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MULTIPLE SOLUTIONS FOR FRACTIONAL SCHRÖDINGER-POISSON SYSTEMS WITH BERESTYCKI-LIONS TYPE NONLINEARITIES

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Abstract. In this paper, we study the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^{\alpha}\psi+\lambda\phi(x)\psi=g(\psi) & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\beta}\phi=\lambda\psi^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\alpha,\beta\in(0,1)$ are constants and $\lambda>0$ is a parameter. By using a truncation technique and an auxiliary functional, we prove the existence of multiple solutions for the above system when g satisfies the Berestycki-Lions type conditions via combining variational methods with genus theory.

Keywords. Berestycki-Lions type conditions; Fractional Laplacian; Genus theory; Multiple solutions; Schrödinger-Poisson system.

1. INTRODUCTION

In this paper, we investigate the multiplicity results for the general fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^{\alpha}\psi+\lambda\phi(x)\psi=g(\psi) & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\beta}\phi=\lambda\psi^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda>0$ is a parameter, $\alpha,\beta\in(0,1)$ and $4\alpha+2\beta>3$, $(-\Delta)^s$ with $s=\alpha,\beta\in(0,1)$ denotes the fractional Laplacian operator which can be defined as

$$(-\Delta)^s v(x)=C(s)\text{P.V.}\int_{\mathbb{R}^3}\frac{v(x)-v(y)}{|x-y|^{3+2s}}\,\mathrm{d}y,$$

where P.V. is the Cauchy principal value and $C(s)$ is a normalization constant. The fractional Laplacian operator $(-\Delta)^s$ has wide applications in several physical phenomena such as fractional quantum mechanics, flames propagation and geophysical fluid dynamics (see, e.g., [10, 14]).

We note that if $\lambda=0$ and replace $\mathbb{R}^3$ with $\mathbb{R}^N(N\geq 2)$, then system (1.1) reduces to the following fractional Schrödinger equation

$$(-\Delta)^{\alpha}\psi=g(\psi)\text{ in }\mathbb{R}^N. \tag{1.2}$$

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Equation (1.2) was studied extensively by using variational methods. For example, Chang and Wang [6] proved the existence of a positive ground state solution for (1.2) by combining the monotonicity trick and the Pohozaev identity. In [5], Byeon et al. studied equation (1.2) and obtained the existence, symmetry, regularity and decay property of a mountain pass solution. Subsequently, Alves et al. [1] proved the existence of ground state solutions for (1.2) with the critical nonlinearity.

In recent years, fractional Schrödinger-Poisson systems have attracted a lot of attention due to their widespread applications in interdisciplinary fields such as condensed matter physics, quantum optics, optimization, finance, and image processing; see, e.g., [10, 11]. On the other hand, due to the presence of multiple nonlocal terms in the fractional Schrödinger-Poisson systems, it brings inherent mathematical difficulties, which has attracted the attention of many mathematical researchers. For example, in [16], Shen and Yao considered the fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^{\alpha}\psi+V(x)\psi+\phi(x)\psi=|\psi|^{q-2}\psi & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\beta}\phi=\psi^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $3<q<2_{\alpha}^{*}=\frac{6}{3-2\alpha}$, $\alpha,\beta\in(0,1)$ with $\alpha<\beta$ and $2\alpha+2\beta>3$. Under certain assumptions on $V(x)$, by using the Nehari-Pohozaev manifold method, they showed that system (1.3) has a ground state solution for $q\in(3,2_{\alpha}^{*})$. Teng [17] studied system (1.3) for the case $\alpha=\beta$ by using the monotone trick and a global compactness result, and the existence of ground state solutions for (1.3) was obtained under different assumptions on $V(x)$. Similarly, using the same method as in [17], Teng [18] studied the existence of nontrivial ground state solutions for the following critical fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^{\alpha}\psi+V(x)\psi+\phi(x)\psi=|\psi|^{2_{\alpha}^{*}-2}\psi+\mu|\psi|^{q-2}\psi & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\beta}\phi=\psi^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $q\in(2,2_{\alpha}^{*})$, $\mu>0$ is a parameter, $\alpha,\beta\in(0,1)$ with $2\alpha+2\beta>3$.

In [22], Zhang et al. investigated system (1.1) as the general nonlinearity f with subcritical or critical growth. Using a perturbation approach, they obtained the existence of positive radial solutions for (1.1) if $\lambda>0$ small enough. For more related results concerning fractional Schrödinger-Poisson system, we refer the interested readers to [7, 12, 19, 20, 21] and the references therein. Up to our knowledge, in the literature, there are few results on the multiplicity results for generalized fractional Schrödinger-Poisson system except works [12, 13]. In [13], the authors studied the following system

$$\begin{cases} \varepsilon^{2\alpha}(-\Delta)^{\alpha}\psi+V(x)\psi+\phi(x)\psi=f(\psi) & \text{in } \mathbb{R}^N, \\ \varepsilon^{\delta}(-\Delta)^{\beta}\phi=\gamma_{\beta}\psi^2 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $\delta\in(0,2\beta)$ and $\gamma_{\beta}=\frac{\pi^{\frac{N}{2}}2^{2\beta}\Gamma(\beta)}{\Gamma(\frac{N-2\beta}{2})}$ is a constant, f has subcritical growth and satisfies the monotonicity condition that $\frac{f(t)}{t^3}$ is strictly increasing in $(0,+\infty)$ and the Ambrosetti-Rabinowitz (AR) type condition: there exists $K>4$ such that $0<K\int_0^t f(\tau)\,\mathrm{d}\tau\leq tf(t)$ for all $t>0$. They established the multiplicity results of (1.4) for suitably small $\varepsilon>0$ via the Ljusternick-Schnirelmann category theory. This result has been subsequently improved by Liu and Zhang

[12] to the critical nonlinearity case: $f(t)$ is replaced by $g(t)+|t|^{2_\alpha^*-2}t$, where $g(t)=o(t^3)$ as $t\to 0$ and $\frac{g(t)}{t^3}$ is strictly increasing for $t>0$.

Inspired by the papers mentioned above, in the present paper, we consider the multiplicity results for fractional Schrödinger-Poisson system (1.1) with the Berestycki-Lions type nonlinearity g. More precisely, we assume that g satisfies the following conditions:
$(\mathbb{G}_0)$ $g\in C(\mathbb{R},\mathbb{R})$ and g is an odd function;
$(\mathbb{G}_1)$ $-\infty<\liminf\limits_{t\to 0^+}\dfrac{g(t)}{t}\le\limsup\limits_{t\to 0^+}\dfrac{g(t)}{t}=-\omega<0$;
$(\mathbb{G}_2)$ $-\infty\le\limsup\limits_{t\to+\infty}\frac{g(t)}{t^{2_\alpha^*-1}}\le 0$, where $2_\alpha^*=\frac{6}{3-2\alpha}$ is the fractional Sobolev critical exponent for 3-dimension;
$(\mathbb{G}_3)$ there exists a constant $\vartheta>0$ such that $G(\vartheta):=\int_0^\vartheta g(t)\,\mathrm{d}t>0$.

Now we state our main result as follows.

Theorem 1.1. *Suppose that $(\mathbb{G}_0)-(\mathbb{G}_3)$ hold. Then, for each $m\in\mathbb{N}^+$, there exists $\lambda_m>0$ such that, for any $\lambda\in(0,\lambda_m)$, problem (1.1) admits at least m couples of radial symmetry solutions in $H^\alpha(\mathbb{R}^3)\times\mathscr{D}^{\beta,2}(\mathbb{R}^3)$.*

Remark 1.2. Compared with existing related research, theorem 1.1 can be seen as an extension of the results presented in [12, 13]. Due to the absence of any monotonicity or (AR) type conditions in the conditions $(\mathbb{G}_0)-(\mathbb{G}_3)$ of this paper, our problem becomes more challenging. It is worth noting that the methods we used in the present paper are different from that in [12, 13].

We emphasize that, in the general assumptions $(\mathbb{G}_0)-(\mathbb{G}_3)$, we do not assume two standard conditions, namely $\frac{g(t)}{t^3}$ is increasing in $(0,+\infty)$ and the (AR) type condition: there exists $k>4$ such that $0<kG(t)\le g(t)t$ for any $t>0$, which bring about two obstacles when we use the mountain-pass arguments both in checking the geometrical conditions in the corresponding energy functional and in proving the boundedness of its Palais-Smale((PS) for short) sequences. On the other hand, since g does not have any homogeneity property, we cannot use the usual arguments as in the pure power case.

