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REGULARITY OF SOLUTIONS FOR DOUBLE-PHASE ELLIPTIC EQUATIONS INVOLVING MEASURES

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Abstract. In this paper, we study regularity for the double-phase problem

$$-\mathrm{div}\left(a(x)|\nabla u|^{p-2}\nabla u+|\nabla u|^{q-2}\nabla u\right)=\mu \text{ in } \mathscr{D}'(\Omega),$$

where p and q are positive constants satisfying $p>q\geq 2$, $a(x)$ is locally Hölder continuous and has positive lower and upper bounds, μ is a nonnegative Radon measure satisfying $\mu(B_s)\leq Cs^m$ with some constant $C>0$ for all balls $\overline{B}_s\subset\Omega$, Ω is a bounded domain in $\mathbb{R}^n(n\geq 2)$, and $m\geq n-1$ is a constant. For different m, we prove Hölder continuity with different exponents for the locally bounded weak solutions, as well as their gradients by using the De Giorgi-Nash-Moser iteration and a freezing coefficient argument.

Keywords. Double-phase elliptic equations; De Giorgi-Nash-Moser iteration; Hölder continuity; Regularity; Radon measure.

1. INTRODUCTION

Through in-depth investigation of strongly anisotropic materials, Zhikov revealed that their hardening characteristics exhibit significant variations depending on spatial location, a phenomenon now termed the Lavrentiev phenomenon (see [32, 33, 34, 35]). To describe this phenomenon, Zhikov proposed a functional having a form:

$$\int_\Omega \frac{a(x)}{p}|\nabla u|^p+\frac{1}{q}|\nabla u|^q\mathrm{d}x, \tag{1.1}$$

where the ellipticity rate of the integrand varies over the spatial domain $\Omega\subset\mathbb{R}^n(n\geq 2)$. The modulating function $a(x)$ coordinates the blending process between two distinct materials with power-law hardening exponents p and q respectively. Thus, the problem associated with a functional under the form (1.1) is called double-phase problem. For a comprehensive review of recent developments on this topic, we refer to the work of Mingione and Rădulescu [23].

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To investigate the double-phase problem, variational analysis can be performed and the corresponding Euler-Lagrange equation, combined with additional nonlinear terms and homogeneous Dirichlet boundary conditions, can be given under the following form:

$$\begin{cases} -\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u+|\nabla u|^{q-2}\nabla u\right)=f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $f(x,u)$ is a nonlinear function.

It is worth mentioning that most of literature focus on studying the existence of solutions to the equation (1.2). For example, in [26], under the assumption that $a(x)$ is a nonnegative Lipschitz continuous function on $\overline{\Omega}$, using the Morse theory, Perera and Squassina proved that there is a nontrivial solution of problem (1.2) when $f(x,u)=\lambda|u|^{q-2}u+|u|^{r-2}u+h(x,u)$ and h is a Carathéodory function on $\Omega\times\mathbb{R}$ satisfying $|h(x.u)|\leq C\left(|u|^{\rho-1}+|u|^{\sigma-1}\right)$, where $\lambda\in\mathbb{R}$ is a parameter and C, r, ρ, and σ are positive constants satisfying

$$1<q<p<\min\left\{n,r,q+\frac{q}{n}\right\},\quad q<\sigma<\rho<r<q^*:=\frac{nq}{n-q}.$$

By applying variational methods, Liu and Dai proved various existence and multiplicity results for solutions to problem (1.2) when $f(x,u)$ satisfies different structural conditions; see [16]. Besides, in [17], Liu and Dai proved the existence of three ground state solutions for the double-phase problem. Under the framework of Musielak-Orlicz-Sobolev spaces, Papageorgiou, Pudełko, and Rădulescu conducted a spectral analysis of the double-phase operator and then studied equation (1.2), proving the existence and multiplicity properties of solutions to the equation involving asymptotic resonance; see [25]. For existence results of solutions to the double-phase problem under more general form, we refer to [18, 19, 20, 21, 28], just cite a few.

Although significant progress has been achieved in developing the existence theory for the double-phase problem, the regularity theory remains less studied except [23], where several integrability results of gradients of solutions have been presented. In particular, the results on the Hölder continuity of solutions and their gradients of the double-phase problem involving measures have not been reported in the literature yet.

It is worth noting that many physical source terms, such as point sources or mass distributions on lower-dimensional sets, cannot be described by locally L^p functions and are typically modeled by Radon measures, e.g., Dirac measures [11]. Studying partial differential equations (PDEs) involving Radon measures generalizes the classical framework from integrable function spaces to a broader class of distributional data. Regularity theory of PDEs reveals a quantitative correspondence between the "size" of a Radon measure and the "smoothness" of the solution: suitable growth or density conditions on the measure imply certain regularity of the solution [1, 3, 30]; conversely, the regularity of the solution can impose structural constraints on the measure [3, 12, 13]. This bidirectional link highlights the fundamental role of Radon measures in connecting the given data with the resulting solution properties, thereby underscoring their theoretical significance in the analysis of PDEs.

In this paper, we consider a generic case of equation (1.2) by replacing the nonlinear term $f(x,u)$ with a Radon measure μ and prove the Hölder continuity of solutions and their gradients. More precisely, we study the following double-phase problem:

$$-\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u+|\nabla u|^{q-2}\nabla u\right)=\mu \text{ in } \mathscr{D}'(\Omega), \tag{1.3}$$

where Ω is an open bounded domain of $\mathbb{R}^n (n \geq 2)$, $2 \leq q < p < +\infty$, μ is a nonnegative Radon measure satisfying $\mu(B_s) \leq Cs^m$ with constants $C > 0$ and $m \geq n-1$ whenever the closed ball $\overline{B}_s \subset \Omega$, and $a(x)$ satisfies the following conditions:

- $a(x)$ is locally β-Hölder continuous with $\beta \in (0,1)$, namely, for any compact subset $\overline{\Omega'} \subset \Omega$, there exists a constant $L > 0$ such that
$$|a(x) - a(y)| \leq L|x-y|^{\beta}, \forall x, y \in \Omega';$$
- there exist positive constants a_0, a_1 such that $a_0 \leq a(x) \leq a_1$, for all $x \in \Omega$.

It is worth noting that equation (1.3) is a generic case of the p-Laplace equation involving the Radon measure μ:

$$-\Delta_p u = -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \mu \text{ in } \mathscr{D}'(\Omega), \tag{1.4}$$

for which Kilpeläinen proved that if there exists a positive constant C such that $\mu(B_s) \leq Cs^{n-p+\alpha(p-1)}$ with $1 < p \leq n$ and $\alpha \in (0,1]$, then any solution is $C^{0,\theta}_{\text{loc}}$-continuous for every $\theta \in (0,\alpha)$; see [12]. Subsequently, Kilpeläinen and Zhong showed that if $\mu(B_s) \leq Cs^{n-p+\alpha(p-1)}$ with $1 < p < +\infty$, and $\alpha \in (0,1)$, then every solution to equation (1.4) is Hölder continuous with the same exponent α; see [13]. Furthermore, Challal, Lyaghfouri, and Zheng et al. extended the Hölder continuity results of the equation (1.4) to the following G-Laplacian under the framework of Orlicz-Sobolev spaces:

$$-\Delta_G u = -\text{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu \text{ in } \mathscr{D}'(\Omega), \tag{1.5}$$

where $g \in C^1([0,+\infty);[0,+\infty))$ satisfying $g(0) = 0$ and

$$\delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \forall t > 0 \tag{1.6}$$

with positive constants δ and g_0; see [2, 3, 29, 30]. Notably, the structural condition (1.6) allows the equation (1.5) to include the (p,q)-Laplace case, which corresponds to equation (1.3) with $a(x)$ being a constant.

It is worth mentioning that under the framework of Orlicz-Sobolev spaces, Chlebicka systematically studied the existence and regularity of solutions to elliptic equations involving measures, particularly achieving important results in the study of integrability of solutions and their gradients; see [4, 5, 6]. Nevertheless, proving Hölder continuity of solutions and their gradients is more challenging for the double-phase problem (1.3) involving measures due to the presence of the phase transition function $a(x)$.

This paper aims to establish various regularity results for the double-phase equation (1.3) involving a Radon measure when m takes different values. The main contributions include:

- establishing the local Hölder continuity of locally bounded weak solutions, as well as their gradients, under suitable conditions on p, q, $a(x)$, and the measure μ;
- extending the regularity results of the G-Laplace equation to the double-phase setting with a phase transition function $a(x)$.

The main techniques employed in this paper are the De Giorgi-Nash-Moser iteration and the freezing coefficient method as used in [9], where local Hölder continuity of almost minimizers was proved for a class of semilinear free boundary problems with variable coefficients.

