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INFINITELY MANY POSITIVE PERIODIC SOLUTIONS FOR TWO CLASSES OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the existence of positive periodic solutions to two classes of neutral functional differential equations. By using Schauder's fixed point theorem and Krasoselskii's fixed point theorem in cones, we demonstrate that the problems have infinitely many positive periodic solutions under appropriate conditions. Some examples are presented to illustrate our results.

Keywords. Fixed point theorem; Neutral functional differential equation; Positive periodic solution.

1. Introduction

In this paper, we are concerned about the existence of positive periodic solution for the following neutral differential equations

$$(x(t)-cx(t-\tau))' = f(x(t-\tau_1)) - f(x(t-\tau_2)) + p(t), \tag{1.1}$$

$$(x(t)-cx(t-\tau))'' = f(x(t-\tau_1)) - f(x(t-\tau_2)) + p(t), \tag{1.2}$$

where $c,\tau,\tau_1,\tau_2 \in \mathbb{R}$, $\tau_1 \neq \tau_2$, $p \not\equiv 0$ is a continuous ω-periodic function with $\int_0^\omega p(t)dt = 0$, and $\omega > 0$ is a constant.

The existence of periodic solutions has always been a hot topic in the field of functional differential equations; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8] and the references therein. In recent years, the existence of periodic solutions for neutral functional differential equations (NFDEs) has attracted the interest due to their various real applications. In [9], Serra investigated the existence of periodic solutions for $x'(t)+ax'(t-\tau) = f(t,x(t))$ by using the continuation theorem. Afonso, Bonotto and Silva [10] studied the existence and uniqueness of periodic solutions for $(x(t)-A(t,x_t))' = f(t,x_t)$ with the aid of a fixed point theorem. In [11], using the coincidence degree theory, Lu and Ge investigated the existence and uniqueness of periodic solutions for the equation

$$(x(t)-cx(t-\tau)))'' = g(u(t)) + g_2(u(t-\delta)) + p(t).$$

Recently, numerous authors studied the existence of positive periodic solutions for NFDEs. In [12], Luo, Wang, and Shen discussed the existence of positive periodic solutions of the following neutral differential equation

$$(x(t)-cx(t-h(t)))' + a(t)x(t) = f(t,x(t-h(t)))$$

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by using Krasoselskii's fixed point theorem. The results presented in [12] were further extended by [13] to the following equation with variable coefficients

$$(x(t)-c(t)x(t-\tau))'+a(t)x(t)=f(t,x(t-\tau)).$$

Similar methods were applied for the following equations

$$(x(t)-cx(t-h(t)))''\pm a(t)x(t)=\pm f(t,x(t-h(t))),$$

$$(x(t)-c(t)x(t-h(t)))''\pm a(t)x(t)=\pm f(t,x(t-h(t))),$$

and

$$\left[x(t)-\sum_{i=1}^{2}c_i(t)x(t-\tau_i)\right]'+a(t)x(t)=\sum_{i=1}^{2}f_i(t,x(t),x(t-\tau_i))-H(\dots)$$

by Cheung et al. [14], Candan [15], and Gözen [16]. In the references mentioned above, the basic idea is to constructed two appropriate operators from the linear term $a(t)x(t)$: One is completely continuous and the other is contractive. In [17], Wu and Wang considered existence, multiplicity, and nonexistence of positive periodic solutions for the following second-order NFDE

$$(x(t)-cx(t-\tau))''+a(t)x(t)=\lambda b(t)f(x(t-\tau))).$$

Wang and Shen [18] studied the existence of positive periodic solutions of the NFDE

$$(x(t)-cx(t-\tau))'=k(t)g(x(t-\delta(t)))-b(t).$$

Yang and Li [19] discussed the existence of positive periodic solutions for the NFDE with delayed derivative terms

$$(x(t)-cx(t-\tau))'''+a(t)x(t)=f(t,x(t-\tau_0),x'(t-\tau_1),x''(t-\tau_2)))$$

under the condition that f satisfies superlinear or sublinear growth. The key step in those papers is the application of the properties of the operator $(Ax)(t)=x(t)-cx(t-\tau)$, and converted the associated problem into a non-neutral functional differential equation. The other results in this field can be found in [20, 21, 22]. It is noticed that almost all the results in literature only discussed the existence of a finite number of positive periodic solutions for NFDEs.

The aim of this paper is to study the existence of infinitely many positive periodic solutions for (1.1) and (1.2). With proper variable substitution, we construct appropriate operators whose fixed point is closely related to the positive periodic solution of the original problem. Under weak conditions, we demonstrated that (1.1) or (1.2) has infinitely many positive periodic solutions.

2. Preliminaries

Let $E_\omega=\{u\in C(\mathbb{R},\mathbb{R}):u(t+\omega)=u(t),t\in\mathbb{R}\}$ with the norm $\|u\|=\max_{0\le t\le\omega}|u(t)|$. It is clear that E_ω is a Banach space. Let $h,\ k\in E_\omega$ and consider the differential equation

$$x'(t)=-h(t)x(t)+k(t). \tag{2.1}$$

Lemma 2.1. [18] *If $\int_0^\omega h(s)\neq 0$, then (2.1) has a unique ω-periodic solution*

$$x(t)=\int_t^{t+\omega}\frac{\exp\int_t^s h(r)dr}{\exp\int_0^\omega h(r)dr-1}k(s)ds.$$

We need the following known Schauder's fixed point theorem and Krasoselskii's fixed point theorem in cone in our arguments.

Lemma 2.2. [23] *Let E be a Banach space, and let $D \subset E$ be closed and convex set. If $\Phi : D \to D$ is a completely continuous operator, then Φ has a fixed point in D.*

Lemma 2.3. [23] *Let P be a cone in Banach space E. Assume that Ω_1 and Ω_2 are open and bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and $\Phi : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that*

(1) $\|\Phi u\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$, *and* $\|\Phi u\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$, *or*
(2) $\|\Phi u\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$, *and* $\|\Phi u\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$.

