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BETA FUNCTION FORMALISM FOR GRUSHIN OPERATORS (2)

RIJU BASAK[1], DER-CHEN CHANG[2,*], JONATHAN RIESS[3], JEN-CHIH YAO[4]

[1]*Department of Mathematics, National Taiwan Normal University, Taipei city, Taiwan*
[2]*Department of Mathematics and Statistics, Georgetown University, Washington D.C. 20057, USA and Graduate Institute of Business Administration, College of Management, Fu Jen Catholic University, New Taipei City 242, Taiwan*
[3]*Department of Mathematics and Statistics and Department of Physics, Georgetown University, Washington DC, 20057, USA*
[4]*Center for General Education, China Medical University, Taichung, Taiwan and Academy of Romanian Scientists, 50044 Bucharest, Romania*

Abstract. In this paper, we revisit the Grushin operator $\Delta_G = \frac{1}{2}(\partial_x^2 + x^{2k}\partial_y^2), k \geq 1$ and study various geometric properties associated with the Grushin manifold. In particular, we employ a Beta function formalism to derive the heat kernel for the step 2 Grushin operator $\Delta_G = \frac{1}{2}(\partial_x^2 + x^2\partial_y^2)$. This approach is novel and proves to be effective in the analysis of the Grushin operator. Furthermore, the results presented here reaffirm existing findings in the literature.

Keywords. Beta function theory; Heat kernel; Horizontal vector fields; Hamilton-Jacobi mechanics; Grushin operator; Missing directions.

1. INTRODUCTION

In this paper, we examine the operator $\Delta_G = \frac{1}{2}(\partial_x^2 + x^2\partial_y^2)$, which we refer to as the Grushin operator with $k = 1$. This operator was introduced in Chang-Riess [7]. Through several examples in that paper, we have become acquainted with the underlying sub-Riemannian geometry associated with this subelliptic operator, known as Grushin manifolds. Based on these considerations and by applying the Chow–Rashevskii theorem [10], we can ascertain that geodesics exist between any two points on the underlying Grushin manifold. The primary objective of this section is to first determine these geodesics using the Beta function formalism. Following the algorithm outlined in [6] and [9], we then count the number of geodesics connecting any two points and derive the heat kernel for Δ_G.

We begin our study with the Grushin operator for $k = 1$, as this case is not only more manageable but also benefits from extensive prior work by various authors in finding geodesic solutions, counting the number of geodesics, and determining the heat kernel. Before applying the new Beta function formalism, we first review these previous results from [3, 4, 6, 7], and related references. This foundational understanding will allow us to apply the Beta function formalism

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[*] Corresponding author.

E-mail address: chang@georgetown.edu (D.-C. Chang).

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effectively and demonstrate that our new method yields results consistent with those previously obtained by Catlin, Chang, Greiner, Furutani, Iwasaki, Li, Tie, and others. We start by solving for the geodesic solutions and show how they align with the results from [9] for the case when $x_0 = 0$. Subsequently, we extend these findings using the Beta function method to arbitrary starting positions $x_0 \neq 0$. Finally, we illustrate how to derive the heat kernel using this Beta function approach.

2. REVIEW OF PREVIOUS RESULTS

In this section, we review previously obtained results for the Grushin operator with $k = 1$, as presented in [5, 9], and other sources. Our discussion draws heavily from [3, Section 11.2] and [4, Section 5.2].

2.1. **Geodesics.**

Definition 2.1. The Grushin operator Δ_G, also called the Grushin-Laplacian, with $k = 1$ is defined using the vector fields $X_1 = \partial_x, X_2 = x\partial_y$ as $\Delta_G = \frac{1}{2}(X_1^2 + X_2^2) = \frac{1}{2}(\partial_x^2 + x^2\partial_y^2)$.

This operator Δ_G is elliptic on $\mathbb{R}^2 \backslash \{x = 0\}$ and subelliptic (and therefore hypoelliptic) on all of $\mathbb{R}^2$. We sometimes therefore say that Δ_G is *degenerate-elliptic* on $\mathbb{R}^2$, with the degeneracy occurring whenever $x = 0$, *i.e.* the y-axis, where the coefficient for ∂_y^2 vanishes. This is also the singular set $S = \{x = 0\}$. The degeneracy of the vector field X_2 can be physically interpreted as telling us that when $x = 0$, motion in the y-direction is blocked. An important property that we will utilize is that the vector fields X_1 and X_2 are invariant under translations of the y variable. However, we do not have the same type of invariance for translations of x due to the coefficient of ∂_y^2. Consequently, throughout this chapter, we will assume a starting point of $y_0 = 0$ to leverage the y-translation invariance. As mentioned above, since $[X_1, X_2] = \partial_y$, the bracket-generating condition is satisfies and so by Chow's theorem we know that every pair of points in $\mathbb{R}^2$ can be joined by a piecewise smooth curve $\gamma(s)$, moreover geodesics which are given by the Hamiltonian system, whose tangent vectors $\dot{\gamma}(s)$ at a point along the curve can always be written as a linear combination of X_1, X_2. We thus can find these geodesics connecting any two points (x_0, y_0) and (x_1, y_1) by studying the Hamiltonian associated to the Grushin operator:

$$H(x, y, \xi, \theta) = \frac{1}{2}(\xi^2 + x^2\theta^2)$$

where ξ, θ are the momenta associated to the the coordinates x, y. The geodesics connecting points (x_0, y_0) and (x_1, y_1) are the projections on the (x, y)-plane of the solution of Hamilton's equations:

$$\begin{aligned}
\dot{x} &= H_\xi = \xi, \\
\dot{y} &= H_\theta = \theta x^2, \\
\dot{\xi} &= -H_x = -\theta^2 x, \\
\dot{\theta} &= -H_y = 0 \implies \theta \text{ is constant along geodesics},
\end{aligned}$$

with boundary conditions: $x(0) = x_0$, $y(0) = y_0$, $x(1) = x_1$, $y(1) = y_1$. The derivative is with respect to the parameter time, which we will denote using s, *i.e.* $\dot{x} = dx/ds$ and $x(s = 0) = x_0$.

We are interested in solving for the (normal) geodesics, *i.e.*, solving this set of differential equations. We summarize the following previous results for Δ_G with $k = 1$ in [3] and [9]:

(1) **Case 1 - Horizontal lines:** For any two points (x_0, y_0) and (x_1, y_0) on the same horizontal line $y = y_0$, *i.e.*, $y_1 = y_1$, there exists only one geodesics connecting them given by:

$$\begin{cases} x(s) = s(x_1 - x_0) + x_0, \\ y(s) = y_0, \end{cases}$$

for $s \in [0, 1]$.

(2) **Case 2 - On the y-axis:** Let $(x_0, y_0) = (0, 0)$ and $(x_1, y_1) = (0, y_1)$ with $y_1 > 0$. Then there exist (countably) infinitely many geodesics connecting the two points $(0, 0)$ and $(0, y_1 > 0)$, given by:

$$\begin{cases} x_m(s) = \sqrt{\frac{2y_1}{m\pi}} \sin(m\pi s), \\ y_m(s) = y_1 \left(s - \frac{\sin(2m\pi s)}{2m\pi} \right), \end{cases}$$

where $m = 1, 2, 3, \ldots \in \mathbb{N}$ labels the geodesic and $\theta = \theta_m = m\pi$. The mth geodesic has length $\ell_m = \sqrt{2m\pi y_1}$. Note that $\lim_{m\to\infty} \ell_m = \infty$.

(3) **Case 3 - Arbitrary geodesics:** For arbitrary boundary conditions $x(0) = x_0$, $y(0) = y_0$, $x(1) = x_1$, $y(1) = y_1$ gives us geodesics:

$$\begin{cases} x(s) = x_0 \cos(\theta s) + \frac{x_1 - x_0 \cos(\theta)}{\sin\theta} \sin(\theta s), \\ y(s) = y_0 + \frac{x_0^2[2\theta s + \sin(2\theta s)]}{4} + \left(\frac{x_1 - x_0 \cos(\theta)}{\sin\theta} \right)^2 \frac{2\theta s - \sin(2\theta s)}{4} + \frac{x_0[x_1 - x_0 \cos(\theta)]}{\sin(\theta)} \sin^2(\theta s). \end{cases}$$

We can use these relations to count the number of geodesics. We have the following sub-results:

(a) **Case 3a - Connecting to the y-axis:** (Theorem 11.2.4 in [3]). Let $x_1 \neq 0$, then there are only finitely many geodesics connecting the points $(0, y_0)$ and $(x_1 \neq 0, y_1)$. The

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Note: In the Riemannian setting, we do not need to distinguish between the characteristic curves of the Hamiltonian H and the minimizing geodesics, in the sub-Riemannian setting we do. This is because we can have normal geodesics, which are the solutions of the Hamiltonian system. All normal geodesics are locally length minimizing. However, the converse of this statement is not true. In the sub-Riemannian setting, there exist length-minimizing geodesics that are not normal, *i.e.*, solutions to H, which we call **abnormal geodesics**. This was first pointed out by Liu and Sussmann [13]. However, we can constrain the set on which the abnormal geodesics lie. Namely, on the set of regular points $M \backslash S$, the manifold is Riemannian and the local existence and uniqueness of geodesics holds. Furthermore, we know that on $M \backslash S$, all locally length-minimizing curves are normal geodesics. As a result, all abnormal geodesics must be contained in the singular set S. Furthermore, we can show that the uniqueness property of geodesics fails when we want to connect two singular points. There could be multiple, even infinitely many geodesics, connecting two singular points. As a result, the solvability of the operator $\Delta_X = \frac{1}{2} \sum_i X_i^2$ might not hold on the set S. For example, the $k = 1$ Grushin operator Δ_G defined above is not solvable along the set $S = \{(0, y); y \in \mathbb{R}\}$. Thus, we know that the non-solvability of an operator is connected to the geodesics induced by the geometry of the vector fields X_i.

geodesics are given by:

$$x_m(s) = \frac{\sin(\theta_m s)}{\sin(\theta_m)} x_1,$$
$$y_m(s) = y_0 + \frac{x_1^2(\theta_m s - \frac{1}{2}\sin(2\theta_m s))}{2\sin^2(\theta_m)},$$

where $m \in \{1,2,...N\}$ and θ_m is a solution to $\frac{2(y_1-y_0)}{x_1^2} = \mu(\theta)$, where $\mu(z) = \frac{z}{\sin^2(z)} - \cot(z)$.

(b) **Case 3b:** (Theorem 11.2.5 in [3]). Let $x_0x_1 \neq 0$. Then there exist only finitely many geodesics connecting (x_0,y_0) and (x_1,y_1).

