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APPROXIMATE SOLUTIONS TO VECTOR VARIATIONAL INEQUALITIES BASED ON IMPROVEMENT SETS: EXISTENCE, CHARACTERIZATION, AND OPTIMALITY CONDITIONS

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Abstract. In this paper, we use the improvement set as a key tool to formulate new approximate vector variational inequality models and introduce novel concepts of approximate solutions to vector variational inequalities in Banach spaces. By using the KKM-Fan theorem and Brouwer's fixed-point theorem, two existence results of such approximate solutions are established, respectively. In the convex setting, linear scalarization characterizations of the proposed approximate solutions are derived through the classical convex separation theorem, and optimality conditions for these approximate solutions are obtained via Ekeland's variational principle and the Fermat rule. In the nonconvex setting, we refine a key property of the Clarke subdifferential of a specialized nonlinear scalarization function, namely the oriented distance function. By utilizing this improved property together with Ekeland's variational principle, we establish optimality conditions for the proposed approximate solutions through the nonlinear scalarization approach.

Keywords. Approximate solutions; Existence; Improvement set; Optimality conditions; Vector variational inequalities.

1. Introduction

As a fundamental branch of variational inequality theory, Vector Variational Inequalities (VVIs, for short) provide a powerful framework for dealing with vector optimization problems. Their importance arises from their significant applications, such as analyzing the existence of solutions, deriving optimality conditions, and investigating the stability and sensitivity of solution sets in vector optimization. The theory of vector variational inequalities originated from the pioneering work of Giannessi [1] in 1980, which generalized classical scalar variational inequalities to the vector setting. Since then, various types of VVIs have been extensively investigated by many researchers in both finite and infinite-dimensional spaces. For further details, we refer to [2, 3, 4, 5] and the references therein.

Over the past four decades, many significant results on VVIs have been achieved, particularly in the existence of solutions and optimality conditions. Chen and Cheng [6] established

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the first result on the existence of weakly efficient solutions to VVIs in infinite-dimensional spaces without the monotonicity assumption. By using the linear scalarization technique and a fixed-point theorem, they derived the result under conditions of convexity and compactness. Subsequently, under the assumption of monotonicity, Chen and Yang [7] presented an existence theorem for weakly efficient solutions to VVIs in infinite-dimensional spaces, by employing the extended linearization lemma and the KKM-Fan theorem. Chen [8] further investigated the existence of weakly efficient solutions for VVIs with a variable ordering structure. Without monotonicity assumptions, Fang and Huang [9] utilized Brouwer's and Browder's fixed point theorems to establish two existence theorems for efficient solutions to VVIs in Banach spaces, respectively. They also demonstrated the solvability of efficient solutions to VVIs under generalized monotonicity conditions by using the KKM-Fan theorem. Han and Huang [10] presented three existence theorems for strongly efficient solutions, weakly efficient solutions, and efficient solutions to generalized vector quasiequilibrium problems, by utilizing the KKM-Fan theorem and the linear scalarization approach. Yao and Zheng [11] applied Fan-Glicksberg-Kakutani theorem to establish existence results for (weakly) efficient solutions to VVIs without requiring monotonicity assumption.

As is well known, in many real-world problems, such as large-scale optimization, nonlinear programming, and multi-objective optimization, complexity and high dimensionality make exact solutions impractical. Additionally, limited time and computing resources further constrain algorithms, leading to approximate solutions. Therefore, studying approximate solutions is crucial for advancing optimization theory and balancing efficiency with solution quality, ensuring practical applicability. Chen and Craven [12] first introduced the concept of approximate solutions to VVI, with the aim of describing approximate Pareto solutions for vector optimization problems. Yang and Zheng [13] presented a new notion of ε-solutions for VVIs and established linear scalarization optimality conditions under cone convexlikeness, along with sufficient conditions under feasible set convexity. Furthermore, they derived optimality conditions for exact efficient solutions to weak vector variational inequality using the Clarke normal cone. Sun and Li [14] introduced a generalized multivalued ε-vector variational inequality, and established the existence theorem for it by using the KKM-Fan theorem. Gao et al. [15] further generalized the concept of approximate solutions to VVIs, which were introduced in [13], to the case of the coradiant sets. They established linear and nonlinear scalarization results for the new approximate solutions by using convex and nonconvex separation theorems, and then established optimality conditions for both convex and nonconvex situations.

It is worth noting that Gutiérrez et al. [16] introduced a more general framework for approximate solutions to vector optimization problems by the tool of the improvement set. Building on this idea, it is natural to extend the notion of approximate solutions to VVIs through the same approach. In this paper, we introduce new approximate solutions to VVIs in Banach spaces via the improvement set, and establish their existence. Furthermore, we explore the corresponding optimality conditions by using the techniques from both linear and nonlinear scalarization.

The structure of this paper is as follows. Section 2 presents the fundamental definitions and notations essential for this study, and introduces new concepts of approximate solutions to vector variational inequalities within the framework of the improvement set. In Section 3, we establish the existence results for the proposed approximate solutions to vector variational inequalities. Section 4 derives sufficient and necessary conditions for the proposed approximate solutions using convex and nonsmooth analysis, based on linear scalarization. Furthermore, in the nonconvex setting, we present an improved property of the Clarke subdifferential of the nonlinear scalarization function $\Delta(\cdot)$, as introduced by the Hiriart-Urruty [17]. This property, together with Ekeland's variational principle, is then used to establish optimality conditions for the proposed approximate solutions. Finally, Section 5 summarizes our main results.

2. Preliminaries

Let X and Y denote real Banach spaces with their topological duals X^* and Y^*, respectively. $\mathbb{B}_X$ and $\mathbb{B}_{X^*}$ denote the unite ball in X and X^*, respectively. Let $\mathbb{B}_X(x,r)$ be the closed ball of X centered at x with radius $r \geq 0$. $\mathbb{B}$ and $\mathbb{B}^*$ denote the closed unit ball in Y and Y^*, respectively. Additionally, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^n_+$ signifies the nonnegative orthant of $\mathbb{R}^n$.

Consider A as a nonempty subset of X. The notations $\text{int}(A), \text{cl}(A), \text{bd}(A), X\backslash A$, and $\text{diam}(A)$ refer to the interior, closure, boundary, complement of A, and the diameter of A, respectively. The cone generated by a set A is defined as $cone(A) := \bigcup_{\alpha \geq 0} \alpha A$. The set A is said to be a cone if $A = cone(A)$. It is called proper if $\emptyset \neq A \neq X$, solid if $\text{int}(A) \neq \emptyset$, and pointed if $A \cap (-A) \subset \{0\}$. For a given nonempty set $D \subset Y$, the positive dual of D is defined as

$$D^+ := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in D\},$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical bilinear form of Y^*. Suppose that A is a closed subset of X with $\alpha \in A$. Let $T(A,a)$ denote the Clarke tangent cone of A at a, defined by

$$T(A,a) = \liminf_{\substack{x \xrightarrow{A} a \\ t \to 0^+}} \frac{A-x}{t},$$

where the notation $x \xrightarrow{A} a$ signifies that $x \to a$ with $x \in A$. Therefore, $v \in T(A,a)$ if and only if, for every sequence $\{a_n\}$ in A converging to a and every sequence $\{t_n\}$ in $(0,\infty)$ decreasing to 0, there exists a sequence $\{v_n\}$ in X converging to v such that $a_n + t_n v_n \in A$ for all $n \in \mathbb{N}$. Let $N(A,a)$ denote the Clarke normal cone, that is,

$$N(A,a) = \{x^* \in X^* : \langle x^*, h\rangle \leq 0, \forall h \in T(A,a)\}.$$