The paper is organized as follows. In Section 2, we introduce some notations, set the variational framework for problem (1.1) and provide some preliminary lemmas which will be used later. In Section 3, the last section, we give the proof of Theorem 1.1.

2. PRELIMINARIES AND MODIFIED FUNCTIONAL

Throughout this paper, $C,C_0,C_1,\cdots$ denote some positive constants that may vary from line to line; $\|w\|_q=(\int_{\mathbb{R}^3}|w|^q\,\mathrm{d}x)^{\frac{1}{q}}$ denote the usual norm of $L^q(\mathbb{R}^3)$, $1\le q<\infty$; $\mathscr{D}^{s,2}(\mathbb{R}^3)$ is completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|w\|_{\mathscr{D}^{s,2}(\mathbb{R}^3)}=(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}w|^2\,\mathrm{d}x)^{\frac{1}{2}}$. The fractional Sobolev space $H^\alpha(\mathbb{R}^3)$ is defined by

$$H^\alpha(\mathbb{R}^3)=\left\{w(x)\in L^2(\mathbb{R}^3):\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|w(x)-w(y)|^2}{|x-y|^{3+2\alpha}}\,\mathrm{d}x\mathrm{d}y<\infty\right\},$$

endowed with the natural norm

$$\|w\|_{H^\alpha(\mathbb{R}^3)}=\left([w]^2_{H^\alpha(\mathbb{R}^3)}+\|w\|_2^2\right)^{\frac{1}{2}},$$

where the term

$$[w]_{H^{\alpha}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|w(x)-w(y)|^2}{|x-y|^{3+2\alpha}}\,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}$$

is the so-called Gagliardo semi-norm of w. Via Fourier transform, we have

$$\widehat{(-\Delta)^{\alpha}}w(\xi) = |\xi|^{2\alpha}\widehat{w}(\xi) \text{ for } \xi \in \mathbb{R}^3,$$

where the symbol $\widehat{\ }$ stands for Fourier transform. Therefore, by the Fourier transform, $H^{\alpha}(\mathbb{R}^3)$ can be equivalently defined as

$$H^{\alpha}(\mathbb{R}^3) = \left\{w(x) \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2\alpha}|\widehat{w}(\xi)|^2\,\mathrm{d}\xi < \infty\right\},$$

and the norm can be equivalently written as

$$\|w\|_{H^{\alpha}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\xi|^{2\alpha}|\widehat{w}(\xi)|^2\,\mathrm{d}\xi + \|w\|_2^2\right)^{\frac{1}{2}}.$$

From [14, Proposition 3.4 and Proposition 3.6], for all $w \in H^{\alpha}(\mathbb{R}^3)$, we have

$$\|(-\Delta)^{\frac{\alpha}{2}}w\|_2^2 = \frac{C_\alpha}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|w(x)-w(y)|^2}{|x-y|^{3+2\alpha}}\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} |\xi|^{2\alpha}|\widehat{w}(\xi)|^2\,\mathrm{d}\xi,$$

where $C_\alpha = \left(\int_{\mathbb{R}^3} \frac{1-\cos\zeta_1}{|\zeta|^{3+2\alpha}}\,\mathrm{d}\zeta\right)^{-1}, \zeta = (\zeta_1,\zeta_2,\zeta_3)$.

Now, we recall some embedding results for fractional Sobolev spaces.

Lemma 2.1. *(See [14]) Let $0 < s < 1$. Then*

- *(i) $\mathscr{D}^{s,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2_s^*}(\mathbb{R}^3)$, i.e., for every $\psi \in \mathscr{D}^{s,2}(\mathbb{R}^3)$, there exists a sharp constant $S > 0$ depending on s such that $\|\psi\|_{2_s^*}^2 \le S\|(-\Delta)^{\frac{s}{2}}\psi\|_2^2$.*
- *(ii) $H^s(\mathbb{R}^3)$ is continuously embedded into $L^q(\mathbb{R}^3)$ for any $q \in [2, 2_s^*]$ and compactly embedded into $L^q_{loc}(\mathbb{R}^3)$ for any $q \in [1, 2_s^*)$.*

Note that if $4\alpha + 2\beta > 3$, then $2 < \frac{12}{3+2\beta} < \frac{6}{3-2\alpha} = 2_\alpha^*$ and thus $H^{\alpha}(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2\beta}}(\mathbb{R}^3)$. For each $\psi \in H^{\alpha}(\mathbb{R}^3)$, denoting $\mathscr{L}_\psi(\varphi)$ the linear functional in $\mathscr{D}^{\beta,2}(\mathbb{R}^3)$ by

$$\mathscr{L}_\psi(\varphi) = \int_{\mathbb{R}^3} \psi^2\varphi\,\mathrm{d}x,$$

we obtain by Lemma 2.1 that, for any $\varphi \in \mathscr{D}^{\beta,2}(\mathbb{R}^3)$,

$$\begin{aligned}|\mathscr{L}_\psi(\varphi)| &\le \left(\int_{\mathbb{R}^3} |\psi|^{\frac{12}{3+2\beta}}\,\mathrm{d}x\right)^{\frac{3+2\beta}{6}} \left(\int_{\mathbb{R}^3} |\varphi|^{2_\beta^*}\,\mathrm{d}x\right)^{\frac{1}{2_\beta^*}} \\ &\le C\|\psi\|^2_{\frac{12}{3+2\beta}}\|\varphi\|_{\mathscr{D}^{\beta,2}(\mathbb{R}^3)} \\ &\le C_1\|\psi\|^2_{H^{\alpha}(\mathbb{R}^3)}\|\varphi\|_{\mathscr{D}^{\beta,2}(\mathbb{R}^3)} < +\infty.\end{aligned}$$

In view of the Lax-Milgram theorem, we can obtain a unique $\phi_\psi^\beta \in \mathscr{D}^{\beta,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} (-\Delta)^{\beta}\phi_\psi^\beta\varphi\,\mathrm{d}x = \int_{\mathbb{R}^3} (-\Delta)^{\frac{\beta}{2}}\phi_\psi^\beta(-\Delta)^{\frac{\beta}{2}}\varphi\,\mathrm{d}x = \lambda\int_{\mathbb{R}^3} \psi^2\varphi\,\mathrm{d}x, \quad \forall\, \varphi \in \mathscr{D}^{\beta,2}(\mathbb{R}^3),$$

so ϕ_{ψ}^{β} solves the equation $(-\Delta)^{\beta}\phi=\lambda\psi^2$ in $\mathbb{R}^3$, and ϕ_{ψ}^{β} can be expressed by

$$\phi_{\psi}^{\beta}(x)=c(\beta)\int_{\mathbb{R}^3}\frac{\lambda\psi^2(y)}{|x-y|^{3-2\beta}}\,\mathrm{d}y, \tag{2.1}$$

where the constant $c(\beta)=\frac{\Gamma(\frac{3-2\beta}{2})}{\pi^{\frac{3}{2}}2^{2\beta}\Gamma(\beta)}$. Moreover, ϕ_{ψ}^{β} has the following properties, which can be found in [16, 18, 22].

Lemma 2.2. *Let $\alpha,\beta\in(0,1)$ and $4\alpha+2\beta>3$. Then, for all $\psi\in H^{\alpha}(\mathbb{R}^3)$,*

- *(i) $\phi_{\psi}^{\beta}\geq 0$ and $\|\phi_{\psi}^{\beta}\|^2_{\mathscr{D}^{\beta,2}(\mathbb{R}^3)}=\lambda\int_{\mathbb{R}^3}\phi_{\psi}^{\beta}\psi^2\,\mathrm{d}x$;*
- *(ii) $\int_{\mathbb{R}^3}\phi_{\psi_t}^{\beta}\psi_t^2\,\mathrm{d}x=t^{3+2\beta}\int_{\mathbb{R}^3}\phi_{\psi}^{\beta}\psi^2\,\mathrm{d}x$ for any $t>0$, where $\psi_t(x)=\psi(t^{-1}x)$;*
- *(iii) there exist $C_1,C_2>0$ independent of $\psi\in H^{\alpha}(\mathbb{R}^3)$ such that*

$$\|\phi_{\psi}^{\beta}\|_{\mathscr{D}^{\beta,2}(\mathbb{R}^3)}\leq C_1\lambda\|\psi\|^2_{\frac{12}{3+2\beta}}\quad\text{and}\quad\int_{\mathbb{R}^3}\phi_{\psi}^{\beta}\psi^2\,\mathrm{d}x\leq C_2\lambda\|\psi\|^4_{\frac{12}{3+2\beta}};$$

