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LEVITIN-POLYAK WELL-POSEDNESS FOR SET OPTIMIZATION WITH A VARIABLE SET STRUCTURE

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Abstract. In this paper, we aim to elaborate on some notions of Levitin-Polyak well-posedness for set optimization problems with a variable set structure and well-posedness of the corresponding scalar optimization problem by employing a nonlinear scalarization function. We categorize these notions into two classes including pointwise and global Levitin–Polyak well-posedness. Some necessary and sufficient conditions for these well-posedness are established. Additionally, we characterize LP well-posedness for set optimization problems in terms of the upper Hausdorff convergence and Painlevé-Kuratowski convergence of approximate solution sets. Furthermore, we explore the interrelationships among these well-posedness concepts. Finally, we explore some applications of the obtained results to multi-criteria traffic network equilibrium problems.

Keywords. Levitin-Polyak well-posedness; Painlevé-Kuratowski convergence; Set optimization; Upper Hausdorff convergence; Variable structure.

1. Introduction

Set-valued optimization problems have been intensively studied in recent years on account of their wide-ranging applications in various fields like optimal control problems, differential inclusions, game theory, robust optimization, image processing problems, and mathematical economics; see, e.g., [6, 13, 25, 31, 33, 36] and the references therein.

In the field of set optimization research, set relations play one of the most essential roles since they act as preference relations that provide a natural way to compare the values of the set-valued objective mapping. Based on a fixed convex cone, there is a variety of set relations known in the literature and several authors have obtained rich results by studying different set relations; see, e.g., [15, 35, 38]. However, sometimes the set ordering structure relies on a set-valued mapping; see [10, 27]. In 1974, Yu [34] introduced the concept of variable set structure by using several different cones in vector optimization. Motivated by applications in medical image registration [8, 9], variable domination structures in vector optimization have gained recognition as they allow the introduction of a specification of the decision-maker's preferences into the model. During the past two decades, the interest in vector optimization and set optimization problems with variable set structures has increased; see, e.g., [5, 19, 26]. In

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this paper, we consider an upper set less relation (see [19]), which is equipped with a variable domination structure.

In the field of set optimization, although the existence of solutions to set optimization problems remains a core focus, the stability of optimization problems is also an important research content. Many researchers have studied the stability theory of perturbed set optimization problems from various perspectives. Luc et al. [24] investigated the convergence of the solution sets of perturbed vector optimization problems in the sense of Painlevé-Kuratowski convergence when both the ordering cone and the feasible set are perturbed. Gutiérrez [13] considered the perturbation of the feasible set and established external and internal stability of the solution sets of a set optimization problem by using the notions of Hausdorff and Painlevé-Kuratowski set convergence. Han and Huang [14] established characterizations for two generalized well-posedness of set optimization problems by using the Hausdorff upper semicontinuity of solution mappings and investigated the semicontinuity of solution mappings to parametric set optimization problems. Most of the previous studies on the stability theory of set optimization are based on a fixed set relation. This naturally raises a question: can we use a variable set relation to study the stability of set optimization problems?

In the study of the sensitivity and stability of optimization problems, well-posedness plays an important role, see [21, 23, 37]. The well-posedness of an optimization problem means that when the values of the objective function approach the optimal value, the corresponding independent variables will also be close to the solution of the optimization problem. In 1966, Tykhonov [29] introduced the concept of well-posedness for scalar optimization problems which ensures the convergence of minimizing sequences to the unique solution of the problem. Since then, numerous researchers have extended the notion of well-posedness to different kinds of optimization problems. Levitin and Polyak [22] introduced an alternative notion of well-posedness, which strengthened the Tykhonov one. In recent years, LP well-posedness for optimization problems has been widely discussed in many studies, see [7, 11]. Vui et al. [30] investigated some characterizations for pointwise LP well-posedness of set optimization problems with respect to various set order relations. Gupta et al. [12] introduced Levitin-Polyak well-posedness in the set and scalar sense, established some relationships among them using the upper less relation and characterized these LP well-posedness by invoking Painlevé-Kuratowski set convergence. Tahu et al. [28] also established characterizations of LP well-posedness for a parametric set optimization problem in terms of upper Hausdorff convergence and Painlevé-Kuratowski convergence of sequences of approximate solution sets.

Inspired by these works, in this paper, we study LP well-posedness for set optimization problems in set sense and scalar sense, respectively, based on pointwise and global notions. In Section 2, we provide some notions that will be used in later sections. Then, we recall a set relation with respect to a variable structure and some properties of a set scalarization functional of type sup-inf, which is an extension of Gerstewitz's functional. In Section 3, we introduce some pointwise notions of LP well-posedness in set sense and scalar sense for a set optimization with a variable structure. Some necessary and sufficient conditions are obtained for these LP well-posedness in terms of upper semicontinuity, compactness and the measure of the approximate solution mappings. We then establish the upper Hausdorff convergence and the Painlevé-Kuratowski convergence of sequences of approximate solution sets for these pointwise LP well-posedness. In Section 4, we define some global notions of LP well-posedness for a set optimization problem in set sense and scalar sense with variable structure, respectively. We also give some necessary and sufficient conditions for these LP well-posedness and establish upper Hausdorff convergence and Painlevé-Kuratowski convergence of sequences of approximate solution sets. In Section 5, we establish the link between different LP well-posedness. In Section 6, the last section, we apply the results to investigate traffic network problems with interval-valued cost functions.

2. Preliminaries

Throughout this paper, let X, Y be real normed spaces. The zero vector and the family of all nonempty subsets of Y are denoted by $\mathbf{0}$ and $\mathscr{P}_0(Y)$, respectively. The open and closed ball centered at a with radius r are denoted by $\mathbb{B}(a,r)$ and $\bar{\mathbb{B}}(a,r)$, respectively. Moreover, we denote by $\bar{\mathbb{B}}$ the closed unit ball in Y. For two nonempty subsets A and B of Y, we denote the sum of A and B by $A+B := \{a+b | a \in A, b \in B\}$. Let $\mathbb{R}^n$ denote n-dimensional Euclidean space, $\mathbb{R}^n_+ := \{(x_1,\ldots,x_n) \in \mathbb{R}^n | x_1 \geq 0,\ldots,x_n \geq 0\}$ and $\mathbb{R}^n_{++} := \{(x_1,\ldots,x_n) \in \mathbb{R}^n | x_1 > 0,\ldots,x_n > 0\}$. Given a nonempty subset $A \subseteq Y$, we denote the topological interior and the topological closure of A by $\mathrm{int}A$ and $\mathrm{cl}A$, respectively. Let $\mathscr{K} : Y \rightrightarrows Y$ be a set-valued mapping satisfying $\mathrm{int}\mathscr{K}(y) \neq \emptyset$ for all $y \in Y$. Let $k^0 \in Y \setminus \{\mathbf{0}\}$ satisfying $\mathscr{K}(y) + (0,+\infty)k^0 \subseteq \mathscr{K}(y)$, for all $y \in Y$. For a set $A \in \mathscr{P}_0(Y)$, we define $\tilde{\mathscr{K}}(A) := \bigcup_{a\in A} \mathscr{K}(a)$ and $\bar{\mathscr{K}}(A) := \bigcap_{a\in A} \mathscr{K}(a)$. To study set optimization problems with variable structures, we recall the following set relation which is appeared in [19] and further studied in [2, 3].

Definition 2.1. [19] Let $A, B \in \mathscr{P}_0(Y)$. The variable generalized upper less relation ($\preccurlyeq_u^{\mathscr{K}}$) is defined by $A \preccurlyeq_u^{\mathscr{K}} B \Leftrightarrow A \subseteq \bigcup_{b\in B}(b - \mathscr{K}(b))$.

Similarly, we introduce the following concept.

Definition 2.2. Let $A, B \in \mathscr{P}_0(Y)$. The variable generalized strictly upper less relation ($\prec_u^{\mathscr{K}}$) is defined by $A \prec_u^{\mathscr{K}} B \Leftrightarrow A \subseteq \bigcup_{b\in B}(b - \mathrm{int}\mathscr{K}(b))$.

We say that $A \sim B \Longleftrightarrow A \preccurlyeq_u^{\mathscr{K}} B$ and $B \preccurlyeq_u^{\mathscr{K}} A$. In order to investigate some properties of variable set relations, we use three kinds of assumptions concerning the set-valued mapping $\mathscr{K} : Y \rightrightarrows Y$:

$$\mathbf{0} \in \mathscr{K}(y), \forall y \in Y. \tag{2.1}$$

$$\mathscr{K}(y) + \mathscr{K}(y) \subseteq \mathscr{K}(y), \forall y \in Y. \tag{2.2}$$

$$\mathscr{K}(y-d) \subseteq \mathscr{K}(y), \forall y \in Y, d \in \mathscr{K}(y). \tag{2.3}$$

Following the notations introduced in [16], for simplicity, we say that $-\mathscr{K}$ is transitive if (2.2) and (2.3) are satisfied, and $-\mathscr{K}$ is reflexive if (2.1) is satisfied. In addition, we give the assumption: (H_1) $\mathscr{K}(y) + (0,+\infty)k^0 \subseteq \mathrm{int}\mathscr{K}(y)$ for all $y \in Y$.

Lemma 2.1. [16] (i) The relation $\preccurlyeq_u^{\mathscr{K}}$ is reflexive if and only if $-\mathscr{K}$ is reflexive.
(ii) If $-\mathscr{K}$ is transitive, then $\preccurlyeq_u^{\mathscr{K}}$ is transitive.

In the following, we introduce the concept of the weakly maximal point of a nonempty subset of Y.

Definition 2.3. Let $A \in \mathscr{P}_0(Y)$. An element $a_0 \in A$ is called a weakly maximal point of A with respect to $\mathscr{K}$, denoted by $a_0 \in \mathrm{WMax}(A, \mathscr{K})$, if $(A - a_0) \cap \mathrm{int}\tilde{\mathscr{K}}(A) = \emptyset$.

Let $Q \in \mathscr{P}_0(Y)$, $k \in Y \setminus \{\mathbf{0}\}$. We consider a nonlinear scalarization function $z^{Q,k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ given by $z^{Q,k}(y) := \inf\{t \in \mathbb{R} | y \in tk - Q\}$, $y \in Y$. Köbis et al. [19] introduced some new nonlinear scalarization functions: $g^{\preccurlyeq_l^{\mathscr{K}}}(\cdot,\cdot)$, $g^{\preccurlyeq_u^{\mathscr{K}}}(\cdot,\cdot)$, $g^{\preccurlyeq_{cl}^{\mathscr{K}}}(\cdot,\cdot)$, $g^{\preccurlyeq_{cu}^{\mathscr{K}}}(\cdot,\cdot)$, $g^{\preccurlyeq_{pl}^{\mathscr{K}}}(\cdot,\cdot)$, $g^{\preccurlyeq_{pu}^{\mathscr{K}}}(\cdot,\cdot)$. We recall the following concept.

Definition 2.4. [19] Let $A,B \in \mathscr{P}_0(Y)$. The nonlinear scalarization function $g^{\preccurlyeq_u^{\mathscr{K}}} : \mathscr{P}_0(Y) \times \mathscr{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$ is defined as $g^{\preccurlyeq_u^{\mathscr{K}}}(A,B) := \sup_{a\in A} \inf_{b\in B} z^{-b+\mathscr{K}(b),k^0}(a)$.

Lemma 2.2. [19] Let $A,B \in \mathscr{P}_0(Y)$.
(i) If $g^{\preccurlyeq_u^{\mathscr{K}}}(A,B) \le r$, then $\bigcup_{t>r}(A - tk^0) \subseteq \bigcup_{b\in B}(b - \mathscr{K}(b))$, i.e., $\bigcup_{t>r}(A - tk^0) \preccurlyeq_u^{\mathscr{K}} B$.
(ii) If $A - rk^0 \subseteq \bigcup_{b\in B}(b - \mathscr{K}(b))$, then $g^{\preccurlyeq_u^{\mathscr{K}}}(A,B) \le r$.

Lemma 2.3. [19] Let $A,B \in \mathscr{P}_0(Y)$. If $\bigcup_{b\in B}(b - \mathscr{K}(b))$ is closed, then $A \preccurlyeq_u^{\mathscr{K}} B \Leftrightarrow g^{\preccurlyeq_u^{\mathscr{K}}}(A,B) \le 0$.

Let $\phi : Y \to \mathbb{R} \cup \{\pm\infty\}$ be defined by $\phi(y) := \inf_{b\in B} z^{-b+\mathscr{K}(b),k^0}(y)$, $\forall y \in Y$. In the following, we present some properties of the relation $\prec_u^{\mathscr{K}}$.

Proposition 2.1. Let $A,B \in \mathscr{P}_0(Y)$ and (H_1) hold.
(i) If $g^{\preccurlyeq_u^{\mathscr{K}}}(A,B) < r$, then $A - rk^0 \prec_u^{\mathscr{K}} B$.
(ii) If $\sup_{a\in A}\phi(a)$ is attained at some $a_0 \in A$, then $A - rk^0 \prec_u^{\mathscr{K}} B$ implies $g^{\preccurlyeq_u^{\mathscr{K}}}(A,B) < r$.

Proof. (i) Let $g^{\preccurlyeq_u^{\mathscr{K}}}(A,B) < r$. Then $\sup_{a\in A}\inf_{b\in B} z^{-b+\mathscr{K}(b),k^0}(a) < r$. Therefore, for any $a \in A$, there exists $b_a \in B$, such that $z^{-b_a+\mathscr{K}(b_a),k^0}(a) < r$. Then there exists $t_0 < r$ such that

$$a \in t_0k^0 + b_a - \mathscr{K}(b_a) = rk^0 + b_a + (t_0 - r)k^0 - \mathscr{K}(b_a) \subseteq rk^0 + b_a - \operatorname{int}\mathscr{K}(b_a).$$

Then $A \subseteq \bigcup_{b\in B}(b - \operatorname{int}\mathscr{K}(b)) + rk^0$, which implies $A - rk^0 \prec_u^{\mathscr{K}} B$.