The structure of this paper is organized as follows. In Section 2, we introduce the concepts of Musielak-Orlicz-Sobolev spaces used in this paper. In Section 3, we present some preliminary results needed in the proof of the main results. In Section 4, we present and prove the main results regarding the local Hölder continuity of solutions and their gradients under appropriate conditions.

2. THE MUSIELAK-ORLICZ-SOBOLEV SPACE $W^{1,\mathscr{P}}(\Omega)$

In this section, we collect some relevant results on the Musielak-Orlicz space $L^{\mathscr{P}}(\Omega)$ and Musielak-Orlicz-Sobolev space $W^{1,\mathscr{P}}(\Omega)$. For more detail, please see references [1, 7, 10, 24]. Let $N(\Omega)$ denote the set of all generalized N-function and $\mathscr{P}:\Omega\times[0,+\infty)\to[0,+\infty)$ denote the mapping $(x,t)\mapsto a(x)t^p+t^q$ for $x\in\Omega$ and $t\geq 0$. It is clear that $\mathscr{P}\in N(\Omega)$ and is locally integrable. The Orlicz space $L^{\mathscr{P}}(\Omega)$ is defined by

$$L^{\mathscr{P}}(\Omega):=\{u:\Omega\to\mathbb{R}\text{measurable}:\rho_{\mathscr{P}}(u)<+\infty\}$$

and is equipped with the norm

$$\|u\|_{L^{\mathscr{P}}(\Omega)}:=\inf\left\{\lambda>0:\rho_{\mathscr{P}}\left(\frac{u}{\lambda}\right)\leq 1\right\},$$

where $\rho_{\mathscr{P}}(u):=\int_\Omega \mathscr{P}(x,|u|)\mathrm{d}x=\int_\Omega a(x)|u|^p+|u|^q\mathrm{d}x$ is called the $\mathscr{P}$-modular of u. The space $L^{\mathscr{P}}(\Omega)$ is a separable, uniformly convex Banach space. We denote by $\|\cdot\|_{L^p(\Omega)}$ the norm in $L^p(\Omega)$ and let

$$L^p_a(\Omega):=\left\{u:\Omega\to\mathbb{R}\text{ measurable}:\|u\|_{p,a}:=\left(\int_\Omega a(x)|u|^p\mathrm{d}x\right)^{\frac{1}{p}}<+\infty\right\}.$$

Note that the following continuous embeddings hold true:

$$L^p(\Omega)\hookrightarrow L^{\mathscr{P}}(\Omega)\hookrightarrow L^q(\Omega)\cap L^p_a(\Omega). \tag{2.1}$$

The proof of (2.1) is standard and can be found in Appendix. The space $W^{1,\mathscr{P}}(\Omega)$ is defined by

$$W^{1,\mathscr{P}}(\Omega):=\left\{u\in L^{\mathscr{P}}(\Omega):|\nabla u|\in L^{\mathscr{P}}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{W^{1,\mathscr{P}}(\Omega)}:=\|u\|_{\mathscr{P}(\Omega)}+\|\nabla u\|_{\mathscr{P}(\Omega)}.$$

We denote by $W^{1,\mathscr{P}}_0(\Omega)$ the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathscr{P}}(\Omega)$. The spaces $W^{1,\mathscr{P}}(\Omega)$ and $W^{1,\mathscr{P}}_0(\Omega)$ are reflexive and separable Banach spaces (see [7, Proposition 2.14] and [16, Proposition 2.2] for details). We call a solution of (1.3) any function $u\in W^{1,\mathscr{P}}_{\text{loc}}(\Omega)$ that satisfies

$$\int_\Omega\left(a(x)|\nabla u|^{p-2}\nabla u+|\nabla u|^{q-2}\nabla u\right)\cdot\nabla\phi\mathrm{d}x=\int_\Omega\phi\mathrm{d}\mu,\forall\phi\in\mathscr{D}(\Omega).$$

3. PRELIMINARY RESULTS

This section presents several lemmas that form the foundation for the subsequent proof of the main theorems.

First, for any $x_0 \in \Omega$ and $R_* \in \left(0, \frac{1}{2}\right)$ such that $\overline{B}_{4R_*}(x_0) \subset \Omega$, denote by u_s^* the weak solution of the $\mathscr{P}$-harmonic equation over the ball $B_s(x_0)$:

$$\begin{cases} -\operatorname{div}\left(a(x)\nabla u_s^*|^{p-2}\nabla u_s^* + |\nabla u_s^*|^{q-2}\nabla u_s^*\right) = 0 & \text{in } B_s(x_0), \\ u_s^* = u & \text{on } \partial B_s(x_0). \end{cases} \tag{3.1}$$

In the sequel, we denote $B_r := B_r(x_0)$ for any $r \in (0, 4R_*]$ without special statements. In addition, we always let $u \in W^{1,\mathscr{P}}_{\text{loc}}(\Omega)$ be a locally bounded solution of equation (1.3). The following lemma establishes the relationship between u and its $\mathscr{P}$-harmonic replacement u_s^*.

Lemma 3.1. *There exists a positive constant C depending only on p, q, a_0, and $\|u\|_{L^\infty(B_{4R_*}(x_0))}$ such that*

$$\int_{B_s(x_0)} |\nabla u - \nabla u_s^*|^p + |\nabla u - \nabla u_s^*|^q \mathrm{d}x \le C s^m, \forall s \in (0, 2R_*].$$

Proof. Since $u - u_s^* \in W_0^{1,\mathscr{P}}(B_s)$, we deduce from (1.3) and (3.1) that

$$\begin{aligned} \int_{B_s} u - u_s^* \mathrm{d}\mu &= \int_{B_s} a(x)\left(|\nabla u|^{p-2}\nabla u - |\nabla u_s^*|^{p-2}\nabla u_s^*\right) \cdot (\nabla u - \nabla u_s^*)\,\mathrm{d}x \\ &\quad + \int_{B_s} \left(|\nabla u|^{q-2}\nabla u - |\nabla u_s^*|^{q-2}\nabla u_s^*\right) \cdot (\nabla u - \nabla u_s^*)\,\mathrm{d}x \\ &\ge \int_{B_s} \frac{a(x)}{2^p}|\nabla u - \nabla u_s^*|^p + \frac{1}{2^q}|\nabla u - \nabla u_s^*|^q \mathrm{d}x \\ &\ge \int_{B_s} \frac{a_0}{2^p}|\nabla u - \nabla u_s^*|^p + \frac{1}{2^q}|\nabla u - \nabla u_s^*|^q \mathrm{d}x \\ &\ge C_0 \int_{B_s} |\nabla u - \nabla u_s^*|^p + |\nabla u - \nabla u_s^*|^q \mathrm{d}x, \end{aligned} \tag{3.2}$$

where $C_0 := \min\left\{\frac{a_0}{2^p}, \frac{1}{2^q}\right\}$ and in the first inequality we used the fact that (see [15, p. 97])

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \ge \left(\frac{1}{2}\right)^p |\xi - \eta|^p, \forall \xi, \eta \in \mathbb{R}^n, p \ge 2. \tag{3.3}$$

Now, we prove that $\|u_s^*\|_{L^\infty(B_s)}$ can be bounded by $\|u\|_{L^\infty(B_{4R_*})}$. Indeed, let $k_0 := \|u\|_{L^\infty(B_{4R_*})}$, define the function $u_{k_0} : B_s \to \mathbb{R}$ by

$$u_{k_0} := \begin{cases} k_0 \cdot \operatorname{sgn}(u_s^*) & \text{if } |u_s^*| > k_0, \\ u_s^* & \text{if } |u_s^*| \le k_0, \end{cases}$$

where $\operatorname{sgn}(u_s^*) := 1$ if $u_s^* \ge 0$ and $\operatorname{sgn}(u_s^*) := -1$ if $u_s^* < 0$. Let $S_{k_0} := \{|u_s^*| > k_0\}$. We have $\nabla u_{k_0} = \nabla u_s^*$ in $S_{k_0}^c$ and $\nabla u_{k_0} = 0$ in S_{k_0}. Since $u = u_s^* = u_{k_0}$ on ∂B_s, it follows that $u_s^* - u_{k_0} \in$

