), we find that, for all $k \in \mathbb{N} \cup \{0\}, j \in [\omega_k]$,

$$\Theta_e \left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k \right) < 0.$$

Therefore,

$$0 < -\sum_{k=0}^{\bar{k}} \tau_k \max_{j \in [\omega_k]} \Theta_e \left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k \right) \leq \frac{(\Upsilon \circ F_U)(x_0) - (\Upsilon \circ F_U)(x_{\bar{k}+1})}{\gamma}.$$

Now, by taking the limit of the previous inequality as $\bar{k} \to \infty$, we deduce that

$$0 \leq -\sum_{k=0}^{\infty} \tau_k \max_{j \in [\omega_k]} \Theta_e \left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k \right) < \infty.$$

In particular, we have

$$\lim_{k \to \infty} \tau_k \max_{j \in [\omega_k]} \Theta_e \left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k \right) = 0$$

$$\text{i.e., } \lim_{k \to \infty} \tau_k \phi(x_k) = 0. \tag{4.11}$$

However, since $\bar{x}$ is an accumulation point of $\{x_k\}_{k \geq 0}$, one sees that there exists a subsequence $\mathcal{K}$ in $\mathbb{N}$ such that $x_k \xrightarrow{\mathcal{K}} \bar{x}$. Since there are only a finite number of subsets of $[p]$ and $\bar{x}$ is regular for F_U, we can utilize [2, Lemma 4.2]. From this, it follows that, without any loss of generality, there exists a subset $A \subseteq P_{\bar{x}}$ and an element $\bar{\beta} \in A$ such that, for large k, $\omega_k = \bar{\omega}$, $P_{x_k} = A$, and $\beta^k = \bar{\beta}$. Furthermore, since $\{\tau_k\}$ and $\{p_k\}$ are bounded by Theorem 4.2, we can find $\bar{\tau} > 0$ and $\bar{p} \in \mathbb{R}^n$ such that $\tau_k \xrightarrow{k \in \mathcal{K}} \bar{\tau}$ and $p_k \xrightarrow{k \in \mathcal{K}} \bar{p}$. Assume that $\bar{x}$ is nonstationary, according to Proposition 4.1. It implies that $\phi(\bar{x}) < 0$ and $\bar{p} \neq 0$. By applying Theorem 4.3 (ii), it follows that there exists an integer q such that

$$\Upsilon \circ \left(F_U(\bar{x} + 2^{-q} \bar{p}) - F_U(\bar{x}) \right) < \gamma \, 2^{-q} \phi(\bar{x}). \tag{4.12}$$

Since ϕ and Υ are continuous within their respective domains, we have

$$\lim_{k \to \infty} p_k = \bar{p} \text{ and } \lim_{k \to \infty} \phi(x_k) = \phi(\bar{x}) < 0. \tag{4.13}$$

For large enough k, we see from (4.12) that $(\Upsilon \circ F_U)(x_k + 2^{-q} p_k) - (\Upsilon \circ F_U)(x_k) < \gamma \, 2^{-q} \phi(x_k)$. Considering the definition of Υ and Step 4 of Algorithm 1, we see that, for sufficiently large k, $\tau_k \geq 2^{-q}$. By taking into account the second limit in (4.13), we deduce $\liminf_{k \to \infty} \tau_k |\phi(x_k)| > 0$, which contradicts (4.11). Thus the desired result follows. $\qquad \square$

Before we start analysing the convergence properties of the proposed quasi-Newton Algorithm 1, we recall some results confirming the superlinear convergence of the BFGS method. The first result, based on [3], states that if $\{x_k\}$ is a sequence of nonstationary points converging to $\bar{x}$, then

$$\lim_{k \to \infty} \frac{\left\| \left(B(x_k) - \nabla^2 F(x_k) \right) s_k \right\|}{\|s_k\|} = 0,$$

where F is twice continuously differentiable such that $\nabla F(\bar{x}) = 0$ and $\nabla^2 F(\bar{x})$ is positive definite, $s_k := x_{k+1} - x_k = \tau_k p_k$. Moreover, B_k is a BFGS approximation at iteration k. Now, from [30], for some $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,

$$\lim_{k \to \infty} \frac{\left\| (B(x_k, \xi_{\beta_j^k}) - \nabla^2 F(\bar{x}, \xi_{\bar{\beta}_j^k})) p_k \right\|}{\|p_k\|} < \varepsilon \text{ for every } j \in [\omega(x_k)].$$

Lemma 4.1. [2, 4] *Let $\bar{x}$ be a regular point of the function F_U.*

(i) Then, there exists $\delta > 0$, such that, for every $x \in \mathcal{N}(\bar{x}, \delta)$, $\omega(x) = \omega(\bar{x})$.
(ii) Let $\varepsilon > 0$ and $\delta > 0$ such that, for any $x, y \in S$ with $\|y - x\| < \delta$,

$$\left\| \nabla^2 F(y, \xi_{\beta_j}) - \nabla^2 F(x, \xi_{\beta_j}) \right\| < \frac{\varepsilon}{2} \text{ for all } j = 1, 2, \ldots, [\omega(\bar{x})]. \tag{4.14}$$

For any $x, y \in S$ such that $\|y - x\| < \delta$,
(a)

$$\left\| \nabla F(y, \xi_{\beta_j}) - [\nabla F(x, \xi_{\beta_j}) + \nabla^2 F(x, \xi_{\beta_j})(y - x)] \right\| < \frac{\varepsilon}{2} \|y - x\|, \text{ and} \tag{4.15}$$

(b)

$$\left\| F(y, \xi_{\beta_j}) - \left[F(x, \xi_{\beta_j}) + \nabla F(x, \xi_{\beta_j})^\top (y - x) + \frac{1}{2}(y - x)^\top \nabla^2 F(x, \xi_{\beta_j})(y - x) \right] \right\| < \frac{\varepsilon}{4} \|y - x\|^2 \tag{4.16}$$

for all $j = 1, 2, \ldots, [\omega(\bar{x})]$.

Next, we modify Lemma 4.1 to assess the approximation error by using the BFGS approximation of the second-order derivative term Hessian.