(ii) Suppose that $A - rk^0 \prec_u^{\mathscr{K}} B$. Since $a_0 \in A$, there exists $b_{a_0} \in B$ such that $a_0 \in b_{a_0} - \operatorname{int}\mathscr{K}(b_{a_0}) + rk^0$. Consequently, there exists $t_0 > 0$ such that $a_0 - b_{a_0} - rk^0 + t_0k^0 \in -\mathscr{K}(b_{a_0})$. Thus, we have $\inf\{t \in \mathbb{R} | a_0 \in tk^0 + b_{a_0} - \mathscr{K}(b_{a_0})\} \le r - t_0 < r$, which implies that

$$\inf_{b\in B} z^{-b+\mathscr{K}(b),k^0}(a_0) < r.$$

Since $\sup_{a\in A}\phi(a)$ is attained at $a_0 \in A$, we have $g^{\preccurlyeq_u^{\mathscr{K}}}(A,B) = \sup_{a\in A}\phi(a) < r$. □

As a consequence of Proposition 2.1, we get the following corollary.

Corollary 2.1. Let $A,B \in \mathscr{P}_0(Y)$ and (H_1) hold.
(i) If $g^{\preccurlyeq_u^{\mathscr{K}}}(A,B) < 0$, then $A \prec_u^{\mathscr{K}} B$.
(ii) If $\sup_{a\in A}\phi(a)$ is attained at some $a_0 \in A$, then $A \prec_u^{\mathscr{K}} B$ implies $g^{\preccurlyeq_u^{\mathscr{K}}}(A,B) < 0$.

Let D be a nonempty subset of X and $F : X \rightrightarrows Y$ be a set-valued mapping. Throughout this paper, it is assumed that $F(x) \neq \emptyset$ for every $x \in D$. The graph of F is defined as $\operatorname{gr}F := \{(x,y) \in X \times Y : y \in F(x)\}$, the image of F under D is denoted by $F(D) := \bigcup_{x\in D} F(x)$.

We consider the following set optimization:

$$\text{(SOP)} \quad \min F(x) \quad \text{subject to } x \in D.$$

Definition 2.5. ([[4], Definitions 3.1.7 and 3.1.12]) Let $x_0 \in X$. The mapping F is said to be

(i) upper semicontinuous (u.s.c.) at x_0 if, for every open set $V \subseteq Y$ with $F(x_0) \subseteq V$, there exists a neighborhood U of x_0 such that $F(x) \subseteq V$, for every $x \in U$;

(ii) lower semicontinuous (l.s.c.) at $x_0 \in X$ if, for any open set $V \subseteq Y$ with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$, for every $x \in U$;

(iii) Hausdorff upper semicontinuous (H-u.s.c.) at $x_0 \in X$ if, for any neighborhood U of the origin in Y, there exists a neighborhood W of x_0 such that $F(x) \subseteq F(x_0) + U$, for every $x \in W$;

(iv) closed at x_0 if, for any sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{gr}F$ which converges to (x_0, y_0), $y_0 \in F(x_0)$;

(v) compact at x_0 if, for every sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \mathrm{gr}F$ with $x_n \to x_0$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0 \in F(x_0)$.

The mapping F is said to be upper semicontinuous (l.s.c., H-u.s.c., closed, compact) on a subset $S \subseteq X$ if F is upper semicontinuous (l.s.c., H-u.s.c., closed, compact) at every point $x \in S$.

Lemma 2.4. Let $F : X \rightrightarrows Y$ be a set-valued mapping. Then the following statements hold.

(i) [17] If $x_0 \in X$ and $F(x_0)$ is compact, then F is u.s.c. at x_0 if and only if, for every sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$ and for any $y_n \in F(x_n)$, there exist $y_0 \in F(x_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$.

(ii) [4] F is lower semicontinuous at $x_0 \in X$ if and only if, for every sequence $\{x_n\} \subseteq X$ with $x_n \to x_0$ and for any $y_0 \in F(x_0)$, there exist $y_n \in F(x_n)$ such that $y_n \to y_0$.

In the following, we introduce the concepts of minimal and weakly minimal solutions of (SOP).

Definition 2.6. An element $x_0 \in D$ is said to be

(i) a minimal solution of (SOP) with respect to the variable generalized upper less relation if $x \in D$ with $F(x) \preccurlyeq_u^{\mathscr{K}} F(x_0)$ implies that $F(x_0) \preccurlyeq_u^{\mathscr{K}} F(x)$;

(ii) a weakly minimal solution of (SOP) with respect to the variable generalized upper less relation if $x \in D$ with $F(x) \prec_u^{\mathscr{K}} F(x_0)$ implies that $F(x_0) \prec_u^{\mathscr{K}} F(x)$.

Let $\mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$ and $\mathrm{WMin}(D, F, \preccurlyeq_u^{\mathscr{K}})$ denote the sets of all minimal solutions and weakly minimal solutions of (SOP) with respect to the variable generalized upper less relation, respectively.

Theorem 2.1. Let $-\mathscr{K}$ be transitive. Then $\mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}}) \subseteq \mathrm{WMin}(D, F, \preccurlyeq_u^{\mathscr{K}})$.

Proof. Let $x_0 \in \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$. If $F(x) \not\prec_u^{\mathscr{K}} F(x_0)$ for all $x \in D$, then $x_0 \in \mathrm{WMin}(D, F, \preccurlyeq_u^{\mathscr{K}})$. If there exists $x_1 \in D$ such that $F(x_1) \prec_u^{\mathscr{K}} F(x_0)$, then

$$F(x_1) \subseteq \bigcup_{y \in F(x_0)} (y - \mathrm{int}\mathscr{K}(y)). \tag{2.4}$$

Thus $F(x_1) \preccurlyeq_u^{\mathscr{K}} F(x_0)$. It follows from $x_0 \in \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$ that $F(x_0) \preccurlyeq_u^{\mathscr{K}} F(x_1)$, which implies

$$F(x_0) \subseteq \bigcup_{y \in F(x_1)} (y - \mathscr{K}(y)). \tag{2.5}$$

Then, for any $y \in F(x_0)$, there exists $y_1 \in F(x_1)$ such that $y \in y_1 - \mathscr{K}(y_1)$. From (2.4), there exists $y_0 \in F(x_0)$ such that $y_1 \in y_0 - \text{int}\mathscr{K}(y_0)$. Hence

$$y \in y_0 - \text{int}\mathscr{K}(y_0) - \mathscr{K}(y_0 - \text{int}\mathscr{K}(y_0)) \subseteq y_0 - \text{int}\mathscr{K}(y_0) - \mathscr{K}(y_0) \subseteq y_0 - \text{int}\mathscr{K}(y_0).$$

It follows from (2.5) that there exists $y_2 \in F(x_1)$ such that $y_0 \in y_2 - \text{int}\mathscr{K}(y_2)$. Then

$$\begin{aligned} y &\in y_2 - \mathscr{K}(y_2) - \text{int}\mathscr{K}(y_2 - \mathscr{K}(y_2)) \\ &\subseteq y_2 - \mathscr{K}(y_2) - \text{int}\mathscr{K}(y_2) \\ &\subseteq y_2 - \text{int}\mathscr{K}(y_2). \end{aligned}$$

Thus $F(x_0) \subseteq \bigcup_{y \in F(x_1)}(y - \text{int}\mathscr{K}(y))$. Therefore, $F(x_0) \prec_u^{\mathscr{K}} F(x_1)$. This implies that $x_0 \in \text{WMin}(D, F, \preccurlyeq_u^{\mathscr{K}})$. □

Theorem 2.2. Let $x_0 \in D$ and $-\mathscr{K}$ be transitive. If $\text{WMax}(F(x_0), \mathscr{K}) \neq \emptyset$, then $x_0 \in \text{WMin}(D, F, \preccurlyeq_u^{\mathscr{K}})$ if and only if $F(x) \not\prec_u^{\mathscr{K}} F(x_0)$ for all $x \in D$.

Proof. By Definition 2.6(ii), the sufficiency holds. We now prove the necessity. Assume that $x_1 \in D$ satisfies $F(x_1) \prec_u^{\mathscr{K}} F(x_0)$. It follows from $x_0 \in \text{WMin}(D, F, \preccurlyeq_u^{\mathscr{K}})$ that $F(x_0) \prec_u^{\mathscr{K}} F(x_1)$. Thus

$$F(x_1) \subseteq \bigcup_{q \in F(x_0)} (q - \text{int}\mathscr{K}(q)) \tag{2.6}$$

and

$$F(x_0) \subseteq \bigcup_{p \in F(x_1)} (p - \text{int}\mathscr{K}(p)). \tag{2.7}$$

From (2.7), we can infer that for any $y \in F(x_0)$, there exists $y_1 \in F(x_1)$ such that $y \in y_1 - \text{int}\mathscr{K}(y_1)$. It follows from (2.6) that there exists $y_0 \in F(x_0)$ and $k_1 \in \text{int}\mathscr{K}(y_0)$, such that $y_1 = y_0 - k_1$. Then $\text{int}\mathscr{K}(y_1) = \text{int}\mathscr{K}(y_0 - k_1) \subseteq \text{int}\mathscr{K}(y_0)$. Hence $y \in y_0 - \text{int}\mathscr{K}(y_0) - \text{int}\mathscr{K}(y_1) \subseteq y_0 - \text{int}\mathscr{K}(y_0)$. Therefore, $(F(x_0) - y) \cap (\text{int}\tilde{\mathscr{K}}(F(x_0))) \neq \emptyset$ for any $y \in F(x_0)$, which implies $\text{WMax}(F(x_0), \mathscr{K}) = \emptyset$. This is a contradiction. □

Let $M_n \subseteq X, n \in \mathbb{N}$. We now recall the notion of Painlevé-Kuratowski convergence (see [17]). Denote

$$\text{Li } M_n := \{x \in X : x_n \to x, x_n \in M_n, \text{for sufficiently large } n\}$$

and

$$\text{Ls } M_n := \{x \in X : x_{n_k} \to x, x_{n_k} \in M_{n_k}, \{n_k\} \text{ is an increasing sequence of integers}\}.$$

We say that $\{M_n\}$ converges to $M \subseteq X$ in the sense of Painlevé-Kuratowski(P.K.), denoted by $M_n \xrightarrow{K} M$, if $\text{Ls } M_n \subseteq M \subseteq \text{Li } M_n$.

For two nonempty sets A and B of X, the diameter of A, denoted by $\text{diam}A$, is defined as

$$\text{diam}A := \sup_{a_1, a_2 \in A} \|a_1 - a_2\|.$$

The excess functional of A over B, denoted by $ex(A,B)$, is defined as

$$ex(A,B) := \sup_{a \in A} d(a,B),$$

where $d(a,B) := \inf_{b \in B} \|a - b\|$.

The sequence $\{M_n\} \subseteq X$ converges to $M \subseteq X$ in the sense of upper Hausdorff set convergence [20], denoted by $M_n \xrightarrow{H} M$, if $ex(M_n, M) \to 0$ as $n \to +\infty$.

3. Pointwise LP well-posedness of (SOP)

In this section, we introduce some notions of pointwise LP well-posedness for (SOP) with respect to variable set relation. Throughout this section, $\mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$ is assumed to be nonempty. Let $v \in \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$. Consider the following scalar optimization:

$$\text{(P)} \min g^{\preccurlyeq_u^{\mathscr{K}}}(F(x), F(v)) \text{ subject to } x \in D.$$

Let $\mathrm{argmin}(D, g, v)$ be the set of all minimal solutions of (P).

Motivated by [12, Definition 3.3] and [18, Definition 2.6], we introduce the following concepts for set optimization with a variable structure.

Definition 3.1. Let $\hat{v} \in \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$. A sequence $\{x_n\} \subseteq X$ is called

(i) a pointwise scalar-LP minimizing sequence at $\hat{v}$, if there exists a real sequence $\varepsilon_n \to 0^+$, such that $d(x_n, D) \leq \varepsilon_n$ and $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_n), F(\hat{v})) \leq \varepsilon_n$;

(ii) a pointwise LP minimizing sequence at $\hat{v}$, if there exists a real sequence $\varepsilon_n \to 0^+$, such that $d(x_n, D) \leq \varepsilon_n$ and $F(x_n) - \varepsilon_n k^0 \preccurlyeq_u^{\mathscr{K}} F(\hat{v})$.

We now present two notions of pointwise LP well-posedness for (SOP).

Definition 3.2. Let $\hat{v} \in \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$. The problem (SOP) is said to be

(i) pointwise scalar-LP well-posed at $\hat{v}$ if, for every pointwise scalar-LP minimizing sequence $\{x_n\}$ at $\hat{v}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \hat{x} \in \mathrm{argmin}(D, g, \hat{v})$;

(ii) pointwise gLP well-posed at $\hat{v}$ if, for every pointwise LP minimizing sequence $\{x_n\}$ at $\hat{v}$, there exists a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to \hat{x}$ such that $d(\hat{x}, D) \leq 0$ and $F(\hat{x}) \preccurlyeq_u^{\mathscr{K}} F(\hat{v})$.

We next introduce the following three approximate solution mappings.