$W_0^{1,\mathscr{P}}(B_s)$. We deduce from (3.1) that

$$
\begin{aligned}
0 =& \int_{B_s} \left(a(x)|\nabla u_s^*|^{p-2}\nabla u_s^* + |\nabla u_s^*|^{q-2}\nabla u_s^*\right) \cdot \left(\nabla u_s^* - \nabla u_{k_0}\right) \mathrm{d}x \\
=& \int_{S_{k_0}} \left(a(x)|\nabla u_s^*|^{p-2}\nabla u_s^* + |\nabla u_s^*|^{q-2}\nabla u_s^*\right) \cdot \left(\nabla u_s^* - \nabla u_{k_0}\right) \mathrm{d}x \\
&+ \int_{S_{k_0}^c} \left(a(x)|\nabla u_s^*|^{p-2}\nabla u_s^* + |\nabla u_s^*|^{q-2}\nabla u_s^*\right) \cdot \left(\nabla u_s^* - \nabla u_{k_0}\right) \mathrm{d}x \\
=& \int_{S_{k_0}} a(x)|\nabla u_s^*|^p + |\nabla u_s^*|^q \mathrm{d}x,
\end{aligned}
$$

which implies that $\|u_s^*\|_{L^\infty(B_s)} \leq k_0$ or $|\nabla u_s^*| = 0$ a.e in S_{k_0}. Note that $u_s^* - u \in W_0^{1,\mathscr{P}}(B_s)$, we always have $\|u_s^*\|_{L^\infty(B_s)} \leq k_0$. Then, by (3.2), we have

$$
\begin{aligned}
\int_{B_s} |\nabla u - \nabla u_s^*|^p + |\nabla u - \nabla u_s^*|^q \mathrm{d}x \leq& \frac{1}{C_0} \int_{B_s} u - u_s^* \mathrm{d}\mu \\
\leq& \frac{1}{C_0} \left(\|u\|_{L^\infty(B_s)} + \|u_s^*\|_{L^\infty(B_s)}\right) \mu(B_s) \\
\leq& C s^m,
\end{aligned}
$$

where the C constant depends only on p, q, a_0, and $\|u\|_{L^\infty(B_{4R_*})}$. □

In the sequel, we freeze the coefficient $a(x)$ at $x_0 \in \Omega$. Let $u_s^\#$ be the weak solution of the following equation:

$$
\begin{cases}
-\operatorname{div}\left(a(x_0)\left|\nabla u_s^\#\right|^{p-2}\nabla u_s^\# + \left|\nabla u_s^\#\right|^{q-2}\nabla u_s^\#\right) = 0 & \text{in } B_s(x_0), \\
u_s^\# = u & \text{on } \partial B_s(x_0).
\end{cases}
\tag{3.4}
$$

Lemma 3.2. *There exist $\sigma \in (0,1)$ and a positive constant C depending only on n, p, q, a_0, and a_1 such that, for any s and R satisfying $0 < s \leq R \leq 2R_*$, there holds*

$$
\begin{aligned}
&\int_{B_s(x_0)} |\nabla u - (\nabla u)_{x_0,s}|^p + |\nabla u - (\nabla u)_{x_0,s}|^q \mathrm{d}x \\
\leq& C\left(\frac{s}{R}\right)^{n+\sigma} \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^p + |\nabla u - (\nabla u)_{x_0,R}|^q \mathrm{d}x \\
&+ C\int_{B_R(x_0)} \left|\nabla u - \nabla u_R^\#\right|^p + \left|\nabla u - \nabla u_R^\#\right|^q \mathrm{d}x,
\end{aligned}
\tag{3.5}
$$

where

$$
(u)_{x_0,s} := \fint_{B_s(x_0)} u \mathrm{d}x.
$$

Proof. Let $g(t) := a(x_0)t^{p-1} + t^{q-1}$, which satisfies the structural condition

$$
0 < q - 1 \leq \frac{tg'(t)}{g(t)} \leq p - 1, \forall t > 0.
$$

Thus, g satisfies the structural condition (1.6). According to [30, Lemma 3.1], by considering $G(t) := \frac{a(x_0)}{p}t^p + \frac{1}{q}t^q$ and $u_R^{\#}$, we have

$$\begin{aligned}
&\int_{B_s} \frac{a(x_0)}{p}|\nabla u - (\nabla u)_{x_0,s}|^p + \frac{1}{q}|\nabla u - (\nabla u)_{x_0,s}|^q \,\mathrm{d}x \\
\le & C\left(\frac{s}{R}\right)^{n+\sigma} \int_{B_R} \frac{a(x_0)}{p}|\nabla u - (\nabla u)_{x_0,R}|^p + \frac{1}{q}|\nabla u - (\nabla u)_{x_0,R}|^q \,\mathrm{d}x \\
&+ C\int_{B_R} \frac{a(x_0)}{p}\left|\nabla u - \nabla u_R^{\#}\right|^p + \frac{1}{q}\left|\nabla u - \nabla u_R^{\#}\right|^q \mathrm{d}x.
\end{aligned}$$

Since $a_0 \le a(x) \le a_1$, we can obtain estimate (3.5) immediately. □

Lemma 3.3. *There exists a positive constant C depending only on p, q, a_0, and L such that*

$$\int_{B_s(x_0)} \left|\nabla u_s^* - \nabla u_s^{\#}\right|^p + \left|\nabla u_s^* - \nabla u_s^{\#}\right|^q \mathrm{d}x \le Cs^{\frac{\beta p}{p-1}} \int_{B_s(x_0)} \left|\nabla u_s^{\#}\right|^p \mathrm{d}x, \forall s \in (0, 2R_*].$$

Proof. Since $u_s^* - u_s^{\#} \in W_0^{1,\mathscr{P}}(B_s)$, we deduce from (3.1), (3.4), (3.3), and the Young's inequality with $\varepsilon > 0$ that

$$\begin{aligned}
&\int_{B_s} \frac{a_0}{2^p}\left|\nabla u_s^* - \nabla u_s^{\#}\right|^p + \frac{1}{2^q}\left|\nabla u_s^* - \nabla u_s^{\#}\right|^q \mathrm{d}x \\
\le & \int_{B_s} \frac{a(x)}{2^p}\left|\nabla u_s^* - \nabla u_s^{\#}\right|^p + \frac{1}{2^q}\left|\nabla u_s^* - \nabla u_s^{\#}\right|^q \mathrm{d}x \\
\le & \int_{B_s} a(x)\left(|\nabla u_s^*|^{p-2}\nabla u_s^* - \left|\nabla u_s^{\#}\right|^{p-2}\nabla u_s^{\#}\right)\cdot\left(\nabla u_s^* - \nabla u_s^{\#}\right)\mathrm{d}x \\
&+ \int_{B_s} \left(|\nabla u_s^*|^{q-2}\nabla u_s^* - \left|\nabla u_s^{\#}\right|^{q-2}\nabla u_s^{\#}\right)\cdot\left(\nabla u_s^* - \nabla u_s^{\#}\right)\mathrm{d}x \\
= & \int_{B_s} \left(a(x)|\nabla u_s^*|^{p-2}\nabla u_s^* + |\nabla u_s^*|^{q-2}\nabla u_s^*\right)\cdot\left(\nabla u_s^* - \nabla u_s^{\#}\right)\mathrm{d}x \\
&- \int_{B_s} \left(a(x)\left|\nabla u_s^{\#}\right|^{p-2}\nabla u_s^{\#} + \left|\nabla u_s^{\#}\right|^{q-2}\nabla u_s^{\#}\right)\cdot\left(\nabla u_s^* - \nabla u_s^{\#}\right)\mathrm{d}x \\
= & -\int_{B_s} \left(a(x)\left|\nabla u_s^{\#}\right|^{p-2}\nabla u_s^{\#} + \left|\nabla u_s^{\#}\right|^{q-2}\nabla u_s^{\#}\right)\cdot\left(\nabla u_s^* - \nabla u_s^{\#}\right)\mathrm{d}x \\
= & \int_{B_s} \left(a(x_0)\left|\nabla u_s^{\#}\right|^{p-2}\nabla u_s^{\#} + \left|\nabla u_s^{\#}\right|^{q-2}\nabla u_s^{\#}\right)\cdot\left(\nabla u_s^* - \nabla u_s^{\#}\right)\mathrm{d}x \\
&- \int_{B_s} \left(a(x)\left|\nabla u_s^{\#}\right|^{p-2}\nabla u_s^{\#} + \left|\nabla u_s^{\#}\right|^{q-2}\nabla u_s^{\#}\right)\cdot\left(\nabla u_s^* - \nabla u_s^{\#}\right)\mathrm{d}x \\
= & \int_{B_s} \left(a(x_0) - a(x)\right)\left|\nabla u_s^{\#}\right|^{p-2}\nabla u_s^{\#}\cdot\left(\nabla u_s^* - \nabla u_s^{\#}\right)\mathrm{d}x \\
\le & \int_{B_s} |a(x_0) - a(x)|\left|\nabla u_s^{\#}\right|^{p-1}\left|\nabla u_s^* - \nabla u_s^{\#}\right|\mathrm{d}x \\
\le & \int_{B_s} C(\varepsilon)|a(x_0) - a(x)|^{\frac{p}{p-1}}\left|\nabla u_s^{\#}\right|^p \mathrm{d}x + \int_{B_s} \varepsilon\left|\nabla u_s^* - \nabla u_s^{\#}\right|^p \mathrm{d}x,
\end{aligned} \tag{3.6}$$