Lemma 4.2. *Let $V \subset S$ be a convex subset, and $\varepsilon > 0$. Let V be such that there exists a $\delta > 0$ for which any $x, y \in V$ holds $\|y - x\| < \delta$. Let $\{x_k\}$ be a sequence generated by Algorithm 1. Assume that, for the given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,*

$$\frac{\left\| (\nabla^2 F(x_k, \xi_{\beta_j^k}) - B(x_k, \xi_{\beta_j^k}))(y - x_k) \right\|}{\|y - x_k\|} < \frac{\varepsilon}{2}. \tag{4.17}$$

Then, for any x_k, $k \geq k_0$, and $y \in V$ with $\|y - x_k\| < \delta$,

$$\left\| \nabla F(y, \xi_{\beta_j}) - [\nabla F(x_k, \xi_{\beta_j^k}) + B(x_k, \xi_{\beta_j^k})(y - x_k)] \right\| < \varepsilon \|y - x_k\|, \tag{4.18}$$

and

$$\left\| F(y, \xi_{\beta_j}) - [F(x_k, \xi_{\beta_j^k}) + \nabla F(x_k, \xi_{\beta_j^k})^\top (y - x_k) + \frac{1}{2}(y - x_k)^\top B(x_k, \xi_{\beta_j^k})(y - x_k)] \right\| < \frac{\varepsilon}{2} \|y - x_k\|^2. \tag{4.19}$$

Proof. From (4.15) of Lemma 4.1, for every $j \in [\omega(x)]$ and $x, y \in S$ such that $\|y - x\| < \delta$, we have

$$\left\| \nabla F(y, \xi_{\beta_j}) - [\nabla F(x_k, \xi_{\beta_j^k}) + B(x_k, \xi_{\beta_j^k})(y - x_k)] \right\|$$

$$\leq \left\| \nabla F(y, \xi_{\beta_j}) - \nabla F(x_k, \xi_{\beta_j^k}) - \nabla^2 F(x_k, \xi_{\beta_j^k})(y - x_k) \right\| + \left\| (\nabla^2 F(x_k, \xi_{\beta_j^k}) - B(x_k, \xi_{\beta_j^k}))(y - x_k) \right\|$$

$$\leq \frac{\varepsilon}{2} \|y - x_k\| + \frac{\varepsilon}{2} \|y - x_k\|, \text{ using (4.15) and (4.17)}$$

$$\leq \varepsilon \|y - x_k\|.$$

Similarly, by (4.16), one has

$$\left\| F(y, \xi_{\beta_j}) - [F(x_k, \xi_{\beta_j^k}) + \nabla F(x_k, \xi_{\beta_j^k})^\top (y - x_k) + \frac{1}{2}(y - x_k)^\top B(x_k, \xi_{\beta_j^k})(y - x_k)] \right\|$$

$$\leq \left\| F(y, \xi_{\beta_j}) - F(x_k, \xi_{\beta_j}^k) - \nabla F(x_k, \xi_{\beta_j}^k)^\top (y - x_k) - \frac{1}{2}(y - x_k)^\top \nabla^2 F(x_k, \xi_{\beta_j}^k)(y - x_k) \right\|$$

$$+ \left\| \frac{1}{2}(y - x_k)^\top (\nabla^2 F(x_k, \xi_{\beta_j^k}) - B(x_k, \xi_{\beta_j^k}))(y - x_k) \right\|$$

$$< \frac{\varepsilon}{4} \|y - x_k\|^2 + \frac{\varepsilon}{4} \|y - x_k\|^2 < \frac{\varepsilon}{2} \|y - x_k\|^2,$$

which confirms (4.19) and the proof is complete. $\square$

Theorem 4.5. (Superlinear convergence). *Let $\{x_k\}$ be a sequence of nonstationary points generated by Algorithm 1 and $\bar{x}$ be one of its accumulation points. Additionally, let $\bar{x}$ be a regular point for F_U. Suppose that S is a neighborhood of $\bar{x}$ and there exists $a > 0, b > 0, \delta > 0$ and $\varepsilon > 0$ for which the following conditions hold:*

(i) $aI \leq B(x, \xi_j) \leq bI$ for all $j \in [\omega(x)]$,
(ii) $\left\| \nabla^2 F(x, \xi_{\beta_j}) - \nabla^2 F(y, \xi_{\beta_j}) \right\| < \varepsilon$ for all $x, y \in S$ with $\|x - y\| < \delta$,
(iii) $\left\| (\nabla^2 F(x_k, \xi_{\beta_j^k}) - B(x_k, \xi_{\beta_j^k}))(y - x_k)) \right\| < \varepsilon$ for all $x_k, y \in S$ with $\|x_k - y\| < \delta$, and
(iv) $\varepsilon \leq a(1 - \gamma)$.

Then, $\tau_k = 1$, for sufficiently large $k \in \mathbb{N}$, and the sequence $\{x_k\}$ converges superlinearly to $\bar{x} \in \mathbb{R}^n$.

Proof. Theorem 4.4 established the convergence of the sequence $\{x_k\}$ to a stationary point $\bar{x}$. Given that $F(\cdot, \xi_{\beta_j})$ functions are twice continuously differentiable, it follows that, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\mathcal{N}(\bar{x}, \delta_\varepsilon) \subset S \text{ and } \left\| B(x, \xi_{\beta_j}) - B(y, \xi_{\beta_j}) \right\| < \varepsilon \text{ for all } x, y \in \mathcal{N}(\bar{x}, \delta_\varepsilon).$$

Now, for $x \in S$, $\lambda_j \geq 0$ and $\sum_{j=1}^{\omega(x)} \lambda_j = 1$, we define a function $\Phi_\lambda : S \times \mathbb{R}^n \to \mathbb{R}^m$ by

$$\Phi_\lambda(x, p) := \sum_{j=1}^{[\omega(x)]} \lambda_j \nabla F(x, \xi_{\beta_j})^\top p + \frac{1}{2} \sum_{j=1}^{[\omega(x)]} \lambda_j p^\top B(x, \xi_{\beta_j}) p.$$

Note that, for any $\beta^k \in P_{x_k}$ and $j \in [\omega(x_k)]$, the matrix $B(x_k, \xi_{\beta_j^k})$ is K positive definite. Furthermore, for any $x_k \in \mathbb{R}^n$ and $\beta^k \in P_{x_k}$, set P_{x_k} is finite. Therefore, function $\Phi_\lambda(x, \cdot)$ is strictly

convex in $\mathbb{R}^n$, and hence it attains its minimum. Observe that Φ_λ attains minimum value when

$$\sum_{j=1}^{[\omega(x_k)]} \lambda_j \nabla F(x_k, \xi_{\beta_j^k}) + \sum_{j=1}^{[\omega(x_k)]} \lambda_j B(x_k, \xi_{\beta_j^k}) p_k = 0. \tag{4.20}$$

$$\implies p_k = -\left[\sum_{j=1}^{[\omega(x_k)]} \lambda_j B(x_k, \xi_{\beta_j^k})\right]^{-1} \sum_{j=1}^{[\omega(x_k)]} \lambda_j \nabla F(x_k, \xi_{\beta_j^k})$$

$$\implies p_k \leq -\frac{1}{a} \sum_{j=1}^{[\omega(x_k)]} \lambda_j \nabla F(x_k, \xi_{\beta_j^k}) \text{ since } aI \leq B(x_k, \xi_{\beta_j^k})$$

$$\implies p_k \leq -\frac{1}{a} \max_{j \in [\omega(x_k)]} \nabla F(x_k, \xi_{\beta_j^k}). \tag{4.21}$$