Let $L(k^0, \cdot, \cdot) : \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}}) \times \mathbb{R}_+ \rightrightarrows X$ be defined as

$$L(k^0, v, \varepsilon) := \left\{ x \in X | d(x, D) \leq \varepsilon, F(x) - \varepsilon k^0 \preccurlyeq_u^{\mathscr{K}} F(v) \right\};$$

$w(\cdot, \cdot) : \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}}) \times \mathbb{R}_+ \rightrightarrows X$ be defined as

$$w(v, \varepsilon) := \left\{ x \in X | d(x, D) \leq \varepsilon, g^{\preccurlyeq_u^{\mathscr{K}}}(F(x), F(v)) \leq \varepsilon \right\}$$

and $\mathscr{D}_1 : \mathbb{R}_+ \rightrightarrows X$ be defined as

$$\mathscr{D}_1(\varepsilon) := \bigcup_{v \in \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})} \left\{ x \in X | d(x, D) \leq \varepsilon, F(x) - \varepsilon k^0 \preccurlyeq_u^{\mathscr{K}} F(v) \right\}.$$

Proposition 3.1. The following statements hold:

(i) If D is closed and $-\mathscr{K}$ is transitive, then $\mathscr{D}_1(0) \subseteq \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$.

(ii) If, for any $x \in \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$, there exists $\tilde{x} \in \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$ such that $F(\tilde{x}) \sim F(x)$, then $\mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}}) \subseteq \mathscr{D}_1(0)$.

Proof. (i) Let $x_0 \in \mathscr{D}_1(0)$. Then there exists $y_0 \in \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$ such that $d(x_0, D) \leq 0$ and $F(x_0) \preccurlyeq_u^{\mathscr{K}} F(y_0)$. For any $x \in D$ satisfying $F(x) \preccurlyeq_u^{\mathscr{K}} F(x_0)$, since $-\mathscr{K}$ is transitive, from Lemma 2.1(ii), we obtain that $F(x) \preccurlyeq_u^{\mathscr{K}} F(y_0)$. It follows from $y_0 \in \mathrm{Min}(D, F, \preccurlyeq_u^{\mathscr{K}})$ that

$F(y_0) \preccurlyeq_u^{\mathscr{K}} F(x)$, which together with the transitivity of $-\mathscr{K}$ gives $F(x_0) \preccurlyeq_u^{\mathscr{K}} F(x)$. Then $x_0 \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$.

(ii) Let $\bar{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Then $d(\bar{x},D) \leq 0$. Since there exists $\hat{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $F(\hat{x}) \sim F(\bar{x})$, we have $\bar{x} \in \mathscr{D}_1(0)$. □

Remark 3.1. If $-\mathscr{K}$ is reflexive, then $F(x) \sim F(x)$ for all $x \in D$. Thus $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) \subseteq \mathscr{D}_1(0)$.

From Proposition 3.1, we obtain the following corollary.

Corollary 3.1. Let D be closed. If $-\mathscr{K}$ is reflexive and transitive, then

$$\mathscr{D}_1(0) = \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}).$$

Next, we give a characterization for the pointwise gLP well-posedness.

Theorem 3.1. Let $\bar{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Then (SOP) is pointwise gLP well-posed at $\bar{x}$ if and only if $L(k^0,\bar{x},\cdot)$ is u.s.c. at 0 and $L(k^0,\bar{x},0)$ is compact.

Proof. Necessity. Let $\{x_n\} \subseteq L(k^0,\bar{x},0)$. Then $d(x_n,D) \leq 0$ and $F(x_n) \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$, which implies $F(x_n) \subseteq \bigcup_{y\in F(\bar{x})}(y - \mathscr{K}(y))$. Hence, for any $\varepsilon_n \to 0^+$, we have $d(x_n,D) \leq \varepsilon_n$ and

$$F(x_n) - \varepsilon_n k^0 \subseteq \bigcup_{y\in F(\bar{x})} (y - \varepsilon_n k^0 - \mathscr{K}(y)) \subseteq \bigcup_{y\in F(\bar{x})} (y - \mathscr{K}(y)).$$

Therefore, $\{x_n\}$ is a pointwise LP minimizing sequence at $\bar{x}$. Since (SOP) is pointwise gLP well-posed at $\bar{x}$, there exists a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to \hat{x}$, such that $d(\hat{x},D) \leq 0$ and $F(\hat{x}) \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$. Then $\hat{x} \in L(k^0,\bar{x},0)$. Thus $L(k^0,\bar{x},0)$ is compact. Assume that $L(k^0,\bar{x},\cdot)$ is not u.s.c. at $\varepsilon = 0$. Then there exists an open set V_0 containing $L(k^0,\bar{x},0)$. And, for any $\gamma > 0$, there exists $\varepsilon_0 \in [0,\gamma)$ such that $L(k^0,\bar{x},\varepsilon_0) \nsubseteq V_0$. Thus, there exist $\mu_n \in [0,\frac{1}{n})$ and a sequence $\{x_n\}$, such that $x_n \in L(k^0,\bar{x},\mu_n)$ but $x_n \notin V_0$. Let $\varepsilon_n = \mu_n + \frac{1}{2n}$. Then $\mu_n \leq \varepsilon_n$ and $\varepsilon_n \to 0^+$. Since $\mu_n \leq \varepsilon_n$, we have $L(k^0,\bar{x},\mu_n) \subseteq L(k^0,\bar{x},\varepsilon_n)$. Hence, $x_n \in L(k^0,\bar{x},\varepsilon_n)$ but $x_n \notin V_0$. Then

$$d(x_n,D) \leq \varepsilon_n \ \text{ and } \ F(x_n) - \varepsilon_n k^0 \subseteq \bigcup_{y\in F(\bar{x})} (y - \mathscr{K}(y)).$$

Hence, $\{x_n\}$ is a pointwise LP minimizing sequence at $\bar{x}$. Therefore, there exists a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to \hat{x}$ such that $d(\hat{x},D) \leq 0$ and $F(\hat{x}) \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$. Then $\hat{x} \in L(k^0,\bar{x},0) \subseteq V_0$. Thus there exists $m \in \mathbb{N}$, such that for any $k > m$, $x_{n_k} \in V_0$, which contradicts $x_{n_k} \notin V_0$.

Sufficiency. Let $\{x_n\}$ be a pointwise LP minimizing sequence at $\bar{x}$. Then, there exists $\varepsilon_n \to 0^+$ such that $d(x_n,D) \leq \varepsilon_n$ and $F(x_n) - \varepsilon_n k^0 \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$. Hence, $x_n \in L(k^0,\bar{x},\varepsilon_n)$. Since $L(k^0,\bar{x},\cdot)$ is u.s.c. at $\varepsilon = 0$ and $L(k^0,\bar{x},0)$ is compact, from Lemma 2.4(i) there exists $x_{n_k} \to \hat{x} \in L(k^0,\bar{x},0)$. Hence, $F(\hat{x}) \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$ and $d(\hat{x},D) \leq 0$. Therefore, (SOP) is pointwise gLP well-posed at $\bar{x}$. □

Remark 3.2. If, for any $a \in A$, $\mathscr{K}(a) \equiv K$, which is a fixed convex cone, then Theorem 3.1 reduces to Theorem 3.14(a) of [1].

Now, we give an example to illustrate Theorem 3.1.

Example 3.1. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $D = [0,1]$ and $k^0 = 1$. Define $F : X \rightrightarrows Y$ and $\mathscr{K} : Y \rightrightarrows Y$ by $F(x) = [0,|x|]$ and $\mathscr{K}(y) = [|y|,+\infty)$, respectively. If $x_0 = 0$, then $F(0) \sim F(0)$ and $F(x) \nsubseteq \bigcup_{y\in F(0)}(y - \mathscr{K}(y))$ for any $x \in D \setminus \{0\}$. Thus $x_0 = 0 \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. If $x_0 \in (0,1]$, by taking $x_1 = 0$, we can observe that $F(x_1) \preccurlyeq_u^{\mathscr{K}} F(x_0)$, but $F(x_0) \not\preccurlyeq_u^{\mathscr{K}} F(x_1)$, which implies

$x_0 \notin \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Hence, $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) = \{0\}$. Consider $\bar{x} = 0$. Let $\{x_n\}$ be a pointwise LP minimizing sequence at 0. Then there exists $\varepsilon_n \to 0^+$ such that $x_n \in [-\varepsilon_n, \varepsilon_n]$, which implies that there exists a subsequence $x_{n_k} \to 0 \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Thus (SOP) is pointwise gLP well-posed at 0.

We observe that

$$L(k^0,0,\varepsilon) = \begin{cases} \{0\}, & \varepsilon = 0, \\ [-\varepsilon,\varepsilon], & \varepsilon > 0. \end{cases}$$

Clearly, $L(k^0,0,\cdot)$ is u.s.c. at $\varepsilon = 0$ and $L(k^0,0,0)$ is compact.

Theorem 3.2. Let D be closed, $\bar{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ and $-\mathscr{K}$ be transitive. If (H_1) holds and $\mathrm{WMax}(F(\bar{x}),\mathscr{K}) \neq \emptyset$, then (SOP) is pointwise scalar-LP well-posed at $\bar{x}$ if and only if $w(\bar{x},\cdot)$ is u.s.c. at $\varepsilon = 0$ and $w(\bar{x},0)$ is compact.

Proof. Necessity. Let $\{x_n\} \subseteq w(\bar{x},0)$. Then $d(x_n,D) \leq 0$ and $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_n),F(\bar{x})) \leq 0$. Hence, $\{x_n\}$ is a pointwise scalar-LP minimizing sequence at $\bar{x}$. Since (SOP) is pointwise scalar-LP well-posed at $\bar{x}$, there exists a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to \hat{x} \in \mathrm{argmin}(D,g,\bar{x})$. Hence, $g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{x}),F(\bar{x})) \leq g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_{n_k}),F(\bar{x})) \leq 0$ and $d(\hat{x},D) \leq 0$, which implies $\hat{x} \in w(\bar{x},0)$. Then $w(\bar{x},0)$ is compact. If $w(\bar{x},\cdot)$ is not u.s.c. at $\varepsilon = 0$, then there exists an open set V_0 with $w(\bar{x},0) \subseteq V_0$ and for any $\gamma > 0$, there exists $\varepsilon_0 \in [0,\gamma)$ such that $w(\bar{x},\varepsilon_0) \nsubseteq V_0$. Thus there exist $\mu_n \in [0,\frac{1}{n})$ and a sequence $\{x_n\}$ such that $x_n \in w(\bar{x},\mu_n)$ but $x_n \notin V_0$. Let $\varepsilon_n = \mu_n + \frac{1}{2n}$. Then $\mu_n \leq \varepsilon_n$ and $\varepsilon_n \to 0^+$. Since $\mu_n \leq \varepsilon_n$, we have $w(\bar{x},\mu_n) \subseteq w(\bar{x},\varepsilon_n)$. Hence $x_n \in w(\bar{x},\varepsilon_n)$ but $x_n \notin V_0$. Then $d(x_n,D) \leq \varepsilon_n$ and $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_n),F(\bar{x})) \leq \varepsilon_n$. Hence, $\{x_n\}$ is a pointwise scalar-LP minimizing sequence at $\bar{x}$. Therefore, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to \hat{x} \in \mathrm{argmin}(D,g,\bar{x})$. Thus $g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{x}),F(\bar{x})) \leq g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_{n_k}),F(\bar{x})) \leq \varepsilon_{n_k}, \forall k \in \mathbb{N}$. Therefore, $g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{x}),F(\bar{x})) \leq 0$ and $d(\hat{x},D) \leq 0$. Hence, $\hat{x} \in w(\bar{x},0) \subseteq V_0$, which contradicts $x_n \notin V_0$.

Sufficiency. Let $\{x_n\}$ be a pointwise scalar-LP minimizing sequence at $\bar{x}$. Then there exists $\varepsilon_n \to 0^+$ such that $d(x_n,D) \leq \varepsilon_n$ and $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_n),F(\bar{x})) \leq \varepsilon_n$. Hence, $x_n \in w(\bar{x},\varepsilon_n)$. Since $w(\bar{x},\cdot)$ is u.s.c at $\varepsilon = 0$, from Lemma 2.4(i), it follows that there exists a subsequence $\{x_{n_k}\}$, such that $x_{n_k} \to \hat{x} \in w(\bar{x},0)$. Then $g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{x}),F(\bar{x})) \leq 0$ and $\hat{x} \in D$. From $\bar{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ and Theorem 2.1 we obtain that $\bar{x} \in \mathrm{WMin}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Then it follows from Theorem 2.2 and Corollary 2.1(i) that $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x),F(\bar{x})) \geq 0$. Hence,

$$g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{x}),F(\bar{x})) \leq 0 \leq g^{\preccurlyeq_u^{\mathscr{K}}}(F(x),F(\bar{x})), \forall x \in D.$$

Therefore $\hat{x} \in \mathrm{argmin}(D,g,\bar{x})$. Thus (SOP) is pointwise scalar-LP well-posed at $\bar{x}$. □

The following example illustrates Theorem 3.2.

Example 3.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $D = [0,1]$, $k^0 = (1,1)$ and $\mathscr{K}(y) = \mathbb{R}^2_+$ for all $y \in Y$. Define $F : X \rightrightarrows Y$ by

$$F(x) = \begin{cases} [x,1] \times [x,2-x], & |x| \leqslant 1, \\ \{(1,1)\}, & |x| > 1. \end{cases}$$

We observe that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) = \{1\}$. If $0 \le x < 1$, then $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x),F(1)) = 1-x$; if $x=1$, then $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x),F(1)) = 0$. Hence $\mathrm{argmin}(D,g,1) = \{1\}$ and

$$w(1,\varepsilon) = \begin{cases} \{1\}, & \varepsilon = 0, \\ [1-\varepsilon,1], & \varepsilon > 0. \end{cases}$$

Clearly, $w(1,\cdot)$ is u.s.c. at $\varepsilon = 0$ and $w(1,0)$ is compact. We can verify that (SOP) is pointwise scalar-LP well-posed at $\bar{x} = 1$.

We next give a sufficient condition for the pointwise gLP well-posedness for (SOP).

Theorem 3.3. Let $\bar{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ and $-\mathscr{K}$ be reflexive. If $\mathrm{diam}L(k^0,\bar{x},\varepsilon) \to 0$ as $\varepsilon \to 0^+$, then (SOP) is pointwise gLP well-posed at $\bar{x}$.