2.2. **Heat Kernel.** We summarize the action and heat kernels found in [4] and [9]. The heat kernel of the Hermite operator $\frac{1}{2}\partial_x^2 - \frac{1}{2}\xi^2 x^2$ is given by (Theorem 3.16.1 in [4]):

$$K(x_0,x;t) = \frac{1}{\sqrt{2\pi t}}\sqrt{\frac{\xi t}{\sinh(\xi t)}}\, e^{-\frac{1}{2t}\frac{\xi t}{\sinh(\xi t)}\left((x^2+x_0^2)\cosh(\xi t)-2xx_0\right)}, \quad t>0,$$

which implies from the x-component of the geodesics given by solutions to the associated Hamilton's equations (Equation 3.16.54 in [4]):

$$x(s) = \frac{x - x_0\cosh(at)}{\sinh(at)}\sinh(as) + x_0\cosh(as).$$

From the associated classical action (Equation 3.16.56 in [4]), one has

$$S(x_0,x;t) = \frac{a}{\sinh(at)}\left((x^2+x_0^2)\cosh(at) - 2xx_0\right).$$

The heat kernel v of the Grushin operator $\Delta_G = \frac{1}{2}(\partial_x^2 + x^2\partial_y^2)$ is then given by:

$$\begin{aligned} v(x,y;t) &= \frac{1}{2\pi}\int e^{iy\xi}\left(\frac{1}{\sqrt{2\pi t}}\sqrt{\frac{\xi t}{\sinh(\xi t)}}\, e^{-\frac{1}{2t}\frac{\xi t}{\sinh(\xi t)}\left((x^2+x_0^2)\cosh(\xi t)-2xx_0\right)}\right) d\xi \\ &= \frac{1}{(2\pi)^{3/2}\sqrt{t}}\int \sqrt{\frac{\tau}{\sinh\tau}}\, e^{i(yt^{-1/2}\tau^{1/2}-\frac{1}{2}(x^2+x_0^2)\tau\coth\tau)}\, d\tau, \end{aligned}$$

where we set $\tau = \xi t$. Since this integral is not expressible in terms of elementary functions, one says the heat kernel of the Grushin operator is **not of function type**, and it is therefore convention to set $f(x_0,x,y;\tau) = -iyt + \frac{1}{2}(x^2+x_0^2)\tau\coth\tau + \frac{\tau}{\sinh\tau}xx_0$ as the **modified complex action** and $V(\tau) = \sqrt{\frac{\tau}{\sinh\tau}}$, the volume element. Thus, the heat kernel of the Grushin operator can be written as:

$$v(x,y;t) = \frac{1}{(2\pi)^{3/2}\sqrt{t}}\int e^{-f(x_0,x,y;\tau)/t}\, V(\tau)\, d\tau, \quad t>0.$$

2.2.1. *Finding the Hermite Operator.* We consider the operator $L = \frac{1}{2}\left(\frac{d^2}{dx^2} - a^2x^2\right)$, with a constant, called the *Hermite operator*. The associated Hamiltonian is given by $H(p,x) = \frac{1}{2}p^2 - \frac{1}{2}a^2x^2$ and the Hamiltonian system of equations is

$$\begin{aligned}\dot{x} &= H_p = p,\\ \dot{p} &= -H_x = a^2x.\end{aligned}$$

The geodesic between x_0 and x within time t satisfies

$$\begin{aligned}&\ddot{x} = a^2x,\\ &x(0) = x_0, \qquad x(t) = x.\end{aligned} \tag{2.1}$$

The solution is

$$x(s) = \frac{x - x_0\cosh(at)}{\sinh(at)}\sinh(as) + x_0\cosh(as). \tag{2.2}$$

The Lagrangian associated with the previous Hamiltonian is

$$L(x,\dot{x}) = p\dot{x} - H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}a^2x^2. \tag{2.3}$$

Integrating the solution (2.2)) along the Lagrangian (2.3) and performing the same computations as in [4, Sect. 7.8] yields

$$S(x_0,x;t) = \frac{a}{\sinh(at)}\left((x^2 + x_0^2)\cosh(at) - 2xx_0\right).$$

There is an alternate way of finding the action using the Hamilton–Jacobi equation. The conservation of energy law for (2.1) is $\frac{1}{2}\dot{x}^2 - \frac{1}{2}a^2x^2 = E$, where E is the energy constant. This can also be written as

$$\frac{dx}{ds} = \sqrt{2E + a^2x^2} \implies \frac{dx}{\sqrt{2E + a^2x^2}} = ds.$$

Integrating between $s = 0$ and $s = t$, with $x(0) = x_0$ and $x(t) = x$, and solving for the energy yields

$$2E = \frac{a^2(x - x_0\cosh(at))^2}{\sinh(at)^2} - a^2x_0^2.$$

The Hamilton–Jacobi equation becomes

$$\partial_t S = -H(x,S) = \frac{a}{\sinh(at)}\left(\frac{1}{2}(x^2 + x_0^2)\cosh(at) - xx_0\right).$$

Hence,

$$S(x_0,x;t) = \frac{a}{\sinh(at)}\left(\frac{1}{2}(x^2 + x_0^2)\cosh(at) - xx_0\right). \tag{2.4}$$

Since $\partial_x^2 S = a\coth(at)$ is a function of t only, the transport equation becomes

$$V'(t) + \frac{1}{2}a\coth(at)V(t) = 0.$$

Integrating, we obtain the solution

$$V(t) = \frac{c}{\sqrt{\sinh(at)}}, \tag{2.5}$$

where c is a constant which will be determined later. Using (2.4) and (2.5), we obtain the fundamental solution $K = V(t)e^{-S}$ as $K(x_0,x;t) = \frac{c}{\sqrt{\sinh(at)}} e^{\frac{a}{2\sinh(at)}\left((x^2+x_0^2)\cosh(at)-2xx_0\right)}$. In order to determine the constant c, we write

$$K(x_0,x;t) = \frac{c}{\sqrt{\sinh(at)}} e^{\frac{a}{2\sinh(at)}\left((x^2+x_0^2)\cosh(at)-2xx_0\right)}.$$

Since $at/\sinh(at) \to 1$ for $a \to 0$, we can write $K(x_0,x;t) \sim \frac{c}{\sqrt{t}} e^{\frac{1}{2t}(x-x_0)^2}$. By comparison with the fundamental solution for the usual heat operator $\frac{1}{2}\partial_x^2$, we obtain $c = \sqrt{2\pi}$. To conclude, we have the following result:

Theorem 2.1. *Let a be a constant. The heat kernel for the operator* $\frac{1}{2}\left(\frac{d^2}{dx^2} - a^2x^2\right)$ *is*

$$K(x_0,x;t) = \frac{1}{\sqrt{2\pi t}}\sqrt{\frac{at}{\sinh(at)}} e^{-\frac{1}{2t}\frac{at}{\sinh(at)}\left((x^2+x_0^2)\cosh(at)-2xx_0\right)}, \quad t > 0.$$

Using the substitutions $a = -i\omega$ and $\cosh(i\omega t) = \cos(\omega t)$ and $\sinh(2i\omega t) = i\sin(2\omega t)$, we see the following corollary.

Corollary 2.1. *Let a be constant. The heat kernel for the operator* $\frac{1}{2}\left(\frac{d^2}{dx^2} + a^2x^2\right)$ *is*

$$K(x_0,x;t) = \frac{1}{\sqrt{2\pi t}}\sqrt{\frac{2at}{\sin(2at)}} e^{-\frac{1}{2t}\frac{2at}{\sin(2at)}\left((x^2+x_0^2)\cos(2at)-2xx_0\right)}, \quad t > 0.$$

3. BETA FUNCTION FORMALISM FOR Δ_G WITH $k = 1$

3.1. Geodesics for General $x_0 \neq 0$ in Beta Function Form.

Let us now apply the Beta Function Formalism to the Grushin operator $\Delta_G = \frac{1}{2}(\partial_x^2 + x^2\partial_y^2)$ to find the geodesics solutions. Recall that the Hamiltonian, *i.e.*, principal symbol, associated to Δ_G is: $H(x,y,\xi,\theta) = \frac{1}{2}(\xi^2 + x^2\theta^2)$. Hamilton's equations, which give us our *normal* geodesics solutions, are

$$\begin{cases} \dot{x} = \xi, \\ \dot{y} = \theta x^2, \\ \dot{\xi} = -\theta^2 x, \\ \dot{\theta} = 0. \end{cases}$$

We see that the equations partially decouple to give us the equation for x: $\ddot{x} = \dot{\xi} = -\theta^2 x$. As a result, we can use the energy method by defining the energy to be: $E = \frac{1}{2}\dot{x}^2(s) + \frac{1}{2}\theta^2x^2(s)$. Rearranging, we find

$$\frac{dx}{ds} = \sqrt{2E - \theta^2x^2(s)} \implies s = \int_{x_0}^{x_1} \frac{dx}{\sqrt{2E - \theta^2x^2(s)}}. \tag{3.1}$$

This is the famous energy integral which for higher k will in fact be an elliptic integral and not expressible in terms of elementary functions. However, for $k = 1$ as in our case, this integral can easily be expressed in terms of trigonometric functions (see [4, 5, 9]). However, to illustrate the basics and the power of the Beta function formalism which we will then apply to the arbitrary k case in subsequent parts of this series, let us express this integral in terms of the incomplete Beta function defined as $B(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$. Note that we could express the integral (3.1) in terms of other special functions. We will do this in Section 4; however, we can see that we will quickly run into issues if we do so. Hence, we can rewrite (3.1) as:

$$s = \int_{x_0}^{x_1} \frac{dx}{\sqrt{2E - \theta^2 x^2(s)}} = \frac{1}{\sqrt{2E}} \int_{x_0}^{x_1} \frac{1}{\sqrt{1 - \frac{\theta^2 x^2}{2E}}} dx.$$

From $u = \frac{\theta x}{\sqrt{2E}}$, one has $du = \frac{\theta}{\sqrt{2E}} dx$ and $u_0 = \frac{\theta x_0}{\sqrt{2E}}$ gives

$$s = \frac{1}{\theta} \int_{u_0}^{u_1} \frac{1}{\sqrt{1-u^2}} du = \frac{1}{\theta} \int_{u_0}^{u_1} (u^2)^0 (1-u^2)^{-1/2} du.$$

Using the substitution $u^2 = t$, we have $dt = 2udu$, $u_0^2 = \left(\frac{\theta x_0}{\sqrt{2E}}\right)^2 = t_0$ and

$$\begin{aligned} s &= \frac{1}{2\theta} \int_{t_0}^{t_1} \frac{(u^2)^0}{u} (1-t)^{-1/2} dt = \frac{1}{2\theta} \int_{t_0}^{t_1} \frac{(u^2)^0}{(u^2)^{1/2}} (1-t)^{-1/2} dt \\ &= \frac{1}{2\theta} \int_{t_0}^{t_1} t^{-1/2} (1-t)^{-1/2} dt = \frac{1}{2\theta} \int_{t_0}^{t_1} t^{1/2-1} (1-t)^{1/2-1} dt \\ &= \tfrac{1}{2\theta} \left[B_{t_1}(1/2,1/2) - B_{t_0}(1/2,1/2) \right]. \end{aligned}$$

Plugging our expressions for t_0 and t_1, we find

$$\boxed{s = \tfrac{1}{2\theta} \left[B_{\frac{\theta^2 x_1^2}{2E}}\left(\tfrac{1}{2},\tfrac{1}{2}\right) - B_{\frac{\theta^2 x_0^2}{2E}}\left(\tfrac{1}{2},\tfrac{1}{2}\right) \right]} \tag{3.2}$$