- *(iv) if $\psi_n\rightharpoonup\psi$ in $H^{\alpha}(\mathbb{R}^3)$, then $\phi_{\psi_n}^{\beta}\rightharpoonup\phi_{\psi}^{\beta}$ in $\mathscr{D}^{\beta,2}(\mathbb{R}^3)$;*
- *(v) if $\psi_n\to\psi$ in $H^{\alpha}(\mathbb{R}^3)$, then $\phi_{\psi_n}^{\beta}\to\phi_{\psi}^{\beta}$ in $\mathscr{D}^{\beta,2}(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3}\phi_{\psi_n}^{\beta}\psi_n^2\,\mathrm{d}x\to\int_{\mathbb{R}^3}\phi_{\psi}^{\beta}\psi^2\,\mathrm{d}x$;*
- *(vi) if ψ is a radial function, then ϕ_{ψ}^{β} is radial.*

Inserting (2.1) into system (1.1), we see the following equivalent equation

$$(-\Delta)^{\alpha}\psi+\lambda\phi_{\psi}^{\beta}\psi=g(\psi)\ \text{ in }\mathbb{R}^3. \tag{2.2}$$

The energy functional associated to equation (2.2) is defined by

$$\mathscr{E}_{\lambda}(\psi)=\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\psi|^2\,\mathrm{d}x+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{\psi}^{\beta}\psi^2\,\mathrm{d}x-\int_{\mathbb{R}^3}G(\psi)\,\mathrm{d}x,$$

where $G(\psi):=\int_0^{\psi}g(t)\,\mathrm{d}t$. It is standard to show that $\mathscr{E}_{\lambda}\in C^1(H^{\alpha}(\mathbb{R}^3),\mathbb{R})$ under $(\mathbb{G}_0)-(\mathbb{G}_3)$. Clearly, if ψ is a critical point of the energy functional $\mathscr{E}_{\lambda}$, then $(\psi,\phi_{\psi}^{\beta})\in H^{\alpha}(\mathbb{R}^3)\times\mathscr{D}^{\beta,2}(\mathbb{R}^3)$ is a weak solution to (1.1).

Similar as that in [3], without loss of generality, we assume that

$$\vartheta=\inf\{t\in(0,+\infty):G(t)>0\},$$

where ϑ is given in $(\mathbb{G}_3)$. Let $t_0=\min\{t\in(\vartheta,+\infty):g(t)=0\}$($t_0=+\infty$ if $g(t)>0$ for any $t\geq\vartheta$), and define $\tilde{g}:\mathbb{R}\to\mathbb{R}$,

$$\tilde{g}(t)=\begin{cases}g(t),\ \text{if }0\leq t\leq t_0,\\ 0,\ \text{if }t\geq t_0,\\ -\tilde{g}(-t)\ \text{for }t<0.\end{cases}$$

Observe that $\tilde{g}$ is an odd function and satisfies $(\mathbb{G}_0)-(\mathbb{G}_3)$. We note that if $\psi\in H^{\alpha}(\mathbb{R}^3)$ solves (2.2) for $\tilde{g}$, then, we obtain by the strong maximum principle for fractional Laplacian that ψ is positive and $\psi(x)\leq t_0$ for all $x\in\mathbb{R}^3$, i.e., ψ solves the original problem (2.2) for g. Hence, we can replace g by $\tilde{g}$, but still use the same notation g. With this modification, g satisfies the stronger condition

$(\mathbb{G}_2)'$ $\lim\limits_{t\to+\infty}\frac{g(t)}{t^{2^*_\alpha-1}}=0$.
In addition, for $t\geq 0$, define

$$g_1(t)=\max\{g(t)+\omega t,0\}=:(g(t)+\omega t)^+,\quad g_2(t)=g_1(t)-g(t), \tag{2.3}$$

and extend g_1, g_2 as odd functions for $t<0$. Then, by (2.3), we have that $g=g_1-g_2$ with $g_1,g_2\geq 0$ on $\mathbb{R}^+$, and

$$g_2(t)\geq \omega t,\ \forall\, t\geq 0, \tag{2.4}$$

and the following limits hold:

$$\lim_{t\to 0}\frac{g_1(t)}{t}=0,\quad \lim_{t\to+\infty}\frac{g_1(t)}{t^{2^*_\alpha-1}}=0. \tag{2.5}$$

Moreover, from (2.5) by some computations, for each $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$g_1(t)\leq \varepsilon g_2(t)+C_\varepsilon t^{2^*_\alpha-1},\ \ t\geq 0. \tag{2.6}$$

Let $G_i(t)=\int_0^t g_i(s)\,\mathrm{d}s, i=1,2$. From (2.6), we have

$$G_1(t)\leq \varepsilon G_2(t)+C_\varepsilon|t|^{2^*_\alpha},\ \ \forall\, t\in\mathbb{R}. \tag{2.7}$$

In order to verify the mountain-pass geometry and the boundedness of the (PS) sequences for $\mathscr{E}_\lambda$, following [9], we construct a truncation function $\Lambda(t)\in C_0^\infty([0,+\infty),\mathbb{R})$ as below

$$\begin{cases}\Lambda(t)=1,\ \text{for } t\in[0,1],\\ \Lambda(t)=0,\ \text{for } t\in[2,+\infty),\\ 0\leq\Lambda(t)\leq 1,\ \text{for } t\in(1,2)\end{cases}\qquad \text{and } \|\Lambda'(t)\|_\infty\leq 2.$$

For all $d>0$, let

$$T_d(\psi)=\Lambda\Big(\frac{\|\psi\|_\theta^\theta}{d^\theta}\Big),\quad \theta=\frac{12}{3+2\beta}.$$

We will study the following modified functional $\mathscr{E}_\lambda^d: H^\alpha(\mathbb{R}^3)\to\mathbb{R}$ defined by

$$\mathscr{E}_\lambda^d(\psi)=\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\psi|^2\,\mathrm{d}x+\frac{\lambda}{4}T_d(\psi)\int_{\mathbb{R}^3}\phi_\psi^\beta\psi^2\,\mathrm{d}x-\int_{\mathbb{R}^3}G(\psi)\,\mathrm{d}x.$$

Under our conditions, it is easy to see that $\mathscr{E}_\lambda^d\in C^1(H^\alpha(\mathbb{R}^3),\mathbb{R})$, and, for every $\psi,\varphi\in H^\alpha(\mathbb{R}^3)$, one has

$$\begin{aligned}\big\langle(\mathscr{E}_\lambda^d)'(\psi),\varphi\big\rangle&=\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}\psi(-\Delta)^{\frac{\alpha}{2}}\varphi\,\mathrm{d}x+\lambda T_d(\psi)\int_{\mathbb{R}^3}\phi_\psi^\beta\psi\varphi\,\mathrm{d}x\\&\quad+\frac{\lambda\theta}{4d^\theta}\Lambda'\Big(\frac{\|\psi\|_\theta^\theta}{d^\theta}\Big)\int_{\mathbb{R}^3}|\psi|^{\theta-2}\psi\varphi\,\mathrm{d}x\int_{\mathbb{R}^3}\phi_\psi^\beta\psi^2\,\mathrm{d}x-\int_{\mathbb{R}^3}g(\psi)\varphi\,\mathrm{d}x.\end{aligned}$$

Clearly, if ψ is a critical point of $\mathscr{E}_\lambda^d$ and satisfies $\|\psi\|_\theta\leq d$, then ψ is a solution of to equivalent equation (2.2).

In the following, we introduce two compactness results obtained in [2, 3] and [6], respectively.