Then Φ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Next, we define an operator $A : E_\omega \to E_\omega$ by $(Ax)(t) = x(t) - cx(t-\tau)$, $x \in E_\omega$.

Lemma 2.4. [11] *If $|c| \neq 1$, then A has a bounded inverse A^{-1} on E_ω and, for all $x \in E_\omega$,*

$$(A^{-1}x)(t) = \begin{cases} \sum_{j\geq 0} c^j x(t-\tau j), & |c| < 1, \\ -\sum_{j\geq 1} c^{-j} x(t+\tau j), & |c| > 1. \end{cases}$$

Lemma 2.5. *Let $x \in E_\omega$ with $0 < \alpha \leq x \leq \beta$. Then,*

$$\frac{\alpha}{1-c} \leq A^{-1}x \leq \frac{\beta}{1-c} \quad \text{if} \quad c \in [0,1)$$

and

$$\frac{\alpha - |c|\beta}{1-c^2} \leq A^{-1}x \leq \frac{\beta - |c|\alpha}{1-c^2} \quad \text{if} \quad c \in (-1,0].$$

Lemma 2.6. *Let $h \in E_\omega$ with $\int_0^\omega h(s)ds = 0$. Then there exists a unique $H \in E$ such that $H'(t) = h$ and $\int_0^\omega H(s)ds = 0$.*

Since the proof of two lemmas mentioned above is trivial, we omit it here.

3. Positive Periodic Solutions of (1.1)

If $x \in E_\omega$ is a solution of (1.1), then $y(t) = x(t) - cx(t-\tau)$ satisfies

$$y'(t) = \left(\int_{t-\tau_2}^{t-\tau_1} f(x(s))ds + p_1(t) \right)',$$

where $p_1' = p$ and $\int_0^\omega p_1(s)ds = 0$. Consider the following equation

$$u(t) = \int_{t-\tau_2}^{t-\tau_1} f[(A^{-1}u)(s)]ds + p_1(t) + c_n, \tag{3.1}$$

where $c_n > 0$ is to be determined. It is easy to check that $A^{-1}u$ is the periodic solution of (1.1) if u is a periodic solution of (3.1). Define the operator T in E_ω by

$$(Tu)(t) = \int_{t-\tau_2}^{t-\tau_1} f[(A^{-1}u)(s)]ds + p_1(t) + c_n.$$

Clearly, the fixed point u of T is the periodic solution of (3.1).

In order to obtain infinitely many positive periodic solutions of (1.1), we introduce the following conditions.

(H_1) $(\tau_2-\tau_1)f:(a,+\infty)\to(b,+\infty)$ or $(-\infty,b)$ is continuous and

$$\lim_{x\to+\infty}\frac{|f(x)|}{x}<\frac{|1-|c||}{|\tau_1-\tau_2|},$$

(H_2) $f:(a,+\infty)\to\mathbb{R}$ is continuous and

$$\lim_{x\to+\infty}\frac{|f(x)|}{x}<\frac{|1-|c||}{2|\tau_1-\tau_2|},$$

where $a,b\in\mathbb{R}$.

Theorem 3.1. *Let (H_1) or (H_2) holds and $c\neq\pm1$. Then (1.1) has a sequence $\{x_n\}$ of periodic solutions satisfying $x_n\to+\infty$ as $n\to\infty$.*

Proof. First, we consider the case $|c|<1$. If $(\tau_2-\tau_1)f:(a,+\infty)\to(b,+\infty)$, then there exist $0<k_1<\frac{1-|c|}{|\tau_1-\tau_2|}$ and $k_2>0$ such that

$$|f(t)|\le k_1|t|+k_2,\quad (\tau_2-\tau_1)f(t)\ge b,\quad \forall t>a. \tag{3.2}$$

Let n be sufficiently large, and let

$$\Omega_n=\{x\in E_\omega:\ n^n=b_n\le x\le b_{n+1}\}\ \text{ if }\ c\in[0,1)$$

and

$$\Omega_n=\left\{x\in E_\omega:\ \frac{1}{n|c|^n}=b_n\le x\le b_{n+1}\right\}\ \text{ if }\ c\in(-1,0).$$

For $u\in\Omega_n$, it follows from Lemma 2.5 and (3.2) that

$$A^{-1}u\ge\frac{b_n}{1-c}>a\ \text{ for }\ c\in[0,1),$$

$$A^{-1}u\ge\frac{1}{1-c^2}(b_n-|c|b_{n+1})>a\ \text{ for }\ c\in(-1,0),$$

and

$$\begin{aligned}-|b|\le\int_{t-\tau_2}^{t-\tau_1}f[(A^{-1}u)(s)]ds&\le\left|\int_{t-\tau_2}^{t-\tau_1}(k_1|A^{-1}u(s)|+k_2)ds\right|\\ &\le\begin{cases}\frac{k_1b_{n+1}}{1-c}|\tau_1-\tau_2|+k_2|\tau_1-\tau_2|, & c\in[0,1),\\ \frac{k_1|\tau_1-\tau_2|}{1-c^2}(b_{n+1}-|c|b_n)+k_2|\tau_1-\tau_2|, & c\in(-1,0),\end{cases}\end{aligned}$$

which implies that, for $u\in\Omega_n$,

$$-|b|-\bar p+c_n\le Tu\le\begin{cases}\frac{k_1b_{n+1}}{1-c}|\tau_1-\tau_2|+k_2|\tau_1-\tau_2|+\bar p+c_n, c\in[0,1),\\ \frac{k_1|\tau_1-\tau_2|}{1-c^2}(b_{n+1}-|c|b_n)+k_2|\tau_1-\tau_2|+\bar p+c_n, c\in(-1,0),\end{cases}$$