where $C(\varepsilon)$ is a positive constant depending only on ε and p. Choosing $\varepsilon < \frac{a_0}{2^p}$ in (3.6) and using the fact that $a(x)$ is β-Hölder continuous, we obtain

$$\begin{aligned}\int_{B_s} \left|\nabla u_s^* - \nabla u_s^\#\right|^p + \left|\nabla u_s^* - \nabla u_s^\#\right|^q \mathrm{d}x \leq & C\int_{B_s} |a(x_0) - a(x)|^{\frac{p}{p-1}} \left|\nabla u_s^\#\right|^p \mathrm{d}x \\ \leq & C\int_{B_s} \left(L|x_0 - x|^\beta\right)^{\frac{p}{p-1}} \left|\nabla u_s^\#\right|^p \mathrm{d}x \\ \leq & Cs^{\frac{\beta p}{p-1}} \int_{B_s} \left|\nabla u_s^\#\right|^p \mathrm{d}x,\end{aligned}$$

where C is a positive constant depending only on p, q, a_0, and L. □

Define

$$\omega(x_0, r) := \omega(u, x_0, r) := \left(\fint_{B_r(x_0)} |\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}}, \forall r \in (0, 2R_*]. \tag{3.7}$$

The next lemma gives us a crucial estimate used in the iteration argument.

Lemma 3.4. *There exist constants $A_1 > 1$, $A_2 > 0$, and $A_3 > 0$, all of which depend only on p, q, n, a_0, a_1, L, and $\|u\|_{L^\infty(B_{4R_*}(x_0))}$, such that for any s and R satisfying $0 < s \leq R \leq 2R_*$, there holds*

$$\omega(x_0, s) \leq A_1\left(\left(\frac{R}{s}\right)^{\frac{n}{p}} R^{\theta_1} + 1\right)\omega(x_0, R) + A_2\left(\frac{R}{s}\right)^{\frac{n}{p}} R^{\theta_2} + A_3, \tag{3.8}$$

where

$$\theta_1 := \frac{\beta}{p-1} \quad \text{and} \quad \theta_2 := \min\left\{\frac{m-n}{p}, \frac{\beta}{p-1}\right\}.$$

Proof. Note that $\omega(x_0, s) = |B_s|^{-\frac{1}{p}}\|\nabla u\|_{L^p(B_s)}$. We first derive an estimate for $\|\nabla u\|_{L^p(B_s)}$. Indeed, using Lemmas 3.1 and 3.3, we obtain

$$\begin{aligned}\|\nabla u\|_{L^p(B_s)} &\leq \|\nabla u - \nabla u_R^*\|_{L^p(B_s)} + \|\nabla u_R^* - \nabla u_R^\#\|_{L^p(B_s)} + \|\nabla u_R^\#\|_{L^p(B_s)} \\ &\leq \|\nabla u - \nabla u_R^*\|_{L^p(B_R)} + \|\nabla u_R^* - \nabla u_R^\#\|_{L^p(B_R)} + \|\nabla u_R^\#\|_{L^p(B_s)} \\ &\leq CR^{\frac{m}{p}} + CR^{\frac{\beta}{p-1}}\|\nabla u_R^\#\|_{L^p(B_R)} + \|\nabla u_R^\#\|_{L^p(B_s)},\end{aligned} \tag{3.9}$$

where C is a positive constant depending only on p, q, a_0, L, and $\|u\|_{L^\infty(B_{4R_*})}$.

Now, we estimate $\|\nabla u_R^\#\|_{L^p(B_R)}$. Since $u_s^\# - u \in W_0^{1,\mathscr{P}}(B_s)$, we multiply equation (3.4) by $u_s^\# - u$ and integrate over B_s, and then apply the Young's inequality with $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, respectively, to obtain

$$\begin{aligned}\int_{B_s} a(x_0)\left|\nabla u_s^\#\right|^p + \left|\nabla u_s^\#\right|^q \mathrm{d}x =& \int_{B_s} a(x_0)\left|\nabla u_s^\#\right|^{p-2}\nabla u_s^\# \cdot \nabla u + \left|\nabla u_s^\#\right|^{q-2}\nabla u_s^\# \cdot \nabla u \mathrm{d}x \\ \leq& \int_{B_s} \varepsilon_1 a(x_0)\left|\nabla u_s^\#\right|^p + C(\varepsilon_1)a(x_0)|\nabla u|^p \mathrm{d}x \\ &+ \int_{B_s} \varepsilon_2\left|\nabla u_s^\#\right|^q + C(\varepsilon_2)|\nabla u|^q \mathrm{d}x,\end{aligned} \tag{3.10}$$

where $C(\varepsilon_1)$ is a positive constant depending only on ε_1 and p, and $C(\varepsilon_2)$ is a positive constant depending only on ε_2 and q.

Choosing appropriate $\varepsilon_1, \varepsilon_2$ in (3.10) and using the inequality $t^q \le 1 + t^p$ for $t \ge 0$, we obtain

$$\begin{aligned}\int_{B_s} \left|\nabla u_s^{\#}\right|^p + \left|\nabla u_s^{\#}\right|^q \mathrm{d}x &\le C \int_{B_s} |\nabla u|^p + |\nabla u|^q \mathrm{d}x && (3.11)\\ &\le C \int_{B_s} |\nabla u|^p + 1 + |\nabla u|^p \mathrm{d}x \\ &\le C \int_{B_s} |\nabla u|^p \mathrm{d}x + C s^n, && (3.12)\end{aligned}$$

where C is a positive constant depending only on p, q, a_0, and a_1.

Putting $s = R$ in (3.12), we obtain

$$\begin{aligned}\|\nabla u_R^{\#}\|_{L^p(B_R)} &\le \left(\int_{B_R} \left|\nabla u_R^{\#}\right|^p + \left|\nabla u_R^{\#}\right|^q \mathrm{d}x\right)^{\frac{1}{p}} \\ &\le C\left(\int_{B_R} |\nabla u|^p \mathrm{d}x + C R^n\right)^{\frac{1}{p}} \\ &\le C\|\nabla u\|_{L^p(B_R)} + C R^{\frac{n}{p}}, && (3.13)\end{aligned}$$

where C is a positive constant depending only on p, q, a_0, and a_1.

Next, we estimate $\|\nabla u_R^{\#}\|_{L^p(B_s)}$. According to [14, p. 345], by considering $G(t) = \frac{a(x_0)}{p} t^p + \frac{1}{q} t^q$ and $u_R^{\#}$, we obtain

$$\begin{aligned}\int_{B_s} \left|\nabla u_s^{\#}\right|^p + \left|\nabla u_s^{\#}\right|^q \mathrm{d}x \le& C \int_{B_s} \frac{a(x_0)}{p} \left|\nabla u_R^{\#}\right|^p + \frac{1}{q}\left|\nabla u_R^{\#}\right|^q \mathrm{d}x \\ \le& C\left(\frac{s}{R}\right)^n \int_{B_R} \frac{a(x_0)}{p} \left|\nabla u_R^{\#}\right|^p + \frac{1}{q}\left|\nabla u_R^{\#}\right|^q \mathrm{d}x \\ \le& C\left(\frac{s}{R}\right)^n \int_{B_R} \left|\nabla u_R^{\#}\right|^p + \left|\nabla u_R^{\#}\right|^q \mathrm{d}x, && (3.14)\end{aligned}$$

where C is a positive constant depending only on n, p, q, a_0, and a_1.

By (3.12) and (3.14), we have

$$\begin{aligned}\|\nabla u_R^{\#}\|_{L^p(B_s)} &\le C\left(\frac{s}{R}\right)^{\frac{n}{p}} \left(\int_{B_R} \left|\nabla u_R^{\#}\right|^p + \left|\nabla u_R^{\#}\right|^q \mathrm{d}x\right)^{\frac{1}{p}} \\ &\le C\left(\frac{s}{R}\right)^{\frac{n}{p}} \left(\int_{B_R} |\nabla u|^p \mathrm{d}x + C R^n\right)^{\frac{1}{p}} \\ &\le C\left(\frac{s}{R}\right)^{\frac{n}{p}} \|\nabla u\|_{L^p(B_R)} + C s^{\frac{n}{p}}, && (3.15)\end{aligned}$$

where C is a positive constant depending only on n, p, q, a_0, and a_1.