Lemma 2.3. *(see [2, 3]) Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\lim_{|t|\to+\infty} \frac{P(t)}{Q(t)} = 0.$$

Let w be a bounded measurable function and $\{v_n\}$ be a sequence of measurable functions: $\mathbb{R}^N \to \mathbb{R}$ such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} |Q(v_n(x))w|\,\mathrm{d}x < +\infty \ \text{ and } \ P(v_n(x)) \to v(x) \text{ for a.e. } x \in \mathbb{R}^N.$$

Then, for any bounded Borel set $\mathscr{B}$,

$$\lim_{n\to\infty} \int_{\mathscr{B}} |(P(v_n(x)) - v(x))w|\,\mathrm{d}x = 0.$$

Moreover, if

$$\lim_{t\to 0} \frac{P(t)}{Q(t)} = 0 \ \text{ and } \ \lim_{|x|\to+\infty} \sup_{n\in\mathbb{N}} |v_n(x)| = 0,$$

then

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |(P(v_n(x)) - v(x))w|\,\mathrm{d}x = 0.$$

Lemma 2.4. *(see [6]) Let $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ be a Banach space, $\mathscr{X}$ be continuously embedded into $L^r(\mathbb{R}^N)$ for $r \in [r_1, r_2]$ and compactly embedded into $L^r(\mathbb{R}^N)$ for $r \in (r_1, r_2)$, where $r_1, r_2 \in (0, +\infty)$. Assume that $P \in C(\mathbb{R}, \mathbb{R})$, $\{v_n\} \subset \mathscr{X}$ and $v : \mathbb{R}^N \to \mathbb{R}$ is a measurable function satisfying*

$$\lim_{|t|\to 0} \frac{P(t)}{|t|^{r_1}} = 0, \ \lim_{|t|\to\infty} \frac{P(t)}{|t|^{r_2}} = 0, \ \sup_{n\in\mathbb{N}} \|v_n\|_{\mathscr{X}} < +\infty, \ P(v_n(x)) \to v(x) \text{ for a.e. } x \in \mathbb{R}^N.$$

Then, up to a subsequence, $P(v_n) \to v$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$.

3. PROOF OF THE MAIN RESULT

This section is devoted to the proof of Theorem 1.1. We will try to find critical point of $\mathscr{E}_\lambda$ on $H^\alpha_{\text{rad}}(\mathbb{R}^3)$, where $H^\alpha_{\text{rad}}(\mathbb{R}^3)$ is the subspace of $H^\alpha(\mathbb{R}^3)$ containing only the radial functions. In this way, if $u \in H^\alpha_{\text{rad}}(\mathbb{R}^3)$, then also ϕ^β_ψ is radial by (vi) of Lemma 2.2. We note that the embedding $H^\alpha_{\text{rad}}(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is compact for any $q \in (2, 2^*_\alpha)$; see, e.g., [14, 1].

In what follows, we set

$$\mathbb{D}_m = \{\sigma = (\sigma_1, \cdots, \sigma_m) \in \mathbb{R}^m : |\sigma| \le 1\}, \quad \partial\mathbb{D}_m = \{\sigma = (\sigma_1, \cdots, \sigma_m) \in \mathbb{R}^m : |\sigma| = 1\}.$$

First, we show that $\mathscr{E}^d_\lambda$ satisfies the symmetric mountain-pass geometry.

Lemma 3.1. *Suppose that $(\mathbb{G}_0) - (\mathbb{G}_3)$ hold. Then*

- *(i) there exist constants $\rho_0, \delta_0 > 0$ such that $\mathscr{E}^d_\lambda(\psi) \ge \delta_0$ for $\|\psi\|_{H^\alpha(\mathbb{R}^3)} = \rho_0$ and $\mathscr{E}^d_\lambda(\psi) \ge 0$ for $\|\psi\|_{H^\alpha(\mathbb{R}^3)} \le \rho_0$;*
- *(ii) for each $m \in \mathbb{N}$, there exists an odd continuous mapping $\gamma_{0m} : \partial\mathbb{D}_m \to H^\alpha_{\text{rad}}(\mathbb{R}^3)$ such that $\mathscr{E}^d_\lambda(\gamma_{0m}(\sigma)) < 0$ for all $\sigma \in \partial\mathbb{D}_m$.*

Proof. (i) By the definition of $\mathscr{E}_\lambda^d(\psi)$, using (2.7) for $\varepsilon = 1$, (2.4) and Lemma 2.1(ii), we have

$$\begin{aligned}\mathscr{E}_\lambda^d(\psi) =&\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\psi|^2\,\mathrm{d}x+\frac{\lambda}{4}T_d(\psi)\int_{\mathbb{R}^3}\phi_\psi^\beta\psi^2\,\mathrm{d}x-\int_{\mathbb{R}^3}(G_1(\psi)-G_2(\psi))\,\mathrm{d}x\\ \geq&\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\psi|^2\,\mathrm{d}x+\frac{\omega}{2}\int_{\mathbb{R}^3}|\psi|^2\,\mathrm{d}x-\frac{C_0}{2_\alpha^*}\int_{\mathbb{R}^3}|\psi|^{2_\alpha^*}\,\mathrm{d}x\\ \geq&\frac{1}{2}\min\{1,\omega\}\|\psi\|_{H^\alpha(\mathbb{R}^3)}^2-C\|\psi\|_{H^\alpha(\mathbb{R}^3)}^{2_\alpha^*}.\end{aligned}$$

Note that $2_\alpha^* > 2$. Thus there exist $\rho_0, \delta_0 > 0$ such that $\mathscr{E}_\lambda^d(\psi) \geq \delta_0 > 0$ for $\|\psi\|_{H^\alpha(\mathbb{R}^3)} = \rho_0$ small enough.

(ii) Arguing as [4, Theorem 10], for every $m \in \mathbb{N}$, we can find a continuous mapping $\pi_m : \partial\mathbb{D}_m \to H_{\mathrm{rad}}^\alpha(\mathbb{R}^3)$ with the properties

$$0 \notin \pi_m(\partial\mathbb{D}_m),\ \ \pi_m(-\sigma) = -\pi_m(\sigma),\ \ \int_{\mathbb{R}^3}G(\pi_m(\sigma))\,\mathrm{d}x \geq 1 \text{ for all } \sigma \in \partial\mathbb{D}_m.$$

Set $\gamma_{0m}(\sigma)(x) := \pi_m(\sigma)(t^{-1}x)$ for $t > 0$. Then, for sufficiently large $t \geq \big(\frac{2d^\theta}{\|\pi_m(\sigma)\|_\theta^\theta}\big)^{\frac{1}{3}} > 0$,

$$\begin{aligned}\mathscr{E}_\lambda^d(\gamma_{0m}(\sigma)) =&\frac{t^{3-2\alpha}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\pi_m(\sigma)|^2\,\mathrm{d}x+\frac{\lambda t^{3+2\beta}}{4}\Lambda\Big(\frac{t^3\|\pi_m(\sigma)\|_\theta^\theta}{d^\theta}\Big)\int_{\mathbb{R}^3}\phi_{\pi_m(\sigma)}^\beta\pi_m(\sigma)^2\,\mathrm{d}x\\ &-t^3\int_{\mathbb{R}^3}G(\pi_m(\sigma))\,\mathrm{d}x\\ \leq&\frac{t^{3-2\alpha}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\pi_m(\sigma)|^2\,\mathrm{d}x-t^3\\ <&0\ \text{ for all } \sigma \in \partial\mathbb{D}_m.\end{aligned}$$

Thus, for sufficiently large $t = t_m > \max\big\{1, \big(\frac{2d^\theta}{\|\pi_m(\sigma)\|_\theta^\theta}\big)^{\frac{1}{3}}\big\}$, γ_{0m} has the desired property. ☐

For every $m \geq 1$, by Lemma 3.1, we can define

$$c_m = \inf_{\gamma\in\Gamma_m}\max_{\sigma\in\mathbb{D}_m}\mathscr{E}_\lambda^d(\gamma(\sigma)), \tag{3.1}$$

where

$$\Gamma_m = \left\{\gamma \in C(\mathbb{D}_m, H_{\mathrm{rad}}^\alpha(\mathbb{R}^3))\,\middle|\,\begin{matrix}\gamma(-\sigma) = -\gamma(\sigma) \text{ for all } \sigma \in \mathbb{D}_m\\ \gamma(\sigma) = \gamma_{0m}(\sigma) \text{ for all } \sigma \in \partial\mathbb{D}_m\end{matrix}\right\},$$

and γ_{0m} is given in Lemma 3.1(ii). We remark that $\Gamma_m \neq \emptyset$, since $\gamma_m \in \Gamma_m$, where

$$\gamma_m(\sigma) = \begin{cases}|\sigma|\gamma_{0m}(\frac{\sigma}{|\sigma|}), \text{ if } \sigma \in \mathbb{D}_m\setminus\{0\},\\ 0, \text{ if } \sigma = 0.\end{cases}$$

Thus, by the Ekeland's principle, there exists a (PS) sequence at the mountain-pass level c_m for $\mathscr{E}_\lambda^d$. That is, a sequence $\{\psi_n\} \subset H_{\mathrm{rad}}^\alpha(\mathbb{R}^3)$ such that

$$\mathscr{E}_\lambda^d(\psi_n) \to c_m,\quad (\mathscr{E}_\lambda^d)'(\psi_n) \to 0 \ \text{ as } \ n \to \infty,$$

where c_m is the minimax level of functional $\mathscr{E}_\lambda^d$ given in (3.1).