If A is convex, then

$$N(A,a) = \{x^* \in X^* : \langle x^*, x-a\rangle \leq 0, \forall x \in A\}.$$

If $g : X \to \mathbb{R} \cup \{\pm\infty\}$ is a locally Lipschitz function on X, for $x \in \text{dom}(g)$, the Clarke subdifferential of g at x is defined as

$$\partial g(x) = \{\xi \in X^* : (\xi, -1) \in N(\text{epi}(g), (x, g(x)))\},$$

where $\text{epi}(g) := \{(x,\alpha) \in X \times \mathbb{R} : g(x) \leq \alpha\}$. Especially, if g is a convex function, then the subdifferential of g at $x \in \text{dom}(g)$ reduces to

$$\partial g(x) = \{\xi \in X^* : g(y) - g(x) \geq \langle \xi, y-x\rangle, \forall y \in \text{dom}(g)\}.$$

A proper closed convex cone $K \subset Y$ with nonempty interior induces the following ordering relationships in Y:

$$\begin{aligned} y_1 \leq_K y_2 &\Longleftrightarrow y_2 - y_1 \in K, \\ y_1 \leq_{K\setminus\{0\}} y_2 &\Longleftrightarrow y_2 - y_1 \in K\setminus\{0\}, \\ y_1 \leq_{\mathrm{int}(K)} y_2 &\Longleftrightarrow y_2 - y_1 \in \mathrm{int}(K), \\ y_1 \nleq_K y_2 &\Longleftrightarrow y_2 - y_1 \notin K, \\ y_1 \nleq_{K\setminus\{0\}} y_2 &\Longleftrightarrow y_2 - y_1 \notin K\setminus\{0\}, \\ y_1 \nleq_{\mathrm{int}(K)} y_2 &\Longleftrightarrow y_2 - y_1 \notin \mathrm{int}(K). \end{aligned}$$

In the subsequent part of this section, we consider A as a closed subset of X, $K \subset Y$ as a closed, convex, proper cone with nonempty interior, and $F : X \to L(X,Y)$ as a mapping, where $L(X,Y)$ represents the set of all continuous linear operators from X to Y.

The classical formulations of vector variational inequality problems are given as follows:

$$\text{find } x \in A \text{ such that } \langle F(x), z - x\rangle \nleq_{\mathrm{int}(K)} 0\ ,\ \forall z \in A, \tag{WVVI}$$

and

$$\text{find } x \in A \text{ such that } \langle F(x), z - x\rangle \nleq_{K\setminus\{0\}} 0\ ,\ \forall z \in A. \tag{VVI}$$

It is evident that $\bar{x} \in A$ is a solution of (WVVI) (resp. (VVI)) if and only if

$$\langle F(\bar{x}), A - \bar{x}\rangle \subset Y\setminus(-\mathrm{int}(K))$$
$$(\text{ resp. } \langle F(\bar{x}), A - \bar{x}\rangle \subset Y\setminus(-K\setminus\{0\})\ ).$$

In recent decades, numerous researchers have explored various notions of approximate solutions for vector variational inequality problems. By allowing a certain margin of error, Yang and Zheng [13] introduced the concept of approximate solutions for vector variational inequality problems via using proper closed convex cones.

Definition 2.1. [13] Let $\varepsilon \geq 0$. The vector $\bar{x} \in A$ is an ε-solution of ((WVVI)) (resp. ((VVI))) if

$$\langle F(\bar{x}), A - \bar{x}\rangle \subset Y\setminus(-\mathrm{int}(K)) + \varepsilon\mathbb{B}_Y$$
$$(\text{resp. } \langle F(\bar{x}), A - \bar{x}\rangle \subset Y\setminus(-K\setminus\{0\}) + \varepsilon\mathbb{B}_Y),$$

where $\mathbb{B}_Y$ denotes the closed unit ball of Y.

Recently, by using the concept of co-radiant sets, Gao et al. [15] introduced a new definition of approximate solutions for vector variational inequality problems. For completeness, we first recall the definition of a co-radiant set.

Definition 2.2. [15] The set $C \subset Y$ is said to be a co-radiant set if $\alpha d \in C$ for all $d \in C$ and $\alpha > 1$. Let $C(\varepsilon) := \varepsilon C$, $\forall \varepsilon > 0$, and $C(0) := \bigcup_{\varepsilon>0} C(\varepsilon)$.

Definition 2.3. [15] Let $\varepsilon \geq 0$ and $C \subset K$ be a co-radiant set with nonempty interior. The vector $\bar{x} \in A$ is an ε-solution of (WVVI) (resp. (VVI)) with respect to C if

$$\langle F(\bar{x}), A - \bar{x}\rangle \subset Y\setminus(-\mathrm{int}(C(\varepsilon)))$$
$$(\text{ resp. } \langle F(\bar{x}), A - \bar{x}\rangle \subset Y\setminus(-(C(\varepsilon)\setminus\{0\}))\ ).$$

Motivated by the definitions of approximate solutions to vector optimization problems in [16], we further employ the concept of improvement sets to introduce new models of vector variational inequality problems and propose novel concepts of solutions for these problems. For clarity, we recall the definition of an improvement set.

Definition 2.4. [16] Let $E \subset Y$ be a nonempty set and K be a convex cone in Y. If $E+K=E$, then E is said to be free disposal with respect to K. If, in addition, $0 \notin E$, then E is called an improvement set with respect to K.

Definition 2.5. Let E be an improvement set with respect to a convex cone $K \subseteq Y$ with $\mathrm{int}E \neq \emptyset$. Consider the general format of vector variational inequality (VVI) problems:

$$\text{find } \bar{x} \in A \text{ such that } \langle F(\bar{x}), x-\bar{x}\rangle \not\leq_{\mathrm{int}(E)} 0, \forall x \in A, \tag{EWVVI}$$

and

$$\text{find } \bar{x} \in A \text{ such that } \langle F(\bar{x}), x-\bar{x}\rangle \not\leq_{E} 0, \forall x \in A. \tag{EVVI}$$

We say that

(i) $\bar{x} \in A$ is a solution of (EWVVI) if

$$\langle F(\bar{x}), A-\bar{x}\rangle \subset Y \backslash (-\mathrm{int}E);$$

(ii) $\bar{x} \in A$ is a solution of (EVVI) if

$$\langle F(\bar{x}), A-\bar{x}\rangle \subset Y \backslash (-E).$$

For the sake of simplicity, we denote the solution sets of (EWVVI) and (EVVI) as Sol(EWVVI) and Sol(EVVI), respectively.

Now, we discuss the connections between our proposed approximate solutions in Definition 2.5 and the existing solutions for vector variational inequalities, which are formulated by using the tools of co-radiant sets and convex cones.

Remark 2.1. (i) Definition 2.5 covers the classical notions of vector variational inequalities (VVIs). If K is solid, then by setting $E=\mathrm{int}K$, the models (EWVVI) and (EVVI) reduce to the weak VVI in [6] and in VVI [7], respectively. In addition, when $X=\mathbb{R}^n$, $Y=\mathbb{R}^p$ and $E=\mathbb{R}^p_+\backslash\{0\}$, the formulations (EWVVI) and (EVVI) correspond to the classical VVI problems proposed by Giannessi [1] in finite dimension Euclidean space.

(ii) The main existing concepts of approximate solutions for VVI problems can also be seen as special cases of Definition 2.5 by selecting appropriate improvement sets. Some examples illustrate this point.

(a) Conisder $E=e+K$, where $e \in \mathrm{int}K$. In this case, the solution of (EWVVI) becomes e-solution of (WVVI) presented by Chen and Craven [12].

