

## CONSERVATIVE WEAK SOLUTIONS TO A RADially SYMMETRIC VARIATIONAL WAVE SYSTEM OUTSIDE A BALL

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**Abstract.** We study the initial-boundary value problem for a radially symmetric variational wave system arising from the theory of nematic liquid crystals. Based on the characteristic reflection method, we introduce the energy-dependent coordinates into the semi-infinite interval to transform the original problem into a new boundary value problem. By regularizing the governing system and deriving the a priori estimates of solutions, we apply the Young measure theory to show the strong convergence of the approximate solution sequence and then establish the global existence of weak solutions for the new boundary value problem. The global conservative weak solution of the original initial-boundary value problem is constructed by returning the solution in terms of energy-dependent coordinate variables to the physical plane.

**Keywords.** Conservative solution; Initial-boundary value problem; Radial symmetry; Variational wave system.

### 1. INTRODUCTION

Nematic liquid crystals are one of the most important members of liquid crystals and have been extensively studied theoretically, numerically, and experimentally. In theory, the mean orientation of the long molecules of nematic liquid crystals can be described by the so-called director field  $\mathbf{n}(\mathbf{x}, t)$  at a spatial location  $\mathbf{x}$  and time  $t$ . In the regime in which inertia effects dominate viscosity, that is, ignoring the kinetic energy of the director field, Hunter and Saxton [27] modeled the propagation of the orientation waves in the director field by the least action principle

$$\delta \int \left( \frac{1}{2} \partial_t \mathbf{n} \cdot \partial_t \mathbf{n} - W(\mathbf{n}, \nabla \mathbf{n}) \right) dx dt = 0, \quad \mathbf{n} \cdot \mathbf{n} = 1, \quad (1.1)$$

where  $W(\mathbf{n}, \nabla \mathbf{n})$  is the well-known Oseen-Frank potential energy density which takes the following form

$$W(\mathbf{n}, \nabla \mathbf{n}) = \frac{1}{2} k_1 (\nabla \cdot \mathbf{n})^2 + \frac{1}{2} k_2 (\mathbf{n} \cdot \nabla \times \mathbf{n})^2 + \frac{1}{2} k_3 |\mathbf{n} \times (\nabla \times \mathbf{n})|^2.$$

Here  $k_1, k_2$  and  $k_3$  are the splay, twist and bend elastic positive constants of the liquid crystal, respectively. They [27] considered plane deformations depending on a single space variable  $x$

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and then took the director field  $\mathbf{n}$  as the special form  $\mathbf{n} = (\cos u(x, t), \sin u(x, t), 0)$  to obtain the Lagrangian density of (1.1)

$$\frac{1}{2} \partial_t \mathbf{n} \cdot \partial_t \mathbf{n} - W(\mathbf{n}, \nabla \mathbf{n}) = \frac{1}{2} u_t^2 - \frac{1}{2} c^2(u) u_x^2,$$

with  $c^2(u) = k_1 \sin^2 u + k_3 \cos^2 u$ . The corresponding Euler-Lagrange equation is

$$u_{tt} - c(u)(c(u)u_x)_x = 0, \quad (1.2)$$

which is also called the one-dimensional variational wave equation. The multi-dimensional generalization of variational wave equation (1.2) reads that [8, 21, 33]

$$u_{tt} - c(u) \nabla \cdot (c(u) \nabla u) = 0. \quad (1.3)$$

To characterize the propagation of twist waves, Ali and Hunter [2] considered three-dimensional deformations depending on a single space variable  $x$ . By taking the director field  $\mathbf{n}$  as the form

$$\mathbf{n} = (\cos u(x, t), \sin u(x, t) \cos v(x, t), \sin u(x, t) \sin v(x, t)),$$

they acquired the corresponding Lagrangian density

$$\frac{1}{2} \partial_t \mathbf{n} \cdot \partial_t \mathbf{n} - W(\mathbf{n}, \nabla \mathbf{n}) = \frac{1}{2} [u_t^2 - c_1^2(u) u_x^2] + \frac{1}{2} a^2(u) [v_t^2 - c_2^2(u) v_x^2],$$

with

$$c_1^2(u) = k_1 \sin^2 u + k_3 \cos^2 u, \quad c_2^2(u) = k_2 \sin^2 u + k_3 \cos^2 u, \quad a^2(u) = \sin^2 u,$$

and then derived the Euler-Lagrange equations

$$\begin{cases} u_{tt} - c_1(u)(c_1(u)u_x)_x = a(u)a'(u)[v_t^2 - c_2^2(u)v_x^2] - a^2(u)c_2(u)c_2'(u)v_x^2, \\ (a^2(u)v_t)_t - [a^2(u)c_2^2(u)v_x]_x = 0. \end{cases} \quad (1.4)$$

The multi-dimensional generalization of variational wave system (1.4) reads that

$$\begin{cases} u_{tt} - c_1(u) \nabla \cdot (c_1(u) \nabla u) = a(u)a'(u)[v_t^2 - c_2^2(u)|\nabla v|^2] - a^2(u)c_2(u)c_2'(u)|\nabla v|^2, \\ (a^2(u)v_t)_t - \nabla \cdot [a^2(u)c_2^2(u) \nabla v] = 0, \end{cases} \quad (1.5)$$

which can be obtained from the general variational principle whose action is a quadratic function of the derivatives of the field with coefficients depending only on the field

$$\delta \int A_{\mu\nu}^{ij}(\mathbf{u}) \frac{\partial u^\mu}{\partial x_i} \frac{\partial u^\nu}{\partial x_j} d\mathbf{x} = 0, \quad (1.6)$$

where the summation convention is employed, see [1, 27, 28] for more background information. In (1.6),  $\mathbf{x} \in \mathbb{R}^{d+1}$  are the space-time independent variables and  $\mathbf{u} : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^n$  are the dependent variables. The coefficients  $A_{\mu\nu}^{ij} : \mathbb{R}^{d+1} \times \mathbb{R}^n \rightarrow \mathbb{R}$  are smooth functions and satisfy  $A_{\mu\nu}^{ij} = A_{\nu\mu}^{ij} = A_{\mu\nu}^{ji}$ . One can choose the coefficients  $A_{\mu\nu}^{ij}$  in (1.6) such that the Lagrangian density takes the classical form

$$A_{\mu\nu}^{ij}(\mathbf{u}) \frac{\partial u^\mu}{\partial x_i} \frac{\partial u^\nu}{\partial x_j} = \frac{1}{2} [u_t^2 - c_1^2(u) |\nabla u|^2] + \frac{1}{2} a^2(u) [v_t^2 - c_2^2(u) |\nabla v|^2].$$

Then the corresponding Euler-Lagrange equations are system (1.5).

The study on the multi-dimensional variational wave systems (1.3) and (1.5) are still very limited now. As a bridge for exploring the behavior of multi-dimensional systems and with their own physical significance, the radially symmetric variational wave systems have also deserved

to be broadly regarded and explored. By setting  $(u, v) = (u(t, r), v(t, r)), r = |\mathbf{x}|$  where  $\mathbf{x} = (x_1, \dots, x_d)$  ( $d > 1$ ), system (1.5) can be reduced to

$$\begin{cases} u_{tt} - c_1(u)(c_1(u)u_r)_r = \frac{2\kappa c_1^2(u)u_r}{r} + a(u)a'(u)[v_t^2 - c_2^2(u)v_r^2] - a^2(u)c_2(u)c_2'(u)v_r^2, \\ (a^2(u)v_t)_t - [a^2(u)c_2^2(u)v_r]_r = \frac{2\kappa a^2(u)c_2^2(u)v_r}{r}, \end{cases} \quad (1.7)$$

where  $\kappa = (d - 1)/2$ . In the present paper, we consider system (1.7) outside the unit ball and supplement (1.7) with the initial data

$$\begin{aligned} (u(0, r), v(0, r)) &= (u_0(r), v_0(r)), \quad (u_t(0, r), v_t(0, r)) = (u_1(r), v_1(r)), \\ r^\kappa u_0'(r), r^\kappa u_1(r) &\in L^2([1, \infty)), \quad r^\kappa v_0'(r), r^\kappa v_1(r) \in L^\infty([1, \infty)), \end{aligned} \quad (1.8)$$

and the homogeneous Dirichlet boundary conditions

$$u(t, 1) = v(t, 1) = 0. \quad (1.9)$$

We here assume that the compatibility conditions  $u_0(0) = u_1(0) = v_0(0) = v_1(0) = 0$  are satisfied. The homogeneous boundary conditions in (1.9) result in the disappearance of boundary energies on  $r = 1$ , which leads to the conservation of energy within the interval  $[1, \infty)$ . In the present paper, we shall study the global existence of conservative weak solutions to the initial-boundary value problem (1.7)-(1.9).