Proof. Let $\{x_n\}$ be a pointwise LP minimizing sequence at $\bar{x}$. Then, there exists $\varepsilon_n \to 0^+$ such that $d(x_n,D) \le \varepsilon_n$ and $F(x_n) - \varepsilon_n k^0 \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$. Then $x_n \in L(k^0,\bar{x},\varepsilon_n)$. Since $-\mathscr{K}$ is reflexive, we have $F(\bar{x}) \subseteq \bigcup_{y\in F(\bar{x})}(y-\mathscr{K}(y))$. Hence,

$$F(\bar{x}) - \varepsilon_n k^0 \subseteq \bigcup_{y\in F(\bar{x})} (y-\mathscr{K}(y)) - \varepsilon_n k^0 \subseteq \bigcup_{y\in F(\bar{x})} (y-\mathscr{K}(y)),$$

which implies that $\bar{x} \in L(k^0,\bar{x},\varepsilon_n)$. Combining this with $x_n \in L(k^0,\bar{x},\varepsilon_n)$, we obtain that $d(x_n,\bar{x}) \le \mathrm{diam}(L(k^0,\bar{x},\varepsilon_n)) \to 0$. Thus $x_n \to \bar{x}$. Since $-\mathscr{K}$ if reflexive, one has $F(\bar{x}) \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$. Therefore (SOP) is pointwise gLP well-posed at $\bar{x}$. □

Remark 3.3. (i) If we replace $\mathscr{K}(y) = [|y|,+\infty)$ by $\mathscr{K}(y) = [|y|,+\infty) \cup \{0\}$ for any $y \in Y$ in Example 3.1, then $-\mathscr{K}$ is reflexive. We observe that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) = \{0\}$ and $\mathrm{diam}(L(k^0,\bar{x},\varepsilon)) \to 0$ as $\varepsilon \to 0^+$. Similar to Example 3.1, we can verify that (SOP) is pointwise gLP well-posed at 0.

(ii) If $-\mathscr{K}$ is not reflexive, then Theorem 3.3 may not hold. For instance, if we replace $\mathscr{K}(y) = [|y|,+\infty)$ by $\mathscr{K}(y) = (|y|,+\infty)$ for any $y \in Y$ in Example 3.1, then $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) = [0,1]$ and $\mathrm{diam}(L(k^0,0,\varepsilon)) \to 0$ as $\varepsilon \to 0^+$, but (SOP) is not pointwise gLP well-posed at 0.

We now give a characterization of the pointwise scalar-LP well-posedness for (SOP).

Theorem 3.4. Let $\bar{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, $\mathrm{WMax}(F(\bar{x}),\mathscr{K}) \neq \emptyset$, $-\mathscr{K}$ be reflexive and transitive. If $\mathrm{diam}(w(\bar{x},\varepsilon)) \to 0$ as $\varepsilon \to 0^+$, then (SOP) is pointwise scalar-LP well-posed at $\bar{x}$.

Proof. Let $\{x_n\}$ be a pointwise scalar-LP minimizing sequence at $\bar{x}$. Then there exists $\varepsilon_n \to 0^+$ such that $d(x_n,D) \le \varepsilon_n$ and $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_n),F(\bar{x})) \le \varepsilon_n$. Hence, $x_n \in w(\bar{x},\varepsilon_n)$. Since $-\mathscr{K}$ is reflexive, we have $F(\bar{x}) \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$. Then, it follows from Lemma 2.2(ii) that $g^{\preccurlyeq_u^{\mathscr{K}}}(F(\bar{x}),F(\bar{x})) \le 0$. Therefore $\bar{x} \in w(\bar{x},\varepsilon_n)$, which implies

$$0 \le d(x_n,\bar{x}) \le \mathrm{diam}(w(\bar{x},\varepsilon_n)) \to 0.$$

Then, we have $x_n \to \bar{x}$. Since $\bar{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, by Theorems 2.1 and 2.2, we have $F(x) \not\prec_u^{\mathscr{K}} F(\bar{x})$ for any $x \in D$. Hence $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x),F(\bar{x})) \ge 0$. Thus $\bar{x} \in \mathrm{argmin}(D,g,\preccurlyeq_u^{\mathscr{K}})$. □

Remark 3.4. Example 3.2 can also be used to illustrate Theorem 3.4.

Proposition 3.11 in [23] establishes the relationship between the generalized l-well-posedness with perturbation and the l-well-set property with perturbation. By adopting a proof strategy analogous to that employed in this proposition, in what follows, we give a characterization of the pointwise gLP well-posedness in terms of upper Hausdorff convergence of sequences of approximate solution sets.

Theorem 3.5. Let $\bar{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. The following assertions hold.
(i) If (SOP) is pointwise gLP well-posed at $\bar{x}$, then $ex(L(k^0,\bar{x},\varepsilon),L(k^0,\bar{x},0)) \to 0$ as $\varepsilon \to 0^+$.
(ii) If $L(k^0,\bar{x},0)$ is compact and $ex(L(k^0,\bar{x},\varepsilon),L(k^0,\bar{x},0)) \to 0$ as $\varepsilon \to 0^+$, then (SOP) is pointwise gLP well-posed at $\bar{x}$.

Proof. (i) Assume that there exists $\varepsilon_n \to 0^+$ such that $ex(L(k^0,\bar{x},\varepsilon_n),L(k^0,\bar{x},0)) \nrightarrow 0$. Then, there exist $\delta > 0$ and $\varepsilon_{n_k} \to 0^+$ such that

$$L(k^0,\bar{x},\varepsilon_{n_k}) \nsubseteq L(k^0,\bar{x},0) + \mathbb{B}(0,\delta).$$

Therefore, there exists $x_{n_k} \in L(k^0,\bar{x},\varepsilon_{n_k})$ such that $x_{n_k} \notin L(k^0,\bar{x},0) + \mathbb{B}(0,\delta)$. Then $\{x_{n_k}\}$ is a pointwise LP minimizing sequence at $\bar{x}$. Since (SOP) is pointwise gLP well-posed at $\bar{x}$, there exists $x_{n_{k_l}} \to \hat{x}$ such that $d(\hat{x},D) \le 0$ and $F(\hat{x}) \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$, which imply that $\hat{x} \in L(k^0,\bar{x},0)$. Hence $x_{n_{k_l}} \in L(k^0,\bar{x},0) + \mathbb{B}(0,\delta)$ for sufficiently large l, which is a contradiction.

(ii) Let $\{x_n\}$ be a pointwise LP minimizing sequence at $\bar{x}$. Then there exists $\varepsilon_n \to 0^+$ such that $x_n \in L(k^0,\bar{x},\varepsilon_n)$. Since $ex(L(k^0,\bar{x},\varepsilon),L(k^0,\bar{x},0)) \to 0$ as $\varepsilon \to 0$, then, for any $\delta > 0$, $ex(L(k^0,\bar{x},\varepsilon_n),L(k^0,\bar{x},0)) < \delta$ for sufficiently large n. Then $d(x_n,L(\bar{x},k^0,0)) < \delta$ for any $x_n \in L(\bar{x},k^0,\varepsilon_n)$, which implies that there exists $u_n \in L(\bar{x},k^0,0)$ such that $\|x_n - u_n\| < \delta$. As $L(\bar{x},k^0,0)$ is compact, there exists a subsequence $u_{n_k} \to \hat{x} \in L(\bar{x},k^0,0)$. From the arbitrariness of δ, we have $x_{n_k} \to \hat{x} \in L(\bar{x},k^0,0)$. Hence, $d(\hat{x},D) \le 0$ and $F(\hat{x}) \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$. Thus (SOP) is pointwise gLP well-posed at $\bar{x}$. □

The following example justifies that Theorem 3.5(ii) may not hold in the absence of the compactness of $L(k^0,\bar{x},0)$.

Example 3.3. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $D = [0,2]$ and $k^0 = (1,1)$. Define $F : X \rightrightarrows Y$ and $\mathscr{K} : Y \rightrightarrows Y$ by

$$F(x) = \begin{cases} \{(1,1)\}, & x \le 0, \\ (0,1)\times(0,1), & 0 < x \le 1, \\ [0,x)\times[0,x), & x > 1, \end{cases}$$

and

$$\mathscr{K}(y) = \begin{cases} \mathbb{R}^2_+, & y = (y_1,y_2) \in [0,1)\times[0,1), \\ \{(s,t) \in \mathbb{R}^2 | s \ge 0, t \ge |y_2 - 1|\}, & y = (y_1,y_2) \notin [0,1)\times[0,1), \end{cases}$$

respectively. A direct calculation gives $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) = (0,1]$. For any $\bar{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, we observe that

$$L(k^0,\bar{x},\varepsilon) = \begin{cases} [-\varepsilon, 1+\varepsilon], & \varepsilon > 0, \\ (0,1], & \varepsilon = 0. \end{cases}$$

Clearly, $ex(L(k^0,\bar{x},\varepsilon),L(k^0,\bar{x},0)) \to 0$ as $\varepsilon \to 0^+$. If $\varepsilon_n \to 0^+$ and $x_n \in (-\varepsilon_n, 0)$, then $\{x_n\}$ is a pointwise LP minimizing sequence at $\bar{x}$. For any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $x_{n_k} \to 0$ as $\varepsilon_{n_k} \to 0$, but $F(0) \not\preccurlyeq_u^{\mathscr{K}} F(\bar{x})$.

Combining Theorems 3.1 and 3.5, we obtain the following corollary.

Corollary 3.2. Let $\bar{x} \in \text{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ and $L(k^0,\bar{x},0)$ be compact. If $ex(L(k^0,\bar{x},\varepsilon),L(k^0,\bar{x},0)) \to 0$ as $\varepsilon \to 0^+$, then $L(k^0,\bar{x},\cdot)$ is u.s.c. at $\varepsilon = 0$.

Next, we give a characterization of pointwise gLP well-posedness for (SOP) in terms of the Painlevé-Kuratowski convergence of sequences of approximate solution sets.

Theorem 3.6. Let $\bar{x} \in \text{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ and $\{\varepsilon_n\}$ be a sequence of positive numbers with $\varepsilon_n \to 0^+$. If (SOP) is pointwise gLP well-posed at $\bar{x}$, then $\text{Ls}\ (L(k^0,\bar{x},\varepsilon_n)) \subseteq L(k^0,\bar{x},0)$. Moreover, if $L(k^0,\bar{x},\cdot)$ is l.s.c. at $\varepsilon = 0$, then $L(k^0,\bar{x},\varepsilon_n) \xrightarrow{K} L(k^0,\bar{x},0)$.

Proof. Let $x \in \text{Ls}\ (L(k^0,\bar{x},\varepsilon_n))$. Then there exists $x_{n_k} \in L(k^0,\bar{x},\varepsilon_{n_k})$ such that $x_{n_k} \to x$. It can be seen that $\{x_{n_k}\}$ is a pointwise LP minimizing sequence at $\bar{x}$. Since (SOP) is pointwise gLP well-posed at $\bar{x}$, there exists $x_{n_{k_l}} \to x$ such that $d(x,D) \le 0$ and $F(x) \preccurlyeq_u^{\mathscr{K}} F(\bar{x})$. Thus $x \in L(k^0,\bar{x},0)$.

Assume that $x_0 \in L(k^0,\bar{x},0)$. Since $L(k^0,\bar{x},\cdot)$ is l.s.c. at $\varepsilon = 0$, it follows from Lemma 2.4(ii) that there exists $x_n \in L(k^0,\bar{x},\varepsilon_n)$ such that $x_n \to x_0$. Then $x_0 \in \text{Li}\ L(k^0,\bar{x},\varepsilon_n)$. Thus $L(k^0,\bar{x},0) \subseteq \text{Li}\ L(k^0,\bar{x},\varepsilon_n)$. □

In what follows, we present a characterization of pointwise scalar-LP well-posedness.

Theorem 3.7. Let $\bar{x} \in \text{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ and $\mathbf{0} \in \bar{\mathscr{K}}(F(\bar{x}))$. If (SOP) is pointwise scalar-LP well-posed at $\bar{x}$ and $\varepsilon_n \to 0^+$, then $\text{Ls}\ w(\bar{x},\varepsilon_n) \subseteq w(\bar{x},0)$. Moreover, if $w(\bar{x},\cdot)$ is l.s.c. at $\varepsilon = 0$, then $w(\bar{x},\varepsilon_n) \xrightarrow{K} w(\bar{x},0)$.

Proof. Let $x_0 \in \text{Ls}\ w(\bar{x},\varepsilon_n)$. Then there exists $x_{n_k} \in w(\bar{x},\varepsilon_{n_k})$ such that $x_{n_k} \to x_0$. It follows from $x_{n_k} \in w(\bar{x},\varepsilon_{n_k})$ that $\{x_{n_k}\}$ is a pointwise scalar-LP minimizing sequence at $\bar{x}$. Since (SOP) is pointwise scalar-LP well-posed at $\bar{x}$, there exists a subsequence of $\left\{x_{n_{k_l}}\right\}$ converging to some $\hat{x} \in \text{argmin}(D,g,\bar{x})$. Then $x_0 = \hat{x} \in \text{argmin}(D,g,\bar{x})$. Hence, $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_0),F(\bar{x})) \le g^{\preccurlyeq_u^{\mathscr{K}}}(F(\bar{x}),F(\bar{x})) \le 0$, which implies $x_0 \in w(\bar{x},0)$. Assume that $\tilde{x} \in w(\bar{x},0)$. Since $w(\bar{x},\cdot)$ is l.s.c. at $\varepsilon = 0$, it follows from Lemma 2.4(ii) that there exists $x_n \in w(\bar{x},\varepsilon_n)$ such that $x_n \to \tilde{x}$. Then $\tilde{x} \in \text{Li}\ w(\bar{x},\varepsilon_n)$. Thus $w(\bar{x},0) \subseteq \text{Li}\ w(\bar{x},\varepsilon_n)$. □

4. Global LP well-posedness of (SOP)

Gupta et al. [12] introduced Levitin-Polyak well-posedness in the set and scalar senses, respectively. Motivated by Definitions 3.1-3.5 in [12], in this section, we introduce some notions of global LP well-posedness for set optimization with respect to a variable set relation.