where $\bar p=\max_{t\in\mathbb{R}}|p_1(t)|$. Choosing

$$c_n=\begin{cases}\frac12\left(1-\frac{k_1|\tau_1-\tau_2|}{1-c}\right)b_{n+1}, & c\in[0,1),\\ (|c|+\varepsilon_1)b_{n+1}, & 0<\varepsilon_1<1-|c|-k_1|\tau_1-\tau_2|,\quad c\in(-1,0),\end{cases}$$

we have

$$\frac{c_n}{b_n} \to +\infty, \ \ \frac{1}{b_{n+1}}\left(\frac{k_1 b_{n+1}}{1-c}|\tau_1-\tau_2|+c_n\right)<1$$

for $c \in [0,1)$ and

$$\frac{c_n}{b_n} \to +\infty, \quad \frac{1}{b_{n+1}}\left[c_n+\frac{k_1|\tau_1-\tau_2|}{1-c^2}b_{n+1}-|c|b_n\right]<1$$

for $c \in (-1,0)$, which implies that

$$b_n \le Tu \le b_{n+1}, \ \ \forall u \in \Omega_n. \tag{3.3}$$

If $(\tau_2-\tau_1)f : [a,+\infty) \to (-\infty,b)$, then there exist $k_3 \in (0, \frac{1-|c|}{|\tau_1-\tau_2|})$ and $k_4 \in (0,+\infty)$ such that

$$f(t) \ge -k_3|t|-k_4, \quad (\tau_1-\tau_2)f(t) \le |b|, \quad \forall t > a.$$

For $u \in \Omega_n$, we have

$$A^{-1}u \ge \frac{b_n}{1-c} > a, \quad c \in [0,1),$$
$$A^{-1}u \ge \frac{1}{1-c^2}(b_n-|c|b_{n+1}) > a, \quad c \in (-1,0),$$

and

$$|b| \ge \int_{t-\tau_2}^{t-\tau_1} f[(A^{-1}u)(s)]ds \ge \begin{cases} \frac{-k_3 b_{n+1}}{1-c}|\tau_1-\tau_2|-k_4|\tau_1-\tau_2|, c \in [0,1), \\ \frac{-k_3|\tau_1-\tau_2|}{1-c^2}(b_{n+1}-|c|b_n)-k_4|\tau_1-\tau_2|, c \in (-1,0). \end{cases}$$

Choosing

$$c_n = \begin{cases} rb_{n+1}, \ \ r \in \left(\frac{k_3|\tau_1-\tau_2|}{1-|c|},1\right) \quad \text{for} \quad c \in [0,1), \\ \frac{|c|+k_3|\tau_1-\tau_2|+\varepsilon_2}{|c|}b_n, \ \ \varepsilon_2 \in (0,1-|c|-k_5|\tau_1-\tau_2|) \ \text{for} \quad c \in (-1,0), \end{cases}$$

one can easily check that

$$b_n \le Tu \le b_{n+1}, \ \ \forall u \in \Omega_n. \tag{3.4}$$

If $f : (a,+\infty) \to \mathbb{R}$, then there exist $k_5 \in (0, \frac{1-|c|}{2|\tau_1-\tau_2|})$ and $k_6 > 0$ such that $|f(t)| \le k_5|t|+k_6$, for all $t > a$. Hence, for $u \in \Omega_n$,

$$\left|\int_{t-\tau_2}^{t-\tau_1} f[(A^{-1}u)(s)]ds\right| \le \begin{cases} \frac{k_5 b_{n+1}}{1-c}|\tau_1-\tau_2|+k_6|\tau_1-\tau_2|, c \in [0,1), \\ \frac{k_5|\tau_1-\tau_2|}{1-c^2}(b_{n+1}-|c|b_n)+k_6|\tau_1-\tau_2|, c \in (-1,0). \end{cases}$$

Choosing

$$c_n = \begin{cases} rb_{n+1}, \ \ \frac{k_5|\tau_1-\tau_2|}{1-c} < r < 1-\frac{k_5|\tau_1-\tau_2|}{1-c}, \ \ c \in [0,1), \\ (1-k_5|\tau_1-\tau_2|-\varepsilon_3)b_{n+1}, \ \ 0 < \varepsilon_3 < 1-|c|-2k_5|\tau_1-\tau_2|, \ \ c \in (-1,0), \end{cases}$$

we have

$$b_n \le Tu \le b_{n+1}, \ \ \forall u \in \Omega_n. \tag{3.5}$$

Next, we show that $T(\Omega_n) \subset \Omega_n$ is compact. For any $u \in \Omega_n$, $A^{-1}u \in E_\omega$, $f(A^{-1}u) \in E_\omega$,

$$\begin{aligned} (Tu)(t+\omega) &= \int_{t+\omega-\tau_2}^{t+\omega-\tau_1} f[(A^{-1}u)(s)]ds+p_1(t+\omega)+c_n \\ &= \int_{t-\tau_2}^{t-\tau_1} f[(A^{-1}u)(r)]dr+p_1(t)+c_n = (Tu)(t), \end{aligned}$$

$$T(\Omega_n) \subset \Omega_n, \quad (\text{ see } (3.3)-(3.5))$$

and

$$(Tu)'(t) = f[(A^{-1}u)(t-\tau_1)] - f[(A^{-1}u)(t-\tau_2)] + p.$$

For any $D \in \Omega_n$, one sees that there exist $M > 0$ such that $|(Tu)'| \leq M$ and $|(Tu)(t_1) - (Tu)(t_2)| \leq M|t_1 - t_2|$. Hence, $T(\Omega_n) \subset \Omega_n$ is compact. By using Schauder's fixed point theorem, one obtains that there exists $y_n \in \Omega_n$ such that $Ty_n = y_n$. Let $x_n = A^{-1}y_n$. Then

$$x_n \geq \frac{b_n}{1-c} \to +\infty, \quad c \in [0,1),$$

$$x_n \geq \frac{1}{1-c^2}(b_n - |c|b_{n+1}) \to +\infty, \quad c \in (-1,0)$$

and x_n is positive periodic solution of (1.1). If $|c| > 1$, we rewrite (1.1) as

$$\left(x(t) - \frac{1}{c}x(t+\tau)\right)' = F(x(t+\tau-\tau_2)) - F(x(t+\tau-\tau_1)) - \frac{p(t+\tau)}{c},$$

where $F(t) = f(t)/c$. We use the conclusion of the case $|c| < 1$ to derive the result. □