Note that $\sum_{j=1}^{[\omega(x)]} \lambda_j b_j \leq \max\{b_1, b_2, \ldots, b_{[\omega(x)]}\}$ holds, where $(\lambda_1, \lambda_2, \ldots, \lambda_{[\omega(x)]}) \in \mathbb{R}_+^{\omega(x)}$ with $\sum_{j=1}^{[\omega(x)]} \lambda_j = 1$ and b_j's in $\mathbb{R}$. This identity is applied in obtaining (4.21). For sequence $\{x_k\}$, using Lemma 4.1, we have

$$\left\| \sum_{j=1}^{[\omega(x_{k+1})]} \lambda_j \nabla F(x_k + p_k, \xi_{\beta_j^k}) - \left[\sum_{j=1}^{[\omega(x_k)]} \lambda_j \nabla F(x_k, \xi_{\beta_j^k}) + \sum_{j=1}^{[\omega(x_k)]} \lambda_j B(x_k, \xi_{\beta_j^k}) p_k\right] \right\| \leq \varepsilon \|p_k\|$$

i.e., $\left\| \sum_{j=1}^{[\omega(x_{k+1})]} \lambda_j \nabla F(x_k + p_k, \xi_{\beta_j^k}) \right\| \leq \varepsilon \|p_k\|$, i.e., $\left\| \max_{j \in [\omega(x_{k+1})]} \nabla F(x_k + p_k, \xi_{\beta_j^k}) \right\| \leq \varepsilon \|p_k\|$

i.e., $\|p_{k+1}\| \leq \frac{\varepsilon}{a} \|p_k\|$ using (4.21). $\tag{4.22}$

As $\{x_k\}$ converges to $\bar{x}$, one sees that there exists $k_\varepsilon \in \mathbb{N}$ such that, for all $k \geq k_\varepsilon$, $x_k, x_k + p_k \in \mathcal{N}(\bar{x}, \delta_\varepsilon)$. Now, using the second-order Taylor expansion for $j = 1, 2, \ldots, [\omega(x_k)]$, we have

$$F(x_k + p_k, \xi_{\beta_j^k})$$

$$\leq F(x_k, \xi_{\beta_j^k}) + \gamma \left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k\right)$$

$$+ (1 - \gamma)\left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k\right) + \frac{\varepsilon}{2} \|p_k\|^2$$

$$\leq F(x_k, \xi_{\beta_j^k}) + \gamma \left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k\right)$$

$$+ (1 - \gamma)\left(-p_k^\top \left(\sum_{j=1}^{[\omega(x_k)]} \lambda_j B(x_k, \xi_{\beta_j^k})\right) p_k\right)$$

$$+ \frac{1}{2} p_k^\top \left(\sum_{j=1}^{[\omega(x_k)]} \lambda_j B(x_k, \xi_{\beta_j^k})\right) p_k\right) + \frac{\varepsilon}{2} \|p_k\|^2 \text{ using (4.20)}$$

$$\leq F(x_k, \xi_{\beta_j^k}) + \gamma \left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k\right) - \frac{a(1 - \gamma)}{2} \|p_k\|^2 + \frac{\varepsilon}{2} \|p_k\|^2$$

$$\leq F(x_k, \xi_{\beta_j^k}) + \gamma \left(\nabla F(x_k, \xi_{\beta_j^k})^\top p_k + \frac{1}{2} p_k^\top B(x_k, \xi_{\beta_j^k}) p_k\right) + \frac{\varepsilon - a(1 - \gamma)}{2} \|p_k\|^2,$$

which together with condition (iv) concludes

$$\varepsilon \le a(1-\gamma) \implies \frac{\varepsilon - a(1-\gamma)}{2} \le 0,$$

and $\tau_k = 1$ holds for $k \ge k_\varepsilon$. Now, for $k \ge k_\varepsilon$, we have

$$\|x_{k+1} - x_{k+2}\| = \|p_{k+1}\| \le \frac{\varepsilon}{a}\|p_k\| = \frac{\varepsilon}{a}\|x_k - x_{k+1}\|, \text{ using (4.22)}.$$

For any $k \ge 1$ and $j \ge 1$, we obtain

$$\|x_{k+j} - x_{k+j+1}\| \le \frac{\varepsilon}{a}\|x_{k+j-1} - x_{k+j}\| \le (\frac{\varepsilon}{a})^2\|x_{k+j-2} - x_{k+j-1}\| \le \cdots \le (\frac{\varepsilon}{a})^j\|x_k - x_{k+1}\|.$$

Now, to prove superlinear convergence, we take $t \in (0,1)$ and define

$$\bar{\varepsilon} := \min\{a(1-\gamma), \frac{t}{1+2t}a\}.$$

If we take $\varepsilon < \bar{\varepsilon}$ and $k \ge k_\varepsilon$, then, by convergence of sequence $\{x_k\}$ to $\bar{x}$, we have

$$\|\bar{x} - x_{k+1}\| \le \sum_{j=1}^{\infty} \|x_{k+j} - x_{k+j+1}\| \le \sum_{j=1}^{\infty} (\frac{t}{1+2t})^j\|x_k - x_{k+1}\| = \frac{t}{1+t}\|x_k - x_{k+1}\|. \quad (4.23)$$

Therefore, we

$$\|\bar{x} - x_k\| \ge \|x_k - x_{k+1}\| - \|x_{k+1} - \bar{x}\| \ge \frac{1}{1+t}\|x_k - x_{k+1}\|. \quad (4.24)$$

Combining (4.23) and (4.24), we can conclude that if $\varepsilon < \bar{\varepsilon}$ and $k \ge k_\varepsilon$, then

$$\frac{\|\bar{x} - x_{k+1}\|}{\|\bar{x} - x_k\|} \le t.$$

As t lies in $(0,1)$, we conclude that sequence $\{x_k\}$ converges superlinear to $\bar{x}$.   □

5. NUMERICAL ILLUSTRATION

In this section, we show the performance of the proposed quasi-Newton Algorithm 1 on several numerical examples. The testing of Algorithm 1 is conducted in MATLAB R2023b. The software is installed on an IOS system with an 8-core CPU and 8 GB RAM. For the numerical execution of the Algorithm 1, we selected the following parameter values:

- In our tests, the cone K is chosen as the standard ordering cone, specifically $K = \mathbb{R}_+^2$ for all cases except in Example 5.1. Additionally, we set the parameter $e = (1,1,\ldots,1)^\top \in \text{int}(K)$ for the scalarizing function Θ_e.
- The parameter γ in Step 4 of the line search in Algorithm 1 was set to $\gamma = 0.1$.
- For the stopping condition, we choose $\|p_k\| \le 0.0001$, or a maximum number of 1000 iterations are reached.
- To find the set $\text{Max}(F_U(x_k), K)$ with respect to upper set less relation at the k-th iterations in Step 1 of Algorithm 1, we implemented the common method of pair-wise comparison the elements in $F_U(x_k)$.