Using (3.9), (3.13), and (3.15), we get

$$\|\nabla u\|_{L^p(B_s)} \le C\left(R^{\frac{\beta}{p-1}} + \left(\frac{s}{R}\right)^{\frac{n}{p}}\right)\|\nabla u\|_{L^p(B_R)} + C\left(R^{\frac{m}{p}} + R^{\frac{n}{p}+\frac{\beta}{p-1}}\right) + C s^{\frac{n}{p}},$$

where C is a positive constant depending only on n, p, q, a_0, a_1, L, and $\|u\|_{L^\infty(B_{4R_*})}$.

Furthermore, in view of the definition of $\omega(x_0,s)$ (see (3.7)), we have

$$\omega(x_0,s) \le C\left(\left(\frac{R}{s}\right)^{\frac{n}{p}} R^{\frac{\beta}{p-1}}+1\right)\omega(x_0,R)+C\left(\frac{R}{s}\right)^{\frac{n}{p}}\left(R^{\frac{m-n}{p}}+R^{\frac{\beta}{p-1}}\right)+C, \tag{3.16}$$

where C is a positive constant depending only on n, p, q, a_0, a_1, L, and $\|u\|_{L^\infty(B_{4R_*})}$.

Letting $\theta_1 := \frac{\beta}{p-1} > 0$ and $\theta_2 := \min\left\{\frac{m-n}{p}, \frac{\beta}{p-1}\right\} > 0$, we deduce from (3.16) that (3.8) holds true for some positive constants A_1, A_2, and A_3 depending only on p, q, n, a_0, a_1, L, and $\|u\|_{L^\infty(B_{4R_*})}$. ☐

We now start to iterate the inequality (3.8). For a constant $\Lambda > A_1 > 1$, which will be determined later (see (4.1)), and any $R \in (0, 2R_*]$, we define the sequence

$$R_j := \Lambda^{-j}R,\ j=0,1,2\dots$$

Then,

$$R_{j+1} := \Lambda^{-1}R_j < R_j \text{ and } \frac{R_j}{R_{j+1}} = \Lambda,\ j=0,1,2\dots$$

Applying Lemma 3.4 with $s = R_{j+1}$ and $R = R_j$, we obtain

$$\begin{aligned}\omega(x_0,R_{j+1}) \le& A_1\left(\Lambda^{\frac{n}{p}}R_j^{\theta_1}+1\right)\omega(x_0,R_j)+A_2\Lambda^{\frac{n}{p}}R_j^{\theta_2}+A_3\\ \le& A_1\left(\Lambda^{\frac{n}{p}}R_j^{\theta_1}+1\right)\omega(x_0,R_j)+C,\ j=0,1,2\dots\end{aligned} \tag{3.17}$$

where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$.

Lemma 3.5. *There exists a positive constant C depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*}(x_0))}$ such that, for any $R \in (0, 2R_*]$, there holds*

$$\omega(x_0,R_{j+1}) \le CA_1^{j+1}\omega(x_0,R)+C\frac{1-A_1^{j+1}}{1-A_1}, \tag{3.18}$$

where A_1 is the same as in Lemma 3.4.

Proof. For $j = 0$, estimate (3.18) follows directly from (3.17). For $j = 1,2,3,\dots$, we infer from iteration (3.17) that

$$\begin{aligned}\omega(x_0,R_{j+1}) \le& A_1^{j+1}\left(1+\Lambda^{\frac{n}{p}}R^{\theta_1}\right)\omega(x_0,R)\prod_{l=0}^{j-1}\left(1+\Lambda^{\frac{n}{p}}R_{j-l}^{\theta_1}\right)\\ &+C\sum_{k=1}^{j}A_1^k\prod_{l=0}^{k-1}\left(1+\Lambda^{\frac{n}{p}}R_{j-l}^{\theta_1}\right)+C,\end{aligned} \tag{3.19}$$

where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$. Let P be a positive constant depending on n, p, θ_1, and Λ such that

$$\prod_{l=1}^{\infty}\left(1+\Lambda^{\frac{n}{p}}R_l^{\theta_1}\right) \le P < +\infty.$$

Then, we deduce from (3.19) that

$$\begin{aligned}\omega(x_0,R_{j+1}) &\leq CPA_1^{j+1}\omega(x_0,R)+CP\sum_{k=1}^{j}A_1^k+C\\ &\leq CPA_1^{j+1}\omega(x_0,R)+C(1+P)\frac{1-A_1^{j+1}}{1-A_1}\\ &\leq CA_1^{j+1}\omega(x_0,R)+C\frac{1-A_1^{j+1}}{1-A_1},\end{aligned}$$

where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$. The proof of the lemma is complete. ☐

Lemma 3.6. *Let $x_0 \in \Omega$ be a Lebesgue point of u. Then, there exists a positive constant C depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$ such that*

$$|u(x_0)-u_0| \leq CR(\omega(x_0,R)+1), \forall R\in(0,2R_*],$$

where

$$u_0 := \fint_{B_R(x_0)} u\mathrm{d}y.$$

Proof. On one hand, following the argument for [8, (2.13)], we find that

$$\begin{aligned}|u(x_0)-u_0| &\leq \sum_{l=0}^{\infty}|u_{l+1}-u_l|\\ &\leq \sum_{l=0}^{\infty}\fint_{B_{R_{l+1}}}|u-u_l|\,\mathrm{d}x\\ &\leq \Lambda^n\left(\fint_{B_R}|u-u_0|\,\mathrm{d}x+\sum_{l=1}^{\infty}\fint_{B_{R_l}}|u-u_l|\,\mathrm{d}x\right)\\ &\leq \Lambda^n\left(\left(\fint_{B_R}|u-u_0|^p\,\mathrm{d}x\right)^{\frac{1}{p}}+\sum_{l=1}^{\infty}\left(\fint_{B_{R_l}}|u-u_l|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\right),\end{aligned} \tag{3.20}$$

where

$$u_l := \fint_{B_{R_l}(x_0)} u\mathrm{d}x, l=1,2,3\ldots$$

On the other hand, from the generalized Poincaré's inequality ([27, p. 23]), we have

$$\left(\fint_{B_R}|u-u_0|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \leq CR\left(\fint_{B_R}|\nabla u|^p\,\mathrm{d}x\right)^{\frac{1}{p}} = CR\omega(x_0,R),$$

$$\left(\fint_{B_{R_l}}|u-u_l|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \leq CR_l\left(\fint_{B_{R_l}}|\nabla u|^p\,\mathrm{d}x\right)^{\frac{1}{p}} = CR_l\omega(x_0,R_l), l=1,2,3\ldots$$

where C is a positive constant depending only on n and p.

Therefore, Lemma 3.5 and (3.20) imply that

$$
\begin{aligned}
|u(x_0)-u_0| &\le C\Lambda^n\left(R\omega(x_0,R)+\sum_{l=1}^{\infty}R_l\omega(x_0,R_l)\right)\\
&\le \Lambda^n\left(CR\omega(x_0,R)+\sum_{l=1}^{\infty}\left(CR_lA_1^l\omega(x_0,R)+CR_l\frac{1-A_1^l}{1-A_1}\right)\right),
\end{aligned}
\tag{3.21}
$$

where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$.

Note that

$$
\sum_{l=1}^{\infty}R_lA_1^l\omega(x_0,R)=R\omega(x_0,R)\sum_{l=1}^{\infty}\left(\frac{A_1}{\Lambda}\right)^l=R\frac{\Lambda}{\Lambda-A_1}\omega(x_0,R)=R\omega(x_0,R),
$$

and, since $\Lambda>A_1>1$, we have

$$
\begin{aligned}
\sum_{l=1}^{\infty}R_l\frac{1-A_1^l}{1-A_1} &= \frac{R}{1-A_1}\sum_{l=1}^{\infty}\Lambda^{-l}\left(1-A_1^l\right)\\
&= \frac{R}{1-A_1}\left(\sum_{l=1}^{\infty}\Lambda^{-l}-\sum_{l=1}^{\infty}\left(\frac{A_1}{\Lambda}\right)^l\right)\\
&= \frac{R}{1-A_1}\frac{1}{\Lambda-1}-\frac{R}{1-A_1}\frac{A_1}{\Lambda-A_1}\\
&\le -\frac{R}{1-A_1}\frac{A_1}{\Lambda-A_1}.
\end{aligned}
$$

We then go back to (3.21) to conclude that

$$
\begin{aligned}
|u(x_0)-u_0| &\le \Lambda^n\left(CR\omega(x_0,R)+\frac{R}{1-A_1}\frac{-A_1}{\Lambda-A_1}\right)\\
&\le CR\left(\omega(x_0,R)+1\right),
\end{aligned}
$$

where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$. ☐

4. LOCAL REGULARITY OF BOUNDED SOLUTIONS

This section builds upon the Lemmas presented in Section 3 to derive various local regularity results for locally bounded solutions to the equation (1.3) involving measures in different cases of the parameter m.