The Cauchy problem for the one-dimensional variational wave equation (1.2) and its related single models have been widely investigated. We refer the reader to the works on the singularity formation [18, 21, 30], on the existence of dissipative weak solutions [7, 33, 34, 35], on the existence of conservative weak solutions [8, 22, 25], on the uniqueness, stability and generic regularity of conservative weak solutions [3, 4, 5, 6, 10, 11]. In [19], the authors investigated the singularity formation of smooth solutions for the spherically symmetric variational wave equation. For the one-dimensional variational wave system (1.4) with  $c_1(\cdot) = c_2(\cdot)$  and  $a(\cdot) \geq Const > 0$ , Zhang and Zheng [36] applied the method of energy-dependent coordinates introduced by Bressan and Zheng [8] to establish the global existence of conservative solutions to its Cauchy problem for initial data of finite energy. Subsequently, they [37] investigated the global existence of conservative solutions to system (1.4) with  $c_1(\cdot) \neq c_2(\cdot)$  and  $a(\cdot) \geq Const > 0$ . To remove the restriction of  $a(\cdot) \geq Const > 0$ , Chen, Zhang, and Zheng [17] derived a one-dimensional nonlinear wave system from (1.1) by considering the director field  $\mathbf{n}$  in its natural three-component form and established its global existence of conservative solutions under the assumption  $c_1(\cdot) = c_2(\cdot)$ , also see [9] for its uniqueness and generic regularity of conservative solutions and [12] for the Lipschitz continuous dependence. In [23, 24, 26], the author considered the Cauchy problem to some one-dimensional nonlinear wave systems arising from the general variational principle (1.6).

It is meaningful that the energy-dependent coordinate method can be used to study the Poiseuille flow of one-dimensional hyperbolic-parabolic Ericksen-Leslie model of nematic liquid crystals, see the recent works by Chen etc. [14, 15, 16]. In [13], Chen, Hu, and Zhang utilized the energy-dependent coordinate method to the initial-boundary value problem by adopting the characteristic method with reflections on the boundaries. They established the global existence of Hölder continuous solution of the initial-boundary value problems with different types of boundary conditions for the one-dimensional hyperbolic-parabolic Ericksen-Leslie model.

In the current paper, we apply the characteristic reflection method to construct the corresponding energy-dependent coordinates for the initial-boundary value problem (1.7)-(1.9) to overcome the difficulty arising from the possible concentration of energy. The appropriate weighted variables are introduced to handle non-homogeneous terms in the governing system (1.7) caused by radial symmetry. Similar to the case of the one-dimensional system (1.4), the new hyperbolic system in terms of the energy-dependent coordinate variables may degenerate at some points. Following the previous work by Zhang and Zheng [37], we regularize the system by adding some suitable positive perturbation terms and then adopt the Young measure theory [31, 32] to establish the precompactness of a sequence of solutions for the regularization system. A global conservative weak solution to the initial-boundary value problem (1.7)-(1.9) is then constructed by transforming the solution in the energy-dependent coordinates back to the original independent variables.

We assume that the smooth functions  $c_1, c_2$  and  $a$  fulfill

$$\begin{aligned} 0 < c_0 \leq c_1(z), c_2(z), a(z) \leq C_0, \\ |c_1'(z)|, |c_2'(z)|, |a'(z)| \leq C_0, \quad \forall z \in \mathbb{R}, \end{aligned} \quad (1.10)$$

for some positive constants  $c_0$  and  $C_0$ , and further require that  $c_1$  and  $c_2$  satisfy

$$b(z) := \frac{c_2(z)}{c_1(z)} \leq b_0 < 1, \quad \forall z \in \mathbb{R}, \quad (1.11)$$

for some positive number  $b_0$ . We comment that in fact the assumptions on  $a(u)$  and  $b(u)$  are not very satisfactory since the extreme case  $\sin u = 0$  in nematic liquid crystals is excluded. Our main result of this paper can be stated below.

**Theorem 1.1.** *Let conditions (1.10) and (1.11) be satisfied. Then the initial-boundary value problem (1.7)-(1.9) admits a global weak solution  $(u, v)(t, r)$  defined for all  $(t, r) \in [0, \infty) \times [1, \infty)$ , as follows:*

(i) *The functions  $(u, v)$  are locally Hölder continuous with exponent  $1/2$  in  $[0, \infty) \times [1, \infty)$ . The vector function  $t \mapsto (u, v)(t, \cdot)$  is continuously differentiable as a map with values in  $L_{\text{loc}}^\theta$  for all  $1 \leq \theta < 2$ . Moreover, it is Lipschitz continuous with respect to the  $L^2$  distance, that is, there exists a constant  $L$  such that*

$$\|u(t, \cdot) - u(s, \cdot)\|_{L^2([1, \infty))} + \|v(t, \cdot) - v(s, \cdot)\|_{L^2([1, \infty))} \leq L|t - s| \quad (1.12)$$

for all  $t, s \in [0, \infty)$ .

(ii) *The functions  $(u, v)$  take on the initial condition in (1.8) pointwise, while its temporal derivative holds in  $L^\theta$  for  $\theta \in [1, 2)$ .*

(iii) *The boundary conditions in (1.9) are satisfied pointwise.*

(iv) System (1.7) is satisfied in the distributional sense, that is,

$$\int_0^\infty \int_1^\infty \left\{ \phi_t u_t - (\phi c_1)_r c_1 u_r + \phi \left( \frac{2\kappa c_1^2(u) u_r}{r} + a(u) a'(u) [v_t^2 - c_2^2(u) v_r^2] - a^2(u) c_2(u) c_2'(u) v_r^2 \right) \right\} dr dt - \int_0^\infty \phi c_1^2 u_r(t, 1) dt = 0,$$

$$\int_0^\infty \int_1^\infty \left\{ \phi_t a^2(u) v_t - \phi_r a^2(u) c_2^2(u) v_r + \phi \frac{2\kappa a^2(u) c_2^2(u) v_r}{r} \right\} dr dt - \int_0^\infty \phi a^2 c_2^2 v_r(t, 1) dt = 0,$$
(1.13)

for all test functions  $\phi \in \mathcal{F}$ , where

$$\mathcal{F} := \left\{ f \in C^\infty(\mathbb{R}^+ \times (1, \infty)) : \partial_t^i \partial_r^j f \Big|_{t=0, \infty} = 0, \partial_t^i \partial_r^j f \Big|_{r=\infty} = 0 \right\}.$$

Moreover, there holds

$$\mathcal{E}(t) \leq \mathcal{E}(0) =: \mathcal{E}_0, \tag{1.14}$$

where

$$\mathcal{E}(t) := \frac{1}{2} \int_1^\infty \left( r^{2\kappa} [u_t^2(t, r) + c_1^2 u_r^2(t, r)] + r^{2\kappa} a^2 [v_t^2(t, r) + c_2^2 v_r^2(t, r)] \right) dr,$$

is the total energy. In addition, the solution  $(u, v)(t, r)$  constructed above is conservative in the sense that the total energy represented by a Radon measure is conserved in time.

The paper is organized as follows. Section 2 is devoted to formulating the original initial-boundary value problem in the new energy-dependent coordinates. In Subsection 2.1, we use the characteristic reflection method to introduce the energy-dependent coordinates into the semi-infinite interval  $[1, \infty)$ . In Subsection 2.2, we present the governing system in terms of energy-dependent coordinate variables. In Subsection 2.3, we analyze the boundary conditions of the governing system in energy-dependent coordinates. Section 3 is devoted to solving the new boundary value problem in energy-dependent coordinate plane. In Subsection 3.1, we regularize the governing system and establish the global existence of bounded solutions for the regularization system. In Subsection 3.2, we establish the strong convergence of the approximate solution subsequence by the Young measure theory. In Section 4, the last section, we construct a global weak solution of the initial-boundary value problem by returning the solution in the energy-dependent coordinate variables to the original physical variables, and then complete the proof of Theorem 1.1.