It is relativity straightforward to reduce this result to results in Section 2 by using certain identities relating the Beta function to trigonometric functions. We will do this in Section 3.2. For now, let us solve for $x = x_1$ in equation (3.2). We can do this by using the inverse incomplete Beta function as follows:

$$\begin{aligned} B_{\frac{\theta^2 x_1^2}{2E}}\left(\tfrac{1}{2},\tfrac{1}{2}\right) &= 2\theta s + B_{\frac{\theta^2 x_0^2}{2E}}\left(\tfrac{1}{2},\tfrac{1}{2}\right), \\ \frac{\theta^2 x_1^2}{2E} &= B^{-1}_{2\theta s + B_{\frac{\theta^2 x_0^2}{2E}}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{1}{2},\tfrac{1}{2}\right), \\ x_1 &= \sqrt{\frac{2E}{\theta^2} B^{-1}_{2\theta s + B_{\frac{\theta^2 x_0^2}{2E}}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{1}{2},\tfrac{1}{2}\right)}. \end{aligned}$$

Hence, the x-component of the geodesics solutions can be parametrized as

$$\boxed{x(s) = \sqrt{\frac{2E}{\theta^2} B^{-1}_{2\theta s + B_{\frac{\theta^2 x_0^2}{2E}}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{1}{2},\tfrac{1}{2}\right)}} \tag{3.3}$$

To obtain the y-component, we can use $y(s)=\theta\int_0^s x^2(\tau)d\tau$. Plugging (3.3) into the integral for $y(s)$, we obtain

$$y(s)=\theta\int_0^s \frac{2E}{\theta^2}B^{-1}_{2\theta\tau+B_{\frac{\theta^2x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})}(\tfrac{1}{2},\tfrac{1}{2})\,d\tau=\frac{2E}{\theta}\int_0^s B^{-1}_{2\theta\tau+B_{\frac{\theta^2x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})}(\tfrac{1}{2},\tfrac{1}{2})\,d\tau.$$

Using the substitution $u=2\theta\tau+B_{\frac{\theta^2x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})$ with $du=2\theta d\tau$ and $u_0=B_{\frac{\theta^2x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})$ and $u_1=2\theta s+B_{\frac{\theta^2x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})$, the integral becomes:

$$\frac{2E}{\theta}\int_0^s B^{-1}_{2\theta\tau+B_{\frac{\theta^2x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})}(\tfrac{1}{2},\tfrac{1}{2})\,d\tau=\frac{E}{\theta^2}\int_{u_0}^{u_1}B_u^{-1}(\tfrac{1}{2},\tfrac{1}{2})\,du.$$

We can evaluate this by recalling the formula for the integral of the inverse incomplete beta function: $\int B_z^{-1}(a,b)\,dz=B_{B_z^{-1}(a,b)}(a+1,b)+c$

$$\begin{aligned}
&=\frac{E}{\theta^2}\left[B_{B_u^{-1}(\frac{1}{2},\frac{1}{2})}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)\right]_{u_0}^{u_1}=\frac{E}{\theta^2}\left[B_{B_{u_1}^{-1}(\frac{1}{2},\frac{1}{2})}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)-B_{B_{u_0}^{-1}(\frac{1}{2},\frac{1}{2})}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)\right]\\
&=\frac{E}{\theta^2}\left[B_{B^{-1}_{2\theta s+B_{\frac{\theta^2x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})}(\frac{1}{2},\frac{1}{2})}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)-B_{B^{-1}_{B_{\frac{\theta^2x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})}(\frac{1}{2},\frac{1}{2})}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)\right]\\
&=\frac{E}{\theta^2}\left[B_{B^{-1}_{2\theta s+B_{\frac{\theta^2x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})}(\frac{1}{2},\frac{1}{2})}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)-B_{\frac{\theta^2x_0^2}{2E}}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)\right].
\end{aligned}$$

Hence, the y-component of the geodesics can be parametrized as

$$\boxed{y(s)=\frac{E}{\theta^2}\left[B_{B^{-1}_{2\theta s+B_{\frac{\theta^2x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})}(\frac{1}{2},\frac{1}{2})}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)-B_{\frac{\theta^2x_0^2}{2E}}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)\right]}\tag{3.4}$$

Thus we can state the following theorem.

Theorem 3.1. *Geodesics of Δ_G for $k=1$ with Arbitrary x_0 in Beta Function Form. Using the Beta function formalism, we find that the geodesic solutions to the Grushin operator Δ_G with*

$k = 1$ and arbitrary starting point x_0 are given by:

$$x(s) = \sqrt{\frac{2E}{\theta^2} B^{-1}_{2\theta s + B_{\frac{\theta^2 x_0^2}{2E}}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{1}{2},\tfrac{1}{2}\right),}$$

$$y(s) = \frac{E}{\theta^2}\left[B_{B^{-1}_{2\theta s + B_{\frac{\theta^2 x_0^2}{2E}}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right) - B_{\frac{\theta^2 x_0^2}{2E}}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)\right].$$

3.2. **Recovering Previous Results for $x_0 = 0$.** In this section, we aim to reduce the parametrizations of the x- and y-components, equations (3.3) and (3.4), in terms of the Beta functions found in Section 3.1 to the previously obtained results in Section 2. Therefore, let us begin by reducing the x-component first. Rather than beginning with equation (3.3), it is easier to work with the uninverted version of equation (3.2):

$$s = \frac{1}{2\theta}\left[B_{\frac{\theta^2 x_1^2}{2E}}\left(\tfrac{1}{2},\tfrac{1}{2}\right) - B_{\frac{\theta^2 x_0^2}{2E}}\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right]. \tag{3.5}$$

We can recover previous result in [4] by using the identity:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(x/a) = \frac{1}{2}B_{x^2/a^2}(1/2,1/2) + c.$$

Applying this identity to (3.5), we obtain $s = \frac{1}{\theta}\left[\arcsin\frac{\theta}{\sqrt{2E}}x_1 - \arcsin\frac{\theta}{\sqrt{2E}}x_0\right]$. Rearranging by solving for x_1, we have

$$x_1 = \frac{\sqrt{2E}}{\theta}\sin\left(s\theta + \arcsin\frac{\theta}{\sqrt{2E}}x_0\right). \tag{3.6}$$

Given that $\ell_m = \sqrt{2\pi m y_1} = \sqrt{2E}$ and $\theta = \pi m$, we see that the coefficient in front of the sine is:

$$\sqrt{\frac{2E}{\theta^2}} = \sqrt{\frac{2\pi m y_1}{m^2\pi^2}} = \sqrt{\frac{2y_1}{\pi m}}.$$

As a result, if we further set $x_0 = 0$, we obtain $x_m(s) = \sqrt{\frac{2y_1}{\pi m}}\sin(sm\pi)$. This is precisely the result obtained in [3] as Equation (11.2.10). Now, to reduce the y-component of the geodesics given by equation (3.4):

$$y(s) = \frac{E}{\theta^2}\left[B_{B^{-1}_{2\theta s + B_{\frac{\theta^2 x_0^2}{2E}}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right) - B_{\frac{\theta^2 x_0^2}{2E}}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)\right],$$

we assume $x_0 = 0$, which simplifies the expression drastically since $B_0(a,b) = \int_0^{z=0} t^{a-1}(1-t)^{b-1} = 0$:

$$\boxed{y(s) = \frac{E}{\theta^2}\left[B_{B^{-1}_{2\theta s}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)\right]} \tag{3.7}$$

We can now rewrite this expression in terms of trigonometric functions by $B_x(1/2,1/2) = 2\arcsin(\sqrt{x})$. Inverting this, we find that $B_x^{-1}(1/2,1/2) = u$ should be equal to the unique u

such that $2\arcsin(\sqrt{u}) = x$, *i.e.*, we just invert the arcsine function to obtain $B_x^{-1}(1/2,1/2) = \sin^2(x/2)$. This solves how to express the inverse Beta function in (3.7), namely,

$$B_{2\theta s}^{-1}(1/2,1/2) = \sin^2\left(\frac{2\theta s}{2}\right) = \sin^2(\theta s). \tag{3.8}$$

But how do we express the other Beta function $B_x(3/2,1/2)$ in terms of trigonometric functions in (3.7)? We can do this by simply writing out the integral expression for $B_x(3/2,1/2)$ as:

$$B_x(3/2,1/2) = \int_0^x t^{3/2-1}(1-t)^{1/2-1}dt = \int_0^x t^{1/2}(1-t)^{-1/2}dt.$$

and using the substitution $t = \sin^2 u$ with $dt = 2\sin(u)\cos(u)du$ yields

$$\begin{aligned}
&= \int_0^x \sin(u)(1-\sin^2(u))^{-1/2}dt = \int_0^x \frac{\sin(u)}{\cos(u)}dt \\
&= \int_{u_0}^{u_1} \frac{\sin(u)}{\cos(u)} 2\sin(u)\cos(u)du = 2\int_{u_0}^{u_1} \sin^2(u)du \\
&= 2\left[\frac{u}{2} - \frac{\sin(2u)}{4}\right]_{u_0}^{u_1} = \left[u - \frac{\sin(2u)}{2}\right]_{u_0}^{u_1}.
\end{aligned}$$

Evaluating the limits, we find $u_0 = \arcsin\sqrt{t_0} = \arcsin 0 = 0$ and $u_1 = \arcsin\sqrt{x}$:

$$= \arcsin\sqrt{x} - \frac{\sin(2\arcsin\sqrt{x})}{2}.$$

We can simplify the second term above by recalling the following trigonometric properties: $\sin(2\arcsin\sqrt{x}) = 2(\arcsin\sqrt{x})\cos(\arcsin\sqrt{x}) = 2\sqrt{x}\sqrt{1-x}$. Hence:

$$= \arcsin(\sqrt{x}) - \sqrt{x(1-x)}. \tag{3.9}$$

Thus, plugging (3.8) into (3.9), we find that (3.7) can be written in terms of trigonometric functions as:

$$\begin{aligned}
y(s) &= \frac{E}{\theta^2}\left[B_{B_{2\theta s}^{-1}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{3}{2},\tfrac{1}{2}\right)\right] \\
&= \frac{E}{\theta^2}\left[B_{\sin^2(\theta s)}\left(\tfrac{3}{2},\tfrac{1}{2}\right)\right] \\
&= \frac{E}{\theta^2}\left[\arcsin(\sqrt{\sin^2(\theta s)}) - \sqrt{\sin^2(\theta s)(1-\sin^2(\theta s))}\right].
\end{aligned}$$

If we assume that $s\theta \in [0,\pi/2]$, then $\sqrt{\sin^2(\theta s)} = |\sin(\theta s)| = \sin(\theta s)$ meaning $\arcsin(\sin\theta s) = \theta s$. In addition, we find $\sqrt{\sin^2(\theta s)(1-\sin^2(\theta s))} = |\sin(\theta s)\cos(\theta s)| = \sin(\theta s)\cos(\theta s) = \frac{1}{2}\sin(2\theta s)$. Thus it is

$$\frac{E}{\theta^2}\left[\theta s - \frac{1}{2}\sin(2\theta s)\right]. \tag{3.10}$$

Again, assuming $\theta = m\pi$ and $\ell_m = \sqrt{2E} = \sqrt{2m\pi y_1}$, we see that

$$= \frac{m\pi y_2}{m^2\pi^2}\left[m\pi s - \frac{1}{2}\sin(2m\pi s)\right] = y_1\left[s - \frac{\sin(2m\pi s)}{2m\pi}\right].$$

This result also agrees with the y-component found by in [3] for Equation (11.2.10). We have thus shown that the Beta Function formalism, when applied to the Gurshin operator Δ_G with $k = 1$ is able to reproduce previous results for the geodesics solutions. We now can state the following theorem.