We can easily see that, for all $\gamma\in\Gamma_m$, $\gamma(\mathbb{D}_m)\cap\{\psi\in H^\alpha_{\rm rad}(\mathbb{R}^3):\|\psi\|_{H^\alpha(\mathbb{R}^3)}=\rho_0\}\neq\emptyset$. Then, it follows from Lemma 3.1 that

$$c_m\geq\delta_0>0,\ \forall\, m\in\mathbb{N}^+. \tag{3.2}$$

Moreover, the c_m has the following property.

Lemma 3.2. *$c_m\to+\infty$ as $m\to\infty$, where c_m defined in (3.1).*

Proof. The proof is similar to that in [15, Chapter 9]. Let

$$\Gamma^*_m=\left\{h(\overline{(\mathbb{D}_k\times H^\alpha_{\rm rad}(\mathbb{R}^3))\setminus Y}):h\in\Gamma_k,k\geq m,Y\in\Sigma_k,\ \mathrm{genus}(Y)\leq k-m\right\},$$

where Σ_k is the family of closed sets $A\subset\mathbb{D}_k\times H^\alpha_{\rm rad}(\mathbb{R}^3)$ with the property that $-A=A$, and genus(Y) denotes the Krasnoselskii genus of Y. Then Γ^*_m is stable under deformation. Thus we can define another sequence of minimax values of $\mathscr{E}^d_\lambda$ by

$$c^*_m=\inf_{A\in\Gamma^*_m}\sup_{u\in A}\mathscr{E}^d_\lambda(\psi).$$

Modifying the arguments in [15, Chapter 9], we see
(i) $c^*_m\leq c_m$;
(ii) $c^*_1\leq c^*_2\leq\cdots\leq c^*_m\leq c^*_{m+1}\leq\cdots$;
(iii) $c^*_m\to+\infty$ as $m\to\infty$.
The conclusion of Lemma 3.2 then follows from (i)-(iii). □

From Lemma 3.2, we can assume that the minimax values c_m are strictly monotone increasing.

It seems difficult to show the (PS) compactness condition for $\mathscr{E}^d_\lambda$ directly and it is a main difficulty in showing that c_m is a critical value of $\mathscr{E}^d_\lambda$. Let $\Phi:\mathbb{R}\times H^\alpha(\mathbb{R}^3)\to H^\alpha(\mathbb{R}^3)$ be given by $\Phi(\tau,v)=v(e^{-\tau}x)$, and define an auxiliary functional $\widetilde{\mathscr{E}}^d_\lambda:\mathbb{R}\times H^\alpha(\mathbb{R}^3)\to\mathbb{R}$:

$$\begin{aligned}\widetilde{\mathscr{E}}^d_\lambda(\tau,v)=\mathscr{E}^d_\lambda(\Phi(\tau,v))=&\frac{e^{(3-2\alpha)\tau}}{2}\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x+\frac{\lambda e^{(3+2\beta)\tau}}{4}\Lambda\Big(\frac{e^{3\tau}\|v\|^\theta_\theta}{d^\theta}\Big)\int_{\mathbb{R}^3}\phi^\beta_v v^2\,\mathrm{d}x\\&-e^{3\tau}\int_{\mathbb{R}^3}G(v)\,\mathrm{d}x.\end{aligned}$$

We endow $\mathbb{R}\times H^\alpha(\mathbb{R}^3)$ with the usual norm $\|(\tau,v)\|_{\mathbb{R}\times H^\alpha(\mathbb{R}^3)}=\sqrt{|\tau|^2+\|v\|^2_{H^\alpha(\mathbb{R}^3)}}$.

In view of $(\mathbb{G}_1)$, the functional $\widetilde{\mathscr{E}}^d_\lambda(\tau,\psi)\in C^1(\mathbb{R}\times H^\alpha(\mathbb{R}^3))$ and satisfies

$$\widetilde{\mathscr{E}}^d_\lambda(0,\psi)=\mathscr{E}^d_\lambda(\psi),\quad \widetilde{\mathscr{E}}^d_\lambda(\tau,\psi)=\mathscr{E}^d_\lambda(\psi(e^{-\tau}x)) \tag{3.3}$$

for all $\psi\in H^\alpha(\mathbb{R}^3)$ and $\tau\in\mathbb{R}$. Now, we define a minimax value $\tilde c_m$ for $\widetilde{\mathscr{E}}^d_\lambda$ by

$$\tilde c_m=\inf_{\tilde\gamma\in\tilde\Gamma_m}\max_{\sigma\in\mathbb{D}_m}\widetilde{\mathscr{E}}^d_\lambda(\tilde\gamma(\sigma)), \tag{3.4}$$

where

$$\tilde\Gamma_m=\left\{\tilde\gamma(\sigma)=(\tau(\sigma),\eta(\sigma))\in C(\mathbb{D}_m,\mathbb{R}\times H^\alpha_{\rm rad}(\mathbb{R}^3))\,\middle|\,\begin{array}{l}(\tau(-\sigma),\eta(-\sigma))=(\tau(\sigma),-\eta(\sigma)),\ \forall\,\sigma\in\mathbb{D}_m\\(\tau(\sigma),\eta(\sigma))=(0,\gamma_{0m}(\sigma)),\ \forall\,\sigma\in\partial\mathbb{D}_m\end{array}\right\}.$$

Then we have the following results.

Lemma 3.3. *For every* $m \geq 1$, $\tilde{c}_m = c_m$.

Proof. First, since for any $\gamma(\sigma) \in \Gamma_m$, we can see that $(0, \gamma(\sigma)) \in \widetilde{\Gamma}_m$, that is, $\{0\} \times \Gamma_m \subset \widetilde{\Gamma}_m$. Then, for $\widetilde{\mathscr{E}}^d_\lambda(0, v) = \mathscr{E}^d_\lambda(v)$ we have $\tilde{c}_m \leq c_m$. On the other hand, for any given

$$\widetilde{\gamma}(\sigma) = (\tau(\sigma), \eta(\sigma)) \in \widetilde{\Gamma}_m,$$

setting $\gamma(\sigma)(x) = \eta(\sigma)(e^{-\tau(\sigma)}(x))$, we can verify that $\gamma(\sigma) \in \Gamma_m$, and by (3.3), $\mathscr{E}^d_\lambda(\gamma(\sigma)) = \widetilde{\mathscr{E}}^d_\lambda(\widetilde{\gamma}(\sigma))$ for every $\sigma \in \mathbb{D}_m$, which implies that $\tilde{c}_m \geq c_m$. Hence $\tilde{c}_m = c_m$. □

Lemma 3.4. *Suppose that* $(\mathbb{G}_0)-(\mathbb{G}_3)$ *hold. For any* $m \geq 1$, *there exists a sequence* $\{(\tau_n, \psi_n)\} \subset \mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)$ *such that*
(i) $\tau_n \to 0$;
(ii) $\widetilde{\mathscr{E}}^d_\lambda(\tau_n, \psi_n) \to c_m$;
(iii) $(\widetilde{\mathscr{E}}^d_\lambda)'(\tau_n, \psi_n) \to 0$ *strongly in* $(\mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3))^*$;
(iv) $\partial_\tau \widetilde{\mathscr{E}}^d_\lambda(\tau_n, \psi_n) \to 0$.

To prove Lemma 3.4, we need the following result, which was established in [8] (see Lemma 2.3).

Lemma 3.5. *Let* $\varepsilon > 0$ *and* $\tilde{\gamma}(\sigma) \in \widetilde{\Gamma}_m$ *such that* $\sup_{\sigma \in \mathbb{D}_m} \widetilde{\mathscr{E}}^d_\lambda(\tilde{\gamma}(\sigma)) \leq \tilde{c}_m + \varepsilon$. *Then there exists a pair of* $(\tau_0, v_0) \in \mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)$ *satisfying*
(a) $\widetilde{\mathscr{E}}^d_\lambda(\tau_0, v_0) \in [\tilde{c}_m - \varepsilon, \tilde{c}_m + \varepsilon]$;
(b) $\|\nabla \widetilde{\mathscr{E}}^d_\lambda(\tau_0, v_0)\|_{(\mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3))^*} \leq 2\sqrt{\varepsilon}$, *where* $\nabla \widetilde{\mathscr{E}}^d_\lambda(\tau, v) = \big(\partial_\tau \widetilde{\mathscr{E}}^d_\lambda(\tau, v), (\widetilde{\mathscr{E}}^d_\lambda)'(\tau, v)\big)$;
(c) $dist_{\mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)}((\tau_0, v_0), \tilde{\gamma}(\mathbb{D}_m)) \leq 2\sqrt{\varepsilon}$, *where* $dist_{\mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)}((\tau, v), B)) = \inf\limits_{(t,u) \in B} (|\tau - t|^2 + \|v - u\|^2_{H^\alpha(\mathbb{R}^3)})^{\frac{1}{2}}$ *for* $B \subset \mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)$.