## 2. NEW FORMULATION IN ENERGY-DEPENDENT COORDINATES

This section is devoted to transforming the original initial-boundary value problem into a new boundary problem by introducing the energy-dependent coordinates into the semi-finite interval  $[1, \infty)$ .

2.1. **The energy-dependent coordinates.** Denote

$$\begin{cases} R_1 := r^\kappa(u_t + c_1(u)u_r), & R_2 := r^\kappa a(u)(v_t + c_2(u)v_r), \\ S_1 := r^\kappa(u_t - c_1(u)u_r), & S_2 := r^\kappa a(u)(v_t - c_2(u)v_r), \end{cases} \quad (2.1)$$

so that

$$u_t = \frac{R_1 + S_1}{2r^\kappa}, \quad u_r = \frac{R_1 - S_1}{2r^\kappa c_1(u)}, \quad v_t = \frac{R_2 + S_2}{2r^\kappa a(u)}, \quad v_r = \frac{R_2 - S_2}{2r^\kappa a c_2(u)}. \quad (2.2)$$

By direct calculations, system (1.7) can be transformed to

$$\begin{cases} R_{1t} - c_1(u)R_{1r} = \frac{c_1'(u)}{4c_1 r^\kappa}(R_1^2 - S_1^2) - \frac{\kappa c_1}{r}S_1 - \frac{Q}{r^\kappa}, \\ S_{1t} + c_1(u)S_{1r} = \frac{c_1'(u)}{4c_1 r^\kappa}(S_1^2 - R_1^2) + \frac{\kappa c_1}{r}R_1 - \frac{Q}{r^\kappa}, \\ R_{2t} - c_2(u)R_{2r} = \frac{1}{r^\kappa}[(1+b)R_1 + (1-b)S_1]F - \frac{\kappa c_2}{r}S_2, \\ S_{2t} + c_2(u)S_{2r} = \frac{1}{r^\kappa}[(1-b)R_1 + (1+b)S_1]G + \frac{\kappa c_2}{r}R_2, \\ u_t - c_1(u)u_r = \frac{S_1}{r^\kappa}, \quad v_t - c_2(u)v_r = \frac{S_2}{r^\kappa a}, \end{cases} \quad (2.3)$$

where

$$\begin{aligned} Q &= \frac{c_2'(u)}{4c_2}(R_2 - S_2)^2 - \frac{a'(u)}{a}R_2 S_2, \\ F &= \frac{c_2'(u)}{4c_2}R_2 - \left(\frac{a'(u)}{2a} + \frac{c_2'(u)}{4c_2}\right)S_2, \quad G = \frac{c_2'(u)}{4c_2}S_2 - \left(\frac{a'(u)}{2a} + \frac{c_2'(u)}{4c_2}\right)R_2. \end{aligned} \quad (2.4)$$

Let  $(t, r)$  be any point in  $D := [0, \infty) \times [1, \infty)$ . We define the forward and backward characteristics  $r = r_\pm(s; t, r)$  ( $s \leq t$ ) passing through the point  $(t, r)$  as follows

$$\begin{cases} \frac{dr_\pm(s; t, r)}{ds} = \pm c_1(u(r_\pm(s; t, r), s)), \\ r_\pm(t; t, r) = r. \end{cases}$$

Let us now define the coordinate transformation  $(t, r) \rightarrow (X, Y)$  on  $D$ . We first specify this transformation to transform the line  $r = 1$  with  $t \geq 0$  into the line  $\Gamma_b : Y = X$  with  $X \geq 0$ . For the line  $t = 0$  with  $r \in [1, \infty)$ , we specify that it is transformed to a curve  $\Gamma_0 : Y = \varphi(X)$  ( $X \geq 0$ ) defined through a parametric  $r \in [1, \infty)$

$$X = \int_1^r (1 + R_{10}^2(z)) dz, \quad Y = \int_r^1 (1 + S_{10}^2(z)) dz,$$

where

$$R_{10}(r) = r^\kappa[u_1(r) + c_1(u_0(r))u_0'(r)], \quad S_{10}(r) = r^\kappa[u_1(r) - c_1(u_0(r))u_0'(r)]. \quad (2.5)$$

Due to assumption (1.8), we know that the two functions  $X = X(r)$  and  $Y = Y(r)$  with  $r \in [1, \infty)$  are well-defined and absolutely continuous satisfying that  $X(r)$  is strictly increasing while  $Y(r)$  is strictly decreasing. Thus  $Y = \varphi(X)$  is continuous and strictly decreasing. See Figure 1 for the illustration.

For any point  $(t, r) \in (0, \infty) \times (1, \infty)$ , we now draw the backward characteristic  $r_-(s; t, r)$  up to a point  $(0, r_-(0; t, r))$  on  $t = 0$  and then define

$$X = \int_1^{r_-(0; t, r)} (1 + R_{10}^2(z)) dz. \quad (2.6)$$

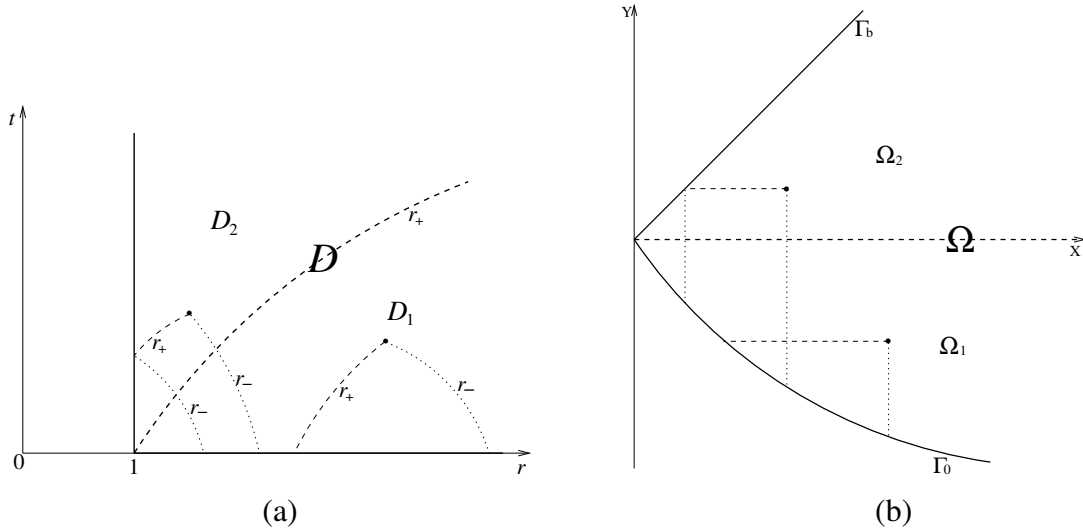


FIGURE 1. The region  $D$  and its image in the coordinate plane  $(X, Y)$ .

To define the quantity  $Y$ , we apply the forward characteristic  $r = r_+(t; 0, 1) (t \geq 0)$  to divide the domain  $D$  into two sub-domains  $D_1$  and  $D_2$ , where

$$D_1 = \{(t, r) \mid t \geq 0, r \geq r_+(t; 0, 1)\}, \quad D_2 = \{(t, r) \mid t \geq 0, 1 \leq r < r_+(t; 0, 1)\}.$$

If  $(t, r) \in D_1$ , we can draw the forward characteristic  $r_+(s; t, r)$  up to a point  $(0, r_+(0; t, r))$  with  $r_+(0; t, r) \geq 1$  on  $t = 0$  and then define

$$Y = \int_{r_+(0; t, r)}^1 (1 + S_{10}^2(z)) \, dz. \tag{2.7}$$

If  $(t, r) \in D_2$ , we draw the forward characteristic  $r_+(s; t, r)$  up to a point  $(\tilde{t}, 1)$  with  $\tilde{t} > 0$  on  $r = 1$ . From the point  $(\tilde{t}, 1)$ , we further draw the backward characteristic  $r_-(s; \tilde{t}, 1)$  up to a point  $(0, r_-(0; \tilde{t}, 1))$  with  $r_-(0; \tilde{t}, 1) > 1$  on  $t = 0$ . Thus we recall the fact that the line  $r = 1$  is transformed into the line  $Y = X$  to define the quantity  $Y(t, r)$  as follows

$$Y = Y(\tilde{t}, 1) = X(\tilde{t}, 1) = \int_1^{r_-(0; \tilde{t}, 1)} (1 + R_{10}^2(z)) \, dz. \tag{2.8}$$