Example 3.1. Consider the equation

$$(x(t) - cx(t-\tau))' = \frac{x(t-1)}{x^2(t-1)+1} - \frac{x(t-2)}{x^2(t-2)+1} + \sin t, \tag{3.6}$$

where $\tau \in \mathbb{R}$ and $c \neq \pm 1$.

In fact, $f(t) = t/(1+t^2)$. It follows from Theorem 3.1 that (3.6) has infinitely many 2π-positive periodic solutions since $f(t) \to 0$ as $t \to +\infty$.

4. Positive Periodic Solutions for (1.2)

In this section, we assume that the following condition is satisfied:

(H) $f \in C(\Lambda, [0,+\infty))$ or $f \in C(\Lambda, (-\infty, 0])$, where $\Lambda \subseteq [0,+\infty)$.

If $y = Ax = x(t) - cx(t-\tau)$, then

$$y''(t) = \left(\int_{t-\tau_2}^{t-\tau_1} f(x(s))ds\right)' + p(t), \tag{4.1}$$

and

$$y'(t) = \int_{t-\tau_2}^{t-\tau_1} f(x(s))ds + p_1(t) + C,$$

where $p_1' = p$ with $\int_0^\omega p_1(s)ds = 0$ and $C \in \mathbb{R}$. Clearly, the periodic solution v of the following equation

$$v'(t) = \int_{t-\tau_2}^{t-\tau_1} f(A^{-1}v)ds + p_1 - d_n \operatorname{sgn}\lambda \tag{4.2}$$

is the periodic solution of (4.1), where $d_n > 0$ is to be determined and $\lambda = (\tau_2 - \tau_1)f(k) \neq 0, k \in \Lambda$. If $u = e^{-v}$, then it follows from (4.2) and Lemma 2.1 that

$$u'(t) = u(d_n \operatorname{sgn}\lambda - p_1) - u\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(s)]ds$$

and

$$u(t)=\int_t^{t+\omega} G(t,s)u(s)\int_{s-\tau_2}^{s-\tau_1} f[A^{-1}(-\ln u)(r)]drds,$$

where

$$G(t,s)=\frac{\int_s^{t+\omega}(d_n\mathrm{sgn}\lambda-p_1)dr}{e^{\int_0^{\omega}(d_n\mathrm{sgn}\lambda-p_1)dr}-1}=\frac{1}{e^{d_n\omega}-1}\begin{cases} e^{\int_s^{t+\omega}(d_n-p_1(r))dr}, & \mathrm{sgn}\lambda>0,\\ -e^{\int_t^{s}(d_n+p_1(r))dr}, & \mathrm{sgn}\lambda<0.\end{cases}$$

It is clear that

$$\frac{1}{e^{d_n\omega}-1}\le |G(t,s)|\le \frac{e^{d_n\omega}}{e^{d_n\omega}-1},\ t\le s\le t+\omega.\ \text{(see [6])}$$

Define the operator B and the cone K in E_ω by

$$\begin{aligned}(Bu)(t) &= \int_t^{t+\omega}|G(u,s)|u(s)\left|\int_{s-\tau_2}^{s-\tau_1} f[A^{-1}(-\ln u)(r)]dr\right|ds,\\ &K=\{x\in E_\omega,\ x(t)\ge k\|x\|\},\ \ k=e^{-d_n\omega}.\end{aligned}$$

Lemma 4.1. [18] (1) *Let* $c\in[0,1), u\in K, \|u\|=r<1$. *Then*

$$0<\frac{-\ln r}{1-c}\le A^{-1}(-\ln u)\le \frac{-\ln kr}{1-c}.$$

(2) *Let* $c\in(-1,0), u\in K, \|u\|=r<k^{\frac{|c|}{1+c}}$. *Then*

$$0<\frac{\ln r^{-1+|c|}k^{|c|}}{1-c^2}\le A^{-1}(-\ln u)\le \frac{\ln k^{-1}r^{-1+|c|}}{1-c^2}.$$

Theorem 4.1. *Let* $c\neq\pm1, a\ge 0$ *and* $\Lambda=[a,+\infty)$. *Let* $f(a)\neq 0$ *and* $\lim_{x\to+\infty}f(x)=0$. *Then* (1.1) *has infinitely many positive periodic solutions.*