Proof of Lemma 3.4. Let $\{\gamma_n(\sigma)\} \subset \Gamma_m$ satisfy $\sup_{\sigma \in \mathbb{D}_m} \mathscr{E}^d_\lambda(\gamma_n(\sigma)) \leq c_m + \frac{1}{n}$. Setting $\tilde{\gamma}_n(\sigma) := (0, \gamma_n(\sigma))$, one has $\tilde{\gamma}_n \in \widetilde{\Gamma}_m$. In view of $\tilde{c}_m = c_m$ (see Lemma 3.3), one obtains

$$\sup_{\sigma \in \mathbb{D}_m} \widetilde{\mathscr{E}}^d_\lambda(\tilde{\gamma}_n(\sigma)) \leq \tilde{c}_m + \frac{1}{n}.$$

By Lemma 3.5, one sees that there exists a sequence $\{(\tau_n, v_n)\}$ in $\mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)$ such that

$$\widetilde{\mathscr{E}}^d_\lambda(\tau_n, v_n) \in \big[\tilde{c}_m - \frac{1}{n}, \tilde{c}_m + \frac{1}{n}\big], \tag{3.5}$$

$$\|\nabla \widetilde{\mathscr{E}}^d_\lambda(\tau_n, v_n)\|_{(\mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3))^*} \leq \frac{2}{\sqrt{n}} \tag{3.6}$$

and

$$\mathrm{dist}_{\mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)}\big((\tau_n, v_n), \tilde{\gamma}_n(\mathbb{D}_m)\big) \leq \frac{2}{\sqrt{n}}. \tag{3.7}$$

Note that $\tilde{\gamma}_n(\mathbb{D}_m) \subset \{0\} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)$. From (3.5)-(3.7) we see that the conclusions of Lemma 3.4 hold. □

Lemma 3.6. *Let* $\{(\tau_n, \psi_n)\} \subset \mathbb{R} \times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)$ *be the sequence given in Lemma 3.4. Then there exist* $d > 0$ *sufficiently large and* $\lambda^* > 0$ *depends on* d, *such that, for any* $\lambda \in (0, \lambda^*)$, *there holds* $\limsup\limits_{n \to \infty} \|\psi_n\|_\theta \leq d$.

Proof. From Lemma 3.4, we see that $3\widetilde{\mathscr{E}}_\lambda^d(\tau_n,\psi_n)-\partial_\tau\widetilde{\mathscr{E}}_\lambda^d(\tau_n,\psi_n)\to 3c_m$, which implies that

$$\begin{aligned}&\alpha e^{(3-2\alpha)\tau_n}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\psi_n|^2\,\mathrm{d}x-\frac{\lambda\beta}{2}e^{(3+2\beta)\tau_n}\Lambda\Big(\frac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big)\int_{\mathbb{R}^3}\phi_{\psi_n}^\beta\psi_n^2\,\mathrm{d}x\\&-\frac{3\lambda e^{(6+2\beta)\tau_n}\|\psi_n\|_\theta^\theta}{4d^\theta}\Lambda'\Big(\frac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big)\int_{\mathbb{R}^3}\phi_{\psi_n}^\beta\psi_n^2\,\mathrm{d}x=3c_m+o_n(1).\end{aligned}\tag{3.8}$$

From (3.8) and Lemma 2.2(iii), we have

$$\begin{aligned}&\alpha e^{(3-2\alpha)\tau_n}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\psi_n|^2\,\mathrm{d}x\\=&3c_m+\frac{\lambda\beta}{2}e^{(3+2\beta)\tau_n}\Lambda\Big(\frac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big)\int_{\mathbb{R}^3}\phi_{\psi_n}^\beta\psi_n^2\,\mathrm{d}x\\&+\frac{3\lambda e^{(6+2\beta)\tau_n}\|\psi_n\|_\theta^\theta}{4d^\theta}\Lambda'\Big(\frac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big)\int_{\mathbb{R}^3}\phi_{\psi_n}^\beta\psi_n^2\,\mathrm{d}x+o_n(1)\\\le&3c_m+\frac{C_2\lambda^2\beta}{2}e^{(3+2\beta)\tau_n}\Lambda\Big(\frac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big)\|\psi_n\|_\theta^4\\&+\frac{3C_2\lambda^2e^{(6+2\beta)\tau_n}}{4d^\theta}\Lambda'\Big(\frac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big)\|\psi_n\|_\theta^{4+\theta}+o_n(1).\end{aligned}\tag{3.9}$$

By the definition of c_m, we obtain

$$\begin{aligned}c_m\le&\max_{\sigma\in\mathbb{D}_m}\mathscr{E}_\lambda^d(\gamma(\sigma))\\\le&\max_{\sigma\in\mathbb{D}_m}\Big(\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\gamma(\sigma)|^2\,\mathrm{d}x-\int_{\mathbb{R}^3}G(\gamma(\sigma))\,\mathrm{d}x\Big)\\&+\max_{\sigma\in\mathbb{D}_m}\Big(\frac{\lambda}{4}\Lambda\Big(\frac{\|\gamma(\sigma)\|_\theta^\theta}{d^\theta}\Big)\int_{\mathbb{R}^3}\phi_{\gamma(\sigma)}^\beta\gamma(\sigma)^2\,\mathrm{d}x\Big)\\=:&A_0+A_1(d).\end{aligned}\tag{3.10}$$

We note that if $\|\gamma(\sigma)\|_\theta^\theta\ge 2d^\theta$, by the definition of Λ, we obtain that $A_1(d)=0$. On the other hand if $\|\gamma(\sigma)\|_\theta^\theta<2d^\theta$, then we conclude by Lemma 2.2(iii) and $\theta=\frac{12}{3+2\beta}$ that

$$A_1(d)\le\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{\gamma(\sigma)}^\beta\gamma(\sigma)^2\,\mathrm{d}x\le\frac{C_2\lambda^2}{4}\|\gamma(\sigma)\|_\theta^4\le C_3\lambda^2d^4.\tag{3.11}$$

Moreover, we can estimate

$$\begin{cases}A_2(d):=\dfrac{C_2\lambda^2\beta}{2}e^{(3+2\beta)\tau_n}\Lambda\Big(\dfrac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big)\|\psi_n\|_\theta^4\le C_4\lambda^2d^4,\\A_3(d):=\dfrac{3C_2\lambda^2e^{(6+2\beta)\tau_n}}{4d^\theta}\Lambda'\Big(\dfrac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big)\|\psi_n\|_\theta^{4+\theta}\le C_5\lambda^2d^4.\end{cases}\tag{3.12}$$

In fact, by the definition of Λ, if $e^{3\tau_n}\|\psi_n\|_\theta^\theta\ge 2d^\theta$, we obtain that $A_2(d)=0$. On the other hand, if $e^{3\tau_n}\|\psi_n\|_\theta^\theta<2d^\theta$, that is, $\|\psi_n\|_\theta<2^{\frac{1}{\theta}}e^{-\frac{3\tau_n}{\theta}}d$, then we find by the fact that $0\le\Lambda(t)\le 1$ for