Combining (2.6), (2.7) and (2.8), we have defined the coordinate transformation  $(t, r) \rightarrow (X, Y)$  on  $D$ . Moreover, according to the construction process, we know that  $X$  and  $Y$  are constants along backward and forward characteristics, respectively, that is,

$$X_t - c_1(u)X_r = 0, \quad Y_t + c_1(u)Y_r = 0,$$

which imply that, for any smooth function  $f$ , there hold

$$f_t + c_1(u)f_r = 2c_1X_rf_X, \quad f_t - c_1(u)f_r = -2c_1Y_rf_Y. \tag{2.9}$$

It suggests by (2.9) that

$$f_t = c_1(X_rf_X - Y_rf_Y), \quad f_r = X_rf_X + Y_rf_Y. \tag{2.10}$$

**2.2. The governing system in energy-dependent coordinates.** In order to deal with the blowup of  $(R_1, S_1)$ , it is convenient to introduce a new set of dependent variables

$$g = \frac{1}{1+R_1^2}, \quad h = \frac{1}{1+S_1^2}, \quad \ell = \frac{R_1}{1+R_1^2}, \quad m = \frac{S_1}{1+S_1^2}, \quad (2.11)$$

from which one has

$$\ell^2 + g^2 = g, \quad m^2 + h^2 = h. \quad (2.12)$$

Furthermore, we introduce

$$p = \frac{1+R_1^2}{X_r}, \quad q = \frac{1+S_1^2}{-Y_r}, \quad (2.13)$$

so that

$$\frac{1}{X_r} = pg, \quad \frac{1}{-Y_r} = qh. \quad (2.14)$$

By using (2.3) and (2.10)-(2.14), we can perform tedious calculations to obtain a governing system for the variables  $(g, h, \ell, m, p, q, R_2, S_2, u, v, r)$  in terms of  $(X, Y)$

$$\begin{cases} g_Y = \frac{q\ell}{c_1} \left\{ \frac{c'_1}{4c_1 r^{\kappa}} (g-h) + \frac{\kappa c_1}{r} mg + \frac{Q}{r^{\kappa}} gh \right\}, \\ h_X = \frac{pm}{c_1} \left\{ \frac{c'_1}{4c_1 r^{\kappa}} (h-g) - \frac{\kappa c_1}{r} lh + \frac{Q}{r^{\kappa}} gh \right\}, \end{cases} \quad (2.15)$$

$$\begin{cases} \ell_Y = \frac{q(1-2g)}{2c_1} \left\{ \frac{c'_1}{4c_1 r^{\kappa}} (g-h) + \frac{\kappa c_1}{r} mg + \frac{Q}{r^{\kappa}} gh \right\}, \\ m_X = \frac{p(1-2h)}{2c_1} \left\{ \frac{c'_1}{4c_1 r^{\kappa}} (h-g) - \frac{\kappa c_1}{r} lh + \frac{Q}{r^{\kappa}} gh \right\}, \end{cases} \quad (2.16)$$

$$\begin{cases} p_Y = \frac{pq}{c_1} \left\{ \frac{c'_1}{4c_1 r^{\kappa}} (m-\ell) - \frac{\kappa c_1}{r} ml - \frac{Q}{r^{\kappa}} lh \right\}, \\ q_X = \frac{pq}{c_1} \left\{ \frac{c'_1}{4c_1 r^{\kappa}} (\ell-m) + \frac{\kappa c_1}{r} ml - \frac{Q}{r^{\kappa}} mg \right\}, \end{cases} \quad (2.17)$$

$$\begin{cases} (1-b)qh\partial_X R_2 + (1+b)pg\partial_Y R_2 = \frac{pq}{c_1} \left\{ \frac{1}{r^{\kappa}} [(1+b)lh + (1-b)mg]F - \frac{\kappa c_2}{r} S_2 gh \right\}, \\ (1+b)qh\partial_X S_2 + (1-b)pg\partial_Y S_2 = \frac{pq}{c_1} \left\{ \frac{1}{r^{\kappa}} [(1-b)lh + (1+b)mg]G + \frac{\kappa c_2}{r} R_2 gh \right\}, \end{cases} \quad (2.18)$$

and

$$\begin{cases} u_Y = \frac{qm}{2c_1 r^{\kappa}} \left( \text{or } u_X = \frac{p\ell}{2c_1 r^{\kappa}} \right), \\ v_Y = \frac{qh}{4ac_2 r^{\kappa}} [(1+b)S_2 - (1-b)R_2] \left( \text{or } v_X = \frac{pg}{4ac_2 r^{\kappa}} [(1+b)R_2 - (1-b)S_2] \right), \\ r_Y = -\frac{qh}{2} \left( \text{or } r_X = \frac{pg}{2} \right). \end{cases} \quad (2.19)$$

In addition, there hold

$$\begin{aligned}
 t_X &= \frac{pg}{2c_1}, \quad t_Y = \frac{qh}{2c_1}, \quad \left(\frac{qh}{c_1}\right)_X = \left(\frac{pg}{c_1}\right)_Y, \\
 (qh)_x + (pg)_y &= 0, \quad \left(\frac{qm}{c_1 r^\kappa}\right)_X = \left(\frac{p\ell}{c_1 r^\kappa}\right)_Y, \\
 \left(\frac{qh}{ac_2 r^\kappa}[(1+b)S_2 - (1-b)R_2]\right)_X &= \left(\frac{pg}{ac_2 r^\kappa}[(1+b)R_2 - (1-b)S_2]\right)_Y,
 \end{aligned} \tag{2.20}$$

and

$$\begin{aligned}
 \partial_Y(g^2 + \ell^2 - g) &= 0, \quad \partial_X(h^2 + m^2 - h) = 0, \\
 \left(q + \frac{1}{2}qh[(1-b)R_2^2 + (1+b)S_2^2]\right)_X &+ \left(p + \frac{1}{2}pg[(1+b)R_2^2 + (1-b)S_2^2]\right)_Y = 0.
 \end{aligned} \tag{2.21}$$

The detailed derivations of (2.15)-(2.21) can be acquired through direct calculations, and a similar process can be referred to the works of Zhang and Zheng [37] and Hu [26].

**2.3. The boundary conditions in energy-dependent coordinates.** Let us consider the boundary conditions of system (2.15)-(2.19) in terms of  $(X, Y)$ , corresponding to (1.8)-(1.9) in the original coordinates  $(t, r)$ .

According to the construction of the coordinate transformation  $(t, r) \rightarrow (X, Y)$  in Subsection 2.1, we see that the curve  $\Gamma_0$  is parameterized by the parameter  $r$ . Thus we can assign the boundary data  $(\hat{g}, \hat{h}, \hat{\ell}, \hat{m}, \hat{p}, \hat{q}, \hat{R}_2, \hat{S}_2, \hat{u}, \hat{v}, \hat{r})$  on  $\Gamma_0$  defined by

$$\begin{cases} \hat{g} = \frac{1}{1+R_{10}^2(r)}, \\ \hat{h} = \frac{1}{1+S_{10}^2(r)}, \end{cases} \begin{cases} \hat{\ell} = R_{10}(r)\hat{h}, \\ \hat{m} = S_{10}(r)\hat{g}, \end{cases} \begin{cases} \hat{p} = 1, \\ \hat{q} = 1, \end{cases} \begin{cases} \hat{R}_2 = R_{20}(r), \\ \hat{S}_2 = S_{20}(r), \end{cases} \begin{cases} \hat{u} = u_0(r), \\ \hat{v} = v_0(r), \\ \hat{r} = r, \end{cases} \tag{2.22}$$

for  $r \geq 1$ , where  $R_{10}(r)$  and  $S_{10}(r)$  are given in (2.5) and

$$\begin{aligned}
 R_{20}(r) &= r^\kappa a(u_0(r))[v_1(r) + c_2(u_0(r))v_0'(r)], \\
 S_{20}(r) &= r^\kappa a(u_0(r))[v_1(r) - c_2(u_0(r))v_0'(r)].
 \end{aligned}$$

The boundary data of  $(p, q)$  come from the definition of  $(p, q)$  in (2.13). It is easy to see by (1.8) that  $(\hat{g}, \hat{h}, \hat{\ell}, \hat{m}, \hat{p}, \hat{q}, \hat{R}_2, \hat{S}_2, \hat{u}, \hat{v}) \in L^\infty$ .