We first show that any locally bounded solution to the equation (1.3) is locally $C^{0,\alpha}$-continuous with some $\alpha\in(0,1)$ for $m\ge n-1$. More precisely, we prove the following theorem, which is the first result obtained in this paper.

Theorem 4.1 (Local Hölder continuity of solution)**.** *The following statements hold true:*

(i) If $m>n$, then $u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ with any $\alpha\in(0,1)$.

(ii) If $n\ge m\ge n-1$, then $u\in C^{0,\theta}_{\mathrm{loc}}(\Omega)$ with any $\theta\in\left(0,\frac{p-1}{p}\right)$.

To prove Theorem 4.1, we need the following iteration lemma.

Lemma 4.1 ([31, Lemma A.3])**.** *Let* ψ *be a non-negative function on an interval* $(0, R_*]$ *with* $R_* \leq 1$. *Suppose that, for all* s *and* R *satisfying* $0 < s \leq R \leq R_*$, *there holds*

$$\psi(s) \leq A\left(\frac{s}{R}\right)^{\alpha} \psi(R) + BR^{\beta},$$

where $A > 1$, B, α, *and* β *are positive constants with* $\alpha > \beta$. *Fix* $\delta \in (\beta, \alpha)$ *and consider* $\lambda \in (0,1)$ *with* $A\lambda^{\alpha} = \lambda^{\delta}$. *Suppose that there exists a constant* $d > 0$ *such that* $\psi(s) \leq d\psi(\lambda^k R)$ *for all non-negative integer* k *and* $s \in [\lambda^{k+1}R, \lambda^k R]$. *Then, there exists a positive constant* C *depending only on* A, α, β, δ, *and* d *such that*

$$\psi(s) \leq C\left(\frac{s}{R}\right)^{\sigma} \left(\psi(R) + BR^{\sigma}\right)$$

holds true for any s, R, *and* σ *satisfying* $0 < s \leq R \leq R_*$ *and* $0 < \sigma \leq \beta$. *Furthermore, there exists a positive constant* D *depending only on* B, C, β, *and* $\psi(R_*)$ *such that* $\psi(s) \leq Ds^{\sigma}$ *holds true for any* $s \in (0, R_*]$ *and* $\sigma \in (0, \beta]$.

Proof of Theorem 4.1. Since we are concerned with local properties of solutions, for simplicity, we restrict our analysis to balls. In the sequel, let $|B_r(\xi)|$ denote the n-dimensional Lebesgue measure of a ball with radius $r > 0$ and center $\xi \in \mathbb{R}^n$.

We first prove the statement of (i). Let $x, y \in \Omega$ be two Lebesgue points of u with $x, y \in B_{R_*}(x_0)$. Assume that

$$R = |x - y| < \frac{R_*}{\Lambda^3}, \quad \Lambda = A_1^{\tau}, \tag{4.1}$$

where $\tau \in \{1, 2, 3 \ldots\}$ will be chosen later. It is possible to choose $j \geq 1$ satisfying

$$\Lambda^{-j-3} R_* < R \leq \Lambda^{-j-2} R_*. \tag{4.2}$$

Letting $z := \frac{x+y}{2}$ and assuming further that $\Lambda > \frac{3}{2}$, we get $B_R(x) \subset B_{\Lambda R}(z)$ and $B_R(y) \subset B_{\Lambda R}(z)$. Using inequality (4.2) and $\Lambda > 1$, we have $\Lambda R \leq \Lambda^{-j-1} R_* \leq R_*$, which implies that

$$B_{\Lambda R}(z) \subset B_{\Lambda^{-j-1}R_*}(z) \subset B_{R_*}(z).$$

Since $z \in B_{R_*}(x_0)$, it follows that $B_{R_*}(z) \subset B_{2R_*}(x_0)$. Therefore, for sufficiently large τ, we obtain

$$B_R(x) \subset B_{\Lambda R}(z) \subset B_{\Lambda^{-j-1}R_*}(z) \subset B_{R_*}(z) \subset B_{2R_*}(x_0), \tag{4.3a}$$

$$B_R(y) \subset B_{\Lambda R}(z) \subset B_{\Lambda^{-j-1}R_*}(z) \subset B_{R_*}(z) \subset B_{2R_*}(x_0). \tag{4.3b}$$

Applying Lemma 3.6 with $x_0 = x$ and $x_0 = y$, respectively, gives us

$$\left| u(x) - \fint_{B_R(x)} u(\xi)\mathrm{d}\xi \right| \leq CR\left(\omega(x,R) + 1\right), \tag{4.4a}$$

$$\left| u(y) - \fint_{B_R(y)} u(\xi)\mathrm{d}\xi \right| \leq CR(\omega(y,R) + 1), \tag{4.4b}$$

where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$. We deduce from (4.4) that

$$
\begin{aligned}
&|u(x)-u(y)| \\
\leq & \left|u(x)-\fint_{B_R(x)} u \mathrm{d}\xi\right|+\left|\fint_{B_R(x)} u \mathrm{d}\xi-M\right|+\left|\fint_{B_R(y)} u \mathrm{d}\xi-M\right|+\left|u(y)-\fint_{B_R(y)} u \mathrm{d}\xi\right| \\
\leq & CR(\omega(x,R)+\omega(y,R)+1)+\left|\fint_{B_R(x)} u \mathrm{d}\xi-M\right|+\left|\fint_{B_R(y)} u \mathrm{d}\xi-M\right|,
\end{aligned}
\tag{4.5}
$$

where

$$
M:=\fint_{B_{\Lambda^{-j-1}R_*}(z)} u \mathrm{d}\xi,
$$

and C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$.

We need to estimate the integrals in the right-hand side of (4.5). Indeed, using the fact that $\Lambda^{-j-3}R_* < R$ and $B_{\Lambda^{-j-3}R_*}(x) \subset B_{R_*}(x) \subset B_{2R_*}(x_0)$ and applying the Hölder's inequality and the Poincaré's inequality, we have

$$
\begin{aligned}
\left|\fint_{B_R(x)} u \mathrm{d}\xi-M\right| = & \left|\frac{1}{|B_R(x)|}\int_{B_R(x)} u \mathrm{d}\xi-\frac{1}{|B_R|}\int_{B_R(x)} M \mathrm{d}\xi\right| \\
\leq & \frac{1}{|B_R(x)|}\int_{B_R(x)} |M-u| \mathrm{d}\xi \\
\leq & \frac{1}{|B_R(x)|}\int_{B_{\Lambda^{-j-1}R_*}(z)} |M-u| \mathrm{d}\xi \\
\leq & \frac{1}{|B_{\Lambda^{-j-3}R_*}(x)|}\left(\int_{B_{\Lambda^{-j-1}R_*}(z)} |M-u|^p \mathrm{d}\xi\right)^{\frac{1}{p}} |B_{\Lambda^{-j-1}R_*}(z)|^{\frac{p-1}{p}} \\
= & \frac{1}{|B_1(0)|\Lambda^{-(j+3)n}R_*^n}\left(\int_{B_{\Lambda^{-j-1}R_*}(z)} |M-u|^p \mathrm{d}\xi\right)^{\frac{1}{p}} |B_{\Lambda^{-j-1}R_*}(z)|^{\frac{p-1}{p}} \\
= & \frac{\Lambda^{2n}}{|B_{\Lambda^{-j-1}R_*}(z)|}\left(\int_{B_{\Lambda^{-j-1}R_*}(z)} |M-u|^p \mathrm{d}\xi\right)^{\frac{1}{p}} |B_{\Lambda^{-j-1}R_*}(z)|^{\frac{p-1}{p}} \\
\leq & \frac{C\Lambda^{-j+1}R_*}{|B_{\Lambda^{-j-1}R_*}(z)|}\left(\int_{B_{\Lambda^{-j-1}R_*}(z)} |\nabla u|^p \mathrm{d}\xi\right)^{\frac{1}{p}} |B_{\Lambda^{-j-1}R_*}(z)|^{\frac{p-1}{p}} \\
= & C\Lambda^{-j+1}R_*\left(\fint_{B_{\Lambda^{-j-1}R_*}(z)} |\nabla u|^p \mathrm{d}\xi\right)^{\frac{1}{p}} \\
= & C\Lambda^{-j+1}R_*\omega(z,\Lambda^{-j-1}R_*) \\
\leq & CR\omega(z,\Lambda^{-j-1}R_*),
\end{aligned}
$$

where C is a positive constant depending only on p, n, and Λ.