Definition 4.1. A sequence $\{x_n\} \subseteq X$ is called

(i) a global scalar-LP minimizing sequence for (SOP) with respect to $\{v_n\} \subseteq \text{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, if there exists a sequence of positive numbers $\{\varepsilon_n\}$ with $\varepsilon_n \to 0^+$ as $n \to +\infty$ such that $d(x_n,D) \le \varepsilon_n$ and $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_n),F(v_n)) \le \varepsilon_n$;

(ii) a global set-LP minimizing sequence for (SOP), if there exist a sequence of positive numbers $\{\varepsilon_n\}$ with $\varepsilon_n \to 0^+$ as $n \to +\infty$ and $v_n \in \text{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $d(x_n,D) \le \varepsilon_n$ and $F(x_n) \preccurlyeq_u^{\mathscr{K}} F(v_n) + \varepsilon_n\bar{\mathbb{B}}$;

(iii) a global LP minimizing sequence for (SOP), if there exist a sequence of positive numbers $\{\varepsilon_n\}$ with $\varepsilon_n \to 0^+$ as $n \to +\infty$ and $v_n \in \text{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $d(x_n,D) \le \varepsilon_n$ and $F(x_n) - \varepsilon_n k^0 \preccurlyeq_u^{\mathscr{K}} F(v_n)$.

Definition 4.2. The problem (SOP) is said to be

(i) globally scalar-LP well-posed if, for every global scalar-LP minimizing sequence $\{x_n\}$ for (SOP) with respect to $\{v_n\} \subseteq \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \to \hat{v}$ and $x_{n_k} \to \hat{x}$ with $\hat{x} \in \mathrm{argmin}(D,g,\hat{v})$;

(ii) globally set-LP well-posed if, for every global set-LP minimizing sequence $\{x_n\}$ for (SOP), there exists a subsequence $\{x_{n_k}\}$ such that $d(x_{n_k}, \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})) \to 0$;

(iii) globally gLP well-posed if, for every global LP minimizing sequence $\{x_n\}$ for (SOP), there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to \hat{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$.

Next, we define $\mathscr{D} : \mathbb{R}_+ \rightrightarrows X$ by

$$\mathscr{D}(\varepsilon) := \bigcup_{y \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})} \left\{ x \in X | d(x,D) \leq \varepsilon, F(x) \preccurlyeq_u^{\mathscr{K}} F(y) + \varepsilon \bar{\mathbb{B}} \right\}.$$

Here, $\mathscr{D} : \mathbb{R}_+ \rightrightarrows X$ and $\mathscr{D}(\varepsilon)$ denote the approximate mapping and the set of approximate solutions, respectively. We observe that $\mathscr{D}(0) = \mathscr{D}_1(0)$.

Next, we give some characterizations of the three types of global LP well-posedness in Definition 4.2 in terms of Hausdorff upper semicontinuity, closedness and compactness of the approximate solution mapping.

Theorem 4.1. The following statements hold.

(i) Let $\mathbf{0} \in \bar{\mathscr{K}}(F(\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})))$. If (SOP) is globally scalar-LP well-posed, then, for any $\hat{v} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, $w(\cdot,\cdot)$ is closed at $(\hat{v},0)$.

(ii) Assume that D is compact and (H_1) holds. Let $-\mathscr{K}$ be transitive and $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ be compact. If $\mathrm{WMax}(F(\hat{v}),\mathscr{K}) \neq \emptyset$ and $w(\cdot,\cdot)$ is closed at $(\hat{v},0)$ for any $\hat{v} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, then (SOP) is globally scalar-LP well-posed.

Proof. (i) Let $(v_n,\varepsilon_n) \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) \times \mathbb{R}_+$ such that $(v_n,\varepsilon_n) \to (\hat{v},0)$, and $x_n \in w(v_n,\varepsilon_n)$ such that $x_n \to \hat{x}$. We next prove $\hat{x} \in w(\hat{v},0)$. Take $\delta_n = \varepsilon_n + \frac{1}{n}$, for any $n \in \mathbb{N}$. From $x_n \in w(v_n,\varepsilon_n)$, we have $d(x_n,D) \leq \varepsilon_n \leq \delta_n$ and $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_n),F(v_n)) \leq \delta_n$. Hence $\{x_n\}$ is a global scalar-LP minimizing sequence for (SOP) with respect to $\{v_n\} \subseteq \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Since (SOP) is globally scalar-LP well-posed, there exist subsequences $\{v_{n_k}\}$ and $\{x_{n_k}\}$, such that $v_{n_k} \to \hat{v}$ and $x_{n_k} \to \hat{x} \in \mathrm{argmin}(D,g,\hat{v})$. Thus $g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{x}),F(\hat{v})) \leq g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{v}),F(\hat{v}))$. It follows from $\mathbf{0} \in \bar{\mathscr{K}}(F(\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})))$ that $F(\hat{v}) \preccurlyeq_u^{\mathscr{K}} F(\hat{v})$. Combining this with Lemma 2.2(ii), we have $g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{v}),F(\hat{v})) \leq 0$. Hence, $\hat{x} \in w(\hat{v},0)$. Thus, $w(\cdot,\cdot)$ is closed at $(\hat{v},0)$.

(ii) Let $\{x_n\}$ be a global scalar-LP minimizing sequence for (SOP) with respect to $\{v_n\} \subseteq \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Then there exists $\varepsilon_n \to 0^+$ such that $d(x_n,D) \leq \varepsilon_n$ and $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_n),F(v_n)) \leq \varepsilon_n$, which implies $x_n \in w(v_n,\varepsilon_n)$. Since $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ is compact, there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and $\hat{v} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $v_{n_k} \to \hat{v}$. Since $d(x_{n_k},D) \leq \varepsilon_{n_k}$ and D is compact, we have that there exists $u_{n_k} \in D$ such that

$$\|x_{n_k} - u_{n_k}\| \leq \varepsilon_{n_k}. \tag{4.1}$$

Since D is compact, there exists $u_{n_{k_l}} \to \hat{x} \in D$. Then, from (4.1), it can be deduced that $x_{n_{k_l}} \to \hat{x}$. Since $w(\cdot,\cdot)$ is closed at $(\hat{v},0)$, one gets $\hat{x} \in w(\hat{v},0)$. Consequently, we obtain $g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{x}),F(\hat{v})) \leq 0$. Since $\hat{v} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, by virtue of Theorems 2.1 and 2.2, we can infer that $F(x) \not\preccurlyeq_u^{\mathscr{K}} F(\hat{v}), \forall x \in D$. Then it follows from Corollary 2.1(i) that $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x),F(\hat{v})) \geq 0$, which implies $\hat{x} \in \mathrm{argmin}(D,g,\hat{v})$. Hence (SOP) is globally scalar-LP well-posed. □

The following example justifies that Theorem 4.1(i) may not hold without the condition $\mathbf{0} \in \bar{\mathscr{K}}(F(\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})))$.

Example 4.1. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $D=[-1,2]$ and $k^0=(1,1)$. Define $F:X\rightrightarrows Y$ and $\mathscr{K}:Y\rightrightarrows Y$ by

$$F(x)=\begin{cases}\{(0,0)\}, & x<-1,\\ \{(1,1)\}, & -1\le x<0,\\ [0,1+x)\times[0,1+x), & x\ge 0,\end{cases}$$

and

$$\mathscr{K}(y)=\begin{cases}\left\{(s,t)\in\mathbb{R}^2_{++}|t>-s+1\right\}, & y\in[0,1)\times[0,1),\\ \mathbb{R}^2_+, & \text{otherwise},\end{cases}$$

respectively. A direct calculation gives that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})=\{0\}$. It is clear that $\mathbf{0}\notin\bar{\mathscr{K}}(F(0))$ and (SOP) is globally scalar-LP well-posed. For any $u\in\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ and $\varepsilon\ge 0$, we observe that

$$w(u,\varepsilon)=\begin{cases}(-1-\varepsilon,-1), & \varepsilon>0,\\ \emptyset, & \varepsilon=0.\end{cases}$$

Hence, $w(\cdot,\cdot)$ is not closed at $(0,0)$.

The following example illustrates that the condition "$-\mathscr{K}$ be transitive" in Theorem 4.1(ii) cannot be omitted.

Example 4.2. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $D=[0,1]$ and $k^0=(1,1)$. Define $F:X\rightrightarrows Y$ and $\mathscr{K}:Y\rightrightarrows Y$, respectively, by

$$F(x)=\begin{cases}[0,1]\times\{1\}, & x\le 0,\\ [0,1]\times\{-1\}, & x=1,\\ \{(3,3)\}, & \text{otherwise},\end{cases}$$

and

$$\mathscr{K}(y)=\begin{cases}\left\{(s,t)\in\mathbb{R}^2|t\ge 0\right\}, & y_2\ge 0,\\ \left\{(s,t)\in\mathbb{R}^2|t\ge 0 \text{ or } s>0\right\}, & y_2<0,\end{cases}$$

where $y=(y_1,y_2)$. We can calculate that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})=\{0,1\}$, $\mathrm{argmin}(D,g,0)=\{1\}$ and $\mathrm{argmin}(D,g,1)=\{0\}$. We observe that $w(0,\varepsilon)=(-\varepsilon,0]\cup\{1\}$, $w(0,0)=\{0,1\}$, $w(1,\varepsilon)=(-\varepsilon,0]\cup\{1\}$ and $w(1,0)=\{0,1\}$, which imply $w(\cdot,\cdot)$ is closed at $(0,0)$ and $(1,0)$. Let $x_n=-\frac{1}{n}$ and $v_n=0$ for any $n\in\mathbb{N}$. Then $\{x_n\}$ is a global scalar-LP minimizing sequence for (SOP) with respect to $\{v_n\}$, but $x_n\to 0\notin\mathrm{argmin}(D,g,0)$.

Next, we give an example to illustrate that the compactness of D in Theorem 4.1(ii) cannot be omitted.

Example 4.3. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $D=[0,2)$ and $k^0=(1,1)$. Define $F:X\rightrightarrows Y$ and $\mathscr{K}:Y\rightrightarrows Y$ by

$$F(x)=\begin{cases}\{(0,0)\}, & x<0,\\ [0,x]\times[0,1-x], & x\in[0,1],\\ \{(x-1,2-x)\}, & x>1,\end{cases}$$

and

$$\mathscr{K}(y) = \begin{cases} \left\{(s,t) \in \mathbb{R}^2 | 0 \le t \le 2s\right\}, & y \in -\mathbb{R}^2_+, \\ \mathbb{R}^2_{++}, & \text{otherwise,} \end{cases}$$

respectively. A direct calculation gives that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) = [0,1]$ and $\mathrm{argmin}(D,g,\hat{v}) = \{\hat{v},\hat{v}+1\}$ for any $\hat{v} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. We observe that for any $(v,\varepsilon) \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) \times \mathbb{R}_+$,

$$w(v,\varepsilon) = \begin{cases} (v-\varepsilon, v+\varepsilon) \cup (-\varepsilon+v+1, v+1+\varepsilon), \varepsilon > 0, \\ \{v, v+1\}, \varepsilon = 0, \end{cases}$$

which implies that $w(\cdot,\cdot)$ is closed at $(v,0)$. Let $x_n = -\frac{1}{n}$ and $v_n = 1-\frac{1}{n}$ for any $n \in \mathbb{N}$. Then $\{x_n\}$ is a global scalar-LP minimizing sequence for (SOP) with respect to $\{v_n\} \subseteq \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, but $x_n \to 0 \notin \mathrm{argmin}(D,g,1)$.

The following example shows that the compactness of $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ in Theorem 4.1(ii) can not be omitted.

Example 4.4. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $D = [0, \frac{3}{2}]$ and $k^0 = (1,1)$. Define $F : X \rightrightarrows Y$ and $\mathscr{K} : Y \rightrightarrows Y$ by

$$F(x) = \begin{cases} \{(1,1)\}, & x \in (-\infty, 0] \cup \{1\}, \\ [0,x) \times [0, 1-x), & x \in (0,1), \\ \{(x-1, 2-x)\}, & x > 1, \end{cases}$$

and

$$\mathscr{K}(y) = \begin{cases} \left\{(s,t) \in \mathbb{R}^2 | 0 \le t \le 2s\right\}, & y \in -\mathbb{R}^2_+, \\ \mathbb{R}^2_+, & \text{otherwise,} \end{cases}$$

respectively. We calculate that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) = (0,1)$ and $\mathrm{argmin}(D,g,\hat{v}) = \{\hat{v},\hat{v}+1\}$, $\forall \hat{v} \in (0,1)$. Further calculation shows that, for any $(v,\varepsilon) \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) \times \mathbb{R}_+$,

$$w(v,\varepsilon) = \begin{cases} (v-\varepsilon, v+\varepsilon) \cup (v+1-\varepsilon, v+1+\varepsilon), \varepsilon > 0. \\ \{v, v+1\}, \varepsilon = 0, \end{cases}$$

which implies that $w(\cdot,\cdot)$ is closed at $(v,0)$. Taking $x_n = \frac{1}{n}+1$ and $v_n = \frac{1}{n}$ for any $n \in \mathbb{N}$, it is clear that $\{x_n\}$ is a global scalar-LP minimizing sequence for (SOP) with respect to $\{v_n\} \subseteq \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, but $v_n \to 0 \notin \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$.

Zhou et al. [37] studied a necessary and sufficient condition for generalized k_0-well-posedness of the set optimization problem with respect to m-upper set less order relation $\preceq_K^{mu}$. We next present the necessary and sufficient conditions for globally gLP well-posedness with the set relation $\preccurlyeq_u^{\mathscr{K}}$.