- In the k-th iteration of Step 2 in Algorithm 1, for each $\beta \in P_k$, we determine a minimizer p of the strictly convex problem $\min\limits_{p \in \mathbb{R}^n} \varphi_x(\beta, p)$. Next, we identify

$$(\beta_k, p_k) = \operatorname*{argmin}_{\beta \in P_k} \varphi_{x_k}(\beta, p)$$

  with the help of the inbuilt function *fmincon* in MATLAB.
- We take some test problems from the literature, while some are freshly introduced in this paper. For each problem, we generate 100 random initial points and make a three-column table. The following values have been collected for each of the examples from Examples 5.1 to 5.4:
  (1) **Initial Points**: The value represents the first column in the table, indicating the number of random initial points used in applying Algorithm 1.
  (2) **Iterations**: The second column with a 6-tuple (Min, Max, Mean, Median, Mode, SD) indicates the minimum, maximum, mean, median, mode, and standard deviation of the number of iterations in which the stopping condition is reached.
  (3) **CPU Time**: The third column comprises another 6-tuple (Min, Max, Mean, Median, Mode, SD) that shows the minimum, maximum, mean, median, least integer greater than or equal to mode, and standard deviation of the CPU time (in seconds) taken by Algorithm 1 in reaching the stopping condition.

Moreover, numerical values are displayed with precision up to four decimal places to enhance clarity. In each problem, the values of F at each iteration are graphically depicted using different colors: the initial points are presented in black and the final points in red. The intermediate points are represented in blue.

Additionally, we evaluate the performance of the proposed quasi-Newton method (abbreviated as QNM) algorithm (Algorithm 1) by comparing it with the existing Newton method (abbreviated as NM) for each of the four Examples 5.1 to 5.4.

Example 5.1. Let the uncertainty set be $U := \{0.1, 0.2, 0.3, 0.4\}$. Consider the UVOP with the bi-objective function $F : \mathbb{R} \times U \to \mathbb{R}^2$ defined as

$$F(x, \xi) := \begin{pmatrix} 2x^2 + e^{\frac{x}{10}} + \frac{(10\xi - 3)}{2} \\ 5x\cos(x) + \frac{(-10\xi + 3)}{2}\sin^2(x) \end{pmatrix}.$$

The ordering cone K is given by

$$K := \{(z_1, z_2)^\top \in \mathbb{R}^2 : 3z_1 - z_2 \geq 0, -z_1 + 3z_2 \geq 0\}.$$

In Figure 1, the outcome of Algorithm 1 is illustrated for a randomly selected starting point within the set $[-4.7, 4.7]$. It can be seen that the points depicted with red color are optimal points of F_U as the set $(1.5680, 0.7815)^\top - K$ does not contain any element of F_U other than $(1.5680, 0.7815)^\top$ for all $x \in [-4.7, 4.7]$.

The performance of Algorithm 1 and comparing it with the Newton method [10] for Example 5.1 is shown in Table 1. As usual, the Newton method performs better than the quasi-Newton method since for each ξ in U, the function $F(\cdot, \xi)$ is strongly convex, and the Newton method has a faster rate (quadratic, see [10]) of convergence than the superlinear rate of the proposed quasi-Newton method.
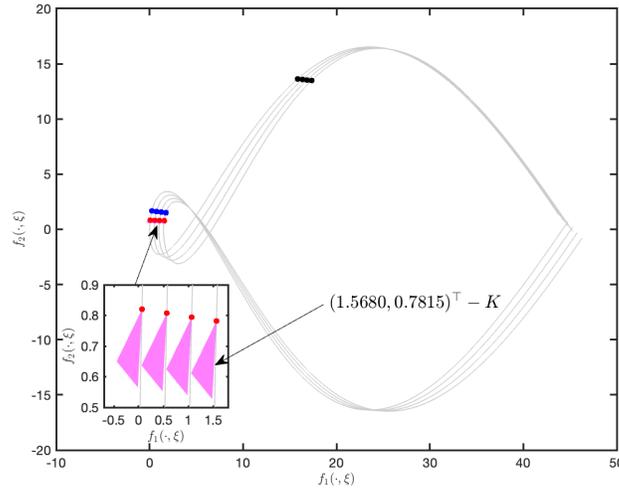
FIGURE 1. The value of F_U at each iteration generated by Algorithm 1 for the initial point $x_0 = -2.8372$ for Example 5.1

TABLE 1. Performance of Algorithm 1 on Example 5.1

Number of successive points	Algorithm	Iterations (Min, Max, Mean, Median, Mode, SD)	CPU time (Min, Max, Mean, Median, Mode, SD)
100	QNM	(0, 4, 0.9800, 0, 0, 1.2551)	(0.1887, 1.4882, 0.5055, 0.2645, 0.1887, 0.3518)
100	NM	(0, 3, 0.7300, 1, 0, 0.8391)	(0.2311, 1.9947, 0.6042, 0.6428, 0.2311, 0.4036)

Example 5.2. Let the uncertainty set be $U := \{0.1, 0.2, \ldots, 4.5\}$. Consider the UMOP with the bi-objective function $F : \mathbb{R}^2 \times U \to \mathbb{R}^2$ defined as

$$F(x,\xi) := \begin{pmatrix} x_1^2 + x_2^2 + 0.5 \sin\left(\frac{2\pi(10\xi-1)}{60}\right) \cos\left(\frac{2\pi(10\xi-1)}{60}\right)^2 + 2e^{(x_1+x_2)} \\ 2x_1^2 + 2x_2^2 + 0.5 \cos\left(\frac{2\pi(10\xi-1)}{60}\right)^2 \end{pmatrix}.$$

In Figure 2, the iterates generated by Algorithm 1 for different initial points taken from the set $[-1.5, 1.5] \times [-1, 0.5]$ are given. The sequence of iterates $\{x_k\}$ and the corresponding $\{F(x_k)\}$ generated by Algorithm 1 for a selected initial point $x_0 = (1.5, 0.5)^\top$ are given in Figure 2a and Figure 2b, respectively.

The performance of Algorithm 1 and comparing it with the Newton method [10] for Example 5.2 is shown in Table 2. In this example also, the Newton method performs better than Algorithm 1 because F is strongly convex.