Analogously, for y, it holds that

$$\left| \fint_{B_R(y)} u \mathrm{d}\xi - M \right| \leq CR\omega(z, \Lambda^{-j-1}R_*),$$

where C is a positive constant depending only on p, n, and Λ. Therefore, (4.5) becomes

$$|u(x) - u(y)| \leq CR\left(\omega(x,R) + \omega(y,R) + 1\right) + CR\omega(z, \Lambda^{-j-1}R_*), \tag{4.6}$$

where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$. It follows from (4.2) and (4.3) that

$$\begin{aligned}
\omega(x,R) + \omega(y,R) &= \frac{1}{(|B_1(0)|R^n)^{\frac{1}{p}}} \left(\left(\int_{B_R(x)} |\nabla u|^p \mathrm{d}\xi \right)^{\frac{1}{p}} + \left(\int_{B_R(y)} |\nabla u|^p \mathrm{d}\xi \right)^{\frac{1}{p}} \right) \\
&\leq \frac{2}{(|B_1(0)|R^n)^{\frac{1}{p}}} \left(\int_{B_{\Lambda^{-j-1}R_*}(z)} |\nabla u|^p \mathrm{d}\xi \right)^{\frac{1}{p}} \\
&= \frac{2|B_{\Lambda^{-j-1}R_*}(z)|^{\frac{1}{p}}}{(|B_1(0)|R^n)^{\frac{1}{p}}} \omega(z, \Lambda^{-j-1}R_*) \\
&= 2\left(\frac{\Lambda^{-j-1}R_*}{R} \right)^{\frac{n}{p}} \omega(z, \Lambda^{-j-1}R_*) \\
&\leq 2\Lambda^{\frac{2n}{p}} \omega(z, \Lambda^{-j-1}R_*) \\
&\leq C\omega(z, \Lambda^{-j-1}R_*),
\end{aligned}$$

which, along with (4.6), yields $|u(x) - u(y)| \leq CR\left(\omega(z, \Lambda^{-j-1}R_*) + 1\right)$, where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$. Note that we have $B_{\Lambda^{-1}R_*}(z) \subset B_{2R_*}(x_0)$ for sufficiently large Λ. Applying Lemma 3.5 for the point z and the radius $\Lambda^{-1}R_*$, we obtain

$$\omega(z, \Lambda^{-j-1}R_*) \leq CA_1^j \omega\left(z, \Lambda^{-1}R_*\right) + C\frac{1 - A_1^j}{1 - A_1},$$

where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$. Then, we get

$$|u(x) - u(y)| \leq CR\left(CA_1^j \omega\left(z, \Lambda^{-1}R_*\right) + C\frac{1 - A_1^j}{1 - A_1} + 1 \right), \tag{4.7}$$

where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$. We infer from (4.2) that

$$A_1^j = \left(\Lambda^{\frac{1}{\tau}}\right)^j = (\Lambda^j)^{\frac{1}{\tau}} \leq \left(\frac{R_*}{\Lambda^2 R} \right)^{\frac{1}{\tau}} \leq \left(\frac{R_*}{\Lambda^2} \right)^{\frac{1}{\tau}} R^{-\frac{1}{\tau}},$$

which, along with (4.7), yields $|u(x) - u(y)| \leq CR^{1-\frac{1}{\tau}}\left(\omega\left(z, \Lambda^{-1}R_*\right) + 1\right)$, where C is a positive constant depending only on p, q, m, n, a_0, a_1, β, Λ, L, and $\|u\|_{L^\infty(B_{4R_*})}$.

Finally, note that

$$\omega\left(z,\Lambda^{-1}R_*\right)=\frac{1}{|B_{\Lambda^{-1}R_*}(z)|}\int_{B_{\Lambda^{-1}R_*}(z)}u\mathrm{d}\xi\leq\frac{(2\Lambda)^n}{|B_{2R_*}(x_0)|}\int_{B_{2R_*}(x_0)}u\mathrm{d}\xi\leq C\omega\left(x_0,2R_*\right)$$

and $R=|x-y|$. Using (**??**) and the standard covering argument, we conclude that the statement of (i) holds true.

Now, we prove the statement of (ii). On one hand, for any s and R satisfying $0<s\leq R\leq R_*$, we deduce from (3.11) and (3.14) that

$$\begin{aligned}\int_{B_s}|\nabla u|^p+|\nabla u|^q\,\mathrm{d}x\leq& C\int_{B_s}\left|\nabla u-\nabla u_R^\#\right|^p+\left|\nabla u-\nabla u_R^\#\right|^q\mathrm{d}x+C\int_{B_s}\left|\nabla u_R^\#\right|^p+\left|\nabla u_R^\#\right|^q\mathrm{d}x\\ \leq& C\int_{B_R}\left|\nabla u-\nabla u_R^\#\right|^p+\left|\nabla u-\nabla u_R^\#\right|^q\mathrm{d}x+C\left(\frac{s}{R}\right)^n\int_{B_R}\left|\nabla u_R^\#\right|^p+\left|\nabla u_R^\#\right|^q\mathrm{d}x\\ \leq& C\int_{B_R}\left|\nabla u-\nabla u_R^\#\right|^p+\left|\nabla u-\nabla u_R^\#\right|^q\mathrm{d}x\\ &+C\left(\frac{s}{R}\right)^n\int_{B_R}|\nabla u|^p+|\nabla u|^q\,\mathrm{d}x,\end{aligned}\tag{4.8}$$

where C is a positive constant depending only on n, p, q, a_0, and a_1.

On the other hand, applying Lemma 3.1 and Lemma 3.3 to u_R^* yields

$$\begin{aligned}\int_{B_R}\left|\nabla u-\nabla u_R^\#\right|^p+\left|\nabla u-\nabla u_R^\#\right|^q\mathrm{d}x\leq& C\int_{B_R}|\nabla u-\nabla u_R^*|^p+|\nabla u-\nabla u_R^*|^q\,\mathrm{d}x\\ &+C\int_{B_R}\left|\nabla u_R^\#-\nabla u_R^*\right|^p+\left|\nabla u_R^\#-\nabla u_R^*\right|^q\mathrm{d}x\\ \leq& CR^m+CR^{\frac{\beta p}{p-1}}\int_{B_R}\left|\nabla u_R^\#\right|^p\mathrm{d}x,\end{aligned}\tag{4.9}$$

where C is a positive constant depending only on p, q, a_0, L, and $\|u\|_{L^\infty(B_{4R_*})}$. Letting $G(t)=\frac{a(x_0)}{p}t^p+\frac{1}{q}t^q$ and applying [29, Lemma 2.4] to $u_R^\#$, we have

$$\int_{B_R}\left|\nabla u_R^\#\right|^p\mathrm{d}x\leq C\int_{B_R}\frac{a(x_0)}{p}\left|\nabla u_R^\#\right|^p+\frac{1}{q}\left|\nabla u_R^\#\right|^q\mathrm{d}x\leq CR^\kappa,\tag{4.10}$$

where κ is an arbitrary constant belonging to $(0,n)$ and C is a positive constant depending only on κ, n, a_0, a_1, p, and q. Choosing $\kappa:=n-\frac{\beta p}{2(p-1)}$ and combining (4.9) and (4.10), we get

$$\int_{B_R}\left|\nabla u-\nabla u_R^\#\right|^p+\left|\nabla u-\nabla u_R^\#\right|^q\mathrm{d}x\leq CR^m+CR^{n+\frac{\beta p}{2(p-1)}},\tag{4.11}$$

where C is a positive constant depending only on n, p, q, a_0, a_1, β, L, and $\|u\|_{L^\infty(B_{4R_*})}$. Since $m\in[n-1,n]$, it follows from (4.8), (4.9), (4.10), and (4.11) that

$$\begin{aligned}\int_{B_s}|\nabla u|^p+|\nabla u|^q\,\mathrm{d}x\leq& C\left(\frac{s}{R}\right)^n\int_{B_R}|\nabla u|^p+|\nabla u|^q\,\mathrm{d}x+CR^m\\ \leq& C\left(\frac{s}{R}\right)^n\int_{B_R}|\nabla u|^p+|\nabla u|^q\,\mathrm{d}x+CR^{n-1},\end{aligned}\tag{4.12}$$

where C is a positive constant depending only on n, p, q, a_0, a_1, β, L, and $\|u\|_{L^\infty(B_{4R_*})}$. Applying Lemma 4.1 to (4.12) gives

$$\int_{B_s} |\nabla u|^p + |\nabla u|^q \,\mathrm{d}x \leq Cs^{n-1}, \forall s \in (0, R_*],$$

where C is a positive constant determined by Lemma 4.1. Therefore, we have

$$\int_{B_s} |\nabla u| \,\mathrm{d}x \leq \left(\int_{B_s} |\nabla u|^p \,\mathrm{d}x \right)^{\frac{1}{p}} |B_s|^{1-\frac{1}{p}} \leq Cs^{n-\frac{1}{p}}, \forall s \in (0, R_*], \tag{4.13}$$

where C is a positive constant independent of the center x_0 and the radius s. According to (4.13) and the Morrey's theorem (see [22, p. 30]), we conclude that $u \in C^{0,\theta}_{\mathrm{loc}}(\Omega)$ with any $\theta \in \left(0, \frac{p-1}{p}\right)$. □

Under the assumption that $m > n$, we have shown in Theorem 4.1 that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ whenever $\alpha \in (0,1)$. Next, under the same assumption, we improve the regularity of solution for $m > n$, showing that the gradient of u is also Hölder continuous. More precisely, we prove the following theorem, which is the second result of the paper.