Proof. Note that there exist $\delta\in(0,1)$ and $M_n>a+1$ such that

$$|f(t)|\ge\frac{1}{2}|f(a)|,\quad a<t<a+\delta$$

and

$$|f(t)|\le\frac{d_n\omega}{e|\tau_1-\tau_2|},\qquad t\ge M_n>a.$$

Case 1. $c\in[0,1)$. Let

$$\alpha=e^{-(a+\frac{\delta}{2})(1-c)},\quad \beta=e^{-\max\{M_n+2,N_0+1\}(1-c)},\quad \text{and}\quad \Omega_r=\{u\in K,\|u\|<r\},$$

where $e^{-N_0(1-c)}<\alpha$ and $r>0$. Clearly, $0<\beta<\alpha<1$. First, we demonstrate that $B:\bar{\Omega}_\alpha\setminus\Omega_\beta\to K$. For $u\in\bar{\Omega}_\alpha\setminus\Omega_\beta$, $\beta\le\|u\|\le\alpha, -\ln\alpha\le-\ln u\le-\ln k\beta$, and

$$A^{-1}(-\ln u)\ge\frac{-\ln\alpha}{1-c}=a+\frac{\delta}{2}>a,$$

which implies that B is well-defined and

$$\mathrm{sgn}\lambda\cdot\int_{s-\tau_2}^{s-\tau_1} f[A^{-1}(-\ln u)(r)]dr\ge 0,\forall s\in\mathbb{R}.$$

Hence, for $u \in \bar{\Omega}_\alpha \setminus \Omega_\beta$,

$$\begin{aligned}
Bu(t) &= \int_t^{t+\omega} G(t,s)u(s)\int_{s-\tau_2}^{s-\tau_1} f[A^{-1}(-\ln u)(r)]drds \geq 0,\\
Bu(t) &\leq \frac{e^{d_n\omega}}{e^{d_n\omega}-1}\int_t^{t+\omega} u(s)\left|\int_{s-\tau_2}^{s-\tau_1} f[A^{-1}(-\ln u)(r)]dr\right|ds,\\
Bu(t) &\geq \frac{1}{e^{d_n\omega}-1}\int_t^{t+\omega} u(s)\left|\int_{s-\tau_2}^{s-\tau_1} f[A^{-1}(-\ln u)(r)]dr\right|ds,\\
Bu(t) &\geq e^{-d_n\omega}\|Bu\|.
\end{aligned}$$

In addition, for $u \in \bar{\Omega}_\alpha \setminus \Omega_\beta$, $\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(r)]dr$ is continuous ω-periodic function. It follow from Lemma 2.1 that $(Bu)(t+\omega) = Bu(t)$ for $u \in \bar{\Omega}_\alpha \setminus \Omega_\beta$. Let $D \in \bar{\Omega}_\alpha \setminus \Omega_\beta$ be bounded set and for $u \in D$,

$$|(Bu)'(t)| \leq |u(d_n \mathrm{sgn}\lambda - p_1)| + \left|u\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(r)]dr\right| \leq M$$

for some $M > 0$, which implies that, for any $t_1, t_2 \in \mathbb{R}$, $|Bu(t_1) - Bu(t_2)| \leq M|t_1 - t_2|$, $u \in D$. Hence, $B : \bar{\Omega}_\alpha \setminus \Omega_\beta \to K$ is compact. Moveover, $B : \bar{\Omega}_\alpha \setminus \Omega_\beta \to K$ is continuous. Let

$$0 < d_n < \min\left\{\frac{(1-c)\delta}{4\omega}, \frac{1}{2\omega}, \frac{|f(a)(\tau_1-\tau_2)|}{4e\omega}\right\}. \tag{4.3}$$

If $u \in \partial\Omega_\alpha$, using Lemma 4.1, we have

$$A^{-1}(-\ln u) \geq \frac{-\ln\alpha}{1-c} = a + \frac{\delta}{2} > a$$

and

$$A^{-1}(-\ln u) \leq \frac{-\ln k\alpha}{1-c} \leq \frac{cd_n\omega + (a+\frac{\delta}{2})(1-c)}{1-c} \leq \frac{d_n\omega}{1-c} + a + \frac{\delta}{2} < a + \delta,$$

which implies that, for $u \in \partial\Omega_\alpha$, $\|u\| = \alpha$ and

$$\left|\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(r)]dr\right| > \frac{|f(a)|}{2}|\tau_1 - \tau_2|,$$

$$\begin{aligned}
Bu(t) &= \int_t^{t+\omega} G(t,s)u(s)\int_{s-\tau_2}^{s-\tau_1} f[A^{-1}(-\ln u)(r)]drds\\
&\geq \frac{\omega\|u\|e^{-d_n\omega}}{e^{d_n\omega}-1}\frac{|f(a)|}{2}|\tau_1-\tau_2| \geq \|u\|,
\end{aligned}$$

where we used the fact

$$e^{d_n\omega}(e^{d_n\omega}-1) < 2d_n\omega e < \frac{|f(a)|}{2}|\tau_1-\tau_2|\omega.$$

For $u \in \partial\Omega_\beta$, $\|u\| = \beta$ and

$$A^{-1}(-\ln u) > \frac{-\ln\beta}{1-c} > M_n + 2, \quad f(A^{-1}(-\ln u)(r)) \leq \frac{d_n}{e|\tau_1-\tau_2|},$$

$$\left|\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(r)]dr\right| \leq \frac{d_n\omega}{e}, \ |Bu| \leq \frac{e^{d_n\omega}}{e^{d_n\omega}-1}\|u\|\frac{d_n\omega}{e} \leq \|u\|.$$

By Lemma 2.2, for each d_n satisfying (4.3), there exists $u \in \bar{\Omega}_\alpha \setminus \Omega_\beta$ such that $Bu = u$ with $\beta \le \|u\| \le \alpha$ and $A^{-1}(-\ln u) > \frac{-\ln\alpha}{1-c} > a$. Let $d_n = \frac{1}{n}$. For n sufficiently large, we can obtain a sequence $\{u_n\} \subset E_\omega : 0 < u_n < 1$ such that $Bu_n = u_n$, that is, u_n satisfies

$$u_n'(t) = u_n(t)(d_n \mathrm{sgn}\lambda - p_1) - u_n(t)\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u_n)(s)]ds,$$

which implies that $u_n \not\equiv u_m$ for $n \ne m$. Hence, $A^{-1}(-\ln u_n)$ is the positive periodic solution of (1.2).