TABLE 2. Algorithm 1 Performance of Algorithm 1 on Example 5.2

Number of successive points	Algorithm	Iterations (Min, Max, Mean, Median, Mode, SD)	CPU time (Min, Max, Mean, Median, Mode, SD)
100	QNM	(0, 11, 4.0220, 4, 3, 2.4266)	(0.0213, 32.3012, 14.2624, 13.2682, 2.3446, 6.8122)
100	NM	(0, 11, 1.4700, 1, 1, 1.5005)	(0.0191, 12.1958, 1.2327, 1.0054, 0.0191, 1.3375)

(A) The value of F_U at each iteration generated by Algorithm 1 for the initial point $x_0 = (1.5, 0.5)$ for Example 5.2

(B) The value of x_k at each iteration generated by Algorithm 1 for the initial point $x_0 = (1.5, 0.5)$ for Example 5.2
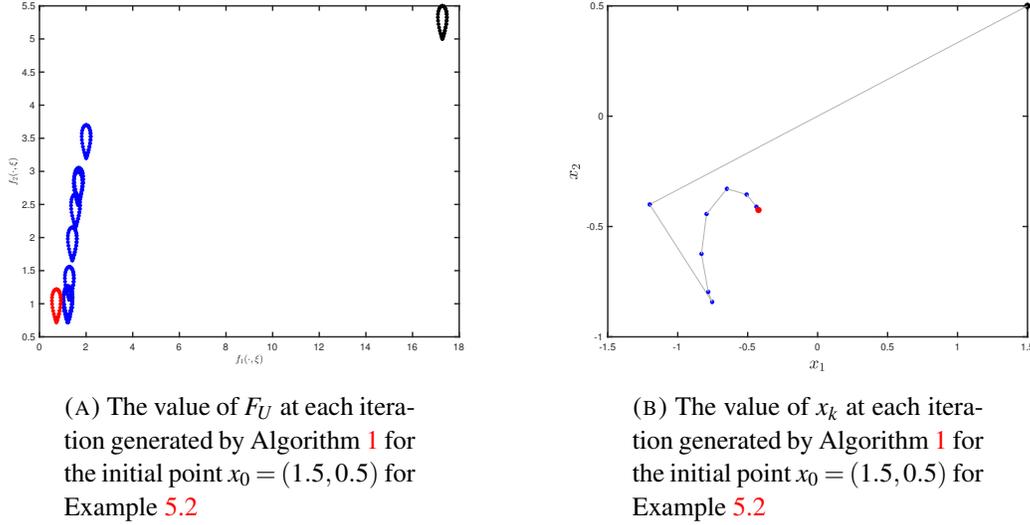
FIGURE 2. Output of Algorithm 1 for Example 5.2

TABLE 3. Performance of Algorithm 1 on Example 5.3

Number of successive points	Algorithm	Iterations (Min, Max, Mean, Median, Mode, SD)	CPU time (Min, Max, Mean, Median, Mode, SD)
100	QNM	(0, 5, 1.6383, 1, 1, 1.5300)	(6.3462, 54.3345, 21.1840, 15.9259, 6.3462, 12.8506)

Example 5.3. Let the uncertainty set be $U := \{0.1, 0.2, \ldots, 5.0\}$. Consider the UMOP with the tri-objective function $F : \mathbb{R}^2 \times U \to \mathbb{R}^3$ defined as

$$F(x, \xi) := \begin{pmatrix} x_1^2 + (0.5) \sin\left(\frac{2\pi(10\xi - 1)}{50}\right) - 0.1\xi x_1 \\ 2x_1^2 + (0.5) \cos\left(\frac{2\pi(10\xi - 1)}{50}\right) + 0.2\xi x_2 \\ x_1^2 + x_2^2 + 10\xi \end{pmatrix}.$$

The sequence of iterates $\{x_k\}$ and the corresponding $\{F_U(x_k)\}$ generated by Algorithm 1 for a selected initial point $x_0 = (1, -1)^\top$ are given in Figure 3a and Figure 3b, respectively.

Note that the objective function of this example does not fulfill the prerequisites of the Newton method given in [10] because Hessian matrices are not positive definite at any given point. Therefore, in Table 3, we provide only the performance of the proposed quasi-Newton quasi-Newton method.

Example 5.4. [10] Consider the UMOP with the tri-objective function $F : \mathbb{R}^2 \times U \to \mathbb{R}^3$ defined as

$$F(x, \xi) := \frac{1}{2} \begin{pmatrix} \|x - l_1 - \xi\|^2 \\ \|x - l_2 - \xi\|^2 \\ \|x - l_3 - \xi\|^2 \end{pmatrix},$$
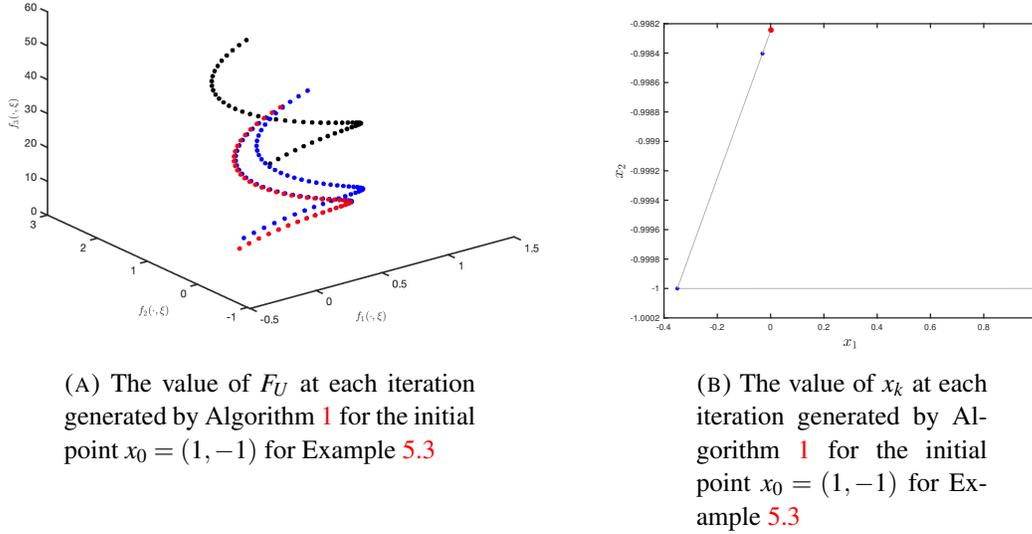
(A) The value of F_U at each iteration generated by Algorithm 1 for the initial point $x_0 = (1, -1)$ for Example 5.3

(B) The value of x_k at each iteration generated by Algorithm 1 for the initial point $x_0 = (1, -1)$ for Example 5.3

FIGURE 3. Output of Algorithm 1 for Example 5.3

TABLE 4. Performance of Algorithm 1 on Example 5.4

Number of successive points	Algorithm	Iterations (Min, Max, Mean, Median, Mode, SD)	CPU time (Min, Max, Mean, Median, Mode, SD)
100	QNM	(0, 2, 1.0800, 1, 1, 0.3075)	(6.1637, 12.7476, 9.8998, 8.5954, 6.1637, 2.0854)
100	NM	(0, 2, 1.0782, 1, 1, 0.3105)	(2.3863, 12.7695, 6.4920, 4.2540, 2.3863, 3.9367)

where $l_1 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, l_2 := \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ and $l_3 := \begin{pmatrix} 0 \\ 8 \end{pmatrix}$. We consider a uniform partition set of 10 points of the interval $[-1, 1]$ given by

$$V := \left\{ -1, -1 + \frac{1}{s}, -1 + \frac{2}{s}, \ldots, -1 + \frac{(2s-1)}{s}, 1 \right\} \text{ with } s = 4.5.$$

A scenario $\xi := (\xi_1, \xi_2)$ is a member of the uncertainty set $U := V \times V$. In Figure 4, 100 initial points were generated within the square $[-50, 50] \times [-50, 50]$. The grey points illustrate the set $(l_1 + \xi) \cup (l_2 + \xi) \cup (l_3 + \xi)$ and the positions of l_1, l_2, l_3 are shown in blue colour. The value of initial and final points generated by the Algorithm 1 are depicted by black and red, respectively, in Figure 4. There are no intermediate points in Figure 4 because the Algorithm 1 has taken only one iteration to reach stationary points from these chosen initial points.