(b) For $\varepsilon \geq 0$, let $M := Y\backslash(-\mathrm{int}K)+\varepsilon\mathbb{B}_Y$. We can conclude that $-(Y\backslash M)$ is improvement with respect to the convex cone K, that is $Y\backslash M-K=Y\backslash M$ and $0 \notin -Y\backslash M$. Indeed, it is easy to show that $0 \notin -Y\backslash M$. Now, let $d \in Y\backslash M$ and $k \in K$. It can be verified that $d-k \in Y\backslash M$. Otherwise, if $d-k \in M$, we have $d=d-k+k \in M+K \subset Y\backslash(-\mathrm{int}K)+\varepsilon\mathbb{B}_Y+K=Y\backslash(-\mathrm{int}K)+\varepsilon\mathbb{B}_Y=M$, which leads to a contradiction. Hence, $Y\backslash M-K=Y\backslash M$. In the case that $\mathrm{int}E=-(Y\backslash M)$, we have

$$Y\backslash(-\mathrm{int}E)=M=Y\backslash(-\mathrm{int}K)+\varepsilon\mathbb{B}_Y,$$

and the solution of (EWVVI) reduces to ε-solution of (WVVI) introduced by Yang and Zheng [13], that is

$$\text{find } \bar{x} \in A \text{ such that } \langle F(\bar{x}), A-\bar{x}\rangle \subset Y\backslash(-\text{int}K)+\varepsilon\mathbb{B}_Y. \qquad \text{(WVVI)}$$

Similarly, in the case where $E=-(Y\backslash M')$, where $M'=Y\backslash(-K\backslash\{0\})+\varepsilon\mathbb{B}_Y$, the solution of (EVVI) reduces to ε-solution of (VVI) given by Yang and Zheng [13], that is

$$\text{find } \bar{x} \in A \text{ such that } \langle F(\bar{x}), A-\bar{x}\rangle \subset Y\backslash(-K\backslash\{0\})+\varepsilon\mathbb{B}_Y. \qquad \text{(VVI)}$$

(c) When E is chosen as a solid convex co-radiant set C in Y, the solutions of (EWVVI) and (EVVI) in Definition 2.5 become the approximate solutions of (WVVI) and (VVI) in Definition 2.3, presented by Gao et al. [15]. That is, the vector $\bar{x}\in A$ is an ε-solution of (WVVI) (resp. (VVI) with respect to C, if

$$\langle F(\bar{x}), A-\bar{x}\rangle \subset Y\backslash(-\text{int}(C(\varepsilon)))$$
$$(\text{ resp. } \langle F(\bar{x}), A-\bar{x}\rangle \subset Y\backslash(-(C(\varepsilon)\backslash\{0\}))\ ),$$

where $C(\varepsilon):=\varepsilon C$ and $\varepsilon\geq 0$.

3. Existence Theorems

In this section, by using the KKM-Fan theorem and Brouwer's fixed point theorem, we establish the existence results of (EWVVI) and (EVVI), respectively. First we recall some concepts and lemmas. Let A be a nonempty subset of a topological vector space X. A multi-valued mapping $T:A\to 2^X$ is called a Knaster-Kuratowski-Mazurkiewieg mapping (KKM mapping) if for each finite subset $\{x_1,\ldots,x_n\}$ of A, we have $\text{conv}\{x_1,\ldots,x_n\}\subset\bigcup_{i=1}^n T(x_i)$, where $\text{conv}\{x_1,\ldots,x_n\}$ the convex hull of $\{x_1,\ldots,x_n\}$.

Lemma 3.1. *(**KKM-Fan Theorem**)*[18] *Let A be a nonempty subset of a topological vector space X, and let $T:A\to 2^X$ be a set-valued mapping that is a* KKM *mapping. Suppose that $T(x)$ is closed for each $x\in A$ and compact for at least one $x\in A$. Then $\bigcap_{x\in A}T(x)\neq\emptyset$.*

Lemma 3.2. *(**Brouwer's Fixed Point Theorem**)* [19] *Let D be a nonempty, compact and convex subset of a finite-dimensional space and $g:D\to D$ be a continuous mapping. Then there exists $x\in D$ such that $g(x)=x$.*

The following theorem provides the existence result for a solution of(EWVVI).

Theorem 3.1. *Let $A\subseteq X$ be a nonempty closed convex set, and let $E\subset Y$ be a nonempty convex improvement set with respect to a pointed convex cone $K\subset Y$ with* $\text{int}(K)\neq\emptyset$*. Assume that $F:A\to L(X,Y)$ is a continuous mapping, the set $\{x\in A: F(x)(q-x)\in Y\backslash(-\text{int}(E))\}$ is compact for some $q\in A$. Then, the vector variational inequality problem* (EWVVI) *is solvable.*

Proof. Since E is improvement with respect to the convex cone K with $\text{int}(K)\neq\emptyset$, it follows that $\text{int}(E)=E+\text{int}(K)\neq\emptyset$. For any $y\in A$, define a mapping $T:A\to 2^A$ by

$$T(y)=\{x\in A:\langle F(x),y-x\rangle\notin -\text{int}(E)\}.$$

We claim that T is a KKM mapping on A. Take $\{x_1,\dots,x_n\} \subset A$, $\{\alpha_1,\dots,\alpha_n\} \subset [0,1]$ with $\sum_{i=1}^n \alpha_i = 1$. Let $x = \sum_{i=1}^n \alpha_i x_i$. If $x \notin \bigcup_{i=1}^n T(x_i)$, then $\langle F(x), x_i - x\rangle \in -\mathrm{int}(E)$ for all $i = 1,\dots,n$. This, together with the convexity of E, implies that

$$\langle F(x), x\rangle = \sum_{i=1}^n \alpha_i \langle F(x), x_i\rangle \in \left(\sum_{i=1}^n \alpha_i\right)[\langle F(x), x\rangle - \mathrm{int}(E)] = \langle F(x), x\rangle - \mathrm{int}(E),$$

which leads to a contradiction due to the fact that E is an improvement set. Hence,

$$\mathrm{conv}\{x_1,\dots,x_n\} \subset \bigcup_{i=1}^n T_1(x_i).$$

Let $y \in A$, and a sequence $\{x_k\} \subset T(y)$ satisfy $\|x_k - x\| \to 0$. Since $F(\cdot)$ is continuous on A, $F(x_k)$ uniformly converge to $F(x)$, which yields that

$$\begin{aligned}
&\|\langle F(x_k), y - x_k\rangle - \langle F(x), y - x\rangle\| \\
&\le \|\langle F(x_k), y - x_k\rangle - \langle F(x), y - x_k\rangle\| + \|\langle F(x), y - x_k\rangle - \langle F(x), y - x\rangle\| \\
&\le \|F(x_k) - F(x)\| \cdot \|y - x_k\| + \|F(x)\| \cdot \|x - x_k\| \to 0 \quad (k \to \infty).
\end{aligned}$$

Since $\{x_k\} \subset T(y)$, it follows that $\langle F(x_k), y - x_k\rangle \in Y \backslash (-\mathrm{int}(E))$. Notice that the set $Y \backslash (-\mathrm{int}(E))$ is closed. We conclude that $\langle F(x), y - x\rangle \in Y \backslash (-\mathrm{int}(E))$, i.e., $x \in T(y)$. This means that the set $T(y)$ is closed. By Lemma 3.1 (KKM-Fan Theorem), there exists some point $\bar{x} \in \bigcap_{y \in A} T(y)$ such that

$$\langle F(\bar{x}), y - \bar{x}\rangle \in Y \backslash (-\mathrm{int}(E)), \quad \forall y \in A.$$

This completes the proof. □

Motivated by the work of Fang and Huang [9], we derive the existence results of the solution to (EVVI) by employing Lemma 3.2, which corresponds to Brouwer's Fixed Point Theorem [19].