Case 2 $c \in (-1, 0)$. Let

$$r = e^{-(1-c)(M_n+N_0+2)}, \quad R = k^{\frac{|c|}{1+c}} e^{-(a+\frac{\delta}{2})(1-c)}$$

and

$$0 < d_n < \min\left\{\frac{1-c^2}{\omega|c|}, \frac{1}{2\omega}, \frac{(1+c)\delta}{4\omega}, \frac{|f(a)||\tau_1-\tau_2|}{4e\omega}\right\}, \tag{4.4}$$

where $e^{-N_0} < R$. Hence, $0 < r < R < k^{\frac{|c|}{1+c}}$ and $B : \bar{\Omega}_R \setminus \Omega_r \to K$ is completely continuous. For $u \in \partial\Omega_r$, we have

$$A^{-1}(-\ln u) \ge \frac{\ln r^{-1+|c|}k^c}{1-c^2} \ge M_n + N_0 + 2 - \frac{b_n\omega|c|}{1-c^2} \ge M_n,$$

$$\left|\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(s)]ds\right| \le \frac{d_n\omega}{e},$$

and

$$|Bu| \le \frac{e^{d_n\omega}}{e^{d_n\omega}-1}\frac{d_n\omega}{e} r \le \|u\|.$$

For $u \in \partial\Omega_R$, we have

$$A^{-1}(-\ln u) \le \frac{\ln k^{-1}R^{-1+|c|}}{1-c^2} = \frac{b_n\omega}{1+c} + a + \frac{\delta}{2} \le a + \delta,$$

$$A^{-1}(-\ln u) \ge \frac{\ln R^{-1+|c|}k^{|c|}}{1-c^2} = a + \frac{\delta}{2},$$

$$\left|\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(s)]ds\right| \ge \frac{|f(a)(\tau_1-\tau_2)|}{2},$$

and

$$|Bu| \ge \frac{\omega}{e^{b_n\omega}-1} k\|u\| \frac{|f(a)||\tau_1-\tau_2|}{2} \ge \frac{\|u\|}{2d_n}\frac{|f(a)||\tau_1-\tau_2|}{2} \ge \|u\|.$$

By Lemma 2.2, for each d_n satisfying (4.4), there exists $u \in \bar{\Omega}_R \setminus \Omega_r$ such that $Bu = u$ with $r \le \|u\| \le R$ and $A^{-1}(-\ln u) > a$. Let $d_n = \frac{1}{n}$. For n sufficiently large, we can obtain a sequence $\{u_n\} \subset E_\omega$ such that $Bu_n = u_n$ with $0 < u_n < 1$, where $A^{-1}(-\ln u_n)$ is the positive periodic solution of (1.2). If $|c| > 1$, then we rewrite (1.2) as

$$(x(t) - c^* x(t+\tau))'' = F(x(t-\tau^1)) - F(x(t-\tau^2)) - \frac{p(t+\tau)}{c}, \tag{4.5}$$

where $c^* = c^{-1}$, $\tau^1 = \tau_2 - \tau$, $\tau^2 = \tau_1 - \tau$, and $F(t) = f(t)/c$. Clearly, $F \in C(\Lambda, [0, +\infty))$ or $F \in C(\Lambda, (-\infty, 0])$, and

$$|c^*| < 1, \quad \tau^1 \ne \tau^2, \quad F(a) \ne 0, \quad \lim_{x\to+\infty} F(x) = 0.$$

We use the conclusion of the case $|c| < 1$ to derive the result. □

Theorem 4.2. *Assume that* $c \neq \pm 1$ *and* $\Lambda = [a,b], 0 \leq a < b < +\infty$. *Further, suppose*

$$f(a) \neq 0, \quad \lim_{t \to b^-} \sup \frac{|f(t)|}{b-t} < \frac{|1-|c||}{|\tau_1 - \tau_2|\omega} \tag{4.6}$$

or

$$f(b) \neq 0, \quad \lim_{t \to a^+} \sup \frac{|f(t)|}{t-a} < \frac{|1-|c||}{|\tau_1 - \tau_2|\omega}.$$

Then (1.2) has infinitely many positive periodic solutions in Λ.

Proof. Here we only show the case (4.6) is satisfied. There exist $\delta_1 > 0, \delta_2 > 0 : \delta_1 + \delta_2 < b - a$, and $0 < \eta < \frac{|1-|c||}{|\tau_1-\tau_2|\omega}$ such that

$$\begin{aligned} &|f(t)| \geq \frac{1}{2}|f(a)|, \ a \leq t \leq a + \delta_1, \\ &|f(t)| \leq \eta(b-t), \ 0 \leq b - t \leq \delta_2. \end{aligned} \tag{4.7}$$

Case 1 $c \in [0,1)$. Let

$$r = e^{-(1-c)(a+\frac{\delta_1}{2})}, \quad R = e^{-b(1-c)+d_n\omega}$$

and

$$\begin{aligned} 0 < d_n < \min\Bigg\{ &\frac{(b-a-\frac{\delta_1}{2})(1-c)}{2\omega}, \frac{\delta_2(1-c)}{2\omega}, \frac{\delta_1(1-c)}{2\omega}, \\ &\frac{\ln(1-c) - \ln \eta|\tau_1 - \tau_2|\omega}{\omega}, \frac{1}{2\omega}, \frac{|f(a)(\tau_1 - \tau_2)|}{4e} \Bigg\}, \end{aligned} \tag{4.8}$$

then $0 < R < r < 1$ and $B : \bar{\Omega}_r \setminus \Omega_R \to K$ is completely continuous. If $u \in \partial\Omega_R$, we obtain from Lemma 4.1 and (4.7) that

$$A^{-1}(-\ln u) \geq \frac{-\ln R}{1-c} = b - \frac{d_n\omega}{1-c}, \quad A^{-1}(-\ln u) \leq \frac{-\ln kR}{1-c} = b,$$

$$0 \leq b - A^{-1}(-\ln u) \leq \frac{d_n\omega}{1-c} < \delta_2,$$

and

$$\left| \int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(s)]ds \right| \leq \frac{d_n\omega}{1-c}\eta|\tau_1 - \tau_2|.$$