Furthermore, we recall the fact that the line  $r = 1(t \geq 0)$  is transformed into the line  $\Gamma_b : Y = X(X \geq 0)$  and apply (1.9) and (2.2) to obtain

$$\ell h + mg = 0, \quad R_2 + S_2 = 0 \quad \text{on } \Gamma_b. \tag{2.23}$$

In view of the relations in (2.12), one has

$$g = h \quad \text{on } \Gamma_b. \tag{2.24}$$

It follows by the fact  $dr = 0$  along  $\Gamma_b$  that

$$0 = dr = r_X dX + r_Y dY = \frac{pg}{2} dX - \frac{qh}{2} dY,$$

which together with (2.24) gives

$$p = q \quad \text{on } \Gamma_b. \tag{2.25}$$

Combining (2.23)-(2.25) yields the boundary conditions on  $\Gamma_b$

$$g = h, \quad \ell + m = 0, \quad R_2 + S_2 = 0, \quad p = q, \quad u = v = 0, \quad r = 1 \quad \text{on } \Gamma_b. \quad (2.26)$$

For convenience, we use  $\Omega_1$  and  $\Omega_2$  to represent the images of  $D_1$  and  $D_2$  in the  $(X, Y)$  plane, respectively, and denote  $\Omega = \Omega_1 \cup \Omega_2$ , see Figure 1 (b).

### 3. SOLUTIONS IN THE ENERGY COORDINATES

In this section, we establish the global existence of solutions for system (2.15)-(2.19) with boundary conditions (2.22) and (2.26) in the energy coordinates  $(X, Y)$ .

**3.1. Solutions of a regularization system.** In order to handle the degeneracy of the equation (2.18) when  $g = 0$  and  $h = 0$  simultaneously, we first regularize the system by adding some terms in the equations of  $(R_2, S_2)$ . Let  $0 < \varepsilon < 1$  be a small number. We consider the following system of the variables  $(g^\varepsilon, h^\varepsilon, \ell^\varepsilon, m^\varepsilon, p^\varepsilon, q^\varepsilon, R_2^\varepsilon, S_2^\varepsilon, u^\varepsilon, v^\varepsilon, r^\varepsilon)$ :

$$\begin{cases} \partial_Y g^\varepsilon = \frac{q^\varepsilon \ell^\varepsilon}{c_1^\varepsilon} \left\{ \frac{(c_1^\varepsilon)'}{4c_1^\varepsilon (r^\varepsilon)^\kappa} (g^\varepsilon - h^\varepsilon) + \frac{\kappa c_1^\varepsilon}{r^\varepsilon} m^\varepsilon g^\varepsilon + \frac{Q^\varepsilon}{(r^\varepsilon)^\kappa} g^\varepsilon h^\varepsilon \right\}, \\ \partial_X h^\varepsilon = \frac{p^\varepsilon m^\varepsilon}{c_1^\varepsilon} \left\{ \frac{(c_1^\varepsilon)'}{4c_1^\varepsilon (r^\varepsilon)^\kappa} (h^\varepsilon - g^\varepsilon) - \frac{\kappa c_1^\varepsilon}{r^\varepsilon} \ell^\varepsilon h^\varepsilon + \frac{Q^\varepsilon}{(r^\varepsilon)^\kappa} g^\varepsilon h^\varepsilon \right\}, \end{cases} \quad (3.1)$$

$$\begin{cases} \partial_Y \ell^\varepsilon = \frac{q^\varepsilon (1-2g^\varepsilon)}{2c_1^\varepsilon} \left\{ \frac{(c_1^\varepsilon)'}{4c_1^\varepsilon (r^\varepsilon)^\kappa} (g^\varepsilon - h^\varepsilon) + \frac{\kappa c_1^\varepsilon}{r^\varepsilon} m^\varepsilon g^\varepsilon + \frac{Q^\varepsilon}{(r^\varepsilon)^\kappa} g^\varepsilon h^\varepsilon \right\}, \\ \partial_X m^\varepsilon = \frac{p^\varepsilon (1-2h^\varepsilon)}{2c_1^\varepsilon} \left\{ \frac{(c_1^\varepsilon)'}{4c_1^\varepsilon (r^\varepsilon)^\kappa} (h^\varepsilon - g^\varepsilon) - \frac{\kappa c_1^\varepsilon}{r^\varepsilon} \ell^\varepsilon h^\varepsilon + \frac{Q^\varepsilon}{(r^\varepsilon)^\kappa} g^\varepsilon h^\varepsilon \right\}, \end{cases} \quad (3.2)$$

$$\begin{cases} \partial_Y p^\varepsilon = \frac{p^\varepsilon q^\varepsilon}{c_1^\varepsilon} \left\{ \frac{(c_1^\varepsilon)'}{4c_1^\varepsilon (r^\varepsilon)^\kappa} (m^\varepsilon - \ell^\varepsilon) - \frac{\kappa c_1^\varepsilon}{r^\varepsilon} m^\varepsilon \ell^\varepsilon - \frac{Q^\varepsilon}{(r^\varepsilon)^\kappa} \ell^\varepsilon h^\varepsilon \right\}, \\ \partial_X q^\varepsilon = \frac{p^\varepsilon q^\varepsilon}{c_1^\varepsilon} \left\{ \frac{(c_1^\varepsilon)'}{4c_1^\varepsilon (r^\varepsilon)^\kappa} (\ell^\varepsilon - m^\varepsilon) + \frac{\kappa c_1^\varepsilon}{r^\varepsilon} m^\varepsilon \ell^\varepsilon - \frac{Q^\varepsilon}{(r^\varepsilon)^\kappa} m^\varepsilon g^\varepsilon \right\}, \end{cases} \quad (3.3)$$

$$\begin{cases} [(1-b^\varepsilon)q^\varepsilon h^\varepsilon + \varepsilon] \partial_X R_2^\varepsilon + [(1+b^\varepsilon)p^\varepsilon g^\varepsilon + \varepsilon] \partial_Y R_2^\varepsilon \\ = \frac{p^\varepsilon q^\varepsilon}{c_1^\varepsilon} \left\{ \frac{1}{(r^\varepsilon)^\kappa} [(1+b^\varepsilon)\ell^\varepsilon h^\varepsilon + (1-b^\varepsilon)m^\varepsilon g^\varepsilon] F^\varepsilon - \frac{\kappa c_2^\varepsilon}{r^\varepsilon} S_2^\varepsilon g^\varepsilon h^\varepsilon \right\}, \\ [(1+b^\varepsilon)q^\varepsilon h^\varepsilon + \varepsilon] \partial_X S_2^\varepsilon + [(1-b^\varepsilon)p^\varepsilon g^\varepsilon + \varepsilon] \partial_Y S_2^\varepsilon \\ = \frac{p^\varepsilon q^\varepsilon}{c_1^\varepsilon} \left\{ \frac{1}{(r^\varepsilon)^\kappa} [(1-b^\varepsilon)\ell^\varepsilon h^\varepsilon + (1+b^\varepsilon)m^\varepsilon g^\varepsilon] G^\varepsilon + \frac{\kappa c_2^\varepsilon}{r^\varepsilon} R_2^\varepsilon g^\varepsilon h^\varepsilon \right\}, \end{cases} \quad (3.4)$$

and

$$\partial_Y u^\varepsilon = \frac{q^\varepsilon m^\varepsilon}{2c_1^\varepsilon (r^\varepsilon)^\kappa}, \quad \partial_Y v^\varepsilon = \frac{q^\varepsilon h^\varepsilon}{4a^\varepsilon c_2^\varepsilon (r^\varepsilon)^\kappa} [(1+b^\varepsilon)S_2^\varepsilon - (1-b^\varepsilon)R_2^\varepsilon], \quad \partial_Y r^\varepsilon = -\frac{q^\varepsilon h^\varepsilon}{2}. \quad (3.5)$$