Theorem 3.2. *Let $A \subset X$ be a nonempty compact convex subset, and let $E \subset Y$ be a nonempty convex improvement set with respect to a pointed convex cone $K \subset Y$ with $\mathrm{int}K \neq \emptyset$. Suppose that $F : A \to L(X,Y)$ is a nonlinear mapping such that, for every $y \in A$, $\{x \in A : \langle F(x), y - x\rangle \le_E 0\}$ is open in A. Then, vector variational inequality problem* (EVVI) *is solvable.*

Proof. Suppose to the contrary that problem (EVVI) is unsolvable. Then, for each $x \in A$, there exists some $y_0 \in A$ such that

$$\langle F(x), y_0 - x\rangle \le_E 0. \tag{3.1}$$

For every $y \in A$, define the set N_y as follows:

$$N_y := \{x \in A : \langle F(x), y - x\rangle \le_E 0\}. \tag{3.2}$$

Clearly, each N_y is open in A and $\bigcup_{y \in A} N_y \subset A$. From (3.1), it is easy to see that for each $x \in A$, there exists $y_0 \in A$ such that $x \in N_{y_0}$, which implies that $A \subset \bigcup_{y \in A} N_y$. That is to say that $\bigcup_{y \in A} N_y$ is an open cover of A. Since A is compact, there exists a finite set $\{y_1, y_2, \dots, y_n\} \subset A$ such that $\bigcup_{i=1}^n N_{y_i} = A$. For $j \in \{1,2,\dots,n\}$, define the function $\beta_j' : A \to \mathbb{R}$ as follows:

$$\beta_j'(x) = d_{Y \backslash N_{y_j}}(x) \cdot \mathbf{1}_{N_{y_j}}(x),$$

where the indicator function $\mathbf{1}_{N_{y_j}}(x)$ is defined by

$$\mathbf{1}_{N_{y_j}}(x) = \begin{cases} 1, & \text{if } x \in N_{y_j}, \\ 0, & \text{if } x \notin N_{y_j}. \end{cases}$$

Denote

$$\beta_j(x) := \frac{\beta_j'(x)}{\sum\limits_{j=1}^{n} \beta_j'(x)}, \ \forall x \in A, \forall j \in \{1,2,\ldots,n\}.$$

Then, each β_j is continuous, $\beta_j(x) \geq 0$ and $\sum_{j=1}^{n} \beta_j(x) = 1$ for all $x \in A$. Let $p : A \to X$ be defined by

$$p(x) := \sum_{j=1}^{n} \beta_j(x) y_j, \quad \forall x \in A.$$

We can see that p is continuous. Denote $S := \operatorname{conv}\{y_1, y_2, \ldots, y_n\} \subset A$. Then S is a compact, convex subset of a finite-dimensional space and p maps S into S. By Lemma 3.2 (i.e., Brouwer's Fixed Point Theorem), there exists some $x_0 \in S$ such that $p(x_0) = x_0$. Now, for any given $x \in A$, let

$$k(x) = \{j : x \in N_{y_j}\} = \{j : \beta_j(x) > 0\}. \tag{3.3}$$

Obviously, $k(x) \neq \emptyset$ and $\sum_{j\in k(x)} \beta_j(x) = 1$. Since $x_0 \in S \subseteq A$ is a fixed point of p, one has $x_0 = p(x_0) = \sum_{j=1}^{n} \beta_j(x_0) y_j$. By (3.2) and (3.3), along with the convexity of E, we deduce that

$$\begin{aligned} 0 &= \langle F(x_0), x_0 - x_0 \rangle \\ &= \left\langle F(x_0), \sum_{j=1}^{n} \beta_j(x_0) y_j - x_0 \right\rangle \\ &= \left\langle F(x_0), \sum_{j \in k(x_0)} \beta_j(x_0) y_j - x_0 \right\rangle \\ &= \sum_{j \in k(x_0)} \beta_j(x_0) \langle F(x_0), y_j - x_0 \rangle \\ &\leq_E 0, \end{aligned}$$

which contradicts the fact that E is improvement. Therefore, there exists $\bar{x} \in A$ such that

$$\langle F(\bar{x}), A - \bar{x} \rangle \subset Y \setminus (-E).$$

This finishes the proof. □

Remark 3.1. By taking appropriate improvement sets in the above existence results, we can also derive the existence of the main existing approximate solutions of (WVVI) and (VVI). Below, we present some examples.

(i) When the improvement set E is taken as the co-radiant set C, following the above proof process, we can also establish the existence of an ε-solution to (WVVI) and (VVI) with respect to the co-radiant set C, which has not been previously obtained in [15].

(ii) As considered in Remark 2.1 (b), let $E = Y\backslash M$, where $M := Y\backslash(-\text{int}(K)) + \varepsilon\mathbb{B}_Y$ (resp. $M := Y\backslash(-K\backslash\{0\}) + \varepsilon\mathbb{B}_Y$). According to Theorems 3.1 and 3.2, the existence results for the approximate solutions of (WVVI) and (VVI), as proposed by Yang and Zheng [13], can be established.

(iii) Let $e \in \text{int}K$. Consider the improvement set $E := e + \text{int}(K)$ (resp. $E := e + K\backslash\{0\}$). Then, Following Theorems 3.1 and 3.2, we can get the existence of an approximate solution to (WVVI) and (VVI). Here, it should be noted that the existence of an approximate solution to (WVVI) does not require the compactness of the set A, thus relaxing the condition imposed by Chen and Crave in [12].

4. Optimality Conditions for Vector Variational Inequality Problems (EWVVI) and (EVVI)

This section is devoted to establishing optimality conditions for approximate solutions to (EWVVI) and (EVVI) by means of the linear and nonlinear scalarization techniques, respectively. First, vector variational inequality problems (EWVVI) and (EVVI) are transformed into some corresponding scalar approximate variational inequality problems. Then, the results concerning optimality conditions for (EWVVI) and (EVVI) are derived based on Ekeland's variational principle.

In the sequel, we assume that K is a proper convex cone in Y with $\text{int}(K) \neq \emptyset$, and E is a nonempty improvement set of Y with respect to the convex cone K. Let $F : X \to L(X,Y)$ be a mapping, where $L(X,Y)$ represents the set of all continuous linear operators from X to Y. For each $x \in X$, the conjugate operator of $F(x)$, denoted by $F(x)^*$, is the unique bounded linear operator $F(x)^* : Y^* \to X^*$ defined by the condition

$$\langle F(x)^* v^*, u\rangle = \langle v^*, F(x)u\rangle, \quad \forall u \in X,\ \forall v^* \in Y^*,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between a Banach space and its dual.

Remark 4.1. As discussed in [20], when E is a nonempty improvement set with respect to the convex cone K such that $\text{int}(K) \neq \emptyset$, we have that $\text{int}(E) = E + \text{int}(K) = \text{int}(E) + \text{int}(K) \neq \emptyset$. This result is used in the subsequent discussion.

Besides, in the following, we need the concept of conjugate operator of $F(\cdot)$.

Definition 4.1. The conjugate operator $F(\bar{x})^*$ of a bounded linear operator $F(\bar{x})$ between Banach spaces is defined as the unique bounded linear operator that acts between the dual spaces and satisfies $\langle F(\bar{x})u, v^*\rangle = \langle u, F(\bar{x})^* v^*\rangle$ for all $u \in X$ and for all $v^* \in Y^*$, where $F(\bar{x}) : X \to Y$ is a bounded linear operator, $F(\bar{x})^* : Y^* \to X^*$ is the conjugate (or adjoint) operator of $F(\bar{x})$, X and Y are Banach spaces, X^* and Y^* are the dual spaces of X and Y, respectively, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between a space and its dual.