Theorem 4.2. (i) Let $-\mathscr{K}$ be transitive and D be closed. If $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ is compact and $\mathscr{D}_1(\cdot)$ is H-u.s.c. at $\varepsilon = 0$, then (SOP) is globally gLP well-posed.

(ii) Assume that for any $x \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, there exists $x_0 \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $F(x_0) \sim F(x)$. If (SOP) is globally gLP well-posed, then $\mathscr{D}_1(\cdot)$ is H-u.s.c. at $\varepsilon = 0$ and $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ is compact.

Proof. (i) Let $\{x_n\}$ be a global LP minimizing sequence for (SOP). Then there exist $\varepsilon_n \to 0^+$ and $\{v_n\} \subseteq \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $d(x_n,D) \le \varepsilon_n$ and $F(x_n) - \varepsilon_n k^0 \preccurlyeq_u^{\mathscr{K}} F(v_n)$. Since $\mathscr{D}_1(\cdot)$ is H-u.s.c. at $\varepsilon = 0$, for any $\varepsilon > 0$, there exists a neighborhood $U(0)$ of 0 such that $\mathscr{D}_1(\alpha) \subseteq \mathscr{D}_1(0) + \mathbb{B}(0,\varepsilon)$ for all $\alpha \in U(0)$. Thus there exists $m \in \mathbb{N}$ such that $\varepsilon_n \in U(0)$ and $\mathscr{D}_1(\varepsilon_n) \subseteq \mathscr{D}_1(0) + \mathbb{B}(0,\varepsilon)$ for all $n > m$. Then $x_n \in \mathscr{D}_1(\varepsilon_n) \subseteq \mathscr{D}_1(0) + \mathbb{B}(0,\varepsilon)$. It follows from $\mathscr{D}_1(0) = \mathscr{D}(0)$ and Proposition 3.1(i) that $x_n \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) + \mathbb{B}(0,\varepsilon)$. Thus $d(x_n, \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})) \le \varepsilon$. From the arbitrariness of ε, we have $d(x_n, \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})) \to 0$. Since $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ is compact, there exists a subsequence $x_{n_k} \to \hat{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Thus, (SOP) is globally gLP well-posed.

(ii) Let $\{x_n\} \subseteq \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. From Proposition 3.1(ii) we have $\{x_n\} \subseteq \mathscr{D}_1(0)$. Thus $\{x_n\}$ is a global LP minimizing sequence for (SOP). Since (SOP) is globally gLP well-posed, there exists a subsequence $x_{n_k} \to \hat{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Hence $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ is compact. Assume that $\mathscr{D}_1(\cdot)$ is not H-u.s.c. at $\varepsilon = 0$. Then there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$, there exists $\alpha_n \in [0, \frac{1}{n})$ such that $\mathscr{D}_1(\alpha_n) \nsubseteq \mathscr{D}_1(0) + \mathbb{B}(0,\varepsilon_0)$. Thus, for any $n \in \mathbb{N}$, there exists $\tilde{x}_n \in \mathscr{D}_1(\alpha_n)$ such that $\tilde{x}_n \notin \mathscr{D}_1(0) + \mathbb{B}(0,\varepsilon_0)$. It follows from $\tilde{x}_n \in \mathscr{D}_1(\alpha_n)$ that $d(\tilde{x}_n, D) \le \alpha_n$ and there exists $y_n \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $F(\tilde{x}_n) - \alpha_n k^0 \preccurlyeq_u^{\mathscr{K}} F(y_n)$. Taking $\beta_n = \alpha_n + \frac{1}{n}$, we obtain $d(\tilde{x}_n, D) \le \beta_n$ and $F(\tilde{x}_n) - \beta_n k^0 \preccurlyeq_u^{\mathscr{K}} F(y_n)$. Thus $\{\tilde{x}_n\}$ is a global LP minimizing sequence for (SOP). Since (SOP) is globally gLP well-posed, there exists a subsequence $\{\tilde{x}_{n_k}\}$ such that $\tilde{x}_{n_k} \to \hat{x} \in \mathscr{D}_1(0)$. Therefore $\tilde{x}_{n_k} \in \mathscr{D}_1(0) + \mathbb{B}(0,\varepsilon_0)$ for sufficiently large k, which is a contradiction. □

Remark 4.1. The condition that D is closed cannot be dropped in Theorem 4.2(i). This can be justified by considering Example 4.3. The next example illustrates that the condition "$-\mathscr{K}$ be transitive" in Theorem 4.2(i) cannot be omitted.

Example 4.5. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $D = [0,1]$ and $k^0 = (1,1)$. Define $F : X \rightrightarrows Y$ and $\mathscr{K} : Y \rightrightarrows Y$, respectively, by

$$F(x) = \begin{cases} [0,1] \times \{1\}, & x \le 0, \\ [1,2) \times \{4\}, & x \in (0,1), \\ [0,2] \times \{-1\}, & x = 1, \\ \{(3,3)\}, & x > 1, \end{cases}$$

and

$$\mathscr{K}(y) = \begin{cases} \mathbb{R}^2_+, & y_2 > 2, \\ \left\{(s,t) \in \mathbb{R}^2 | t \ge 0\right\}, & 0 \le y_2 \le 2, \\ \left\{(s,t) \in \mathbb{R}^2 | t \ge 0 \text{ or } s > 0\right\}, & y_2 < 0, \end{cases}$$

where $y = (y_1, y_2) \in Y$. A direct calculation gives that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) = \{0\}$. We observe that

$$\mathscr{D}_1(\varepsilon) = \begin{cases} (-\varepsilon, 0] \cup \{1\}, \varepsilon > 0, \\ \{0,1\}, \varepsilon = 0, \end{cases}$$

which implies that $\mathscr{D}_1(\cdot)$ is H-u.s.c. at 0. Let $x_n = 1$ for any $n \in \mathbb{N}$. Then $\{x_n\}$ is a global LP minimizing sequence for (SOP), but $1 \notin \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$.

Now, we give an example to illustrate the condition "for any $x \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, there exists $x_0 \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $F(x_0) \sim F(x)$" in Theorem 4.2(ii) cannot be omitted.

Example 4.6. Let $X=\mathbb{R}^2$, $Y=\mathbb{R}^2$, $D=[0,2]\times[0,2]$, $k^0=(1,1)$. Define $F:X\rightrightarrows Y$ and $\mathscr{K}:Y\rightrightarrows Y$, respectively, by

$$F(x)=\begin{cases}\{0\}\times[1,2], & x\in-\mathbb{R}^2_+,\\ [0,1]\times[0,1], & x\in\{(1,1),(2,2)\},\\ \{(1,2)\}, & \text{otherwise},\end{cases}$$

and

$$\mathscr{K}(y)=\begin{cases}\mathbb{R}^2_{++}, & y_1\le 0,\\ \mathbb{R}^2_+, & y_1>0,\end{cases}$$

where $y=(y_1,y_2)\in Y$. A direct calculation gives that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})=\{(0,0),(1,1),(2,2)\}$ and $F(x)\not\preccurlyeq_u^{\mathscr{K}} F((0,0))$ for any $x\in D$. We can verify that (SOP) is globally gLP well-posed. We observe that

$$\mathscr{D}_1(\varepsilon)=\begin{cases}((-\mathbb{R}^2_+)\cap\bar{\mathbb{B}}((0,0),\varepsilon))\cup\{(1,1),(2,2)\},\varepsilon>0,\\ \{(1,1),(2,2)\},\varepsilon=0,\end{cases}$$

which implies that $\mathscr{D}_1(\cdot)$ is not H-u.s.c. at 0.

From Theorem 4.2, we can easily get the following corollary.

Corollary 4.1. Let D be closed and $-\mathscr{K}$ be transitive and reflexive. Then (SOP) is globally gLP well-posed if and only if $\mathscr{D}_1(\cdot)$ is H-u.s.c. at $\varepsilon=0$ and $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ is compact.

Theorem 4.3. (i) Let $-\mathscr{K}$ be transitive and D be closed. If $\mathscr{D}(\cdot)$ is H-u.s.c. at $\varepsilon=0$, then (SOP) is globally set-LP well-posed.

(ii) Let $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ be compact. If for any $x\in\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, there exists $v_0\in\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $F(v_0)\sim F(x)$, then that (SOP) is globally set-LP well-posed implies $\mathscr{D}(\cdot)$ is H-u.s.c. at $\varepsilon=0$.

Proof. The proof is similar to that of Theorem 4.2 and is thus omitted. □

We now present some necessary conditions of global gLP well-posedness for (SOP).

Theorem 4.4. Let F be compact on $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. If (SOP) is globally gLP well-posed, then for any global LP minimizing sequence $\{x_n\}$ for (SOP), there exists a subsequence $\{x_{n_k}\}$ such that $ex(F(x_{n_k}),F(\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})))\to 0$.

Proof. On the contrary, suppose that there exists a global LP minimizing sequence $\{\tilde{x}_n\}$ such that for any subsequence $\{\tilde{x}_{n_k}\}$, $ex(F(\tilde{x}_{n_k}),F(\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})))\nrightarrow 0$. Then there exist $\delta>0$ and a subsequence $\left\{\tilde{x}_{n_{k_l}}\right\}$ such that

$$F(\tilde{x}_{n_{k_l}})\nsubseteq F(\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}))+\mathbb{B}(\mathbf{0},\delta),\forall l\in\mathbb{N}.$$

Consequently, for any $l\in\mathbb{N}$, we can find an element $y_{n_{k_l}}\in F(\tilde{x}_{n_{k_l}})$ satisfying

$$y_{n_{k_l}}\notin F(\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})+\mathbb{B}(\mathbf{0},\delta). \tag{4.2}$$

Since $\left\{\tilde{x}_{n_{k_l}}\right\}$ is global LP minimizing sequencefor (SOP), there is a subsequence of $\left\{\tilde{x}_{n_{k_l}}\right\}$ converging to some $\hat{x}\in\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Without loss of generality, suppose that $\tilde{x}_{n_{k_l}}\to\hat{x}$. From

$y_{n_{k_l}} \in F(\tilde{x}_{n_{k_l}})$ and the compactness of F at $\hat{x}$, we deduce that there exists a subsequence $\left\{y_{n_{k_{l_m}}}\right\}$ of $\left\{y_{n_{k_l}}\right\}$, such that $y_{n_{k_{l_m}}} \to \hat{y} \in F(\hat{x}) \subseteq F(\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}))$. Then, $y_{n_{k_{l_m}}} \in F(\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})) + \mathbb{B}(\mathbf{0},\delta)$ for sufficiently large $m \in \mathbb{N}$, which contradicts with (4.2) □

Theorem 4.5. Let $-\mathscr{K}$ be reflexive and $\varepsilon_n \to 0^+$. If (SOP) is globally gLP well-posed, then $ex(\mathscr{D}_1(\varepsilon_n), \mathscr{D}_1(0)) \to 0$ as $n \to +\infty$.

Proof. On the contrary, if $ex(\mathscr{D}_1(\varepsilon_n), \mathscr{D}_1(0))$ fails to converge to 0, then there exist a positive number δ and a subsequence $\{\varepsilon_{n_k}\}$ such that $\mathscr{D}_1(\varepsilon_{n_k}) \nsubseteq \mathscr{D}_1(0) + \mathbb{B}(0,\delta)$ for any $k \in \mathbb{N}$. Hence, for any $k \in \mathbb{N}$, there exists $x_{n_k} \in \mathscr{D}_1(\varepsilon_{n_k})$ such that

$$x_{n_k} \notin \mathscr{D}_1(0) + \mathbb{B}(0,\delta). \tag{4.3}$$

It follows from $x_{n_k} \in \mathscr{D}_1(\varepsilon_{n_k})$ that $\{x_{n_k}\}$ is a global LP minimizing sequence for (SOP). Since (SOP) is globally gLP well-posed, there is a subsequence of $\{x_{n_k}\}$ converging to some $\hat{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Without loss of generality, we suppose that $x_{n_k} \to \hat{x}$. Owing to the reflexivity of $-\mathscr{K}$ and by Proposition 3.1(ii), we can deduce that $\hat{x} \in \mathscr{D}_1(0)$. Then, $x_{n_k} \in \mathscr{D}_1(0) + \mathbb{B}(0,\delta)$ for sufficiently large $k \in \mathbb{N}$, which contradicts (4.3). □

Theorem 4.6. Let $-\mathscr{K}$ be reflexive and $\varepsilon_n \to 0^+$. If (SOP) is globally gLP well-posed, then $\mathrm{Ls}\ \mathscr{D}_1(\varepsilon_n) \subseteq \mathscr{D}_1(0)$. Moreover, if $\mathscr{D}_1(\cdot)$ is l.s.c. at $\varepsilon = 0$, then $\mathscr{D}_1(\varepsilon_n) \xrightarrow{K} \mathscr{D}_1(0)$.

Proof. Let $\bar{x} \in \mathrm{Ls}\ \mathscr{D}_1(\varepsilon_n)$. Then there exists $x_{n_k} \in \mathscr{D}_1(\varepsilon_{n_k})$ such that $x_{n_k} \to \bar{x}$. Thus there exists $v_{n_k} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $d(x_{n_k}, D) \le \varepsilon_{n_k}$ and $F(x_{n_k}) - \varepsilon_{n_k} k^0 \preccurlyeq_u^{\mathscr{K}} F(v_{n_k})$. Consequently, $\{x_{n_k}\}$ is a global LP minimizing sequence for (SOP). Since (SOP) is globally gLP well-posed, we can find a subsequence of $\{x_{n_k}\}$ converging to some $\hat{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Then, from Remark 3.1, we have $\bar{x} = \hat{x} \in \mathscr{D}_1(0)$. Let $x_0 \in \mathscr{D}_1(0)$. Since $\mathscr{D}_1(\cdot)$ is l.s.c. at $\varepsilon = 0$, then, for any positive sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0^+$, it follows from Lemma 2.4(ii) that there exists $x_n \in \mathscr{D}_1(\varepsilon_n)$ such that $x_n \to x_0$. Then $x_0 \in \mathrm{Li}\mathscr{D}_1(\varepsilon_n)$. Thus $\mathscr{D}_1(0) \subseteq \mathrm{Li}\mathscr{D}_1(\varepsilon_n)$. □

5. The relationships among the different LP well-posedness

In this section, our focus shifts to exploring the relationships among the three global LP well-posedness in Section 4 and the connections between pointwise LP well-posedness and global LP well-posedness. Now, we first present the relationship between the minimal solutions of (SOP) and (P).