Theorem 3.2. *Reduction of Beta Function Form to Trigonometric Form. The x- and y-components of the geodesics solutions (3.3) and (3.4) in Beta function form reduce for* $x_0 = 0$ *to the previously obtained results in* [3]*, Equation (11.2.10):*

$$x(s) = \sqrt{\frac{2E}{\theta^2} B^{-1}_{2\theta s + B_{\frac{\theta^2 x_0^2}{2E}}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{1}{2},\tfrac{1}{2}\right)} \longrightarrow \sqrt{\frac{2y_1}{\pi m}}\sin(sm\pi)$$

$$\begin{aligned} y(s) &= \frac{E}{\theta^2}\left[B_{B^{-1}_{2\theta s + B_{\frac{\theta^2 x_0^2}{2E}}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right) - B_{\frac{\theta^2 x_0^2}{2E}}\left(\tfrac{1}{2}+1,\tfrac{1}{2}\right)\right] \\ &\longrightarrow y_1\left[s - \frac{\sin(2m\pi s)}{2m\pi}\right]. \end{aligned}$$

3.3. **Extending Previous Results to General Starting Point** x_0**.** In Section 3.1 we found the expression for the geodesics for arbitrary starting point x_0 in terms of Beta functions. However, when reducing the Beta functions to trigonometric functions to recover previous results, we assumed $x_0 = 0$. Let's see how our reductions to trigonometric functions would differ if we start from arbitrary x_0 and not assume $x_0 = 0$, *i.e.* let us rewrite the full Beta function form in trigonometric form. In this sense, let us try to extend previous results in [3] to arbitrary starting x-value of x_0. Since we already have derived an expression for $x(s)$ starting at an arbitrary x_0 in equation (3.6)

$$\boxed{x(s) = \frac{\sqrt{2E}}{\theta}\sin\left(s\theta + \arcsin\frac{\theta}{\sqrt{2E}}x_0\right)} \tag{3.11}$$

we can begin by working with the full expression for $y(s)$ as seen in equation (3.4)

$$y(s) = \frac{E}{\theta^2}\left[B_{B^{-1}_{2\theta s + B_{\frac{\theta^2 x_0^2}{2E}}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tfrac{3}{2},\tfrac{1}{2}\right) - B_{\frac{\theta^2 x_0^2}{2E}}\left(\tfrac{3}{2},\tfrac{1}{2}\right)\right]. \tag{3.12}$$

We can rewrite the second term in (3.12) in terms of trigonometric functions as

$$\begin{aligned} B_{\frac{\theta^2 x_0^2}{2E}}\left(\tfrac{3}{2},\tfrac{1}{2}\right) &= \arcsin\left(\sqrt{\frac{\theta^2 x_0^2}{2E}}\right) - \sqrt{\frac{\theta^2 x_0^2}{2E}\left(1 - \frac{\theta^2 x_0^2}{2E}\right)} \\ &= \arcsin\left(\frac{\theta x_0}{\sqrt{2E}}\right) - \frac{\theta x_0}{\sqrt{2E}}\sqrt{1 - \frac{\theta^2 x_0^2}{2E}}. \end{aligned}$$

For the first term in (3.12), we can rewrite

$$B_{\frac{\theta^2 x_0^2}{2E}}\left(\tfrac{1}{2},\tfrac{1}{2}\right) = 2\arcsin\sqrt{\frac{\theta^2 x_0^2}{2E}} = 2\arcsin\frac{\theta x_0}{\sqrt{2E}}. \tag{3.13}$$

Furthermore, for $B_z^{-1}(1/2,1/2)$, we can use

$$B^{-1}_{2\theta s+B_{\frac{\theta^2 x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})}\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \sin^2\left(\frac{2\theta s+B_{\frac{\theta^2 x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})}{2}\right).$$

From (3.13), it is

$$\sin^2\left(\frac{2\theta s+2\arcsin\frac{\theta x_0}{\sqrt{2E}}}{2}\right) = \sin^2\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right).$$

Now we can plug this in for the expression $B_z(3/2,1/2) = \arcsin(\sqrt{x}) - \sqrt{x(1-x)}$, which obtains

$$\begin{aligned}
B_{B^{-1}_{2\theta s+B_{\frac{\theta^2 x_0^2}{2E}}(\frac{1}{2},\frac{1}{2})}(\frac{1}{2},\frac{1}{2})}\left(\tfrac{3}{2},\tfrac{1}{2}\right) &= B_{\sin^2\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)}\left(\tfrac{3}{2},\tfrac{1}{2}\right) \\
&= \arcsin(\sqrt{\sin^2\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)}) \\
&\quad -\sqrt{\sin^2\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)(1-\sin^2\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right))} \\
&= \arcsin\left|\sin\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right| \\
&\quad -\left|\sin\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\cos\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right| \\
&= \arcsin\left|\sin\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right| - \left|\frac{1}{2}\sin\left(2\theta s+2\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right|,
\end{aligned}$$

which is

$$\begin{aligned}
&=\arcsin(\sqrt{\sin^2\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)}) \\
&\quad -\sqrt{\sin^2\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)(1-\sin^2\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right))} \\
&=\arcsin\left|\sin\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right| - \left|\sin\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\cos\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right| \\
&=\arcsin\left|\sin\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right| - \left|\frac{1}{2}\sin\left(2\theta s+2\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right|.
\end{aligned}$$

Thus we obtain for (3.12)

$$\begin{aligned}
y(s) &= \frac{E}{\theta^2}\left[\arcsin\left|\sin\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right| - \left|\frac{1}{2}\sin\left(2\theta s+2\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right|\right.\\
&\quad \left. - \left(\arcsin\left(\frac{\theta x_0}{\sqrt{2E}}\right) - \frac{\theta x_0}{\sqrt{2E}}\sqrt{1-\frac{\theta^2 x_0^2}{2E}}\right)\right]\\
&= \frac{E}{\theta^2}\left[\arcsin\left|\sin\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right| - \left|\frac{1}{2}\sin\left(2\theta s+2\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right|\right.\\
&\quad \left. - \arcsin\left(\frac{\theta x_0}{\sqrt{2E}}\right) + \frac{\theta x_0}{\sqrt{2E}}\sqrt{1-\frac{\theta^2 x_0^2}{2E}}\right]. \qquad (3.14)
\end{aligned}$$

Under the assumption that the arguments in the sine function are between $[0,\pi/2]$, *i.e.*,

$$\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right) \in [0,\pi/2],$$

expression (3.14) simplifies drastically since we can then ignore the absolute value symbols and the $\arcsin\sin(\cdot)$ expressions simply equal their arguments. This yields

$$\begin{aligned}
&\frac{E}{\theta^2}\left[\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right) - \frac{1}{2}\sin\left(2\theta s+2\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right.\\
&\quad \left. - \arcsin\left(\frac{\theta x_0}{\sqrt{2E}}\right) + \frac{\theta x_0}{\sqrt{2E}}\sqrt{1-\frac{\theta^2 x_0^2}{2E}}\right].
\end{aligned}$$

The arcsines cancel gives

$$\begin{aligned}
&= \frac{E}{\theta^2}\left[(\theta s) - \frac{1}{2}\sin\left(2\theta s+2\arcsin\frac{\theta x_0}{\sqrt{2E}}\right) + \frac{\theta x_0}{\sqrt{2E}}\sqrt{1-\frac{\theta^2 x_0^2}{2E}}\right]\\
&\boxed{= \frac{Es}{\theta} - \frac{E}{2\theta^2}\sin\left(2\theta s+2\arcsin\frac{\theta x_0}{\sqrt{2E}}\right) + \frac{\sqrt{E}x_0}{\sqrt{2}\theta}\sqrt{1-\frac{\theta^2 x_0^2}{2E}}} \qquad (3.15)
\end{aligned}$$

Comparing (3.15) with (3.10), we see that starting at an arbitrary x_0 introduces a phase $\varphi = 2\arcsin\frac{\theta x_0}{\sqrt{2E}}$ in the sine function term above (second term) and gives us an additional initial y-displacement of $y_{\text{displacement}} = \frac{\sqrt{E}x_0}{\sqrt{2}\theta}\sqrt{1-\frac{\theta^2 x_0^2}{2E}}$ as the third term above. Thus the two important equations we investigated in this section are the trigonometric versions of x-component equation (3.3) and y-component equation (3.4) which are given in terms of Beta functions.

Theorem 3.3. *Geodesics for Δ_G with $k=1$ and arbitrary x_0 in trigonometric form, the trigonometric versions of (3.3) and (3.4) are respectively*

$$
\begin{aligned}
x(s) &= \frac{\sqrt{2E}}{\theta}\sin\left(s\theta+\arcsin\frac{\theta}{\sqrt{2E}}x_0\right)\\
y(s) &= \frac{E}{\theta^2}\left[\arcsin\left|\sin\left(\theta s+\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right|-\left|\frac{1}{2}\sin\left(2\theta s+2\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)\right|\right.\\
&\qquad\left.-\arcsin\left(\frac{\theta x_0}{\sqrt{2E}}\right)+\frac{\theta x_0}{\sqrt{2E}}\sqrt{1-\frac{\theta^2x_0^2}{2E}}\right]\\
&\to \frac{Es}{\theta}-\frac{E}{2\theta^2}\sin\left(2\theta s+2\arcsin\frac{\theta x_0}{\sqrt{2E}}\right)+\frac{\sqrt{E}x_0}{\sqrt{2}\theta}\sqrt{1-\frac{\theta^2x_0^2}{2E}},
\end{aligned}
$$

where we assumed in the last step that the arguments in the sine function are between $[0,\pi/2]$.

Remark 3.1. In [5] and [6], we found a function μ which is a monotone increasing diffeomorphism of the interval $(-\pi,\pi)$ onto $\mathbb{R}$. On each interval $(m\pi,(m+1)\pi)$, $m=1,2,\ldots$, μ has a unique critical point x_m which satisfies $\tan(x_m)=x_m$. On this interval, μ decreases strictly from $+\infty$ to $\mu(x_m)$ and then increases strictly from $\mu(x_m)$ to $+\infty$. Moreover, $\mu(x_m)+\pi<\mu(x_{m+1})$, $m=1,2,\ldots$, and $0<\left(m+\frac{1}{2}\right)\pi-x_m<\frac{1}{m\pi}$.