4.1. Optimality Conditions via Linear Scalarization.

The following linear scalarization results of (EWVVI) and (EVVI) are obtained through convex separation theorem.

Theorem 4.1. *Let A be a nonempty, closed convex subset of X with $\bar{x} \in A$, and let E be a closed convex subset of Y. If $\bar{x} \in \text{Sol(EWVVI)}$ (or, $\bar{x} \in \text{Sol(EVVI)}$), then there exists a continuous linear functional $c^* \in E^+\backslash\{0\}$ such that*

$$\inf\{\langle F(\bar{x})^*(c^*), x\rangle : x \in A\} \geq \langle F(\bar{x})^*(c^*), \bar{x}\rangle + \sigma_{-\text{int}(E)}(c^*), \tag{4.1}$$

where $F(\bar{x})^*$ denotes the conjugate operator of $F(\bar{x})$.

Proof. By definition, Sol(EVVI) $\subset$ Sol(EWVVI), we only need to prove that if $\bar{x} \in$ Sol(EWVVI), then there exists a continuous linear functional $c^* \in E^+ \setminus \{0\}$ such that (4.1) holds. Let $\bar{x} \in$ Sol(EWVVI). Then, $\langle F(\bar{x}), A - \bar{x}\rangle \cap (-\text{int}(E)) = \emptyset$. By the classical convex separation theorem, there exists $c^* \in Y^* \setminus \{0\}$ such that

$$\langle F(\bar{x})^*(c^*), (x-\bar{x})\rangle = \langle c^*, F(\bar{x})(x-\bar{x})\rangle \geq \langle c^*, -e\rangle, \forall x \in A, \forall e \in E.$$

This shows that $c^* \in E^+ \setminus \{0\}$, and the inequality (4.1) holds. □

Remark 4.2. The linear scalarization characterization in (4.1) for (EWVVI) and (EVVI) can also hold for some $c^* \in K^+ \setminus \{0\}$. Furthermore, under the assumption of generalized convexity, such as cone subconvexlikeness, the characterization in (4.1) for (EWVVI) and (EVVI) could also be established.

Next, we show that the linear scalarization characterization in (4.1) is also sufficient for the solutions of (EWVVI) and (EVVI).

Theorem 4.2. *Let $\bar{x} \in A$. Then the following statements hold.*

(i) *If there exists $c^* \in E^+ \setminus \{0\}$ such that*

$$\inf\{\langle F(\bar{x})^*(c^*), x\rangle : x \in A\} \geq \langle F(\bar{x})^*(c^*), \bar{x}\rangle + \sigma_{-\text{int}(E)}(c^*), \tag{4.2}$$

where $F(\bar{x})^*$ denotes the conjugate operator of $F(\bar{x})$, then $\bar{x} \in$ Sol(EWVVI).

(ii) *Let $0 < \varepsilon < 1$. If there exists $c^* \in \text{int}(E^+)$ such that*

$$\inf\{\langle F(\bar{x})^*(c^*), x\rangle : x \in A\} \geq \langle F(\bar{x})^*(c^*), \bar{x}\rangle + \varepsilon\sigma_{-E\setminus\{0\}}(c^*), \tag{4.3}$$

then $\bar{x} \in$ Sol(EVVI).

Proof. (i) By (4.2), it is easy to verify that, for each $x \in A$ and $e \in \text{int}(E)$,

$$\langle c^*, F(\bar{x})(x-\bar{x}) + e\rangle \geq 0. \tag{4.4}$$

Since E is improvement with respect to the convex cone K, we obtain by [16, Proposition 2.6 (a)] that $E^+ \subset K^+$, which together with $c^* \in E^+ \setminus \{0\}$ yields $c^* \in K^+ \setminus \{0\}$. From (4.4), it turns out that

$$(F(\bar{x})(A-\bar{x}) + \text{int}(E)) \cap (-\text{int}(K)) = \emptyset. \tag{4.5}$$

By Remark 4.1, $\text{int}(E) + \text{int}(K) = \text{int}(E)$, it follows from (4.5) that $(F(\bar{x})(A-\bar{x})) \cap (-\text{int}(E)) = \emptyset$. Consequently, $\bar{x} \in$ Sol(EWVVI).

(ii) Suppose to the contrary that $\bar{x} \notin$ Sol(EVVI). Then there exists $\hat{x} \in A$ such that

$$F(\bar{x})(\hat{x}-\bar{x}) \in -E \setminus \{0\}.$$

This implies that

$$\langle c^*, F(\bar{x})(\hat{x}-\bar{x})\rangle \leq \sigma_{-E\setminus\{0\}}(c^*), \tag{4.6}$$

and

$$\langle c^*, F(\bar{x})(\hat{x}-\bar{x})\rangle < 0, \tag{4.7}$$

as $c^* \in \text{int}(E^+)$. Combining (4.3) with (4.7), we can conclude that $\sigma_{-E\setminus\{0\}}(c^*) < 0$. This together with (4.6) and $0 < \varepsilon < 1$ yields that $\langle c^*, F(\bar{x})(\hat{x}-\bar{x})\rangle < \varepsilon\sigma_{-E\setminus\{0\}}(c^*)$, which is a contradiction to (4.3). Consequently, $\bar{x} \in$ Sol(EVVI). □

Theorem 4.3. *Let A be a nonempty, closed convex subset of X with $\bar{x} \in A$, and let E be a closed convex subset of Y. If $\bar{x} \in \mathrm{Sol}$(EWVVI) (or, $\bar{x} \in \mathrm{Sol}$(EVVI)), then there exists a continuous linear functional $c^* \in E^+ \setminus \{0\}$ such that for any $\lambda > 0$, there exists $\hat{x} \in A \bigcap \mathbb{B}(\bar{x};\lambda)$ satisfying*

$$0 \in F(\bar{x})^* c^* + \frac{\sigma_{-\mathrm{int}(E)}(c^*)}{\lambda} \mathbb{B}_X^* + N(A;\hat{x}), \tag{4.8}$$

where $F(\bar{x})^*$ is the conjugate operator of $F(\bar{x})$.

Proof. Since $\bar{x} \in \mathrm{Sol}$(EWVVI), according to Theorem 4.1, there exists a continuous linear functional $c^* \in E^+ \setminus \{0\}$ such that

$$\langle F(\bar{x})^*(c^*), x\rangle + \left(-\sigma_{-\mathrm{int}(E)}(c^*)\right) \geq \langle F(\bar{x})^*(c^*), \bar{x}\rangle, \ \forall x \in A.$$

Owing to the fact that $c^* \in E^+ \setminus \{0\}$, we have that

$$-\sigma_{-\mathrm{int}(E)}(c^*) = -\sup_{e \in \mathrm{int}(E)} \langle c^*, -e\rangle \geq 0.$$

Applying Ekeland's variational principle to $\langle F(\bar{x})^*(c^*), \cdot\rangle$ on A, we see that, for any $\lambda > 0$, there exists $\hat{x} \in A \bigcap \mathbb{B}(\bar{x};\lambda)$ such that

$$\langle F(\bar{x})^*(c^*), x\rangle + \frac{-\sigma_{-\mathrm{int}(E)}(c^*)}{\lambda} \|x - \hat{x}\| \geq \langle F(\bar{x})^*(c^*), \hat{x}\rangle, \ \forall x \in A.$$

This means that $\hat{x}$ is an optimal solution of $\langle F(\bar{x})^*(c^*), \cdot\rangle + \dfrac{-\sigma_{-\mathrm{int}(E)}(c^*)}{\lambda} \|\cdot - \hat{x}\|$ on A. Consequently, by the Fermat rule, we have that

$$0 \in F(\bar{x})^*(c^*) + \frac{\sigma_{-\mathrm{int}(E)}(c^*)}{\lambda} \mathbb{B}_{X^*} + N(A;\hat{x}).$$

□

Remark 4.3. It should be pointed out that, according to the inequality (4.1), if $E = K$, then $\sigma_{-\mathrm{int}(E)}(c^*) = 0$, and the solutions of (EWVVI) and (EVVI) with respect to the improvement set E reduce to the solutions of (WVVI) and (VVI) with respect to the convex cone K, respectively. The necessary optimality condition (4.8) then becomes $0 \in F(\bar{x})^* c^* + N(A;\bar{x})$, where $c^* \in K^+ \setminus \{0\}$. This result was given by Yang ad Zheng [13].