The performance of Algorithm 1 and comparing it with the Newton method [10] for Example 5.4 is shown in Table 4. As usual, in this example, also the Newton method outperforms the quasi-Newton method since the function F is strongly convex for each ξ, and the Newton Method has a quadratic rate of convergence.

Next, we generate a few test problems and show the performance of the proposed Algorithm 1 on those problems. These test problems are either newly introduced or generated from the commonly used test problems in the literature of multiobjective optimization. The list of the
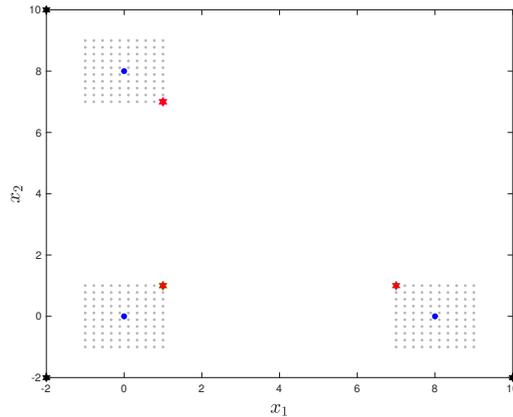
FIGURE 4. Output of Algorithm 1 for Example 5.4: The value of x_k at each iteration for three different initial points: $\{(-2, -2)^\top, (-2, 10)^\top, \text{ and } (10, -2)^\top\}$, as generated by Algorithm 1 in 5.4

generated test problems is given in Table 5. In the last column of the table, we mention the source of the associated multiobjective problem. For example, the problem P3 is generated from the bi-objective test problem BK1 from [13]. In the conventional bi-objective test problem BK1, we have incorporated the uncertainty variables ξ_1 and ξ_2 to generate the uncertain bi-objective test problem P3; note that for $\xi_1 = 1$ and $\xi_2 = 1$, the problem P3 reduces to BK1 problem. Similarly, all other problems, except P2 and P7, are generated from the sources mentioned in the last column of Table 5. Problems P2 and P7 are newly introduced problems for UMOP.

The performance of Algorithm 1 on the test problems from Table 4 are provided in Table 6. In Figure 5, we have shown the generated robust weakly efficient point corresponding to a randomly chosen initial point for test problems taken from Table 5. The black-colored points in each figure collectively present the value of $F_U(x_0)$ at the initial point x_0, blue-colored points present $F_U(x_k)$ for intermediate x_k's, and the red-colored points collectively represent $F_U(\bar{x})$, where $\{x_k\}$ is the sequence generated by Algorithm 1 corresponding to a given initial point x_0 and $x_k \to \bar{x}$. The point $\bar{x}$ is a robust weakly efficient point of the corresponding point.

6. CONCLUSION

In this paper, we have introduced a quasi-Newton method to determine weakly robust efficient solutions for UVOPs with an uncertainty set of finite cardinality. We adopted a set-valued optimization perspective to reformulate the problem as a deterministic one. The deterministic set optimization problem was structured so that its efficient solutions, under the upper set less relation, correspond to the robust efficient solutions of the original UVOP. Utilizing the concept of the partition set, we developed a class of VOPs, which yields efficient solutions to the set optimization problem, thereby identifying robust efficient solutions for the UVOP. In the process of finding a weakly minimal solution of the UVOP, we have approximated the Hessian matrices corresponding to the given functions with the help of BFGS methods for vector optimization. To generate a sequence of β_k from the partition set P_{x_k} of the current iteration at x_k, we evaluated quasi-Newton direction p_k (Step 2) with the help of concepts in [30]. The process of generating iterates by Algorithm 1 continued until the stopping condition (Step 4) was met, and if not,