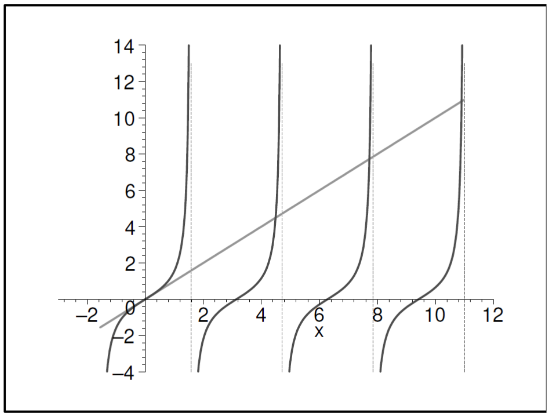

FIGURE 1. On each interval $(m\pi,(m+1)\pi)$, $m\in\mathbb{Z}_+$, μ has a unique critical point x_m which satisfies $\tan(x_m)=x_m$.

In fact, There are *finitely many* geodesics joining the origin and $P(x_1,y_1)$ if and only if $x_1\neq 0$. These geodesics are parametrized by the solutions σ of

$$\frac{y_1}{|x_1|^2}=\mu(2\theta),\quad \text{where}\quad \mu(z)=\frac{z}{\sin^2 z}-\cot z \tag{3.16}$$

Furthermore, let $\theta_1<\theta_2<\cdots<\theta_N$ be the solutions of (3.16). The square of the length of the geodesic associated with the solution θ_m is

$$\ell_m^2=\left(\frac{\theta_m^2}{\sin^2\theta_m}\right)^2|x|^2.$$

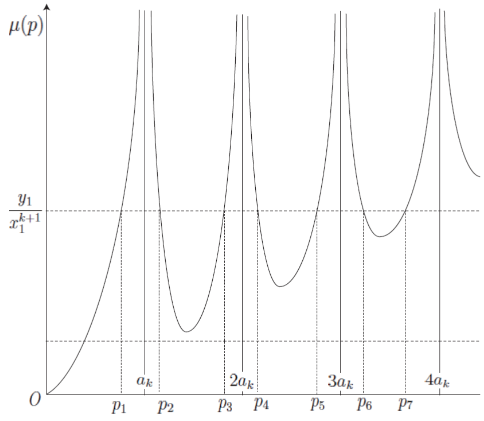

FIGURE 2. There are finitely many geodesics joining the origin and $P(x_1, y_1)$ if and only if $x_1 \neq 0$. This is the graph for the case $k = 1$.

3.4. **Heat Kernel for Δ_G with $k = 1$ in Beta Function Form.** In this section, we use the Beta function formalism to find the heat kernel of the operator $\Delta_G = \frac{1}{2}(\partial_x^2 + x^2\partial_y^2)$. We know from Section 2.2 that the heat kernel v for Δ_G should satisfy

$$\begin{cases} \partial_t v = \Delta_G v, \quad t > 0 \\ \lim_{t\to 0} v(x,y,t) = \delta_{x_0}(x) \otimes \delta_{y_0}(y) \end{cases}$$

As usual, we can perform the partial Fourier transform on y to obtain $u = \mathscr{F}_y(v)$ and $\partial_t u = \frac{1}{2}\left(\partial_x^2 u - x^2\xi^2 u\right)$. The solution is given by the Ansatz

$$u = e^{\frac{t}{2}\left(\partial_x^2 - x^2\xi^2\right)}\left(\delta_{x_0}(x) \otimes I_\xi\right) \tag{3.17}$$

We thus want to first find the heat kernel of the Hermite operator $L = \frac{1}{2}\left(\partial_x^2 - x^2\xi^2\right)$. To do so, we again need to find the Hamiltonian and the associated equations of motion, which are $H(p,x) = \frac{1}{2}p^2 - \frac{1}{2}\xi^2 x^2$ with

$$\begin{cases} \dot{x} = H_p = p, \\ \dot{p} = -H_x = \xi^2 x. \end{cases}$$

As a result, we need to solve the equation $\ddot{x} = \xi^2 x$, which can be done with the energy method using the boundary conditions $x(0) = x_0$ and $x(t) = x_1 = x$:

$$E = \frac{1}{2}p^2 - \frac{1}{2}\xi^2 x^2 \implies ds = \frac{dx}{\sqrt{2E + \xi^2 x^2}}.$$

Hence, we can express the following integral in terms of the incomplete Beta function

$$t = \int_0^t ds = \int_{x_0}^{x_1} \frac{dx}{\sqrt{2E + \xi^2 x^2}} = \frac{1}{\sqrt{2E}} \int_{x_0}^{x_1} \frac{dx}{\sqrt{1 + \frac{\xi^2 x^2}{2E}}}$$

by substituting $u = \frac{\xi x}{\sqrt{2E}}$ with $du = \frac{\xi}{\sqrt{2E}}dx$,

$$\frac{1}{\sqrt{2E}} \int_{u_0}^{u_1} \frac{1}{\sqrt{1+u^2}} \frac{\sqrt{2E}}{\xi} du = \frac{1}{\xi} \int_{u_0}^{u_1} \frac{1}{\sqrt{1+u^2}} du.$$

Now, we can use the substitution $t=\frac{u^2}{1+u^2}$ which gives us $1+u^2=\frac{1}{1-t}$ and hence $\frac{1}{\sqrt{1+u^2}}=\sqrt{1-t}$. The differential can be found as

$$\frac{dt}{du}=\frac{2u}{(1+u^2)^2}=2u(1-t)^2=2\sqrt{\frac{t}{1-t}}(1-t)^2=2t^{1/2}(1-t)^{3/2}.$$

Plugging this into our integral yields

$$\begin{aligned}&=\frac{1}{\xi}\int_{t_0}^{t_1}\sqrt{1-t}\frac{dt}{2t^{1/2}(1-t)^{3/2}}=\frac{1}{2\xi}\int_{t_0}^{t_1}t^{-1/2}(1-t)^{-1}dt\\&=\frac{1}{2\xi}\int_{t_0}^{t_1}t^{1/2-1}(1-t)^{0-1}dt=\frac{1}{2\xi}\left[B_{t_1}(1/2,0)-B_{t_0}(1/2,0)\right].\end{aligned}$$

Plugging in the boundary conditions $t_0=\frac{u_0^2}{1+u_0^2}=\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}$, we obtain

$$\boxed{t=\frac{1}{2\xi}\left[B_{\frac{\left(\frac{\xi x_1}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_1}{\sqrt{2E}}\right)^2}}(1/2,0)-B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)\right]} \tag{3.18}$$

We are faced with a minor technical problem at this point. Namely, although we have written the incomplete Beta function $B_z(a,b)$ with one of its arguments as $b=0$, this value is out of the domain of the Beta function in its usual definition. Therefore, it would be more correct to write $\lim_{b\to0^+}B_z(a,b)$ instead. In this limit when $b\to0$, the incomplete Beta function becomes an arc sinh function, which recovers the previous result for the geodesics of the Hermite operator studied in [4] and [8]. We will show this shortly after rearranging (3.18) to solve for $x_1=x(t)$. Therefore, the notation $B_z(a,0)$ is more-so a technical notational shortcut. In a certain sense, this is also not an issue since, as we have mentioned in [7], it is straightforward to analytically continue in the arguments of the Beta function to any complex a,b.

Let us now solve $x_1=x(t)$ in (3.18). To do so, we can use the inverse incomplete Beta function:

$$\begin{aligned}B_{\frac{\left(\frac{\xi x_1}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_1}{\sqrt{2E}}\right)^2}}(1/2,0)&=2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)\\ \frac{\left(\frac{\xi x_1}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_1}{\sqrt{2E}}\right)^2}&=B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0).\end{aligned}$$

We can invert the fraction by observing:

$$\frac{\eta^2}{1+\eta^2}=f\qquad\Rightarrow\qquad\eta^2-f\eta^2=f\qquad\Rightarrow\qquad\eta=\sqrt{\frac{f}{1-f}}.$$

Hence,

$$\left(\frac{\xi x_1}{\sqrt{2E}}\right)=\sqrt{\frac{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}}$$

$$\boxed{x(t)=x_1=\frac{\sqrt{2E}}{\xi}\left(\frac{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right)^{1/2}} \tag{3.19}$$

We will show that this is in fact the correct result by reducing it to previous work in Section 3.5. However, having found the x-component of the geodesics, we can now find the classical action, which is the next step in our geometric mechanical algorithm for finding the heat kernel. We can find the classical action using several different ways. The only one that seems to work is the standard approach which is to first find the Lagrangian L associated to the Hamiltonian of the Hermite operator. The classical action is then just the integral of L, with L evaluated along the geodesic. Finding L can be done with the help of the inverse Legendre transform and Hamilton's equations, which tell us that $\dot{x}=p$:

$$L(x,\dot{x})=px-H=\dot{x}^2-\frac{1}{2}(p^2-\xi^2x^2)=\frac{1}{2}\dot{x}^2+\xi^2x^2$$

Thus, the classical action S_{cl}, being the integral of the Lagrangian evaluated on the classical path, i.*e.* the geodesics $x_1=x(t)=x_{cl}$, is given by:

$$S_{cl}(x,x_0,t)=\int_0^t L(x_{cl}(s),\dot{x}_{cl}(s))ds=\frac{1}{2}\int_0^t(\dot{x}_{cl}^2+\xi^2x_{cl}^2)ds$$

Therefore, we need to calculate x_{cl}^2 and $\dot{x}_{cl}^2$:

$$x_{cl}^2=x_1^2=\frac{2E}{\xi^2}\left(\frac{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right)$$

and

$$\dot{x}_{cl} = \frac{\sqrt{2E}}{\xi}\frac{d}{dt}\left(\frac{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right)^{1/2}$$

$$= \frac{\sqrt{2E}}{\xi}\frac{1}{2}\left(\frac{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right)^{-1/2} \frac{d}{dt}\left(\frac{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right).$$

Setting $f = B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)$, we see that the derivative from the chain rule is

$$\frac{d}{dt}\frac{f}{1-f} = \frac{f'(1-f)-f(-f')}{(1-f)^2} = \frac{f'}{(1-f)^2}.$$

We can calculate f' with the help of

$$\frac{d}{dt}B_t^{-1}(b_1,b_2) = [B_t^{-1}(b_1,b_2)]^{1-b_1}[1-B_t^{-1}(b_1,b_2)]^{1-b_2}$$

Hence, calculating f' yields:

$$f' = \frac{d}{dt}B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)$$

$$= \left[B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)\right]^{1-1/2}\left[1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)\right]^{1-0}(2\xi),$$

where the 2ξ comes from the chain rule. Thus

$$\dot{x}_{cl} = \frac{\sqrt{2E}}{\xi}\frac{1}{2}\left(\frac{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right)^{-1/2}$$

$$\left[B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)\right]^{1-1/2}\left[1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)\right]^{1-0}(2\xi)$$

$$\left[1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)\right]^{-2}.$$

Remark 3.2. Note however that these cancellations are only a result of the fact that we have $k = 1$. For general k, this cancellation does not occur as we will see later. In fact, we need to develop a new formalism to deal with the general k Hermite operator by using a different integral function than the inverse incomplete Beta function.