In the system,  $(c_1^\varepsilon, c_2^\varepsilon, a^\varepsilon, b^\varepsilon) = (c_1, c_2, a, b)(u^\varepsilon)$  and  $(Q^\varepsilon, F^\varepsilon, G^\varepsilon) = (Q, F, G)(u^\varepsilon, R_2^\varepsilon, S_2^\varepsilon)$ . Moreover, there hold

$$\partial_Y [(g^\varepsilon)^2 + (\ell^\varepsilon)^2 - g^\varepsilon] = 0, \quad \partial_X [(h^\varepsilon)^2 + (m^\varepsilon)^2 - h^\varepsilon] = 0, \quad (3.6)$$

and

$$\begin{aligned} & \left( q^\varepsilon + \frac{1}{2}q^\varepsilon h^\varepsilon [(1-b^\varepsilon)(R_2^\varepsilon)^2 + (1+b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2}[(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right)_X \\ & + \left( p^\varepsilon + \frac{1}{2}p^\varepsilon g^\varepsilon [(1+b^\varepsilon)(R_2^\varepsilon)^2 + (1-b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2}[(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right)_Y = 0. \end{aligned} \quad (3.7)$$

The local existence of solutions to system (3.1)-(3.5) with the boundary value conditions (2.22), (2.26) can be established by the fixed point theorem, see a similar process in Bressan and Zheng [8]. We next mainly derive the a priori estimates of solutions to system (3.1)-(3.5) in  $\Omega_2$ , and the estimates of solutions in  $\Omega_1$  are analogous. For any  $(X, Y) \in \Omega_2$ , we utilize (2.22) and (3.6) to achieve

$$\begin{aligned} & [(g^\varepsilon)^2 + (\ell^\varepsilon)^2 - g^\varepsilon](X, Y) \\ & = [(g^\varepsilon)^2 + (\ell^\varepsilon)^2 - g^\varepsilon](X, \varphi(X)) = \hat{g}^2 + \hat{\ell}^2 - \hat{g} = 0. \end{aligned} \quad (3.8)$$

Moreover, it follows by (3.6), (2.26), and (3.8) and that

$$\begin{aligned} & [(h^\varepsilon)^2 + (m^\varepsilon)^2 - h^\varepsilon](X, Y) = [(h^\varepsilon)^2 + (m^\varepsilon)^2 - h^\varepsilon](Y, Y) \\ & = [(g^\varepsilon)^2 + (\ell^\varepsilon)^2 - g^\varepsilon](Y, Y) = [(g^\varepsilon)^2 + (\ell^\varepsilon)^2 - g^\varepsilon](Y, \varphi(Y)) = 0. \end{aligned}$$

This together with (3.8) yield

$$(g^\varepsilon)^2 + (\ell^\varepsilon)^2 = g^\varepsilon, \quad (h^\varepsilon)^2 + (m^\varepsilon)^2 = h^\varepsilon, \quad \forall (X, Y) \in \Omega_2,$$

which leads to

$$0 \leq g^\varepsilon \leq 1, \quad 0 \leq h^\varepsilon \leq 1, \quad |\ell^\varepsilon| \leq \frac{1}{2}, \quad |m^\varepsilon| \leq \frac{1}{2}. \quad (3.9)$$

In order to estimate  $p^\varepsilon$  and  $q^\varepsilon$  on the domain  $\Omega_2$ , we consider the domain  $\Omega_b$  which is enclosed by a horizontal segment between  $(Y, Y)$  and  $(X, Y)$ , a vertical segment between  $(X, Y)$  and  $(X, \varphi(X))$ ,  $\Gamma_0$  and  $\Gamma_b$ . Thanks to Green's theorem, we obtain by (3.7)

$$\begin{aligned} & \int_{\partial\Omega_b} \left( p^\varepsilon + \frac{1}{2}p^\varepsilon g^\varepsilon [(1+b^\varepsilon)(R_2^\varepsilon)^2 + (1-b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2}[(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right) dX' \\ & - \left( q^\varepsilon + \frac{1}{2}q^\varepsilon h^\varepsilon [(1-b^\varepsilon)(R_2^\varepsilon)^2 + (1+b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2}[(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right) dY' \\ & = - \iint_{\Omega_b} \left( q^\varepsilon + \frac{1}{2}q^\varepsilon h^\varepsilon [(1-b^\varepsilon)(R_2^\varepsilon)^2 + (1+b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2}[(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right)_X \\ & + \left( p^\varepsilon + \frac{1}{2}p^\varepsilon g^\varepsilon [(1+b^\varepsilon)(R_2^\varepsilon)^2 + (1-b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2}[(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right)_Y dX' dY' \\ & = 0, \end{aligned} \quad (3.10)$$

where  $\partial\Omega_b$  is the boundary of  $\Omega_b$ . It concludes by (3.10) that

$$\begin{aligned} & \int_Y^X \left( p^\varepsilon + \frac{1}{2}p^\varepsilon g^\varepsilon [(1+b^\varepsilon)(R_2^\varepsilon)^2 + (1-b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2}[(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right) dX' \\ & + \int_{\varphi(X)}^Y \left( q^\varepsilon + \frac{1}{2}q^\varepsilon h^\varepsilon [(1-b^\varepsilon)(R_2^\varepsilon)^2 + (1+b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2}[(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right) dY' \\ & = \mathcal{I}_1 + \mathcal{I}_2, \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} \mathcal{J}_1 = & \int_{\Gamma_0} \left( p^\varepsilon + \frac{1}{2} p^\varepsilon g^\varepsilon [(1+b^\varepsilon)(R_2^\varepsilon)^2 + (1-b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2} [(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right) dX' \\ & - \left( q^\varepsilon + \frac{1}{2} q^\varepsilon h^\varepsilon [(1-b^\varepsilon)(R_2^\varepsilon)^2 + (1+b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2} [(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right) dY', \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_2 = & - \int_{\Gamma_b} \left( p^\varepsilon + \frac{1}{2} p^\varepsilon g^\varepsilon [(1+b^\varepsilon)(R_2^\varepsilon)^2 + (1-b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2} [(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right) dX' \\ & - \left( q^\varepsilon + \frac{1}{2} q^\varepsilon h^\varepsilon [(1-b^\varepsilon)(R_2^\varepsilon)^2 + (1+b^\varepsilon)(S_2^\varepsilon)^2] + \frac{\varepsilon}{2} [(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] \right) dY'. \end{aligned}$$

By means of boundary conditions (2.26), we find that

$$\mathcal{J}_2 = 0. \quad (3.12)$$

For  $\mathcal{J}_1$ , we apply conditions (2.22) to achieve

$$\begin{aligned} \mathcal{J}_1 = & \int_0^X \left\{ \left( 1 + \frac{1}{2} \hat{g} [(1+\bar{b})\hat{R}_2^2 + (1-\hat{b})\hat{S}_2^2] + \frac{\varepsilon}{2} [\hat{R}_2^2 + \hat{S}_2^2] \right) \right. \\ & \left. - \left( 1 + \frac{1}{2} \hat{h} [(1-\hat{b})\hat{R}_2^2 + (1+\hat{b})\hat{S}_2^2] + \frac{\varepsilon}{2} [\hat{R}_2^2 + \hat{S}_2^2] \right) \varphi'(X') \right\} dX' \\ \leq & [1 + 2(\|\hat{R}_2\|_{L^2([1,\infty))}^2 + \|\hat{S}_2\|_{L^2([1,\infty))}^2)](X - \varphi(X)), \end{aligned} \quad (3.13)$$

where  $\hat{b} = b(\hat{u})$ . Combining (3.11), (3.12), and (3.13) yields

$$\begin{aligned} & \int_Y^X \left( p^\varepsilon + \frac{1}{2} p^\varepsilon g^\varepsilon [(1+b^\varepsilon)(R_2^\varepsilon)^2 + (1-b^\varepsilon)(S_2^\varepsilon)^2] \right) dX' \\ & + \int_{\varphi(X)}^Y \left( q^\varepsilon + \frac{1}{2} q^\varepsilon h^\varepsilon [(1-b^\varepsilon)(R_2^\varepsilon)^2 + (1+b^\varepsilon)(S_2^\varepsilon)^2] \right) dY' \\ \leq & [1 + 2(\|\hat{R}_2\|_{L^2([1,\infty))}^2 + \|\hat{S}_2\|_{L^2([1,\infty))}^2)](X - \varphi(X)). \end{aligned} \quad (3.14)$$