$t \in [0,2)$, and $\theta = \frac{12}{3+2\beta}$ that

$$\begin{aligned} A_2(d) \leq & \frac{C_2\lambda^2\beta}{2} e^{(3+2\beta)\tau_n} \|\psi_n\|_\theta^4 \\ \leq & \frac{C_2\lambda^2\beta}{2} e^{(3+2\beta)\tau_n} \cdot 2^{\frac{4}{\theta}} e^{-\frac{12\tau_n}{\theta}} d^4 \\ = & C_2\beta \cdot 2^{\frac{4}{\theta}-1} \cdot e^{(3+2\beta-\frac{12}{\theta})\tau_n} \cdot \lambda^2 d^4 \\ = & C_2\beta \cdot 2^{\frac{2\beta}{3}} \cdot \lambda^2 d^4 = C_4\lambda^2 d^4, \end{aligned}$$

where $C_4 = C_2\beta \cdot 2^{\frac{2\beta}{3}}$. In addition, $\|\Lambda'(t)\|_\infty \leq 2$ and $\theta = \frac{12}{3+2\beta}$ yield

$$\begin{aligned} A_3(d) \leq & \frac{3C_2\lambda^2 e^{(6+2\beta)\tau_n}}{2d^\theta} \|\psi_n\|_\theta^{4+\theta} \\ \leq & \frac{3C_2\lambda^2 e^{(6+2\beta)\tau_n}}{2d^\theta} \cdot 2^{\frac{4+\theta}{\theta}} \cdot e^{-\frac{3(4+\theta)}{\theta}\tau_n} \cdot d^{4+\theta} \\ = & 3C_2 \cdot 2^{\frac{4}{\theta}} \cdot e^{(6+2\beta-\frac{3(4+\theta)}{\theta})\tau_n} \cdot \lambda^2 d^4 \\ = & 6C_2 2^{\frac{2\beta}{3}} \cdot \lambda^2 d^4 = C_5\lambda^2 d^4, \end{aligned}$$

where $C_5 = 6C_2 2^{\frac{2\beta}{3}}$. Now, it follows by (3.9) that

$$\begin{aligned} \alpha e^{(3-2\alpha)\tau_n} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}}\psi_n|^2 \,\mathrm{d}x \leq & 3(A_0 + A_1(d)) + A_2(d) + A_3(d) + o_n(1) \\ \leq & 3(A_0 + C_3\lambda^2 d^4) + C_4\lambda^2 d^4 + C_5\lambda^2 d^4 + o_n(1). \end{aligned}$$

In view of $\tau_n \to 0$, we obtain that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}}\psi_n|^2 \,\mathrm{d}x \leq C_6(1+\lambda^2 d^4). \tag{3.13}$$

In addition, by the fact that $\partial_\tau \widetilde{\mathscr{E}}_\lambda^d(\tau_n, \psi_n) \to 0$ and (2.7), we deduce that

$$\begin{aligned} & \frac{3-2\alpha}{2} e^{(3-2\alpha)\tau_n} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}}\psi_n|^2 \,\mathrm{d}x + \frac{\lambda(3+2\beta)}{4} e^{(3+2\beta)\tau_n} \Lambda\Big(\frac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big) \int_{\mathbb{R}^3} \phi_{\psi_n}^\beta \psi_n^2 \,\mathrm{d}x \\ & \quad + \frac{3\lambda e^{(6+2\beta)\tau_n}\|\psi_n\|_\theta^\theta}{4d^\theta} \Lambda'\Big(\frac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big) \int_{\mathbb{R}^3} \phi_{\psi_n}^\beta \psi_n^2 \,\mathrm{d}x + 3e^{3\tau_n} \int_{\mathbb{R}^3} G_2(\psi_n) \,\mathrm{d}x \\ = & 3e^{3\tau_n} \int_{\mathbb{R}^3} G_1(\psi_n) \,\mathrm{d}x + o_n(1) \\ \leq & 3e^{3\tau_n} \left(C_\varepsilon \int_{\mathbb{R}^3} |\psi_n|^{2_\alpha^*} \,\mathrm{d}x + \varepsilon \int_{\mathbb{R}^3} G_2(\psi_n) \,\mathrm{d}x \right) + o_n(1). \end{aligned} \tag{3.14}$$

Therefore, by (2.4), (3.14), (3.12) and (3.13), one has

$$\begin{aligned}
&\frac{\omega}{2}(1-\varepsilon)3e^{3\tau_n}\int_{\mathbb{R}^3}\psi_n^2\,\mathrm{d}x\\
\le&(1-\varepsilon)3e^{3\tau_n}\int_{\mathbb{R}^3}G_2(\psi_n)\,\mathrm{d}x\\
\le&3e^{3\tau_n}C_\varepsilon\int_{\mathbb{R}^3}|\psi_n|^{2^*_\alpha}\,\mathrm{d}x-\frac{3\lambda e^{(6+2\beta)\tau_n}\|\psi_n\|_\theta^\theta}{4d^\theta}\Lambda'\Big(\frac{e^{3\tau_n}\|\psi_n\|_\theta^\theta}{d^\theta}\Big)\int_{\mathbb{R}^3}\phi^\beta_{\psi_n}\psi_n^2\,\mathrm{d}x+o_n(1)\\
\le&\bar{C}_\varepsilon\left(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\psi_n|^2\,\mathrm{d}x\right)^{\frac{2^*_\alpha}{2}}+C_5\lambda^2d^4+o_n(1)\\
\le&C(1+\lambda^2d^4)^{\frac{2^*_\alpha}{2}}+C_5\lambda^2d^4+o_n(1).
\end{aligned}\tag{3.15}$$

Next, we claim that $\limsup_n\|\psi_n\|_\theta\le d$ if d is sufficiently large. We suppose by contradiction that $\|\psi_n\|_\theta>d$ (up to extracting a subsequence if necessary). It follows from (3.13) and (3.15) that

$$d^2<\|\psi_n\|_\theta^2\le C\|\psi_n\|^2_{H^\alpha(\mathbb{R}^3)}\le C_7+C_8\lambda^2d^4+C_9(\lambda^2d^4)^{\frac{2^*_\alpha}{2}},$$

which is a contradiction for d large enough and $\lambda^2d^4<1$. In fact, if we choose $d_0>0$ such that $d_0^2>C_7+C_8+C_9$ and $\lambda^*=\lambda^*(d_0)$ such that $\lambda^2d_0^4<1$ for all $\lambda\in(0,\lambda^*)$, then we reach a contradiction. This finishes the proof. ☐

Lemma 3.7. *Let $\{(\tau_n,\psi_n)\}\subset\mathbb{R}\times H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)$ be the sequence given in Lemma 3.4. Then $\{\psi_n\}$ admits a strong convergent subsequence in $H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)$.*

Proof. By Lemma 3.4, $\tau_n\to0$. Now, from Lemma 3.6, we see that $\{\psi_n\}$ is bounded, passing to a subsequence, we may assume that there exists $\psi\in H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)$ such that

$$\begin{cases}\psi_n\rightharpoonup\psi, & \text{in } H^\alpha_{\mathrm{rad}}(\mathbb{R}^3),\\ \psi_n\to\psi, & \text{a.e. in } \mathbb{R}^3,\\ \psi_n\to\psi, & \text{in } L^p(\mathbb{R}^3), 2<p<2^*_\alpha.\end{cases}\tag{3.16}$$

By Lemma 3.6, $\|\psi_n\|_\theta\le d$, so, for any $\varphi\in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}
\langle(\widetilde{\mathscr{E}}^d_\lambda)'(\tau_n,\psi_n),\varphi\rangle=&\langle(\widetilde{\mathscr{E}}_\lambda)'(\tau_n,\psi_n),\varphi\rangle\\
=&e^{(3-2\alpha)\tau_n}\int_{\mathbb{R}^3}(-\Delta)^{\frac{\alpha}{2}}\psi_n(-\Delta)^{\frac{\alpha}{2}}\varphi\,\mathrm{d}x+\lambda e^{(3+2\beta)\tau_n}\int_{\mathbb{R}^3}\phi^\beta_{\psi_n}\psi_n\varphi\,\mathrm{d}x\\
&-e^{3\tau_n}\int_{\mathbb{R}^3}\big(g_1(\psi_n)-(g_2(\psi_n)\big)\varphi\,\mathrm{d}x.
\end{aligned}\tag{3.17}$$

If we apply Lemma 2.3 to $P(t)=g_i(t), i=1,2$, $Q(t)=|t|^{2^*_\alpha-1}$, $v_n=\psi_n$, $v=g_i(\psi), i=1,2$ and $w=\varphi\in C_0^\infty(\mathbb{R}^3)$, by (3.16), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}g_i(\psi_n)\varphi\,\mathrm{d}x=\int_{\mathbb{R}^3}g_i(\psi)\varphi\,\mathrm{d}x,\quad i=1,2.\tag{3.18}$$

Note that $\psi_n\rightharpoonup\psi$ in $H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)$ and $H^\alpha_{\mathrm{rad}}(\mathbb{R}^3)\hookrightarrow L^q(\mathbb{R}^3)$ is compact for any $q\in(2,2^*_\alpha)$. In view of $\frac{12}{3+2\beta}\in(2,2^*_\alpha)$ since $4\alpha+2\beta>3$, we have $\psi_n\to\psi$ in $L^{\frac{12}{3+2\beta}}(\mathbb{R}^3)$, then by continuity