Furthermore, if, in addition, A is bounded, then we can obtain the following sufficient optimality conditions for vector variational inequality problems (EWVVI) and (EVVI) .

Theorem 4.4. *Let A be a closed bounded convex subset of X with $\bar{x} \in A$, and let $L \in (\mathrm{diam}(A), +\infty)$. Then the following statements hold.*

(i) *If there exist $c^* \in E^+ \setminus \{0\}$ and $u^* \in \mathbb{B}_{X^*}$ such that*

$$0 \in F(\bar{x})^*(c^*) + N(A, \bar{x}) + \frac{1}{L} \sigma_{-\mathrm{int}(E)}(c^*) u^*, \tag{4.9}$$

then $\bar{x} \in \mathrm{Sol}$(EWVVI).

(ii) *If there exist $c^* \in \mathrm{int}(E^+)$ and $u^* \in \mathbb{B}_{X^*}$ such that*

$$0 \in F(\bar{x})^*(c^*) + N(A, \bar{x}) + \frac{1}{L} \sigma_{-E \setminus \{0\}}(c^*) u^*, \tag{4.10}$$

then $\bar{x} \in \mathrm{Sol}$(EVVI).

Proof. (i) From (4.9), there exists $a^* \in N(A,\bar{x})$ such that $F(\bar{x})^*(c^*)+a^*+\frac{1}{L}\sigma_{-\text{int}(E)}(c^*)u^*=0$. This combines with the fact that A is a closed convex set, yields that

$$-\left\langle F(\bar{x})^*(c^*)+\frac{1}{L}\sigma_{-\text{int}(E)}(c^*)u^*, x-\bar{x}\right\rangle = \langle a^*, x-\bar{x}\rangle \le 0,\ \forall x \in A.$$

Notice that $\sigma_{-\text{int}(E)}(c^*) \le 0$. Then, the above inequality gives

$$\begin{aligned}\langle F(\bar{x})^*(c^*), x-\bar{x}\rangle &\ge -\left\langle \frac{1}{L}\sigma_{-\text{int}(E)}(c^*)u^*, x-\bar{x}\right\rangle \\ &= -\frac{1}{L}\sigma_{-\text{int}(E)}(c^*)\,\langle u^*, x-\bar{x}\rangle \\ &\ge \frac{1}{L}\sigma_{-\text{int}(E)}(c^*)\|u^*\|\cdot\|x-\bar{x}\| \\ &\ge \sigma_{-\text{int}(E)}(c^*),\ \forall x\in A.\end{aligned}$$

where the last inequality follows from the facts that $L \in (\text{diam}(A), +\infty)$ and $u^* \in \mathbb{B}_{X^*}$. According to Theorem 4.2 (i), we thus have that $\bar{x} \in \text{Sol(EWVVI)}$.

(ii) Similarly to the proof of part (i), by (4.10), we can conclude that

$$\langle F(\bar{x})^*(c^*), x-\bar{x}\rangle > \sigma_{-E\setminus\{0\}}(c^*),\ \forall x \in A. \tag{4.11}$$

It turns out that $\bar{x} \in \text{Sol(EVVI)}$. Otherwise, there exists $\hat{x} \in A$ such that $\langle F(\bar{x})^*, \hat{x}-\bar{x}\rangle \in -E\setminus\{0\}$, which leads to a contradiction with the inequality (4.11). This completes the proof. □

The following example shows that even in the special case, Theorem 4.4 does not hold if the convexity assumption of A is dropped.

Example 4.1. Let $X=Y=\mathbb{R}^2$, and define $E=\{(y_1,y_2)^T \in \mathbb{R}^2 : y_1 \ge 0, y_2 \ge 0\} = \mathbb{R}^2_+$. Consider the mapping $F(x)$ as the identical mapping on $\mathbb{R}^2$ for all $x \in X$, and let

$$A = \{(x_1,x_2)^T \in \mathbb{R}^2 : |x_1|+|x_2| \le 5 \text{ and } x_1^5 \le x_2\}.$$

Take $\bar{x} = (0,0)^T \in A$. Then, $E^+ = E = \mathbb{R}^2_+$. It follows from Example 3.1 in [13] that

$$N(A,\bar{x}) = \{(0,t)^T \in \mathbb{R}^2 : t \le 0\},$$

and the inclusions (4.9) and (4.10) hold with $u^* = (0,0)^T \in \mathbb{B}_{X^*}$ and $L \in (2\sqrt{5}, +\infty)$. Since $F(x)$ is an identical mapping, we have $\langle F(\bar{x}), A-\bar{x}\rangle = A$. Therefore, we obtain

$$\langle F(\bar{x}), A-\bar{x}\rangle \cap -\text{int}(E) = A \cap -\text{int}(E) \ni (-3,-2)^T.$$

This implies that $\bar{x}$ is not a solution to (EWVVI), nor to (EVVI).

4.2. **Optimality Conditions via Nonlinear Scalarization.** In this subsection, we derive necessary conditions for vector variational inequalities (EWVVI) and (EVVI) by using nonlinear scalarization function of the oriented distance function [17].

For a subset $\emptyset \ne S \subset Y$, the oriented distance function associated with S is defined by

$$\Delta_S(y) := d_S(y) - d_{Y\setminus S}(y),$$

where $d_S(y) := \inf_{s\in S}\|y-s\|$.

The function Δ_S has very nice properties, and we list below the properties that would be used in the sequel.

Lemma 4.1. [21] *Let S be a proper subset of Y. The following assertions hold:*

(i) Δ_S *is real-valued;*

(ii) Δ_S *is Lipschitiz of rank* 1*;*

(iii) $\mathrm{cl}(S) = \{y : \Delta_S(y) \leq 0\}$, $\mathrm{int}(S) = \{y : \Delta_S(y) < 0\}$, $\mathrm{bd}(S) = \{y : \Delta_S(y) = 0\}$, $\mathrm{int}(X\backslash S) = \{y : \Delta_S(y) > 0\}$*;*

(iv) If S is a convex set, then Δ_S is convex.

When the associated set is free disposal, we have the following property of the oriented distance function.

Lemma 4.2. [20] *Let $S \subseteq Y$ be free disposal with respect to a convex cone $K \subseteq Y$. If $\mathrm{int}(K) \neq \emptyset$. Then $\Delta_S = \Delta_{\mathrm{cl}(S)} = \Delta_{\mathrm{int}(S)}$.*

As shown above, the real-valued function Δ_S is Lipschitiz, the Clarke subdifferential of Δ_S at any point $y \in Y$, denoted as $\partial\Delta_S(\bar{y})$, is nonempty, convex and compact. The following proposition states that $0 \notin \partial\Delta_S(\bar{y})$ at any $\bar{y} \in Y$ when the set S is free disposal with respect to a convex cone K with $\mathrm{int}(K) \neq \emptyset$. This insightful result is an improved version of the result of [20].