TABLE 5. Test problems

Problem	(m,n,r)	Modified f	Domain for x $lb^\top$	$ub^\top$	Domain for scenarios $lb^\top$	$ub^\top$	Reference for f
P1	(2,1,1)	$f_1(x,\xi) := x^2 + 2(\xi x+1)^2 + 2(\xi x-1)^2 + x^3$ $f_2(x,\xi) := (x-2\xi)^2 + \xi^3 x + \xi x^3$	-1	0.5	-2	2	MOP1 [13]
P2	(2,1,1)	$f_1(x,\xi) := x + \frac{1}{140}\left(10+\exp\left\{\sin\left(\frac{2\pi\xi}{50}\right)\right\} - \sin\left(\frac{4\pi\xi}{50}\right)\right)$ $\qquad + \xi x$ $f_2(x,\xi) := \cos(3x) + \frac{1}{1+\exp\{(2x)\}} + \frac{1}{140}\left(10+\exp\left\{\sin\left(\frac{2\pi\xi}{50}\right)\right\}\right.$ $\qquad \left. - \sin\left(\frac{4\pi\xi}{50}\right)\right)$	-1	-0.7	1	9	New1
P3	(2,2,2)	$f_1(x,\xi) := \xi_1 x_1^2 + \xi_2 x_2^2 + \xi_2(x_2-5)^2$ $\qquad + (\xi_1 x_1-2)^2 + (\xi_2 x_2-2)^2$ $f_2(x,\xi) := \xi_1 x_1^2 + \xi_2 x_2^2 + (\xi_1 x_1-5)^2$	$(-3,-3)$	$(5,5)$	$(-2,-2)$	$(2,2)$	BK1 [13]
P4	(2,2,2)	$f_1(x,\xi) := x_1^2 + \xi_2 x_2^2 + \xi_1 x_1^2 + 8\xi_2 x_2$ $f_2(x,\xi) := \xi_1(x_1+1)^2 + x_2^2 + \xi_1 x_1^2 + \xi_2 x_2^2$	$(-5,-5)$	$(50,50)$	$(0.8,1)$	$(1,1.2)$	LRS1 [13]
P5	(2,2,2)	$f_1(x,\xi) := (\xi_1 x_1^2 - 1)^2 + \xi_2(x_1-x_2)^2$ $f_2(x,\xi) := (\xi_2 x_2 - 3)^2 + \xi_1(x_1-x_2)^2$	$(-1,-1)$	$(5,5)$	$(1,1)$	$(2,2)$	SP1 [13]
P6	(2,2,2)	$f_1(x,\xi) := \frac{\xi_1}{x_1^2 + \xi_2 x_2^2 + 1} + \xi_1^2 x_1^2 + (\xi_2 x_2-1)^2$ $f_2(x,\xi) := \frac{\xi_1 x_1^2 + 3x_2^2 + 1}{\xi_2} + 2\xi_1 x_1 + 2\xi_2 x_2$	$(-3,-3)$	$(3,3)$	$(1,0.5)$	$(1.5,1)$	VU1 [13]
P7	(2,2,2)	$f_1(x,\xi) := \xi_2 + x_1^2 + (\xi_1 x_1-2)^2 + (\xi_2 x_2-2)^2$ $f_2(x,\xi) := \xi_2 + \xi_1^2 + \xi_1 x_1^2 + \xi_2 x_2^2 + (\xi_1 x_1-2)^3$	$(-1,-1)$	$(0,0)$	$(1.2,0.6)$	$(1.5,1)$	New2
P8	(2,2,2)	$f_1(x,\xi) := -1.05\xi_1 x_1^2 - 0.98\xi_2 x_2^2 + (\xi_1 x_1-2)^2 + (\xi_2 x_2+2)^2$ $f_2(x,\xi) := -0.99\xi_1(x_1-3)^2 - 1.03\xi_2(x_2-2.5)^2 + 5\xi_1 x_1^2 + \xi_2 x_2^2$	$(-3,-3)$	$(5,5)$	$(-1.5,-1.5)$	$(0,0)$	Lovison1 [29]
P9	(2,3,3)	$f_1(x,\xi) := 2x_1^2 + \frac{(\xi_1-3)}{2} + 4x_2\frac{(\xi_2-3)}{2}$ $f_2(x,\xi) := \frac{x_1^2}{4}\cos x_2 - \cos^3 x_3\frac{(\xi_1-3)}{2} + \xi_3$	$(-100000,\ldots,-100000)$	$(100000,\ldots,100000)$	$(-2,\ldots,-2)$	$(2,\ldots,2)$	GKZ9 [10]
P10	(2,5,1)	$f_1(x,\xi) := x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + \xi^2$ $f_2(x,\xi) := (3+\xi)x_1 + 2x_2 - \frac{x_3}{3} + (x_4-x_5)^3 + \xi x_1^2$	$(-20,\ldots,-20)$	$(20,\ldots,20)$	0.008	0.012	DD1 [13]
P11	(2,20,20)	$f_1(x,\xi) := \frac{1}{n}\sum_{i=1}^{n}\xi_i^2 x_i$ $f_2(x,\xi) := \frac{1}{n}\sum_{i=1}^{n}(x_i - 2\xi_i)^2 + \xi_i^2$	$(-9,\ldots,-9)$	$(-7,\ldots,-7)$	$(-2,\ldots,-2)$	$(1,\ldots,1)$	Jin1 [17]
P12	(3,2,2)	$f_1(x,\xi) := \frac{(\xi_1 x_1-2)^2}{2} + \frac{(\xi_2 x_2+1)^2}{13} + 3\xi_1 + x_1^2 + x_2^2 + \xi_1\xi_2$ $f_2(x,\xi) := \frac{(\xi_1 x_1+\xi_2 x_2-3)^2}{36} + \frac{(-\xi_1 x_1+\xi_2 x_2+2)^2}{8} - 17\xi_2 + x_1 + x_2 + \xi_2$ $f_3(x,\xi) = \frac{(\xi_1 x_1+2\xi_2 x_2-1)^2}{175} + \frac{(-\xi_1 x_1+2\xi_2 x_2)^2}{17} - 13\xi_1\xi_2 + \xi_2 x_1^2 - 9x_2$	$(-400,-400)$	$(400,400)$	$(-2,-2)$	$(1,1)$	MOP7 [13]
P13	(3,2,2)	$f_1(x,\xi) := x_1 + (x_2-1.5) + \frac{1}{4}\sin\left(\frac{4\pi(\xi_1-1)}{7}\right) + \frac{1}{100}\xi_2 + x_1^2$ $f_2(x,\xi) := 2(x_1-1)^2 + 2x_2^2 + \frac{1}{4}\cos\left(\frac{4\pi(\xi_2-1)}{7}\right) + \frac{2}{100}\xi_1 + x_2^2$ $f_3(x,\xi) := \xi_1 x_1^2 + x_2^2 + \xi_1\xi_2$	$(1,1)$	$(1.5,1.5)$	$(1,1)$	$(1.5,2)$	GKZ6 [10]
P14	(3,2,3)	$f_1(x,\xi) := \xi_1^2 x_1^2 + \xi_2(x_2-1)^2 + x_3 + \xi_1 x_1^2 + x_2^2$ $f_2(x,\xi) := x_1^2 + (\xi_1 x_2-1)^2 + x_1^4 + \xi_2 x_2^2 + 1$ $f_3(x,\xi) := \xi_1(x_1-1)^2 + \xi_2 x_2^2 + \xi_3 + 5\xi_1 x_1 + x_2 + 3$	$(-2,-2)$	$(2,2)$	$(1,0.5,-0.5)$	$(1.5,1,2.5)$	VFM1 [13]
P15	(3,2,3)	$f_1(x,\xi) := (\xi_1 x_1-1)^2 + (\xi_2 x_2-1)^2 + \xi_1 x_1^2 + x_2^2$ $f_2(x,\xi) := (\xi_2 x_1-1.5)^2 + (\xi_1 x_2-1)^2 + x_1^4 + \xi_2 x_2^2$ $f_3(x,\xi) := \xi_2(x_1-1)^2 + \xi_1(\xi_2 x_2-1)^2 + \xi_3 + 5\xi_1 x_1 + x_2$	$(-4,-4)$	$(4,4)$	$(-2.5,-2.5,-0.5)$	$(2.5,2.5,2.5)$	MHHM2 [18]
P16	(3,10,10)	$f_1(x,\xi) := \sum_{i=1}^{n}\xi_i x_i + \xi_i^2$ $f_2(x,\xi) := 1 + 9\sum_{i=1}^{n}\xi_i^2 x_i$ $f_3(x,\xi) := 1 - \sqrt{\frac{f_1(x,\xi)}{f_2(x,\xi)}}$	$(0.2,\ldots,0.2)$	$(0.8,\ldots,0.8)$	$(-1,\ldots,-1)$	$(1,\ldots,1)$	ZDT1 [35]
P17	(3,10,10)	$f_1(x,\xi) := \sum_{i=1}^{n}\xi_i x_i$ $f_2(x,\xi) := 5 + 10\sum_{i=1}^{n}\xi_i x_i^2$ $f_3(x,\xi) := 2 - \left(\frac{f_1(x,\xi)}{f_2(x,\xi)}\right)^2$	$(0.2,\ldots,0.2)$	$(0.6,\ldots,0.6)$	$(-0.8,\ldots,-0.8)$	$(0.7,\ldots,0.7)$	ZDT2 [35]
P18	(3,10,10)	$f_1(x,\xi) := \frac{1}{n}\sum_{i=1}^{n}i(\xi_i x_i - i)^4$ $f_2(x,\xi) := \exp\left(\sum_{i=1}^{n}\frac{x_i}{n}\right)$ $f_3(x,\xi) = \frac{1}{n(n+1)}\sum_{i=1}^{n}i(n-i+1)e^{-\xi_i x_i}$	$(-2,\ldots,-2)$	$(2,\ldots,2)$	$(-1,\ldots,-1)$	$(2,\ldots,2)$	FDS [8]