Theorem 4.2 (Hölder continuity of the gradient). *If $m > n$, then $u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$ with any $\gamma \in \left(0, \min\left\{\frac{\sigma}{1+p}, \frac{m-n}{p}, \frac{\beta}{2(p-1)}\right\}\right)$, where σ is the same as in Lemma 3.2.*

Proof. For any s and R satisfying $0 < s \leq R \leq R_*$, we infer from Lemma 3.2 and (4.11) that

$$\begin{aligned}
&\int_{B_s} |\nabla u - (\nabla u)_{x_0,s}|^p + |\nabla u - (\nabla u)_{x_0,s}|^q \,\mathrm{d}x \\
\leq& C\left(\frac{s}{R}\right)^{n+\sigma} \int_{B_R} |\nabla u - (\nabla u)_{x_0,R}|^p + |\nabla u - (\nabla u)_{x_0,R}|^q \,\mathrm{d}x \\
&+ C\int_{B_R} \left|\nabla u - \nabla u_R^\#\right|^p + \left|\nabla u - \nabla u_R^\#\right|^q \,\mathrm{d}x \\
\leq& C\left(\frac{s}{R}\right)^{n+\sigma} \int_{B_R} |\nabla u - (\nabla u)_{x_0,R}|^p + |\nabla u - (\nabla u)_{x_0,R}|^q \,\mathrm{d}x + C\left(R^m + R^{n+\frac{\beta p}{2(p-1)}}\right) \\
\leq& C\left(\frac{s}{R}\right)^{n+\sigma} \int_{B_R} |\nabla u - (\nabla u)_{x_0,R}|^p + |\nabla u - (\nabla u)_{x_0,R}|^q \,\mathrm{d}x + CR^{n+\gamma p},
\end{aligned} \tag{4.14}$$

where $\gamma := \min\left\{\frac{\sigma}{1+p}, \frac{m-n}{p}, \frac{\beta}{2(p-1)}\right\}$ and C is a positive constant depending only on n, p, q, a_0, a_1, β, L, and $\|u\|_{L^\infty(B_{4R_*})}$. Applying Lemma 4.1 to (4.14) gives

$$\int_{B_s} |\nabla u - (\nabla u)_{x_0,s}|^p + |\nabla u - (\nabla u)_{x_0,s}|^q \,\mathrm{d}x \leq Cs^{n+\gamma p}, \forall 0 < s \leq R \leq R_*,$$

where C is a positive constant independent of the center x_0 and the radius s. Therefore, it holds that

$$\int_{B_s} |\nabla u - (\nabla u)_{x_0,s}|^p \,\mathrm{d}x \leq Cs^{n+\gamma p}, \forall 0 < s \leq R \leq R_*, \tag{4.15}$$

According to (4.15) and the Campanato's embedding theorem, we conclude that $u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$ with any $\gamma \in \left(0, \min\left\{\frac{\sigma}{1+p}, \frac{m-n}{p}, \frac{\beta}{2(p-1)}\right\}\right)$. □

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APPENDIX: PROOF OF (2.1)

First, we prove that $L^p(\Omega) \hookrightarrow L^{\mathscr{P}}(\Omega)$. Indeed, for any $u \in L^p(\Omega)$, due to the facts that $0 < a(x) \le a_1$ and $p > q$, it is clear that $u \in L^{\mathscr{P}}(\Omega)$. Let $\mu := \|u\|_{L^p(\Omega)} \cdot \max\left\{(2a_1)^{\frac{1}{p}}, \left(2|\Omega|^{1-\frac{q}{p}}\right)^{\frac{1}{q}}\right\}$. By the Hölder's inequality, we deduce that

$$\begin{aligned}\rho_{\mathscr{P}}\left(\frac{u}{\mu}\right) &= \int_\Omega a(x)\frac{|u|^p}{\mu^p}\mathrm{d}x + \int_\Omega \frac{|u|^q}{\mu^q}\mathrm{d}x\\ &\le \frac{a_1}{\mu^p}\int_\Omega |u|^p\mathrm{d}x + \frac{1}{\mu^q}\int_\Omega |u|^q\mathrm{d}x\\ &\le \frac{a_1}{\mu^p}\|u\|^p_{L^p(\Omega)} + \frac{|\Omega|^{1-\frac{q}{p}}}{\mu^q}\|u\|^q_{L^p(\Omega)}\\ &\le 1,\end{aligned}$$

which, along with the definition of the norm in $L^{\mathscr{P}}(\Omega)$, implies that $\|u\|_{L^{\mathscr{P}}(\Omega)} \le \mu$. Furthermore, letting $C_0 := \max\left\{(2a_1)^{\frac{1}{p}}, \left(2|\Omega|^{1-\frac{q}{p}}\right)^{\frac{1}{q}}\right\}$, we have $\|u\|_{L^{\mathscr{P}}(\Omega)} \le C_0\|u\|_{L^p(\Omega)}$. Hence, it holds that $L^p(\Omega) \hookrightarrow L^{\mathscr{P}}(\Omega)$.

Next, we prove that $L^{\mathscr{P}}(\Omega) \hookrightarrow L^q(\Omega) \cap L^p_a(\Omega)$. Indeed, for any $u \in L^{\mathscr{P}}(\Omega)$, by the definition of $\rho_{\mathscr{P}}(u)$, we have

$$\int_\Omega |u|^q\mathrm{d}x \le \left(\int_\Omega a(x)|u|^p\mathrm{d}x + \int_\Omega |u|^q\mathrm{d}x\right) = \rho_{\mathscr{P}}(u) < +\infty,$$

$$\int_\Omega a(x)|u|^p\mathrm{d}x \le \left(\int_\Omega a(x)|u|^p\mathrm{d}x + \int_\Omega |u|^q\mathrm{d}x\right) = \rho_{\mathscr{P}}(u) < +\infty.$$

Thus, $u \in L^q(\Omega)$ and $u \in L^p_a(\Omega)$, i.e., $L^{\mathscr{P}}(\Omega) \subset L^q(\Omega) \cap L^p_a(\Omega)$.

Now let $\lambda := \|u\|_{L^{\mathscr{P}}(\Omega)}$. By the definition of the norm in $L^{\mathscr{P}}(\Omega)$, we have $\rho_{\mathscr{P}}\left(\frac{u}{\lambda}\right) \leq 1$, i.e.,

$$\int_{\Omega} a(x)\left|\frac{u}{\lambda}\right|^{p} \mathrm{d}x + \int_{\Omega}\left|\frac{u}{\lambda}\right|^{q} \mathrm{d}x \leq 1,$$

which implies that

$$\|u\|_{L^{q}(\Omega)} \leq \lambda = \|u\|_{L^{\mathscr{P}}(\Omega)}, \ \|u\|_{L^{p}_{a}(\Omega)} \leq \lambda = \|u\|_{L^{\mathscr{P}}(\Omega)}.$$

Therefore, $L^{\mathscr{P}}(\Omega) \hookrightarrow L^{q}(\Omega)$ and $L^{\mathscr{P}}(\Omega) \hookrightarrow L^{p}_{a}(\Omega)$. Note that

$$\|u\|_{L^{q}(\Omega)\cap L^{p}_{a}(\Omega)} = \|u\|_{L^{q}(\Omega)} + \|u\|_{L^{p}_{a}(\Omega)}.$$

We conclude that $L^{\mathscr{P}}(\Omega) \hookrightarrow L^{q}(\Omega) \cap L^{p}_{a}(\Omega)$. Proof of (2.1) is complete.