$$\dot{x}_{cl} = \sqrt{2E}\left(\frac{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right)^{-1/2}\left[B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)\right]^{1/2}$$

$$\left[1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)\right]^{-1} = \sqrt{2E}\left(\frac{1}{1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right)^{1/2},$$

which implies that

$$\dot{x}_{cl} = \sqrt{2E}\left(1 - B^{-1}_{2\xi t + B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)\right)^{-1/2}.$$

Squaring this expression gives

$$\dot{x}^2_{cl} = 2E\left(1 - B^{-1}_{2\xi t + B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)\right)^{-1}.$$

We can see here that if we integrate this expression, the integrand would be of the form $t^a(1-t)^b$, which means that we would again have a Beta function as the integral. This again indicates to us the power of the Beta function formalism as a closed formalism, *i.e.* expressions will often cancel and evaluate to be Beta functions, rather than only expressible using other special functions. Hence, to see how we obtain a Beta function again from the integral of $\dot{x}^2_{cl}$, let us use the substitution again

$$f(s) = B^{-1}_{2\xi s + B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)$$

for clarity the following integral:

$$\int_0^t \dot{x}^2_{cl}(s)ds = 2E\int_0^t (1-f(s))^{-1}\,ds. \tag{3.20}$$

Now change integration variables with the substitution to $f(s)$ by rewriting $ds = df\frac{ds}{df}$ (*i.e.* a u-substitution with $u = f(s)$ and $du = f'ds$ and $u_0 = f(0)$ and $u_1 = f(t)$). (3.20) can be rewritten as

$$\int_0^t \dot{x}^2_{cl}(s)ds = 2E\int_{f(0)}^{f(t)} df\,\frac{1}{f'}(1-f(s))^{-1}.$$

Note that f' may make the integrand no longer of the form $t^a(1-t)^b$. Since f is a inverse incomplete Beta function, we can use the derivative formula for f' to get $f' = f^{1/2}(1-f)^1$.

Technically, we have $f(2\xi s+B)$, which yields $f' = f^{1/2}(1-f)2\xi$ and

$$\begin{aligned}
\int_0^t \dot{x}_{cl}^2(s)ds &= 2E\int_{f(0)}^{f(t)} df\,\frac{1}{2\xi f^{1/2}(1-f)^1}(1-f(s))^{-1} = \frac{E}{\xi}\int_{f(0)}^{f(t)} dff^{-1/2}1-f(s))^{-2} \\
&= \frac{E}{\xi}\int_{f(0)}^{f(t)} dff^{1/2-1}(1-f)^{-1-1} = \frac{E}{\xi}\left[B_{f(t)}(1/2,-1) - B_{f(0)}(1/2,-1)\right] \\
&= \frac{E}{\xi}\left[B_{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}(1/2,-1) - B_{B^{-1}_{B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}(1/2,-1)\right] \\
&= \frac{E}{\xi}\left[B_{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}(1/2,-1) - B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,-1)\right]. \qquad (3.21)
\end{aligned}$$

We can now do the same for integrating the potential $\xi^2 x_{cl}^2$:

$$\int_0^t ds\,\xi^2 x_{cl}^2(s) = 2E\int_0^t ds\left(\frac{B^{-1}_{2\xi s+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi s+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right)$$

Again, we see that this will be of the form of the integrand of the Beta function. To make this clear, let us use the substitution

$$f = B^{-1}_{2\xi s+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)$$

with $df = f'ds = 2\xi f^{1/2}(1-f)ds$:

$$\begin{aligned}
&= 2E\int_0^t df\,\frac{1}{2\xi f^{1/2}(1-f)}\left(\frac{f}{1-f}\right) = \frac{2E}{2\xi}\int_0^t df f^{1/2}(1-f)^{-2} \\
&= \frac{2E}{2\xi}\int_0^t df f^{3/2-1}(1-f)^{-1-1} \\
&= \frac{2E}{2\xi}B_t(3/2,-1). \qquad (3.22)
\end{aligned}$$

Thus we can add together (3.21) and (3.22) to obtain the classical action as

$$\begin{aligned} & S_{cl}(x,x_0;t) \qquad (3.23)\\ &= \frac{1}{2}\int_0^t (\dot{x}_{cl}^2 + \xi^2 x_{cl}^2)ds \\ &= \frac{E}{2\xi}\left[B_{B^{-1}_{2\xi t + B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}(1/2,-1) - B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,-1)\right] + \frac{E}{2\xi}B_t(3/2,-1) \end{aligned}$$

$$\boxed{S_{cl}(x,x_0;t) = \frac{E}{2\xi}\left[B_{B^{-1}_{2\xi t + B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}(1/2,-1) - B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,-1) + B_t(3/2,-1)\right]} \tag{3.24}$$

One could reduce this expression for the action to that found in [4] as equation (3.16.56). We will not do that here. Instead, we move straight to finding the propagator for the Hermite operator, which can be done by using the path integral approach and the van Vleck's formula:

$$V^2(t) = \det\left(-\frac{1}{2\pi}\frac{\partial^2 S_{cl}(x_0,x)}{\partial x_0 \partial x}\right).$$

The heat kernel now can be written in an even simpler form:

$$K(x_0,x;t) = \sqrt{\det\left(-\frac{1}{2\pi}\frac{\partial^2 S_{cl}(x_0,x)}{\partial x_0 \partial x}\right)}e^{-S_{cl}(x_0,x,t)} = V(t)e^{-S_{cl}}. \tag{3.25}$$

Note: It turns out that in many cases, such a second-order elliptic operators with constant, linear, and quadratic potential, the second integral $\int_{\mathfrak{B}_{0,0;t}} e^{-S(\psi,t)} d\mathfrak{m}(\psi)$ is a function of t only, *i.e.* $\int_{\mathfrak{B}_{0,0;t}} e^{-S(\psi,t)} d\mathfrak{m}(\psi) = V(t)$. We use V to denote this function in representation of its **volume element**. It tells us how the geodesics spread away from the initial point x_0 as we move along the parameter of the geodesics. Accordingly, it satisfies a transport equation, usually of the form $V'(t) + V(t)\mathbb{L}(S_{cl}) = 0$. More precisely, the square of $V(t)$ gives us the density of the geodesic flow, *i.e.* how geodesics spread away from a given geodesics. This square is given in many cases by the determinant of the rate of change of the action. It is called **van Vleck's formula** .

With an expression for S_{cl}, we need to find the volume element $V(t)$. Thus, let us find the second derivative of the classical action

$$\frac{\partial^2}{\partial x_0 \partial x} S_{cl}(x,x_0;t)$$
$$= \frac{\partial^2}{\partial x_0 \partial x}\left(\frac{E}{2\xi}\left[B_{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}(1/2,-1) - B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,-1) + B_t(3/2,-1)\right]\right).$$

Notice however that if we were to take naively the derivative with respect to x of this expression for S_{cl}, we would get zero. This would give us a null propagator, which is clearly wrong. The mistake we would've done is to assume that S_{cl}, as it is written, is independent of $x = x_1$, which it is not. We see, in Section 3.5, that the combinations of E, ξ, and x_0 give us expressions for x or x_1. Specially, from the geodesics, we find

$$\frac{\sqrt{(\xi x_0)^2 + 2E}}{\xi} = \frac{[x_1 - x_0 \cosh(\xi t)]}{\sinh(\xi t)}.$$

In view of the arguments in the Beta functions in the action (3.24), we can rewrite

$$\begin{aligned}\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2} &= \frac{(\xi x_0)^2}{(\xi x_0)^2 + 2E} = \left(\frac{\sqrt{(\xi x_0)^2+2E}}{\xi}\right)^{-2} \xi^{-2}(\xi x_0)^2 \\ &= \left(\frac{x_1 - x_0\cosh(\xi t)}{x_0 \sinh(\xi t)}\right)^{-2} = \left(\frac{x_0 \sinh(\xi t)}{x_1 - x_0 \cosh(\xi t)}\right)^2 =: \alpha(x_0, x_1; t). \qquad (3.26)\end{aligned}$$

Combing (3.26) with (3.24) yields

$$\begin{aligned} S_{cl}(x_0,x;t) &= \frac{E}{2\xi}\left[B_{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}(1/2,-1) - B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,-1) + B_t(3/2,-1)\right] \\ &= \frac{E}{2\xi}\left[B_{B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)}(1/2,-1) - B_{\alpha(x_0,x_1;t)}(1/2,-1) + B_t(3/2,-1)\right]. \end{aligned} \qquad (3.27)$$

Remark 3.3. In fact, we do not need to change the parameter in (3.24) from $t \to s$ since to obtain the action we have integrated over the parameter s already from $(0,t)$, thus meaning we already have reached the value $s = t$.

We can therefore take the derivative of this expression now to find

$$\begin{aligned}&\frac{\partial^2}{\partial x_0 \partial x} S_{cl}\\&= \frac{\partial^2}{\partial x_0 \partial x}\left(\frac{E}{2\xi}\left[B_{B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)}(1/2,-1) - B_{\alpha(x_0,x_1;t)}(1/2,-1) + B_t(3/2,-1)\right]\right).\end{aligned} \tag{3.28}$$

This derivative, when it hits the third term above, will give zero since $B_t(3/2,-1)$ is independent of x and x_0. Hence, we can focus on only the first and second term. Since the second term is easier to work with, let us compute its second partial mixed derivative first. Using the chain rule and recalling that $\partial_z B_z(a,b) = z^{a-1}(1-z)^{b-1}$, we find

$$\frac{\partial}{\partial x} B_{\alpha(x_0,x;t)}(1/2,-1) = \frac{\partial \alpha}{\partial x}\alpha^{1/2-1}(1-\alpha)^{-1-1} = \alpha^{-1/2}(1-\alpha)^{-2}\frac{\partial \alpha}{\partial x}.$$

From $\frac{\partial \alpha}{\partial x} = -2\frac{(x_0 \sinh(\xi t))^2}{[x - x_0\cosh(\xi t)]^3}$, we find that it is

$$\begin{aligned}&-2\left(\frac{x - x_0\cosh(\xi t)}{x_0 \sinh(\xi t)}\right)\left(\frac{[x - x_0\cosh(\xi t)]^2}{x_0 - 2xx_0\cosh(\xi t) + x_0^2}\right)^2 \frac{(x_0 \sinh(\xi t))^2}{[x - x_0\cosh(\xi t)]^3}\\&= -2\frac{x_0 \sinh(\xi t)[x - x_0\cosh(\xi t)]^2}{[x^2 - 2xx_0\cosh(\xi t) + x_0^2]^2}.\end{aligned}$$