Theorem 5.1. Let $\hat{v} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ and $-\mathscr{K}$ be transitive.

(i) If (H_1) holds, $\mathrm{WMax}(F(\hat{v}),\mathscr{K}) \neq \emptyset$ and $F(x_1) \sim F(x_2)$ for any $x_1, x_2 \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, then $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}}) \subseteq \mathrm{argmin}(D,g,\hat{v})$.

(ii) If $-\mathscr{K}$ is reflexive and $\bigcup_{y \in F(\hat{v})}(y - \mathscr{K}(y))$ is closed, $\mathrm{argmin}(D,g,\hat{v}) \subseteq \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$.

Proof. (i) Let $x_0 \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Then $F(x_0) \sim F(\hat{v})$, which implies $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_0),F(\hat{v})) \le 0$. Since $\mathrm{WMax}(F(\hat{v})),\tilde{\mathscr{K}}) \neq \emptyset$, by virtue of Theorems 2.1 and 2.2, we can infer that $F(x) \not\prec_u^{\mathscr{K}} F(\hat{v})$ for all $x \in D$. Hence, by Corollary 2.1, one has $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x),F(\hat{v})) \ge 0$. Therefore $x_0 \in \mathrm{argmin}(D,g,\hat{v})$.

(ii) Let $\bar{x} \in \mathrm{argmin}(D,g,\hat{v})$. Then $g^{\preccurlyeq_u^{\mathscr{K}}}(F(\bar{x}),F(\hat{v})) \le g^{\preccurlyeq_u^{\mathscr{K}}}(F(x),F(\hat{v})), \forall x \in D$. Hence,

$$g^{\preccurlyeq_u^{\mathscr{K}}}(F(\bar{x}),F(\hat{v})) \le g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{v}),F(\hat{v})).$$

Since $-\mathscr{K}$ is reflexive, we have $g^{\preccurlyeq_u^{\mathscr{K}}}(F(\hat{v}),F(\hat{v}))\leq 0$. Then it follows from Lemma 2.3 that $F(\bar{x})\preccurlyeq_u^{\mathscr{K}} F(\hat{v})$. From $\hat{v}\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, it follows that $F(\bar{x})\sim F(\hat{v})$. As $-\mathscr{K}$ is transitive, we obtain $\bar{x}\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. □

Theorem 5.2. Let $-\mathscr{K}$ be transitive and reflexive. For any $\hat{v}\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, $\bigcup_{y\in F(\hat{v})}(y-\mathscr{K}(y))$ is closed. If (SOP) is globally scalar-LP well-posed, then (SOP) is globally gLP well-posed.

Proof. Let $\{x_n\}$ be a global LP minimizing sequence for (SOP). Then there exist $\varepsilon_n\to 0^+$ and $v_n\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $d(x_n,D)\leq \varepsilon_n$ and $F(x_n)-\varepsilon_n k^0\preccurlyeq_u^{\mathscr{K}} F(v_n)$. It is clear that $F(x_n)-\varepsilon_n k^0\subseteq \bigcup_{y\in F(v_n)}(y-\mathscr{K}(y))$. Thus, by virtue of Lemma 2.2(ii), we obtain $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_n),F(v_n))\leq \varepsilon_n$. Thus, $\{x_n\}$ is a global scalar-LP minimizing sequence for (SOP) with respect to $\{v_n\}$. Since (SOP) is globally scalar-LP well-posed, we obtain that there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k}\to\hat{v}$ and $x_{n_k}\to\hat{x}$ with $\hat{x}\in \mathrm{argmin}(D,g,\hat{v})$. It follows from Theorem 5.1(ii) that $\hat{v}\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Thus (SOP) is globally gLP well-posed. □

We now give an example to show Theorem 5.2 may not hold, if there exists $x_0\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $\bigcup_{y\in F(x_0)}(y-\mathscr{K}(y))$ is not closed.

Example 5.1. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $D=[-1,2]$ and $k^0=(1,1)$. Define $F:X\rightrightarrows Y$ and $\mathscr{K}:Y\rightrightarrows Y$, respectively, by

$$F(x)=\begin{cases}\{(0,0)\}, & x<-1,\\ \{(1,1)\}, & x\in[-1,0),\\ [0,1+x)\times[0,1+x), & x\geq 0,\end{cases}$$

and

$$\mathscr{K}(y)=\begin{cases}\mathbb{R}^2_{++}\cup\{\mathbf{0}\}, & y_1<1,\\ \mathbb{R}^2_{+}, & y_2\geq 1,\end{cases}$$

where $y=(y_1,y_2)$. We observe that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})=\{0\}$ and $\mathrm{argmin}(D,g,0)=[-1,0]$. Here, $\bigcup_{y\in F(0)}(y-\mathscr{K}(y))$ is not closed, (SOP) is globally scalar-LP well-posed. Take $x_n=-1-\frac{1}{n}$ for any $n\in\mathbb{N}$. Then $\{x_n\}$ is a global LP minimizing sequence for (SOP), but $x_n\to -1\notin \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$.

Theorem 5.3. Let (H_1) hold, $-\mathscr{K}$ be transitive and $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ be compact. Assume that $\mathrm{WMax}(F(v),\mathscr{K})\neq\emptyset$ for any $v\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, and $F(x_1)\sim F(x_2)$ for any $x_1,x_2\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. If (SOP) is globally gLP well-posed, then (SOP) is globally scalar-LP well-posed.

Proof. Let $\{x_n\}$ be a global scalar-LP minimizing sequence for (SOP) with respect to $\{v_n\}\subseteq \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Then there exists $\varepsilon_n\to 0^+$ such that $d(x_n,D)\leq\varepsilon_n$ and $g^{\preccurlyeq_u^{\mathscr{K}}}(F(x_n),F(v_n))\leq\varepsilon_n$. By applying Lemma 2.2(i), we deduce that

$$\bigcup_{t>\varepsilon_n}(F(x_n)-tk^0)\subseteq \bigcup_{y\in F(v_n)}(y-\mathscr{K}(y)).$$

Hence, $F(x_n)-2\varepsilon_n k^0\subseteq \bigcup_{y\in F(v_n)}(y-\mathscr{K}(y))$, i.e., $F(x_n)-2\varepsilon_n k^0\preccurlyeq_u^{\mathscr{K}} F(v_n)$. It is clear that $d(x_n,D)\leq 2\varepsilon_n$ and $2\varepsilon_n\to 0^+$. Hence $\{x_n\}$ is a global LP minimizing sequence for (SOP). Since (SOP) is globally gLP well-posed, there exists a subsequence $\{x_{n_k}\}$ converging to some

$\hat{x} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Owing to the compactness of $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, there exists $v_{n_{k_l}} \to \hat{v} \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. From Theorem 5.1(i) we have $\hat{x} \in \mathrm{argmin}(D,g,\hat{v})$. □

The following example illustrates that Theorem 5.3 may not hold if there exist $x_1,x_2 \in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $F(x_1) \not\sim F(x_2)$.

Example 5.2. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $D=[0,2]$ and $k^0=(1,1)$. Define $F: X \rightrightarrows Y$ and $\mathscr{K}: Y \rightrightarrows Y$, respectively, by

$$F(x)=\begin{cases}\{0\}\times[1,2], & x=0,\\ [0,1]\times[0,1], & x\in\{1,2\}\cup(-\infty,0),\\ \{(1,2)\}, & \text{otherwise},\end{cases}$$

and

$$\mathscr{K}(y)=\begin{cases}\mathbb{R}^2_{++}, & y_1\le 0,\\ \mathbb{R}^2_{+}, & y_1>0,\end{cases}$$

where $y=(y_1,y_2)$. We observe that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})=\{0,1,2\}$. Here, $F(0)\not\sim F(1)$ and (SOP) is globally gLP well-posed. Let $x_n=-\frac{1}{n}$ and $v_n=1$ for any $n\in\mathbb{N}$. Then $\{x_n\}$ is a global scalar-LP minimizing sequence for the problem (SOP) with respect to $\{v_n\}$. For any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, it is clear that $x_{n_k}\to 0\notin \mathrm{argmin}(D,g,1)$, which implies (SOP) is not globally scalar-LP well-posed.

Theorem 5.4. Let D be closed and $-\mathscr{K}$ be transitive. If $F(x_1)\sim F(x_2)$ for any $x_1,x_2\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, then (SOP) is globally gLP well-posed if and only if (SOP) is pointwise gLP well-posed at any $x\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$.

Proof. Necessity. Let $\bar{x}\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ and $\{x_n\}$ be a pointwise LP minimizing sequence at $\bar{x}$. Then there exists $\varepsilon_n\to 0^+$ such that $d(x_n,D)\le\varepsilon_n$ and $F(x_n)-\varepsilon_n k^0\preccurlyeq_u^{\mathscr{K}} F(\bar{x})$. Hence $\{x_n\}$ is a global LP minimizing sequence for (SOP). Thus, we can find a subsequence of $\{x_n\}$ converging to some $\hat{x}\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Then, since $F(\hat{x})\sim F(\bar{x})$, we obtain that (SOP) is pointwise gLP well-posed at $\bar{x}$.

Sufficiency. Let $\{x_n\}$ be a global LP minimizing sequence for (SOP). Then there exist $\varepsilon_n\to 0^+$ and $\{v_n\}\subseteq \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ such that $d(x_n,D)\le\varepsilon_n$ and $F(x_n)-\varepsilon_n k^0\preccurlyeq_u^{\mathscr{K}} F(v_n)$. For any $x\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, we have $F(v_n)\sim F(x)$. Then $F(x_n)-\varepsilon_n k^0\preccurlyeq_u^{\mathscr{K}} F(x)$. Thus, $\{x_n\}$ is a pointwise LP minimizing sequence at x. Since (SOP) is pointwise gLP well-posed at x, there exists a subsequence of $\{x_n\}$ converging to some $\hat{x}$, such that $d(\hat{x},D)\le 0$ and $F(\hat{x})\preccurlyeq_u^{\mathscr{K}} F(x)$. As D is closed, then $\hat{x}\in D$. From $x\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$ one has $F(x)\sim F(\hat{x})$. Then, since $-\mathscr{K}$ is transitive, it follows that $\hat{x}\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$. Hence (SOP) is globally gLP well-posed. □

Remark 5.1. In Theorem 5.4, the condition " $F(x_1)\sim F(x_2)$ for any $x_1,x_2\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$" cannot be relaxed. From Example 4.6, it is clear that (SOP) is globally gLP well-posed but is not pointwise gLP well-posed at $x_0=(0,0)$.

The following example illustrates that the closedness of D in Theorem 5.4 cannot be omitted.

Example 5.3. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $D=(0,1]$ and $k^0=(1,1)$. Define $F: X\rightrightarrows Y$ and $\mathscr{K}: Y\rightrightarrows Y$ by $F(x)=[0,1]\times[0,1]$ for all $x\in X$, and $\mathscr{K}(y)=\mathbb{R}^2_+$ for all $y\in Y$. We observe that $\mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})=(0,1]$. Then (SOP) is pointwise gLP well-posed at any $\hat{v}\in \mathrm{Min}(D,F,\preccurlyeq_u^{\mathscr{K}})$, but it is not globally gLP well-posed.

The following example illustrates that Theorem 5.4 may not hold if $-\mathscr{K}$ is not transitive.

Example 5.4. Let $X = l^\infty$, $Y = \mathbb{R}^2$, $D = \{x = (a_1, a_2, ..., a_m, ...) \in l^\infty | a_m \in [0,1], m = 1,2,...\}$ and $k^0 = (1,1)$. Denote $0_X = (0,0,...,0,...)$ and $1_X = (1,1,...,1,...)$. Define $F : X \rightrightarrows Y$ and $\mathscr{K} : Y \rightrightarrows Y$, respectively, by

$$F(x) = \begin{cases} [0,2) \times \{1\}, & x \in (-l^\infty_+) \setminus \{0_X\}, \\ [0,1] \times \{1\}, & x = 0_X, \\ [0,2) \times \{2\}, & x \in A, \\ [0,2] \times \{-1\}, & x = 1_X, \\ [0,3] \times \{2\}, & \text{otherwise}, \end{cases}$$

and

$$\mathscr{K}(y) = \begin{cases} \{(s,t) \in \mathbb{R}^2 | t \geq 0\}, & y_2 \geq 0, \\ \{(s,t) \in \mathbb{R}^2 | t \geq 0 \text{ or } s > 0\}, & y_2 < 0, \end{cases}$$

where $-l^\infty_+ = \{x = (a_1, a_2, ..., a_m, ...) \in l^\infty | a_m \leq 0, m \in \mathbb{N}\}$, $A = \left\{x = (a_1, 0, ..., 0, ...) | a_1 \in [\frac{1}{2}, \frac{3}{4}]\right\}$ and $y = (y_1, y_2)$. We can verify that $-\mathscr{K}$ is not transitive. A direct calculation gives that $\mathrm{Min}(D, F, \preccurlyeq^{\mathscr{K}}_u) = \{0_X, 1_X\}$. We observe that (SOP) is pointwise gLP well-posed at $x = 1_X$ and $x = 0_X$, but it is not globally gLP well-posed. In fact, by taking $x_n = (\frac{1}{2} + \frac{1}{n}, 0, ..., 0, ...)$, $v_n = 0_X$ and $\varepsilon_n = \frac{1}{n}$ for any $n \in \mathbb{N}$, one obtains that $\{x_n\}$ is a global LP minimizing sequence, but $x_n \to (\frac{1}{2}, 0, ..., 0, ...) \notin \mathrm{Min}(D, F, \preccurlyeq^{\mathscr{K}}_u)$.