It is easy to see by (3.3) and the boundary conditions (2.22), (2.26) that  $p^\varepsilon > 0$  and  $q^\varepsilon > 0$ . Now we denote the equations of  $p^\varepsilon$  and  $q^\varepsilon$  as

$$\frac{\partial_Y p^\varepsilon}{p^\varepsilon} = \mathcal{J}_3, \quad \frac{\partial_X q^\varepsilon}{q^\varepsilon} = \mathcal{J}_4, \quad (3.15)$$

where

$$\begin{aligned} \mathcal{J}_3 = & \frac{q^\varepsilon}{c_1^\varepsilon} \left\{ \frac{(c_1^\varepsilon)'}{4c_1^\varepsilon (r^\varepsilon)^\kappa} (m^\varepsilon - \ell^\varepsilon) - \frac{\kappa c_1^\varepsilon}{r^\varepsilon} m^\varepsilon \ell^\varepsilon - \frac{Q^\varepsilon}{(r^\varepsilon)^\kappa} \ell^\varepsilon h \right\}, \\ \mathcal{J}_4 = & \frac{p^\varepsilon}{c_1^\varepsilon} \left\{ \frac{(c_1^\varepsilon)'}{4c_1^\varepsilon (r^\varepsilon)^\kappa} (\ell^\varepsilon - m^\varepsilon) + \frac{\kappa c_1^\varepsilon}{r^\varepsilon} m^\varepsilon \ell^\varepsilon - \frac{Q^\varepsilon}{(r^\varepsilon)^\kappa} m^\varepsilon g^\varepsilon \right\}. \end{aligned}$$

Recalling (1.10), (1.11) and (2.4), one utilizes (3.9) to obtain

$$\begin{aligned}
 |\mathcal{J}_3| &\leq \frac{q^\varepsilon}{c_0} \left\{ \frac{C_0}{4c_0} + \frac{\kappa C_0}{4} + \frac{1}{2} |Q^\varepsilon| h^\varepsilon \right\} \\
 &\leq \frac{(1 + \kappa c_0) C_0}{4c_0^2} q^\varepsilon + \frac{C_0}{c_0} [(R_2^\varepsilon)^2 + (S_2^\varepsilon)^2] q^\varepsilon h^\varepsilon \\
 &\leq \frac{(1 + \kappa c_0) C_0}{4c_0^2} q^\varepsilon + \frac{C_0}{c_0(1 - b_0)} q^\varepsilon h^\varepsilon [(1 - b^\varepsilon)(R_2^\varepsilon)^2 + (1 + b^\varepsilon)(S_2^\varepsilon)^2] \\
 &\leq \frac{(1 + 8c_0 + \kappa c_0) C_0}{4c_0^2(1 - b_0)} \left\{ q^\varepsilon + \frac{1}{2} q^\varepsilon h^\varepsilon [(1 - b^\varepsilon)(R_2^\varepsilon)^2 + (1 + b^\varepsilon)(S_2^\varepsilon)^2] \right\}, \tag{3.16}
 \end{aligned}$$

and

$$\begin{aligned}
 |\mathcal{J}_4| &\leq \frac{p^\varepsilon}{c_0} \left\{ \frac{C_0}{4c_0} + \frac{\kappa C_0}{4} + \frac{1}{2} |Q^\varepsilon| g^\varepsilon \right\} \\
 &\leq \frac{(1 + \kappa c_0) C_0}{4c_0^2} p^\varepsilon + \frac{C_0}{c_0(1 - b_0)} p^\varepsilon g^\varepsilon [(1 + b^\varepsilon)(R_2^\varepsilon)^2 + (1 - b^\varepsilon)(S_2^\varepsilon)^2] \\
 &\leq \frac{(1 + 8c_0 + \kappa c_0) C_0}{4c_0^2(1 - b_0)} \left\{ p^\varepsilon + \frac{1}{2} p^\varepsilon g^\varepsilon [(1 + b^\varepsilon)(R_2^\varepsilon)^2 + (1 - b^\varepsilon)(S_2^\varepsilon)^2] \right\}. \tag{3.17}
 \end{aligned}$$

We integrate the equation of  $p^\varepsilon$  from  $\varphi(X)$  to  $Y$  and apply (3.14), (3.16) to acquire

$$\exp \left( - \int_{\varphi(X)}^Y |\mathcal{J}_3| dX' \right) \leq p^\varepsilon(X, Y) \leq \exp \left( \int_{\varphi(X)}^Y |\mathcal{J}_3| dX' \right), \tag{3.18}$$

which means that

$$e^{-M(X)} \leq p^\varepsilon(X, Y) \leq e^{M(X)}, \tag{3.19}$$

where

$$M(X) = \frac{(1 + 8c_0 + \kappa c_0) C_0}{4c_0^2(1 - b_0)} [1 + 2(\|\hat{R}_2\|_{L^2([1, \infty))}^2 + \|\hat{S}_2\|_{L^2([1, \infty))}^2)](X - \varphi(X)). \tag{3.20}$$

Integrating the equation of  $q^\varepsilon$  from  $Y$  to  $X$  and using the boundary condition  $p = q$  on  $\Gamma_b$ , we find that

$$q^\varepsilon(X, Y) = q^\varepsilon(Y, Y) \exp \left( \int_Y^X \mathcal{J}_4 dY' \right) = p^\varepsilon(Y, Y) \exp \left( \int_Y^X \mathcal{J}_4 dY' \right), \tag{3.21}$$

from which and (3.19) one obtains

$$e^{-M(Y) - M(X)} \leq q^\varepsilon(X, Y) \leq e^{M(Y) + M(X)}. \tag{3.22}$$

Hence we combine (3.19) and (3.22) to conclude

$$0 < e^{-M(Y) - M(X)} \leq p^\varepsilon(X, Y), q^\varepsilon(X, Y) \leq e^{M(Y) + M(X)}, \quad \forall (X, Y) \in \Omega_2. \tag{3.23}$$

Finally, we estimate  $(R_2^\varepsilon, S_2^\varepsilon)$ . Denote  $v^\varepsilon = ((1 - b^\varepsilon)q^\varepsilon h^\varepsilon + \varepsilon, (1 + b^\varepsilon)p^\varepsilon g^\varepsilon + \varepsilon)$ . Then we can rewrite the equation for  $R_2^\varepsilon$  as

$$\begin{aligned}
 &v^\varepsilon \cdot (\partial_X, \partial_Y) R_2^\varepsilon \\
 &= \frac{1}{c_1^\varepsilon (r^\varepsilon)^\kappa} [(1 + b^\varepsilon) p^\varepsilon q^\varepsilon \ell^\varepsilon h^\varepsilon + (1 - b^\varepsilon) p^\varepsilon q^\varepsilon m g^\varepsilon] F^\varepsilon - \frac{\kappa c_2^\varepsilon}{c_1^\varepsilon r^\varepsilon} S_2^\varepsilon p^\varepsilon q^\varepsilon g^\varepsilon h^\varepsilon. \tag{3.24}
 \end{aligned}$$

It is easily seen by (1.10), (1.11), (3.9), and (3.23) that

$$\begin{aligned} & \frac{|(1+b^\varepsilon)p^\varepsilon q^\varepsilon \ell^\varepsilon h^\varepsilon + (1-b) p^\varepsilon q^\varepsilon m^\varepsilon g^\varepsilon|}{|v^\varepsilon|} \\ & \leq \frac{(1+b) p^\varepsilon}{1-b^\varepsilon} + \frac{(1-b^\varepsilon) q^\varepsilon}{1+b^\varepsilon} \leq \frac{3-b_0}{1-b_0} e^{M(Y)+M(X)}, \end{aligned} \tag{3.25}$$

and

$$\frac{p^\varepsilon q^\varepsilon h^\varepsilon g^\varepsilon}{|v^\varepsilon|} \leq q^\varepsilon \frac{p^\varepsilon g^\varepsilon}{|v^\varepsilon|} \leq q^\varepsilon \leq e^{M(Y)+M(X)}. \tag{3.26}$$

It follows by (3.24)-(3.26) and the expression of  $F^\varepsilon$  that the value of  $R_2^\varepsilon$  can grow exponentially to infinity at most, and the rate of growth is independent of  $\varepsilon$ . The analysis is also valid for  $S_2^\varepsilon$ . Therefore, the variables  $(R_2^\varepsilon, S_2^\varepsilon)$  are bounded on bounded domains and their bounds are independent of  $\varepsilon$ . According to the boundedness of  $(p^\varepsilon, q^\varepsilon)$  and  $(R_2^\varepsilon, S_2^\varepsilon)$ , one can directly derive the bounds of  $u^\varepsilon, v^\varepsilon$  and  $r^\varepsilon$  on bounded domains.