Taking the derivative with respect to x_0, we find

$$\begin{aligned}&\frac{\partial^2}{\partial x_0 \partial x} B_{\alpha(x_0,x;t)}(1/2,-1)\\&= -2\frac{\sinh(\xi t)[x - x_0\cosh(\xi t)]^2}{[x^2 - 2xx_0\cosh(\xi t) + x_0^2]^2} + 4\frac{x_0\sinh(\xi t)[x - x_0\cosh(\xi t)]\cosh(\xi t)}{[x^2 - 2xx_0\cosh(\xi t) + x_0^2]^2}\\&\quad - 4\frac{x_0\sinh(\xi t)[x - x_0\cosh(\xi t)]^2[2x_0 - 2x\cosh(\xi t)]}{[x^2 - 2xx_0\cosh(\xi t) + x_0^2]^3}\\&= \frac{-2\sinh(\xi t)[x - x_0\cosh(\xi t)]}{[x^2 - 2xx_0\cosh(\xi t) + x_0^2]^3}\left([x - 3x_0\cosh(\xi t)][x^2 - 2xx_0\cosh(\xi t) + x_0^2]\right.\\&\qquad\qquad\left. + 4xx_0\cosh(\xi t)[x_0 - x\cosh(\xi t)]\right).\end{aligned} \tag{3.29}$$

We can now compute the derivatives of the first term in (3.28). To make the chain rule clearer, let us define $z(x) = 2\xi t + B_{\alpha(x_0,x_1;t)}(1/2,0)$. Thus

$$\begin{aligned}\frac{\partial}{\partial x} B_{B^{-1}_{z(x)}(1/2,0)}(1/2,-1) &= \frac{\partial z}{\partial x}\left(z^{1/2-1}(1-z)^{0-1}\right)\left((B_z^{-1}(1/2,0))^{1/2-1}(1 - B_z^{-1}(1/2,0))^{-1-1}\right)\\&= \frac{\partial z}{\partial x}\left(z^{-1/2}(1-z)^{-1}\right)\left((B_z^{-1}(1/2,0))^{-1/2}(1 - B_z^{-1}(1/2,0))^{-2}\right).\end{aligned}$$

Therefore, let us compute $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} B_{\alpha(x_0,x_1;t)}(1/2,0) = \frac{\partial \alpha}{\partial x}\frac{\partial}{\partial \alpha} B_{\alpha(x_0,x_1;t)}(1/2,0) = \frac{\partial \alpha}{\partial x}\alpha^{-1/2}(1-\alpha)^{-1}$. Hence, it is

$$\frac{\partial \alpha}{\partial x}\alpha^{-1/2}(1-\alpha)^{-1}\left(z^{-1/2}(1-z)^{-1}\right)\left((B_z^{-1}(1/2,0))^{-1/2}(1 - B_z^{-1}(1/2,0))^{-2}\right) =: (*).$$

Thus

$$\begin{aligned}
(*) =& \left(-2\frac{(x_0\sinh(\xi t))}{[x-x_0\cosh(\xi t)]^2}\right)\left(1-\left(\frac{x_0\sinh(\xi t)}{x_1-x_0\cosh(\xi t)}\right)^2\right)^{-1}\\
&\times\left(\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-1/2}(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0))^{-1}\right)\\
&\times\left((B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-1/2}(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-2}\right).
\end{aligned}$$

Notice that we can rewrite the following middle term

$$\begin{aligned}
&\left(1-\left(\frac{x_0\sinh(\xi t)}{x_1-x_0\cosh(\xi t)}\right)^2\right)^{-1}\\
&=\left(\frac{[x_1-x_0\cosh(\xi t)]^2}{[x_1-x_0\cosh(\xi t)]^2}-\frac{x_0^2\sinh^2(\xi t)}{[x_1-x_0\cosh(\xi t)]^2}\right)^{-1}\\
&=\left(\frac{x_1^2-2x_0x_1\cosh(\xi t)+x_0^2\cosh^2(\xi t)-x_0^2\sinh^2(\xi t)}{[x_1-x_0\cosh(\xi t)]^2}\right)^{-1}\\
&=\frac{[x_1-x_0\cosh(\xi t)]^2}{x_1^2-2x_0x_1\cosh(\xi t)+x_0^2}.
\end{aligned} \tag{3.30}$$

Hence, we have

$$\begin{aligned}
(*) =& \left(-2\frac{(x_0\sinh(\xi t))}{x_1^2-2x_0x_1\cosh(\xi t)+x_0^2}\right)\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-1/2}\\
&\times(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0))^{-1}\\
&\left((B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-1/2}(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-2}\right).
\end{aligned}$$

We can now take the derivative of this expression with respect to x_0. This leads to

$$\begin{aligned}
\frac{\partial}{\partial x_0}(*) =& \frac{\partial}{\partial x_0}\Bigg[\left(-2\frac{(x_0\sinh(\xi t))}{x_1^2-2x_0x_1\cosh(\xi t)+x_0^2}\right)\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-1/2}\\
&\times(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0))^{-1}\\
&\left((B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-1/2}(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-2}\right)\Bigg].
\end{aligned}$$

Applying the derivative to each of the four terms gives

$$
\begin{aligned}
&= \left(-2\frac{x_0\sinh(\xi t)}{x_1^2-2x_0x_1\cosh(\xi t)+x_0^2}\right)\left[-\tfrac{1}{2}\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-3/2}\right]\\
&\times \alpha^{-1/2}(1-\alpha)^{-1}\frac{\partial\alpha}{\partial x_0}\times(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0))^{-1}\\
&\times\left((B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-1/2}(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-2}\right)\\
&+\left(-2\frac{(x_0\sinh(\xi t))}{x_1^2-2x_0x_1\cosh(\xi t)+x_0^2}\right)\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-1/2}\\
&\times(-1)(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0))^{-2}\alpha^{-1/2}(1-\alpha)^{-1}\frac{\partial\alpha}{\partial x_0}\\
&\times\left((B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-1/2}(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-2}\right)\\
&+\left(-2\frac{(x_0\sinh(\xi t))}{x_1^2-2x_0x_1\cosh(\xi t)+x_0^2}\right)\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-1/2}\\
&\times(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0))^{-1}\left(-\tfrac{1}{2}(B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-3/2}\right)\\
&\times\frac{\partial\alpha}{\partial x_0}\left[\alpha^{-1/2}(1-\alpha)^{-1}\right]\\
&\times\left[[B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)]^{1/2}(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{1}\right]\\
&\times(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-2}\\
&+\left(-2\frac{(x_0\sinh(\xi t))}{x_1^2-2x_0x_1\cosh(\xi t)+x_0^2}\right)\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-1/2}\\
&\times(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0))^{-1}\left((B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-1/2}\right)\\
&\times 2(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-3}\\
&\times\frac{\partial\alpha}{\partial x_0}\left[\alpha^{-1/2}(1-\alpha)^{-1}\right]\\
&\times\left[[B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)]^{1/2}(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{1}\right]
\end{aligned}
\tag{3.31}
$$

where in the last two terms we used

$$
\begin{aligned}
&\frac{\partial}{\partial x_0}B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)\\
&\quad=\frac{\partial\alpha}{\partial x_0}\left[\alpha^{-1/2}(1-\alpha)^{-1}\right]\\
&\quad\times\left[[B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)]^{1/2}(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{1}\right].
\end{aligned}
$$

Note that

$$\begin{aligned}\frac{\partial\alpha}{\partial x_0} &= 2\left(\frac{x_0\sinh(\xi t)}{x_1-x_0\cosh(\xi t)}\right)\frac{\sinh(\xi t)[x_1-x_0\cosh(\xi t)]-[-\cosh(\xi t)]x_0\sinh(\xi t)}{[x_1-x_0\cosh(\xi t)]^2}\\ &= 2\left(\frac{x_0\sinh(\xi t)}{x_1-x_0\cosh(\xi t)}\right)\frac{x_1\sinh(\xi t)}{[x_1-x_0\cosh(\xi t)]^2}=\frac{2x_0x_1(\sinh(\xi t))^2}{[x_1-x_0\cosh(\xi t)]^3}.\end{aligned}$$

Using (3.30), one has

$$\begin{aligned}&\frac{\partial\alpha}{\partial x_0}\left[\alpha^{-1/2}(1-\alpha)^{-1}\right]\\ &\quad=\frac{2x_0x_1(\sinh(\xi t))^2}{[x_1-x_0\cosh(\xi t)]^3}\left(\frac{x_0\sinh(\xi t)}{x_1-x_0\cosh(\xi t)}\right)^{-1}\left(1-\left(\frac{x_0\sinh(\xi t)}{x_1-x_0\cosh(\xi t)}\right)^2\right)^{-1}\\ &\quad=\frac{2x_0x_1(\sinh(\xi t))^2}{[x_1-x_0\cosh(\xi t)]^3}\left(\frac{x_0\sinh(\xi t)}{x_1-x_0\cosh(\xi t)}\right)^{-1}\frac{[x_1-x_0\cosh(\xi t)]^2}{x_1^2-2x_0x_1\cosh(\xi t)+x_0^2}\\ &\quad=\frac{2x_1\sinh(\xi t)}{x_1^2-2x_0x_1\cosh(\xi t)+x_0^2}.\end{aligned}\tag{3.32}$$

Thus, plugging (3.32) into (3.31), we obtain

$$\begin{aligned}&\frac{\partial^2}{\partial x_0\partial x}B_{B^{-1}_{z(x)}(1/2,0)}(1/2,-1)\\ &\quad=\left(-2\frac{\sinh(\xi t)[x_1^2-x_0^2]}{[x_1^2-2x_0x_1\cosh(\xi t)+x_0^2]^2}\right)\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-1/2}\\ &\qquad\times(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0))^{-1}\left((B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-1/2}\right)\\ &\qquad+\Bigg[\left[-\tfrac{1}{2}\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-3/2}\right]\\ &\qquad\times(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0))^{-1}\left(B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)\right)^{-1/2}\\ &\qquad-\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-1/2}\left(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-2}\\ &\qquad\times\left(B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)\right)^{-1/2}\\ &\qquad-\tfrac{1}{2}\left(2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)\right)^{-1/2}(1-2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0))^{-1}\\ &\qquad\times\left[B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)\right]^{-1}\left(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)\right)\\ &\qquad+2(1-B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0))^{-2}[B^{-1}_{2\xi t+B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)]^{1/2}\Bigg]\\ &\qquad\times\left(\frac{-4x_0x_1(\sinh(\xi t))^2}{[x_1^2-2x_0x_1\cosh(\xi t)+x_0^2]^2}\right).\end{aligned}\tag{3.33}$$

Thus, we can find the volume element by putting (3.33) together with (3.29) to obtain (3.28):

$$\begin{aligned}\frac{\partial^2}{\partial x_0 \partial x} S_{cl} &= \frac{E}{2\xi}\left[\frac{\partial^2}{\partial x_0 \partial x} B_{B^{-1}_{2\xi t + B_{\alpha(x_0,x_1;t)}(1/2,0)}(1/2,0)}(1/2,-1) - \frac{\partial^2}{\partial x_0 \partial x} B_{\alpha(x_0,x;t)}(1/2,-1)\right] \\ &= \frac{E}{2\xi}\left[(3.33)+(3.29)\right],\end{aligned}$$

which implies

$$V(t) = \sqrt{\det\left(-\frac{1}{2\pi}\frac{\partial^2 S_{cl}(x_0,x)}{\partial x_0 \partial x}\right)} = \sqrt{-\frac{1}{2\pi}\frac{E}{2\xi}\left[(3.33)+(3.29)\right]},$$

where the determinant in the expression or $V(t)$ is only really necessary if we deal with multiple dimensions. Since our derivative result for the classical action is just a simple function, we can ignore [12]. Thus, we find the propagator in Beta function form of the Hermite operator to be

$$\boxed{K(x_0,x;t) = \sqrt{-\frac{1}{2\pi}\frac{E}{2\xi}\left[(3.33)+(3.29)\right]}\, e^{-(3.24)}} \tag{3.34}$$

Following the steps in the algorithm outlined in Section 2.2, we summarize this result in the following theorem.