Theorem 5.5. Let $-\mathscr{K}$ be reflexive, $\mathrm{Min}(D, F, \preccurlyeq^{\mathscr{K}}_u)$ be compact, $\mathscr{K}(y) \subseteq \mathscr{K}(y+d)$ for any $y \in Y$ and $d \in \mathscr{K}(y)$. If (SOP) is globally set-LP well-posed, then (SOP) is globally gLP well-posed.

Proof. Let $\{x_n\}$ be a global LP minimizing sequence for (SOP). Then there exist $\varepsilon_n \to 0^+$ and $\{v_n\} \subseteq \mathrm{Min}(D, F, \preccurlyeq^{\mathscr{K}}_u)$ such that $d(x_n, D) \leq \varepsilon_n$ and $F(x_n) - \varepsilon_n k^0 \preccurlyeq^{\mathscr{K}}_u F(v_n)$. It is clear that $F(x_n) \subseteq \bigcup_{y \in F(v_n)} (y - \mathscr{K}(y)) + \varepsilon_n k^0$. Since $-\mathscr{K}$ is reflexive and $\mathscr{K}(y) + [0, +\infty)k^0 \subseteq \mathscr{K}(y)$, one has $\varepsilon_n k^0 \in \mathscr{K}(y)$ for any $y \in Y$. Then $\mathscr{K}(y) \subseteq \mathscr{K}(y + \varepsilon_n k^0)$, which implies

$$F(x_n) \subseteq \bigcup_{y \in F(v_n)} (y + \varepsilon_n k^0 - \mathscr{K}(y + \varepsilon_n k^0)). \tag{5.1}$$

Take $\delta_n = 2\varepsilon_n \max\left\{1, \left\|k^0\right\|\right\}$. Then $\delta_n \geq \varepsilon_n$ and $\varepsilon_n k^0 \in \delta_n \bar{\mathbb{B}}$. Hence, combining this and (5.1), we obtain $F(x_n) \subseteq \bigcup_{y \in F(v_n), b \in \bar{\mathbb{B}}} (y + \delta_n b - \mathscr{K}(y + \delta_n b))$, thus $F(x_n) \preccurlyeq^{\mathscr{K}}_u F(v_n) + \delta_n \bar{\mathbb{B}}$ and $d(x_n, D) \leq \delta_n$. Hence $\{x_n\}$ is a global set-LP minimizing sequence for (SOP). Since (SOP) is globally set-LP well-posed, there exists a subsequence $\{x_{n_k}\}$ such that $d(x_{n_k}, \mathrm{Min}(D, F, \preccurlyeq^{\mathscr{K}}_u)) \to 0$. Owing to the compactness of $\mathrm{Min}(D, F, \preccurlyeq^{\mathscr{K}}_u)$, there exist a subsequence $\left\{x_{n_{k_l}}\right\}$ of $\{x_{n_k}\}$ and $\hat{x} \in \mathrm{Min}(D, F, \preccurlyeq^{\mathscr{K}}_u)$ such that $x_{n_{k_l}} \to \hat{x}$. Thus, (SOP) is globally gLP well-posed. □

6. An Application to Traffic Network Problem with Interval-valued Cost Functions

As an application of the results obtained in the preceding sections, we consider a traffic network model studied in literature [32] to clarify the viability of the results. Xu et al. [32] investigated a multi-criteria traffic equilibrium network problem with uncertain cost. In the traffic network problem, the traffic cost is in the form of a vector, covering various cost types. In practice, influenced by a series of actual situations such as road conditions and policies, decision-makers will compare these different actual situations. By evaluating the potential impacts of each situation on different cost types, they will choose to focus on certain types of costs according to their specific needs and priorities.

In the traffic flow cost problem, the design of the weight function $\omega(\cdot):\mathbb{R}^n \to \mathbb{R}^n_+$ is crucial for reflecting the decision-maker's preferences among different cost types within a single cost vector $y=(y_1,y_2,\ldots,y_n)$. Each component $\omega_i(y)$ of $\omega(y):=(\omega_1(y),\omega_2(y),...,\omega_n(y))$ represents the decision-maker's relative emphasis on the i-th cost type.

Such a weight at a point y defines a cone of preferred directions by

$$\mathscr{K}(\omega(y)) := \left\{ d\in\mathbb{R}^n \,|\, \sum_{i=1}^{n} sgn(d_i)\omega_i(y) \geq 0 \right\},$$

where

$$sgn(d_i) := \begin{cases} 1, \text{ if } d_i>0, \\ 0, \text{ if } d_i=0, \\ -1, \text{ if } d_i<0. \end{cases}$$

In the traffic network, the set of nodes and the set of directed arcs are denoted by N and E, respectively. We denote by $c_e>0$ the capacity of arc $e\in E$. Let $C=(c_e)_e\in E$ denote the capacity vector. The set of origin-destination (O-D) pairs is denoted by W, and the demand vector is denoted by $D=(d_w)_{w\in W}$, where $d_w>0$ is the demand of traffic flow on O-D pair w. Let P_w denote the set of available paths connecting O-D pair $w\in W$. Let $\sum_{w\in W}|P_w|=m$ and $P=\bigcup_{w\in W}P_w$. For a given path $k\in P_w$, h_k is called the traffic flow and $h=(h_1,h_2,...,h_m)\in\mathbb{R}^m$ is called a path flow. The arc flow needs to meet the capacity constraints $0\le v_e\le c_e, \forall e\in E$, where $v_e=\sum_{w\in W}\sum_{k\in P_w}\delta_{ek}h_k$, $\delta_{ek}=1$ if path k includes arc e; otherwise, $\delta_{ek}=0$, see [32] for more details. The path flow vector h needs to satisfy the demand constraints $\sum_{k\in P_w}h_k=d_w$. Let $g^e(v)=(g_1(v),g_2(v),...,g_s(v))$ be an interval-valued function on the arc e, where $g_i(v)=[g_i^L(v),g_i^R(v)](i=1,2,...,s)$ denotes the different types of interval-valued cost functions such as time, money, etc.

Let $F^k(h)=(f_1^k(h),...,f_s^k(h))$ be the interval-valued cost function along the path k. Then $F^k(h)=\sum_{e\in E}\delta_{ek}g^e(v)$. Now, we consider the case of $s=2$. The minimum cost flow problem of the traffic network equilibrium is defined by

$$\min \sum_{w\in W}\sum_{k\in P_w}F^k(h)$$

$$\text{s.t.} \begin{cases} \sum_{k\in P_w}h_k=d_w, & \forall w\in W, \\ c_e \geqslant \sum_{w\in W}\sum_{k\in P_w}\delta_{ek}h_k \geqslant 0, & \forall e\in E, \end{cases}$$

where $F^k(h)=(f_1^k(h),f_2^k(h))$.

Example 6.1. Consider the network problem depicted in Figure 1. $N=\{1,2,3\}, E=\{e_1,e_2,e_3\}$, $C=(3,2,3)^T, W=\{(1,3)\}$. Let the demand quantity be 4. The cost functions of arcs from $\mathbb{R}^3$ to $2^{\mathbb{R}}$ are defined as follows:

$$g_1^{e_1}(v)=[v_1,1.5v_1], \qquad g_2^{e_1}(v)=[v_1,2v_1],$$

$$g_1^{e_2}(v)=[0.5v_2,2.5v_2], \qquad g_2^{e_2}(v)=[2-(v_2-2)^2,4-2(v_2-2)^2],$$

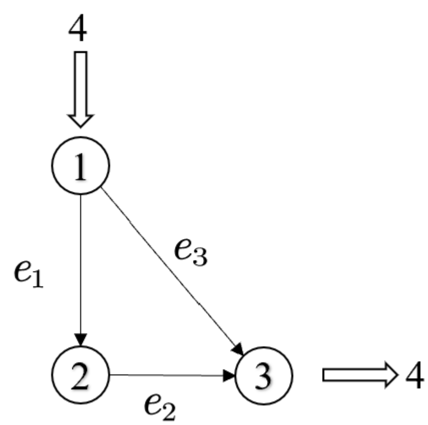

FIGURE 1.

$$g_1^{e_3}(v) = [1.5v_3, 2v_3], \qquad g_2^{e_3}(v) = [v_3, 2v_3],$$

where $v = (v_1, v_2, v_3) \in \mathbb{R}^3$.

Let $A = \left\{(y_1, y_2) \in \mathbb{R}^2 : y_1 \leq 10\right\}$, $B = \left\{(y_1, y_2) \in \mathbb{R}^2 : y_1 > 10, y_2 \leq 10\right\}$, $k^0 = (1,1)$ and $\omega : \mathbb{R}^2 \to \mathbb{R}^2_+$ be defined by

$$\omega(y) = \begin{cases} (\frac{2}{3}, \frac{1}{3}), y \in A, \\ (\frac{1}{3}, \frac{2}{3}), y \in B, \\ (1,0), \text{otherwise.} \end{cases}$$

Then the minimum cost flow problem of the example can be expressed as follows:

$$\text{(SOP)} \qquad \min \Phi(h) = (\Phi_1(h), \Phi_2(h)) = F^1(h) + F^2(h)$$

$$\text{s.t.} \begin{cases} 0 \leq h_1 \leq 2, \\ 0 \leq h_2 \leq 3, \\ h_1 + h_2 = 4, \end{cases}$$

where $F^1(h) + F^2(h) = (f_1^1(h) + f_1^2(h), f_2^1(h) + f_2^2(h))$.

Let S denote the feasible region. Then $S = \operatorname{conv}\{(1,3),(2,2)\}$. We obtain $\Phi_1(h) = f_1^1(h) + f_1^2(h) = g_1^{e_1}(v) + g_1^{e_2}(v) + g_1^{e_3}(v) = [v_1 + 0.5v_2 + 1.5v_3, 1.5v_1 + 2.5v_2 + 2v_3]$, $\Phi_2(h) = f_2^1(h) + f_2^2(h) = g_2^{e_1}(v) + g_2^{e_2}(v) + g_2^{e_3}(v) = [v_1 + v_3 + 2 - (v_2 - 2)^2, 2v_1 + 2v_3 + 4 - 2(v_2 - 2)^2]$, where $h = (h_1, h_2), v = (v_1, v_2, v_3)$ and $h_1 = v_1 = v_2$, $h_2 = v_3$.

$$\mathscr{K}(y) = \begin{cases} \left\{d \in \mathbb{R}^2 | (d_1 \geq 0, d_2 \geq 0) \text{or} (d_1 > 0, d_2 \leq 0)\right\}, y \in A, \\ \left\{d \in \mathbb{R}^2 | (d_1 \geq 0, d_2 \geq 0) \text{or} (d_1 \leq 0, d_2 > 0)\right\}, y \in B, \\ \left\{d \in \mathbb{R}^2 | d_1 \geq 0, d_2 \in \mathbb{R}\right\}, \text{otherwise.} \end{cases}$$

If $\bar{h} = (1,3)$, we observe that $\Phi(\bar{h}) = [6,10] \times [5,10]$ and for any $h \in S \setminus \left\{\bar{h}\right\}$, $\Phi(h) \not\subseteq \bigcup_{y \in \Phi(\bar{h})} (y - \mathscr{K}(y))$; if $\bar{h} \in S \setminus \{(1,3)\}$, taking $\hat{h} = (1,3)$, we observe that $\Phi(\hat{h}) \preccurlyeq_u^{\mathscr{K}} \Phi(\bar{h})$ but $\Phi(\bar{h}) \not\preccurlyeq_u^{\mathscr{K}} \Phi(\hat{h})$.

Then $\text{Min}(S,\Phi,\preccurlyeq_u^{\mathscr{K}})=\{(1,3)\}$. Let $\bar{h}=(1,3)$. Through calculation, we can know that the approximate solution sets $L(k^0,\bar{h},0)=\{(1,3)\}$ and

$$\begin{aligned}L(k^0,\bar{h},\varepsilon)&=\left\{x\in\mathbb{R}^2|d(x,S)\le\varepsilon,\Phi(x)-\varepsilon k^0\preccurlyeq_u^{\mathscr{K}}\Phi(\bar{h})\right\}\\&\subseteq\left\{x\in\mathbb{R}^2|d(x,S)\le\varepsilon,y_1-\varepsilon\le 10,\forall y=(y_1,y_2)\in\Phi(x)\right\}\\&\subseteq(1,3)+\frac{5}{2}\varepsilon\bar{\mathbb{B}},\end{aligned}$$

where $\varepsilon>0$. Thus $\text{diam}L(k^0,\bar{h},\varepsilon)\to 0$ and $ex(L(k^0,\bar{h},\varepsilon),L(k^0,\bar{h},0))\to 0$ as $\varepsilon\to 0^+$. According to Theorem 3.5, we obtain that (SOP) is pointwise gLP well-posed at $\bar{h}$.

In fact, we can verify that (SOP) is pointwise gLP well-posed at $\bar{h}=(1,3)$. Let $\{x_n\}$ be a pointwise LP minimal sequence at $\bar{h}=(1,3)$. Then there exists $\varepsilon_n\to 0^+$ such that $d(x_n,S)\le\varepsilon_n$ and $F(x_n)-\varepsilon_n k^0\preccurlyeq_u^{\mathscr{K}}F(\bar{h})$. Then, we derive that $x_n\in(1,3)+\varepsilon_n\mathbb{B}((0,0),1)$. Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to\bar{h}$, which implies that (SOP) is pointwise gLP well-posed at $\bar{h}=(1,3)$. Meanwhile, according to Theorem 5.4, (SOP) is globally gLP well-posed.

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