Proposition 4.1. *Let $S \subseteq Y$ be free disposal with respect to a convex cone $K \subseteq Y$. If $\mathrm{int}(K) \neq \emptyset$, then $\partial\Delta_S(\bar{y}) \subseteq -K^+\backslash\{0\} \bigcap \mathbb{B}^*_Y, \forall \bar{y} \in Y$.*

Proof. According to Theorem 4.5 in [20], $\partial\Delta_S(\bar{y}) \subseteq -K^+ \bigcap \mathbb{B}^*_Y, \forall \bar{y} \in Y$. Therefore, it suffices to prove that for each $\bar{y} \in Y$ and $k_0 \in \mathrm{int}(K)$, the Clarke generalized directional derivative of Δ_S at $\bar{y}$ in the direction k_0 satisfies that

$$\Delta^o_S(\bar{y};k_0) := \limsup_{\substack{y' \to \bar{y} \\ t \downarrow 0}} \frac{\Delta_S(y' + tk_0) - \Delta_S(y')}{t} < 0. \tag{4.12}$$

Case 1: $\bar{y} \in \mathrm{int}(Y\backslash S)$.

First, we claim that there exists a scalar $\delta > 0$ such that for all $y \in \mathbb{B}(\bar{y};\delta)$ and $t \in (0,\delta)$,

$$d_S(y + tk_0) - d_S(y) \leq -\frac{t\delta}{2}. \tag{4.13}$$

Since $k_0 \in \mathrm{int}(K)$, then there exists a positive scalar δ such that $\mathbb{B}(k_0;\delta) \subseteq \mathrm{int}(K)$ and $\mathbb{B}(\bar{y};\delta) + 2\delta\mathbb{B}(k_0;\delta) \subseteq Y\backslash S$. Since S is free disposal with respect to K, it follows that

$$d_S(y - tk) = d_{S+tk}(y) \geq d_S(y), \forall y \in Y, \forall k \in \mathbb{B}(k_0;\delta), \forall t > 0. \tag{4.14}$$

Now, for each $y \in \mathbb{B}(\bar{y};\delta)$ and $t \in (0,\delta)$, there exists $z_t \in \mathrm{bd}(S)$ such that

$$d_S(y) \geq \|y - z_t\| - \frac{t\delta}{2}, \tag{4.15}$$

Here, $z_t \notin y + tk_0 - t\mathbb{B}(k_0;\delta)$. Otherwise, it follows from (4.14) and $z_t \in \mathrm{bd}(S)$ that $0 = d_S(z_t) \geq d_S(y + tk_0) > 0$, which is a contradiction. Hence there exists $u_t \in y + tk_0 - t\mathbb{B}(k_0;\delta) = y - t\mathbb{B}(0;\delta)$ such that $\|y - u_t\| = t\delta$, and

$$\|y - z_t\| = t\delta + \|u_t - z_t\|. \tag{4.16}$$

Combining with (4.15) and (4.16), we can deduce that

$$d_S(y) \geq d_S(u_t) + \frac{t\delta}{2}, \tag{4.17}$$

as $z_t \in \text{bd}S$. Notice that $u_t \in y + tk_0 - tB(k_0;\delta)$. From (4.14), one has that $d_S(u_t) \geq d_S(y+tk_0)$, which combined with (4.17) yields that (4.13) holds for all $y \in \mathbb{B}(\bar{y};\delta) \cap cl(A^c)$ and $t \in (0,+\infty)$. Since $\mathbb{B}(\bar{y};\delta) + 2\delta\mathbb{B}(k_0;\delta) \subseteq Y\backslash S$, it directly follows from (4.13) that, for all $y \in \mathbb{B}(\bar{y};\delta)$ and $t \in (0,\delta)$, $\Delta_S(y+tk_0) - \Delta_S(y) = d_S(y+tk_0) - d_S(y) \leq -\frac{t\delta}{2}$. This implies that (4.12) holds for $\bar{y} \in \text{int}(Y\backslash S)$.

Case 2: $\bar{y} \in \text{int}S$.

To prove (4.12), it is enough to show that there exists a scalar $\delta > 0$ such that for any $y \in \mathbb{B}(\bar{y};\delta) \subseteq \text{cl}S$ and $t \in (0,\delta)$,

$$y + tk_0 \notin \text{cl}(Y\backslash S), \tag{4.18}$$

and

$$d_{Y\backslash S}(y+tk_0) \geq d_{Y\backslash S}(y) + \frac{t\delta}{2}. \tag{4.19}$$

Since S is free disposal with respect to K, it follows that $Y\backslash S$ is free disposal with respect to $-K$, i.e., $Y\backslash S - K \subseteq Y\backslash S$. Then,

$$d_{Y\backslash S}(y+tk) = d_{(Y\backslash S)-tk}(y) \geq d_{Y\backslash S}(y), \forall y \in Y, \forall k \in K, \forall t > 0, \tag{4.20}$$

and it is easy to verify that for any $y \in \text{cl}(Y\backslash S)$ and $t > 0$, $y + tk_0 \notin \text{cl}(Y\backslash S)$. Hence, (4.18) holds. Note that $\bar{y} \in \text{int}(S)$ and $k_0 \in \text{int}(K)$, then there exists a positive scalar δ such that $\mathbb{B}(k_0;\delta) \subseteq \text{int}(K)$ and $\mathbb{B}(\bar{y};\delta) + t\mathbb{B}(k_0;\delta) \subseteq \text{int}(S)$, for all $t \in [0,\delta]$. Now, for each $y \in \mathbb{B}(\bar{y};\delta) \subseteq \text{int}(S)$ and $t \in (0,\delta)$, there exists $z_t \in \text{bd}(Y\backslash S)$ such that

$$d_{(Y\backslash S)}(y+tk_0) \geq \|y + tk_0 - z_t\| - \frac{t\delta}{2}, \tag{4.21}$$

Here, $z_t \notin y + t\mathbb{B}(k_0;\delta)$. Otherwise, it follows from (4.20) and $z_t \in \text{bd}(Y\backslash S)$ that $0 = d_{(Y\backslash S)}(z_t) \geq d_{(Y\backslash S)}(y) > 0$, which is a contradiction. Hence there exists $u_t \in y + t\mathbb{B}(k_0;\delta) = y + tk_0 + t\mathbb{B}(0;\delta)$ such that $\|y + tk_0 - u_t\| = t\delta$, and

$$\|y - tk_0 - z_t\| = t\delta + \|u_t - z_t\|. \tag{4.22}$$

Combining with (4.21) and (4.22), we can deduce that

$$d_{(Y\backslash S)}(y+tk_0) \geq d_{(Y\backslash S)}(u_t) + \frac{t\delta}{2}, \tag{4.23}$$

as $z_t \in \text{bd}(Y\backslash S)$. Notice that $u_t \in y + t\mathbb{B}(k_0;\delta)$. It follows from (4.20) that $d_{(Y\backslash S)}(u_t) \geq d_{(Y\backslash S)}(y)$, which together with (4.23) implies that (4.19) holds for all $y \in \mathbb{B}(\bar{y};\delta) \subseteq \text{int}(S)$ and $t \in (0,\delta)$. Since $\mathbb{B}(\bar{y};\delta) + tk_0 \subseteq \text{int}(S)$, it directly follows from (4.19) that for all $y \in \mathbb{B}(\bar{y};\delta)$ and $t \in (0,\delta)$, $\Delta_S(y+tk_0) - \Delta_S(y) \leq -\frac{t_0\delta}{2}$. This implies that (4.12) holds for $\bar{y} \in \text{int}(S)$.