Theorem 3.4. *Propagator of the Hermite Operator. The propagator of the Hermite operator* $\mathbb{L} = \frac{1}{2}(\partial_x^2 - x^2\xi^2)$ *in Beta function form is* $K(x_0,x;t) = \sqrt{-\frac{1}{2\pi}\frac{E}{2\xi}\left[(3.33)+(3.29)\right]}\, e^{-(3.24)}$.

The next step in the geometric method is to use the inverse Fourier transform with respect to ξ on equation (3.17). Since $u = \mathscr{F}_y(v)$, this gives us an expression for the heat kernel v of the Grushin operator, $\mathscr{F}_y^{-1}(u) = v$, as

$$v(x,y,t) = \frac{1}{2\pi}\int e^{-iy\xi} K(x,\xi;t)\, d\xi = \frac{1}{2\pi}\int e^{-iy\xi}\sqrt{-\frac{1}{2\pi}\frac{E}{2\xi}\left[(3.33)+(3.29)\right]}\, e^{-(3.24)}\, d\xi.$$

Hence, we can summarize this result as follows.

Theorem 3.5. *Heat Kernel of* Δ_G *with* $k=1$ *in Beta Function Form. The heat kernel of the Grushin operator* Δ_G *with* $k=1$ *in Beta function form is given by*

$$v(x,y,t) = \frac{1}{2\pi}\int e^{-iy\xi}\sqrt{-\frac{1}{2\pi}\frac{E}{2\xi}\left[(3.33)+(3.29)\right]}\, e^{-(3.24)}\, d\xi.$$

3.5. **Reducing Geodesics for the Hermite Operator to Trigonometric Form.** In this section, we show how to reduce expression (3.19) for the x-component of the geodesics for the Hermite operator in Beta function form to the hyperbolic trigonometric form found in [2, Section 10.4] and in [4, Equation (3.16.54)]. To do so, we will use the key identity. The proof is from the inverse of $B_x(1/2,0) = \int_0^x t^{-1/2}(1-t)^{-1}\, dt = 2\ln\frac{1+\sqrt{x}}{\sqrt{1-x}}$, where $0 < x < 1$. The inverse is then the unique $x \in (0,1)$ with $B_x(1/2,0) = 2\ln\frac{1+\sqrt{x}}{\sqrt{1-x}} = z$, $B_z^{-1}(1/2,0) = \tanh^2(z/2)$. As a result, we see that

$$\sqrt{\frac{B_z^{-1}(1/2,0)}{1-B_z^{-1}(1/2,0)}} = \sqrt{\frac{\tanh^2 z/2}{1-\tanh^2 z/2}} = \sqrt{\frac{\tanh^2 z/2}{\operatorname{sech}^2 z/2}} = \sinh z/2 \tag{3.35}$$

In (3.19), we have

$$x(t) = \frac{\sqrt{2E}}{\xi}\left(\frac{B^{-1}_{2\xi t + B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi t + B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right)^{1/2}.$$

Setting $\alpha = \frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2} = \frac{(\xi x_0)^2}{(\xi x_0)^2+2E}$, one sees that the z in (3.35) is

$$z = 2\xi t + B_\alpha(1/2,0) = 2\xi t + 2\ln\frac{1+\sqrt{\alpha}}{\sqrt{1-\alpha}} = 2\ln\left(\frac{1+\sqrt{\alpha}}{\sqrt{1-\alpha}}e^{\xi t}\right).$$

Plugging this into our $\sinh z/2$ expression in (3.35), we can use the identity $\sinh(\ln y) = \frac{1}{2}(y - 1/y)$:

$$\sinh z/2 = \sinh\ln\left(\frac{1+\sqrt{\alpha}}{\sqrt{1-\alpha}}e^{\xi t}\right) = \frac{1}{2}\left(\frac{1+\sqrt{\alpha}}{\sqrt{1-\alpha}}e^{\xi t} - \frac{\sqrt{1-\alpha}}{1+\sqrt{\alpha}}e^{-\xi t}\right).$$

We can now simplify the coefficients in front of the exponents as follows:

$$\frac{1+\sqrt{\alpha}}{\sqrt{1-\alpha}} = \frac{1+\sqrt{\frac{(\xi x_0)^2}{(\xi x_0)^2+2E}}}{\sqrt{1-\frac{(\xi x_0)^2}{(\xi x_0)^2+2E}}} = \frac{1+\frac{\xi x_0}{\sqrt{(\xi x_0)^2+2E}}}{\sqrt{\frac{2E}{(\xi x_0)^2+2E}}} = \frac{\sqrt{(\xi x_0)^2+2E}+\xi x_0}{\sqrt{2E}}.$$

The reciprocal is

$$\frac{\sqrt{1-\alpha}}{1+\sqrt{\alpha}} = \frac{\sqrt{2E}(\sqrt{(\xi x_0)^2+2E}-\xi x_0)}{2E} = \frac{\sqrt{(\xi x_0)^2+2E}-\xi x_0}{\sqrt{2E}}.$$

Thus,

$$\begin{aligned} x(t) &= \frac{\sqrt{2E}}{\xi}\sinh(z/2) = \frac{\sqrt{2E}}{\xi}\frac{1}{2}\left(\frac{1+\sqrt{\alpha}}{\sqrt{1-\alpha}}e^{\xi t} - \frac{\sqrt{1-\alpha}}{1+\sqrt{\alpha}}e^{-\xi t}\right) \\ &= \frac{1}{2\xi}\left(\sqrt{(\xi x_0)^2+2E}(e^{\xi t}-e^{-\xi t}) + \xi x_0(e^{\xi t}+e^{-\xi t})\right). \end{aligned}$$

Using the identities: $e^{\xi t} - e^{-\xi t} = 2\sinh(\xi t)$ and $e^{\xi t} + e^{-\xi t} = 2\cosh(\xi t)$, one has

$$\begin{aligned} &\frac{1}{\xi}\left(\sqrt{(\xi x_0)^2+2E}\sinh(\xi t) + \xi x_0\cosh(\xi t)\right) \\ &= \frac{\sqrt{(\xi x_0)^2+2E}}{\xi}\sinh(\xi t) + x_0\cosh(\xi t). \end{aligned} \tag{3.36}$$

This agrees exactly with the result in [2, Section 10.4]. To get this into the form of [4, equation (3.16.54)], we simply use the fact that

$$\frac{\sqrt{(\xi x_0)^2+2E}}{\xi}=\frac{[x_1-x_0\cosh(\xi t)]}{\sinh(\xi t)},$$

which follows from the initial condition $x(t)=x_1$. Hence, we can plug this into equation (3.36) to see that

$$x(s)=\frac{[x_1-x_0\cosh(\xi t)]}{\sinh(\xi t)}\sinh(\xi s)+x_0\cosh(\xi s).$$

This is precisely the form found in equation (3.16.54), with $\xi=a$. We can thus summarize this section by the following theorem.

Theorem 3.6. *Reduction of Beta Function Form for Geodesic of Hermite Operator to Trigonometric Form. The expression for the x-component of the geodesics of the Hermite operator* $\mathbb{L}=\frac{1}{2}(\partial_x^2-x^2\xi^2)$ *in Beta function form (3.19) can be reduced to the result presented in* [2, Section 10.4] *and to* [4, equation (3.16.54)]*:*

$$\begin{aligned} x(t)&=\frac{\sqrt{2E}}{\xi}\left(\frac{B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}{1-B^{-1}_{2\xi t+B_{\frac{\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}{1+\left(\frac{\xi x_0}{\sqrt{2E}}\right)^2}}(1/2,0)}(1/2,0)}\right)^{1/2}\\ &\longrightarrow \frac{\sqrt{(\xi x_0)^2+2E}}{\xi}\sinh(\xi t)+x_0\cosh(\xi t)\\ &\longrightarrow x(s)=\frac{[x_1-x_0\cosh(\xi t)]}{\sinh(\xi t)}\sinh(\xi s)+x_0\cosh(\xi s).\end{aligned}$$

If one assumes $x_0=0$, both expressions simplify significantly, then the action in Beta function form $S_{cl}=\frac{E}{2\xi}\left[B_{B^{-1}_{2\xi t}(1/2,0)}(1/2,1)+B_t(3/2,-1)\right]$, which could provide a solid starting point.

4. ALTERNATIVE METHODS TO THE BETA FUNCTION FORMALISM

Recall that the foundation of the Beta function formalism was to express the energy or t-integral $\int\frac{dx}{\sqrt{2E+\xi^2x^2}}$ in terms of incomplete Beta functions. However, this integral can also be represented in various ways, including through elliptic functions and hypergeometric functions.

4.1. **Elliptic Function Method.**

One possible alternative formalism we can consider is to express the t-integral in terms of elliptic functions, e.g. the incomplete elliptic integral of the first kind $F(\phi|m)$:

$$F(\phi|m):=\int_0^{\phi}\frac{d\theta}{\sqrt{1-m\sin^2\theta}}.$$

Namely, we can use the identity, for $a>0$ and $m=-b/a$,

$$\int\frac{dx}{\sqrt{a+bx^2}}=\frac{1}{\sqrt{a}}F(\sinh^{-1}(\sqrt{b/a}x)|-b/a)+c,$$

where $a = 2E > 0$ and $b = \xi^2 > 0$. Hence, we have $m = -\xi^2/2E$ and

$$t = \int_{x_0}^{x_1} \frac{dx}{\sqrt{2E + \xi^2 x^2}} = \frac{1}{\sqrt{2E}} F(\sinh^{-1}(\xi x/\sqrt{2E}) \,|\, -\xi^2/2E)\Big|_{x_0}^{x_1}.$$

An intuitive concern regarding the downsides of this formalism is that inverting the function $F(\sinh^{-1}(\cdot)|\cdot)$ may lead to complications when integrating and differentiating. For a reference on elliptic functions, we refer to Beals and Wong [1] and Lawden [11].

4.2. **Hypergeometric Function Method.** An alternative formalism that could be considered is to express $\int \frac{dx}{\sqrt{2E+\xi^2 x^2}}$. Another possible formalism is to express it in terms of hypergeometric functions. We already encountered Gauss's hypergeometric function in [7, Section 8.3.3]. For a more detailed discussion, we refer to [11]. We aim to address the limitations of this formalism in the more general case of arbitrary k in the future. Additionally, it is worth noting that there are other *extended* hypergeometric functions that we could also use to express the t-integral.

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