Hence,

$$\begin{aligned} |Bu| &= \int_t^{t+\omega} |G(t,s)|u(s) \left| \int_{s-\tau_2}^{s-\tau_1} f[A^{-1}(-\ln u)(r)]dr \right| ds \\ &\leq \frac{\omega e^{d_n\omega}}{e^{d_n\omega} - 1}\|u\| \frac{d_n\omega}{1-c}\eta|\tau_1 - \tau_2| \leq \|u\|, \end{aligned}$$

where we used $d_n\omega < e^{d_n\omega} - 1$ and $\omega d_n < \ln(1-c) - \ln \eta|\tau_1 - \tau_2|\omega$. If $u \in \partial\Omega_r$, we have

$$A^{-1}(-\ln u) \geq \frac{-\ln r}{1-c} = a + \frac{\delta_1}{2}, \quad A^{-1}(-\ln u) \leq \frac{-\ln kr}{1-c} = a + \frac{\delta_1}{2} + \frac{d_n\omega}{1-c} \leq a + \delta_1,$$

and

$$\left|\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(s)]ds\right| \geq \frac{|f(a)||\tau_1-\tau_2|}{2},$$

$$|Bu| \geq \frac{\omega}{e^{d_n\omega}-1}e^{-d_n\omega}\|u\|\frac{|f(a)||\tau_1-\tau_2|}{2}| \geq \|u\|$$

since

$$(e^{d_n\omega}-1)e^{d_n\omega} < 2d_n\omega < \frac{|f(a)||\tau_1-\tau_2|\omega}{2}.$$

By Lemma 2.2, for each d_n satisfying (4.8), there exists $u \in \bar{\Omega}_r \setminus \Omega_R$ such that $Bu = u$ with $R \leq \|u\| \leq r$ and

$$a < \frac{-\ln r}{1-c} \leq A^{-1}(-\ln u) \leq \frac{-\ln kR}{1-c} = b.$$

Let $d_n = \frac{1}{n}$. For n sufficiently large, we can obtain a sequence $\{u_n\} \subset E_\omega$ such that $Bu_n = u_n$. Hence, $A^{-1}(-\ln u_n) \in [a,b]$ is the positive periodic solution of (1.2).

Case 2 $c \in (-1,0)$. Let

$$R = k^{\frac{|c|}{1+c}}e^{-(a+\frac{\delta_1}{2})(1-c)}, \quad r = e^{-b(1-c)+\frac{d_n\omega}{1+c}},$$

$$0 < d_n < \min\left\{\frac{(b-a-\frac{\delta_1}{2})(1+c)}{\omega}, \frac{(1+c)\delta_1}{2\omega}, \frac{(1+c)\delta_2}{2\omega}, \frac{1}{2\omega}, \frac{|f(a)(\tau_1-\tau_2)|}{4e}, \frac{1}{\omega}\ln\frac{1+c}{\eta|\tau_1-\tau_2|\omega}\right\}. \tag{4.9}$$

Then $0 < r < R < k^{\frac{|c|}{1+c}}$ and $B : \bar{\Omega}_R \setminus \Omega_r \to K$ is completely continuous. For $u \in \partial\Omega_R$, we have

$$A^{-1}(-\ln u) \leq \frac{\ln k^{-1}R^{-1-c}}{1-c^2} \leq a+\delta_1, \quad A^{-1}(-\ln u) \geq \frac{\ln k^{-c}R^{-1-c}}{1-c^2} = a+\frac{\delta_1}{2},$$

$$\left|\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(s)]ds\right| \geq \frac{|f(a)(\tau_1-\tau_2)|}{2}, \text{ and}$$

$$|Bu| \geq \frac{\omega k\|u\|}{e^{d_n\omega}-1}\frac{|f(a)(\tau_1-\tau_2)|}{2} \geq \|u\|.$$

For $u \in \partial\Omega_r$, we have

$$A^{-1}(-\ln u) \geq \frac{\ln r^{-1-c}k^{-c}}{1-c^2} = b-\frac{d_n\omega}{1+c}, \qquad A^{-1}(-\ln u) \leq \frac{\ln k^{-1}r^{-1-c}}{1-c^2} = b,$$

$$0 < b - A^{-1}(-\ln u) \leq \frac{d_n\omega}{1+c} < \delta_2,$$

$$\left|\int_{t-\tau_2}^{t-\tau_1} f[A^{-1}(-\ln u)(s)]ds\right| \leq \frac{d_n\omega}{1+c}\eta|\tau_1-\tau_2|,$$

and

$$|Bu| \leq \frac{\omega\|u\|e^{d_n\omega}}{e^{d_n\omega}-1}\frac{d_n\omega}{1+c}|\tau_1-\tau_2|\eta \leq \omega e^{d_n\omega}\frac{|\tau_1-\tau_2|\eta}{1+c}\|u\| \leq \|u\|.$$

By Lemma 2.2, for each d_n satisfying (4.9), one sees that there exists $u \in \bar{\Omega}_R \setminus \Omega_r$ such that $Bu = u$ with $r \le \|u\| \le R$ and

$$a < \frac{\ln k^{-c} R^{-1-c}}{1-c^2} \le A^{-1}(-\ln u) \le \frac{\ln k^{-1} r^{-1-c}}{1-c^2} = b.$$