Case 3: $\bar{y} \in \text{bd}(S)$. It was proved by Tang and Gao [20], and hence is omitted here. □

Remark 4.4. When the set S is further strengthened to an improvement set with respect to a convex cone $K \subseteq Y$ with $\text{int}(K) \neq \emptyset$, we also have that

$$\partial\Delta_S(\bar{y}) \subseteq -K^+\backslash\{0\} \bigcap \mathbb{B}^*_Y, \forall \bar{y} \in Y.$$

By Lemma 4.2, it should be mentioned that $\Delta_{-\text{int}(E)} = \Delta_{-E\setminus\{0\}} = \Delta_{-E}$ under the assumption that the set E is improvement with respect to the convex cone K with $\text{int}(K) \neq \emptyset$. Hence, in the following, based on the oriented distance function Δ_{-E}, we consider the following corresponding scalar optimization problem:

$$\min_{x\in A} \Delta_{-E}\left(\langle F(\bar{x}), x-\bar{x}\rangle\right), \tag{$P_{-E,\bar{x}}$}$$

where $\bar{x} \in A$.

We can establish the following complete characterizations of approximate solutions to (EWVVI) and (EVVI). The proof follows similarly to Theorem 5.3 in [20] and is therefore omitted.

Theorem 4.5. *Let $\bar{x} \in A$. Denote $\varepsilon := \Delta_E(0)$. Then the following assertions hold:*

(i) $\bar{x} \in \text{Sol(EWVVI)}$ if and only if it is an ε-optimal solution for ($P_{-E,\bar{x}}$), i.e.,

$$\Delta_{-E}(\langle F(\bar{x}), x-\bar{x}\rangle) \geq \Delta_{-E}(\langle F(\bar{x}), \bar{x}-\bar{x}\rangle) - \varepsilon,\ \forall x \in A. \tag{4.24}$$

(ii) $\bar{x} \in \text{Sol(EVVI)}$ if it is a strictly ε-optimal solution for ($P_{-E,\bar{x}}$), i.e.,

$$\Delta_{-E}(\langle F(\bar{x}), x-\bar{x}\rangle) > \Delta_{-E}(\langle F(\bar{x}), \bar{x}-\bar{x}\rangle) - \varepsilon,\ \forall x \in A\setminus\bar{x}. \tag{4.25}$$

In addition, assume that E is closed. Then $\bar{x} \in \text{Sol(EVVI)}$ if and only if it is a strictly ε-optimal solution for ($P_{-E,\bar{x}}$).

Remark 4.5. Since E is improvement with respect to the convex cone K, the error $\varepsilon = \Delta_E(0) = \inf_{e\in E}\|e\| \geq 0$.

In the end, we establish optimality conditions for (EWVVI) and (EVVI) via the oriented direction function in the nonconvex case.

Theorem 4.6. *Assume that A is a closed subset of X. Denote $\varepsilon = \inf_{e\in E}\|e\|$. If $\bar{x} \in \text{Sol(EWVVI)}$ (or $\bar{x} \in \text{Sol(EVVI)}$), then for any $\lambda > 0$, there exist $\hat{x} \in A\bigcap\mathbb{B}(\bar{x};\lambda)$ and $y^* \in E^+\setminus\{0\}$ such that*

$$0 \in F(\bar{x})^* y^* + \frac{\varepsilon}{\lambda}\mathbb{B}_{X^*} + N(A;\hat{x}), \tag{4.26}$$

where $F(\bar{x})^*$ is the conjugate operator of $F(\bar{x})$.

Proof. By Theorem 4.5, $\bar{x} \in \text{Sol(EWVVI)}$ yields $\Delta_{-E}(F(\bar{x})(x-\bar{x})) \geq \Delta_{-E}(F(\bar{x})(\bar{x}-\bar{x})) - \varepsilon$ for all $x \in A$. Applying Ekeland's variational principle to $\Delta_{-E}(F(\bar{x})(x-\bar{x}))$ on A, we get that for any $\lambda > 0$, there exists $\hat{x} \in A\bigcap\mathbb{B}(\bar{x};\lambda)$ such that

$$\Delta_{-E}(F(\bar{x})(\hat{x}-\bar{x})) \leq \Delta_{-E}(F(\bar{x})(x-\bar{x})) + \frac{\varepsilon}{\lambda}\|x-\hat{x}\|, \forall x \in A.$$

This means that $\hat{x}$ is an optimal solution of $\Delta_{-E}(F(\bar{x})(\cdot-\bar{x})) + \frac{\varepsilon}{\lambda}\|\cdot-\hat{x}\|$ on A. Owing to the calculus rules of the Clarke subdifferential, we obtain that

$$\begin{aligned}
0 &\in \partial\left(\Delta_{-E}(F(\bar{x})(\cdot-\bar{x})) + \frac{\varepsilon}{\lambda}\|\cdot-\hat{x}\|\right)(\hat{x}) + N(A;\hat{x}) \\
&\subseteq \partial\Delta_{-E}\left((F(\bar{x})(\cdot-\bar{x}))\right)(\hat{x}) + \frac{\varepsilon}{\lambda}\mathbb{B}_{X^*} + N(A;\hat{x}) \\
&= F(\bar{x})^*\partial\Delta_{-E}\left(F(\bar{x})(\hat{x}-\bar{x})\right) + \frac{\varepsilon}{\lambda}\mathbb{B}_{X^*} + N(A;\hat{x})
\end{aligned}$$

According to Proposition 4.1, there exists $y^* \in E^+\setminus\{0\}$ such that (4.26) holds. □

Remark 4.6. (i) When the improvement set E is selected as a proper convex co-radiant set C, according to Theorem 4.6, we have that if $\bar{x} \in A$ is an ε-solution of (WVVI) (resp. (VVI)) with respect to C, then for any $\lambda > 0$, there exists $\hat{x} \in A \bigcap B(\bar{x};\lambda)$ and $y^* \in E^+ \backslash \{0\} = (C(0))^+ \backslash \{0\} = (\bigcup_{\varepsilon>0} \varepsilon C)^+ \backslash \{0\}$ such that

$$0 \in F(\bar{x})^* y^* + \frac{\varepsilon}{\lambda} \beta \mathbb{B}_{X^*} + N(A;\hat{x}),$$

where $\beta = \inf_{e\in E} \|e\| = d(0,C)$. This result has been established in [15] by Gao et al.

(ii) Furthermore, when the improvement set E is chosen as the pointed convex cone K, following the proof process of Theorem 4.6, we obtain the following result:

$$\bar{x} \in \text{Sol(WVVI) (or } \bar{x} \in \text{Sol(VVI))} \Longrightarrow 0 \in F(\bar{x})^* y^* + N(A;\bar{x}) \text{ for some } y^* \in K^+ \backslash \{0\},$$

which has been achieved by Yang and Zheng [13].

5. Conclusion

This paper develops new approximate vector variational inequality models based on the improvement set and introduces new concepts of approximate solutions in Banach spaces. Two existence results are established via the KKM-Fan and Brouwer's fixed-point theorems, respectively. Additionally, optimality conditions for the new approximate solutions to vector variational inequalities are derived through the scalarization technique, the classical Ekeland's variational principle, and the Fermat rule.

In this work, the existence of approximate solution to (EVVI) relies on the compactness of the set A. In future studies, it would be worthwhile to consider the existence of approximate solution to (EVVI) in the noncompact case. Furthermore, optimality conditions could be established directly through the vector version of Ekeland's variational principle, avoiding the need for scalarization and cEkeland's variational principle.

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