$\phi_{\psi_n}^{\beta} \to \phi_{\psi}^{\beta}$ in $\mathscr{D}^{\beta,2}(\mathbb{R}^3)$. Furthermore, using Lemma 2.2, we can deduce that

$$\int_{\mathbb{R}^3} \phi_{\psi_n}^{\beta} \psi_n \varphi \,\mathrm{d}x \to \int_{\mathbb{R}^3} \phi_{\psi}^{\beta} \psi \varphi \,\mathrm{d}x \ (n \to \infty). \tag{3.19}$$

From Lemma 3.4(iii), $(\widetilde{\mathscr{E}_\lambda^d})'(\tau_n, \psi_n) \to 0$, then we deduce by(3.16)-(3.19) and $\tau_n \to 0$ that

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} \psi (-\Delta)^{\frac{\alpha}{2}} \varphi \,\mathrm{d}x + \lambda \int_{\mathbb{R}^3} \phi_{\psi}^{\beta} \psi \varphi \,\mathrm{d}x - \int_{\mathbb{R}^3} \big(g_1(\psi) - g_2(\psi)\big) \varphi \,\mathrm{d}x = 0, \ \ \forall\, \varphi \in C_0^{\infty}(\mathbb{R}^3).$$

Since $C_0^{\infty}(\mathbb{R}^3)$ is dense in $H_{\mathrm{rad}}^{\alpha}(\mathbb{R}^3)$, we get $\langle \mathscr{E}_\lambda'(\psi), \varphi \rangle = 0$ for any $\varphi \in H_{\mathrm{rad}}^{\alpha}(\mathbb{R}^3)$. In particular $\langle \mathscr{E}_\lambda'(\psi), \psi \rangle = 0$, that is,

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} \psi|^2 \,\mathrm{d}x + \lambda \int_{\mathbb{R}^3} \phi_{\psi}^{\beta} \psi^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} \big(g_1(\psi) - g_2(\psi)\big) \psi \,\mathrm{d}x = 0. \tag{3.20}$$

Now, taking $\varphi = \psi_n$ in (3.17), we obtain

$$\begin{aligned} o_n(1) =& \langle (\widetilde{\mathscr{E}_\lambda})'(\tau_n, \psi_n), \psi_n \rangle \\ =& e^{(3-2\alpha)\tau_n} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} \psi_n|^2 \,\mathrm{d}x + \lambda e^{(3+2\beta)\tau_n} \int_{\mathbb{R}^3} \phi_{\psi_n}^{\beta} \psi_n^2 \,\mathrm{d}x \\ &+ e^{3\tau_n} \int_{\mathbb{R}^3} g_2(\psi_n) \psi_n \,\mathrm{d}x - e^{3\tau_n} \int_{\mathbb{R}^3} g_1(\psi_n) \psi_n \,\mathrm{d}x. \end{aligned} \tag{3.21}$$

Again, by (3.16) and Lemma 2.2, we obtain as $n \to \infty$

$$\int_{\mathbb{R}^3} \phi_{\psi_n}^{\beta} \psi_n^2 \,\mathrm{d}x \to \int_{\mathbb{R}^3} \phi_{\psi}^{\beta} \psi^2 \,\mathrm{d}x. \tag{3.22}$$

By $(\mathbb{G}_1)$, $(\mathbb{G}_2)$, and (3.16), we apply Lemma 2.4 to $\mathscr{X} = H_{\mathrm{rad}}^{\alpha}(\mathbb{R}^3)$, $q_1 = 2, q_2 = 2_\alpha^*$, $P(t) = g_1(t)t$, $v_n = \psi_n$ and $v = g_1(\psi)\psi$. Then

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} g_1(\psi_n) \psi_n \,\mathrm{d}x = \int_{\mathbb{R}^3} g_1(\psi) \psi \,\mathrm{d}x. \tag{3.23}$$

Moreover, by (3.16) and Fatou's lemma, we obtain

$$\int_{\mathbb{R}^3} g_2(\psi) \psi \,\mathrm{d}x \le \liminf_{n\to\infty} \int_{\mathbb{R}^3} g_2(\psi_n) \psi_n \,\mathrm{d}x. \tag{3.24}$$

Then by (3.21)-(3.24) and the fact that $\tau_n \to 0$, we have

$$\begin{aligned} &\limsup_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} \psi_n|^2 \,\mathrm{d}x \\ =& \limsup_{n\to\infty} e^{(3-2\alpha)\tau_n} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} \psi_n|^2 \,\mathrm{d}x \\ =& \limsup_{n\to\infty} \left(e^{3\tau_n} \int_{\mathbb{R}^3} g_1(\psi_n) \psi_n \,\mathrm{d}x - e^{3\tau_n} \int_{\mathbb{R}^3} g_2(\psi_n) \psi_n \,\mathrm{d}x - \lambda e^{(3+2\beta)\tau_n} \int_{\mathbb{R}^3} \phi_{\psi_n}^{\beta} \psi_n^2 \,\mathrm{d}x \right) \\ \le& \int_{\mathbb{R}^3} g_1(\psi) \psi \,\mathrm{d}x - \int_{\mathbb{R}^3} g_2(\psi) \psi \,\mathrm{d}x - \lambda \int_{\mathbb{R}^3} \phi_{\psi}^{\beta} \psi^2 \,\mathrm{d}x \\ =& \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} \psi|^2 \,\mathrm{d}x. \ (\text{by } (3.20)) \end{aligned} \tag{3.25}$$

On the other hand, by weak lower semi-continuity,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} \psi_n|^2 \,\mathrm{d}x \ge \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} \psi|^2 \,\mathrm{d}x. \tag{3.26}$$

Now, by (3.26) and (3.25), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\psi_n|^2\,\mathrm{d}x=\int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}\psi|^2\,\mathrm{d}x. \tag{3.27}$$

From (3.25) and (3.27) we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}g_2(\psi_n)\psi_n\,\mathrm{d}x=\int_{\mathbb{R}^3}g_2(\psi)\psi\,\mathrm{d}x. \tag{3.28}$$

Recall that g is odd on $\mathbb{R}$. By (2.3), we have

$$g_2(t)=(g(t)+\omega t)^+-g(t)=\omega t+(g(t)+\omega t)^-.$$

Thus $g_2(t)t=\omega t^2+k(t)$, where $k(t)=t(g(t)+\omega t)^-$ is a nonnegative and continuous function. Fatou's lemma yields

$$\liminf_{n\to\infty}\int_{\mathbb{R}^3}|\psi_n|^2\,\mathrm{d}x\geq\int_{\mathbb{R}^3}|\psi|^2\,\mathrm{d}x,\quad \liminf_{n\to\infty}\int_{\mathbb{R}^3}k(\psi_n)\,\mathrm{d}x\geq\int_{\mathbb{R}^3}k(\psi)\,\mathrm{d}x.$$

By the above two inequalities and (3.28), up to a subsequence, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|\psi_n|^2\,\mathrm{d}x=\int_{\mathbb{R}^3}|\psi|^2\,\mathrm{d}x. \tag{3.29}$$

From (3.27) and (3.29) we have $\psi_n\to\psi$ strongly in $H^{\alpha}_{\mathrm{rad}}(\mathbb{R}^3)$. □

Proof of Theorem 1.1. Let $\{(\tau_n,\psi_n)\}\subset\mathbb{R}\times H^{\alpha}_{\mathrm{rad}}(\mathbb{R}^3)$ be a sequence obtained in Lemma 3.4. From Lemma 3.7, we see that there exists $\psi_{0m}\in H^{\alpha}_{\mathrm{rad}}(\mathbb{R}^3)$ such that $\psi_n\to\psi_{0m}$ strongly in $H^{\alpha}_{\mathrm{rad}}(\mathbb{R}^3)$. Then by Lemma 3.4, we have

$$\widetilde{\mathscr{E}}^d_\lambda(0,\psi_{0m})=c_m,\quad (\widetilde{\mathscr{E}}^d_\lambda)'(0,\psi_{0m})=0. \tag{3.30}$$

Moreover, by Lemma 3.6, we see that $\|\psi_{0m}\|_\theta\leq d$ for any $\lambda\in(0,\lambda^*)$. Hence, (3.30) yields

$$\mathscr{E}_\lambda(\psi_{0m})=c_m,\quad \mathscr{E}'_\lambda(\psi_{0m})=0.$$

By (3.2), we see that $c_m\geq\delta_0>0$, which implies $\psi_{0m}\neq0$. Thus c_m is a critical value of $\mathscr{E}_\lambda$. From Lemma 3.2 $c_m\to+\infty$ as $m\to\infty$, we can consider $c_1<c_2<\cdots<c_m$, which completes the proof. □

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