Let $d_n = \frac{1}{n}$. For n sufficiently large, we can obtain a sequence $\{u_n\} \subset E_\omega$ such that $Bu_n = u_n$. Hence, $A^{-1}(-\ln u_n) \in [a,b]$ is the positive periodic solution of (1.2). If $|c| > 1$, we consider (4.5). In view of $F \in C(\Lambda, [0,+\infty))$ or $F \in C(\Lambda, (-\infty, 0])$, and

$$F(a) \neq 0, \quad \lim_{t \to b^-} \sup \frac{|F(t)|}{b-t} < \frac{|1-|c^*||}{|\tau^1 - \tau^2|\omega},$$

we use the conclusion of the case $|c| < 1$ to derive the result. □

Example 4.1. Consider the equation

$$(x(t) - cx(t-\tau))'' = (1 - x(t-\tau_1))^3 - (1 - x(t-\tau_2))^3 + p(t), \tag{4.10}$$

where $\tau, \tau_1, \tau_2 \in \mathbb{R}, \tau_1 \neq \tau_2$, $c \neq \pm 1$, and $p \not\equiv 0$ is a continuous ω-periodic function with $\int_0^\omega p(t)dt = 0$.

In fact, $f(t) = (1-t)^3$. Note that $f(0.5) = f(1.5) \neq 0$ and $\lim_{t \to 1^-} \frac{f(t)}{1-t} = \lim_{t \to 1^+} \frac{f(t)}{t-1} = 0$. It follows from Theorem 4.2 that (4.10) has infinitely many ω-positive periodic solutions in $[0.5, 1]$ and $[1, 0.5]$.

5. Conclusion

In this paper, we investigated the nonlinear NFDEs of first order and second order. We presented three new theorems, which are based on the Schauder's fixed point theorem, Krasnoselskii's fixed point theorem in a cone and the Green's functions, on the infinitely many positive periodic solutions of the considered NFDEs. Finally, we also presented examples to demonstrate and verify an application of our results. The results presented in this paper provide new views to the qualitative theory of NFDEs.

References

[1] S. Ai, Bounded/periodic solutions for a class of scalar delay differential equations, Proc. Amer. Math. Soc. 151 (2023), 3547–3560.

[2] T. Faria, Periodic solutions for a non-monotone family of delayed differential equations with applications to Nicholson systems, J. Differential Equations, 263 (2017), 509–533.

[3] S. Heidarkhani, S. Moradi, Multiple solutions for a quasilinear periodic boundary value problem, J. Nonlinear Funct. Anal. 2024 (2024), 26.

[4] B. Ricceri, Some of the latest applications of certain minimax theorems, Appl. Set-Valued Anal. Optim. 7 (2025), 313–334.

[5] R.O.A. Taie, M.G.A.Alwaleedy, Existence and uniqueness of a periodic solution to a certain third order neutral functional differential equation, Math. Commun. 27 (2022), 255-274.

[6] H. Wang, Positive periodic solutions of functional differential equations, J. Differential Equations, 202 (2004), 354–366.

[7] W. Wang, S. Luo, Infinitely many positive periodic solutions for second order functional differential equations. Appl. Math. Lett. 163 (2025), 109431.

[8] J. Wu,Y. Li, X. Fu, Existence of solutions and periodic solutions of a neutral measure functional differential equation with infinite delay, J. Differential Equations, 410 (2024), 1–45.

[9] E. Serra, Periodic solutions for some nonlinear differential equations of neutral type, Nonlinear Anal. 17 (1991), 139–151.

[10] S.M. Afonso, E.M. Bonotto, M.R. da Silva, Periodic solutions of neutral functional differential equations, J. Differential Equations, 350 (2023), 89–123.

[11] S. Lu, W. Ge, On the existence of periodic solutions for neutral functional differential equation, Nonlinear Anal. 547 (2003), 1285–1306.

[12] Y. Luo, W. Wang, J. Shen, Existence of positive periodic solutions for two kinds of neutral functional differential equations, Appl. Math. Lett. 21 (2008), 581–587.

[13] T. Candan, Existence of positive periodic solutions of first order neutral differential equations with variable coefficients, Appl. Math. Lett. 52 (2016), 142–148.

[14] W. Cheung, J. Ren, W. Han, Positive periodic solution of second-order neutral functional differential equations, Nonlinear Anal. 71 (2009), 3948–3955.

[15] T. Candan, Existence of positive periodic solution of second-order neutral differential equations, Turk J. Math. 42 (2018), 797–806.

[16] M. Gözen, On the existence and uniqueness of positive periodic solutions of neutral differential equations, J. Nonlinear Var. Anal. 7 (2023), 367–379.

[17] J. Wu,Z. Wang, Two periodic solutions of second-order neutral functional differential equations, J. Math. Anal. Appl. 329 (2007), 677–689.

[18] W. Wang, J. Shen, Positive periodic solutions for neutral functional differential equations, Appl. Math. Lett. 102 (2020), 106154.

[19] S. Yang, Y. Li, Positive periodic solutions of third-order neutral differential equations with delayed derivative terms, Bull. Sci. Math. 190 (2024), 03370.

[20] A. Bouakkaz, A. Ardjouni, A. Djoudi, Existence of positive periodic solutions for a second-order nonlinear neutral differential equation by the Krasnoselskii's fixed point theorem, Nonlinear Dyn. Syst. Theory, 17 (2017), 230–238.

[21] L. Mezghiche, R. Khemis, A. Boukkaz, Positive periodic solutions for a neutral differential equation with iterative terms arising in biology and population dynamics, Int. J. Nonlinear Anal. Appl. 13 (2022), 1041–1051.

[22] F. Kong, Z. Luo, S. Lu, Positive periodic solutions for singular high-order neutral functional differential equations, Math. Slovaca, 68 (2018), 379–396.

[23] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, Orlando, 1988.