TABLE 6. Performance of Algorithm 1 for all problems given in Table 5

Problem	Senarios	Time			Iterations count		
		min	mean	max	min	mean	max
P1	40	0.0629	0.5031	1.0282	0	4.76	9
P2	35	1.5480	1.6323	1.9709	3	3.98	4
P3	40	0.3044	1.2164	2.5341	0	1.60	7
P4	40	0.5314	2.8990	62.3406	2	11.70	256
P5	25	0.6717	5.1853	119.0370	1	8.6531	198
P6	30	2.1577	2.2529	3.2651	2	2	2
P7	20	0.8279	6.2513	73.1986	1	9	98
P8	40	0.2294	12.8858	117.0456	0	16.10	149
P9	40	1.609	99.1422	407.6247	7	106	911
P10	25	14.2015	58.6909	144.2981	39	76.7234	155
P11	20	23.8567	25.9025	30.3595	7	7.0392	8
P12	30	0.4093	20.6827	202.9008	1	30.4583	313
P13	30	2.9827	4.8822	10.6499	2	4.36	8
P14	20	0.4212	2.3219	26.9343	1	8.58	136
P15	40	0.1858	1.9971	28.8318	0	4.62	89
P16	10	1.6223	4.0545	15.1023	0	2.53	12
P17	10	0.6223	3.0545	16.1023	1	1.54	10
P18	10	0.2532	23.6702	221.7030	0	30.5319	297

then we find a step length τ_k in (Step 5) to progress to the next iterate. It has been established that if a weakly robust efficient point is a regular point, the method's sequence exhibits global convergence (Theorem 4.4) and has a local superlinear convergence rate (see Theorem 4.5).

In future research, we aim to explore the following directions:

- Extending these results and methods with respect to the other set less relations and through different scalarization functions.
- The goal is to create techniques for solving UMOP with uncertainty sets that are either infinite or continuous in nature.
- Deriving results without requiring regularity assumption on the accumulation points of the sequence generated by the proposed algorithm.

Acknowledgments

(A) Problem P1

(B) Problem P3

(C) Problem P5

(D) Problem P6

(E) Problem P7

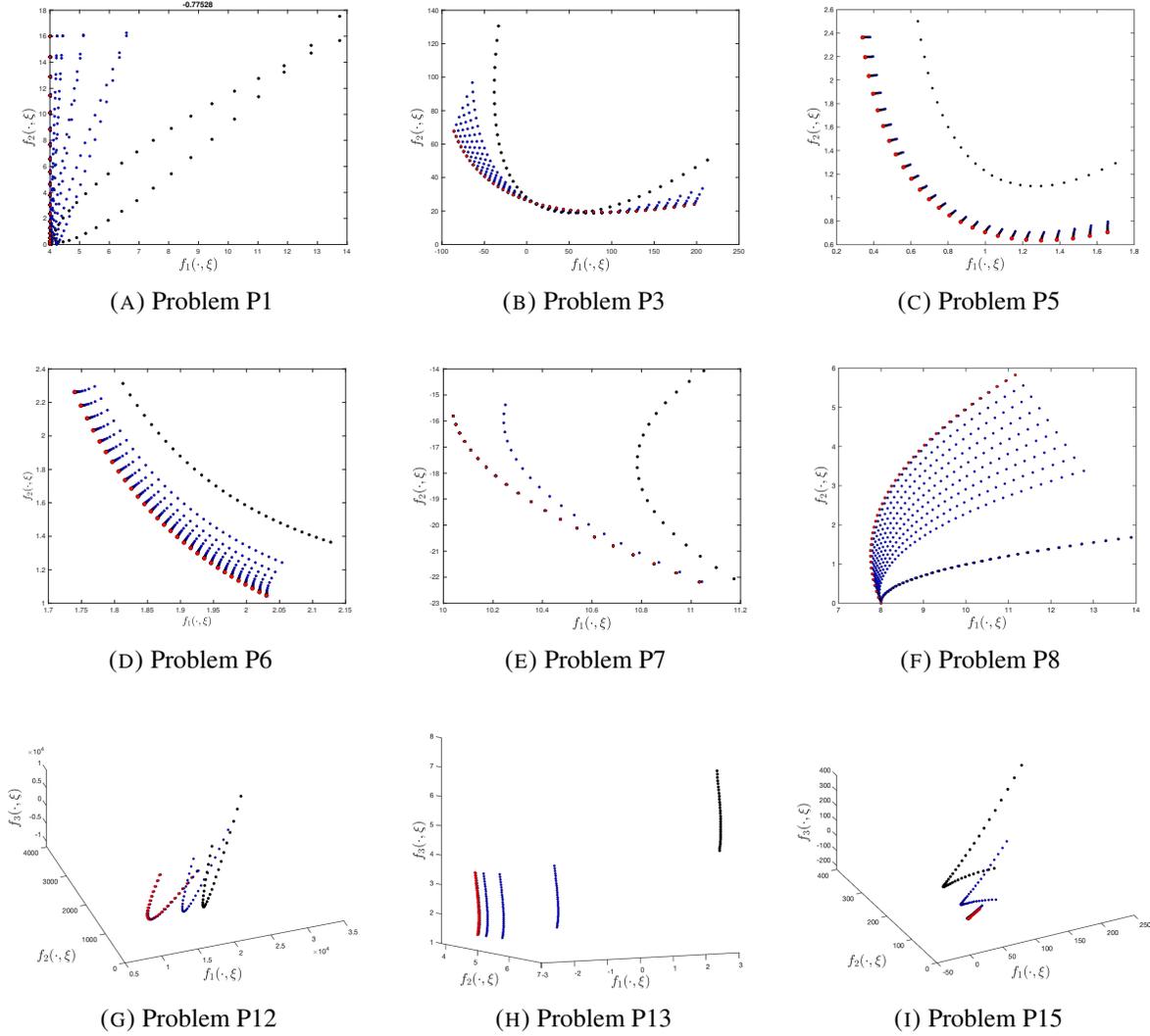(F) Problem P8

(G) Problem P12

(H) Problem P13

(I) Problem P15

FIGURE 5. Robust weakly efficient points generated by Algorithm 1 for a few test problems from Table 4

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