Based on the above a priori estimates, we can extend the local solution to the entire domain  $\Omega$ . Thus we have the global existence theorem.

**Theorem 3.1.** *Let conditions (1.8) and (1.9) hold. Then the problem (3.1)-(3.5) with boundary conditions (2.22), (2.26) admits a unique global solution  $(g^\varepsilon, h^\varepsilon, \ell^\varepsilon, m^\varepsilon, p^\varepsilon, q^\varepsilon, R_2^\varepsilon, S_2^\varepsilon, u^\varepsilon, v^\varepsilon, r^\varepsilon)$  such that*

$$\begin{aligned} 0 \leq g^\varepsilon(X, Y), h^\varepsilon(X, Y) \leq 1, \quad |\ell^\varepsilon(X, Y)|, |m^\varepsilon(X, Y)| \leq \frac{1}{2}, \\ \tilde{c}_0 \leq p^\varepsilon(X, Y), q^\varepsilon(X, Y) \leq \tilde{C}_0, \quad |R_2^\varepsilon(X, Y)|, |S_2^\varepsilon(X, Y)| \leq \tilde{C}_0, \quad \forall (X, Y) \in \hat{\Omega}, \end{aligned} \tag{3.27}$$

for some positive constants  $\tilde{c}_0, \tilde{C}_0$  depending only on  $c_0, C_0, b_0, \kappa$  and  $(\hat{X}, \hat{Y})$ , where  $(\hat{X}, \hat{Y})$  is any point in  $\Omega$  and  $\hat{\Omega}$  is the domain bounded by  $\Gamma_0, \Gamma_b, X = \hat{X}$  and  $Y = \hat{Y}$ .

**3.2. Weak solutions in the energy coordinates.** In this subsection, we mollify the boundary data on  $\Gamma_0$  and then pass the limit  $\varepsilon \rightarrow 0^+$  in regularization system (3.1)-(3.5) to establish the global existence of weak solutions of boundary value problem (2.15)-(2.19) with (2.22), (2.26).

The main result is as follows.

**Theorem 3.2.** *Let conditions (1.8) and (1.9) be fulfilled. Then problem (2.15)-(2.19) with boundary conditions (2.22) and (2.26) admits a global weak solution  $(g, h, p, q, \ell, m, R_2, S_2, u, v, r)$  such that there hold*

$$g^2 + \ell^2 = g, \quad h^2 + m^2 = h, \tag{3.28}$$

$$(qh)_X + (pg)_Y = 0, \quad \left(\frac{qh}{c_1}\right)_X - \left(\frac{pg}{c_1}\right)_Y = 0, \quad \left(\frac{qm}{c_1 r^\kappa}\right)_X = \left(\frac{p\ell}{c_1 r^\kappa}\right)_Y, \tag{3.29}$$

and

$$\left(q + \frac{1}{2}qh[(1-b)R_2^2 + (1+b)S_2^2]\right)_X + \left(p + \frac{1}{2}pg[(1+b)R_2^2 + (1-b)S_2^2]\right)_Y = 0, \tag{3.30}$$

in the sense of distribution.

To show Theorem 3.2, the main tool used here is the propagation of precompactness for appropriate approximate solution sequence, see [29, 34, 37] for details. We denote the weak \* limit to a subsequence of  $\{f^\varepsilon\}_{\varepsilon>0}$  by  $\bar{f}$ . Then according to Theorem 3.1, we can find  $(\bar{g}, \bar{h}, \bar{p}, \bar{q}, \bar{\ell}, \bar{m}) \in L^\infty_{\text{loc}}(\Omega)$  and  $(u, v, r) \in \text{Lip}_{\text{loc}}(\Omega)$ , and a suitable subsequence of

$$\{g^\varepsilon, h^\varepsilon, p^\varepsilon, q^\varepsilon, \ell^\varepsilon, m^\varepsilon, u^\varepsilon, v^\varepsilon, r^\varepsilon\},$$

still denoted by  $\{g^\varepsilon, h^\varepsilon, p^\varepsilon, q^\varepsilon, \ell^\varepsilon, m^\varepsilon, u^\varepsilon, v^\varepsilon, r^\varepsilon\}$ , such that

$$\begin{aligned} \{g^\varepsilon, h^\varepsilon, p^\varepsilon, q^\varepsilon, \ell^\varepsilon, m^\varepsilon\} &\rightharpoonup (\bar{g}, \bar{h}, \bar{p}, \bar{q}, \bar{\ell}, \bar{m}) \quad \text{weak * in } L^\infty_{\text{loc}}(\Omega), \\ \{u^\varepsilon, v^\varepsilon, r^\varepsilon\} &\rightarrow \{u, v, r\} \quad \text{locally uniformly on } \Omega. \end{aligned} \quad (3.31)$$

We next establish the strong convergence of  $\{g^\varepsilon, h^\varepsilon, p^\varepsilon, q^\varepsilon, \ell^\varepsilon, m^\varepsilon\}$  and of  $\{R_2^\varepsilon, S_2^\varepsilon\}$  with some weights. By (3.1) and (3.3), one directly calculates

$$\begin{cases} \partial_Y(p^\varepsilon g^\varepsilon) = \frac{(c_1^\varepsilon)'}{4(c_1^\varepsilon)^2(r^\varepsilon)^\kappa} (p^\varepsilon q^\varepsilon m^\varepsilon g^\varepsilon - p^\varepsilon q^\varepsilon h^\varepsilon \ell^\varepsilon), \\ \partial_X(q^\varepsilon h^\varepsilon) = \frac{(c_1^\varepsilon)'}{4(c_1^\varepsilon)^2(r^\varepsilon)^\kappa} (p^\varepsilon q^\varepsilon h^\varepsilon \ell^\varepsilon - p^\varepsilon q^\varepsilon m^\varepsilon g^\varepsilon), \end{cases} \quad (3.32)$$

which together with (3.1), (3.3) and the div-curl lemma [20, 29] leads to

$$\begin{cases} \partial_Y(\bar{p}\bar{g}) = \frac{c_1'}{4c_1^2 r^\kappa} (\bar{p}\bar{g} \cdot \bar{q}\bar{m} - \bar{p}\bar{\ell} \cdot \bar{q}\bar{h}), \\ \partial_X(\bar{q}\bar{h}) = \frac{c_1'}{4c_1^2 r^\kappa} (\bar{p}\bar{\ell} \cdot \bar{q}\bar{h} - \bar{p}\bar{g} \cdot \bar{q}\bar{m}). \end{cases} \quad (3.33)$$

Moreover, we utilize (3.4) to gain

$$\begin{aligned} &\partial_X[(c_1^\varepsilon - c_2^\varepsilon)q^\varepsilon h^\varepsilon R_2^\varepsilon + \varepsilon c_1^\varepsilon R_2^\varepsilon] + \partial_Y[(c_1^\varepsilon + c_2^\varepsilon)p^\varepsilon g^\varepsilon R_2^\varepsilon + \varepsilon c_1^\varepsilon R_2^\varepsilon] \\ &= \left( \frac{(c_1^\varepsilon)'}{2c_1^\varepsilon(r^\varepsilon)^\kappa} + \frac{(c_2^\varepsilon)'}{4c_2^\varepsilon(r^\varepsilon)^\kappa} \right) [(1 - b^\varepsilon)h^\varepsilon \ell^\varepsilon + (1 + b^\varepsilon)m^\varepsilon g^\varepsilon] p^\varepsilon q^\varepsilon R_2^\varepsilon \\ &\quad - \left( \frac{(a^\varepsilon)'}{2a^\varepsilon(r^\varepsilon)^\kappa} + \frac{(c_2^\varepsilon)'}{4c_2^\varepsilon(r^\varepsilon)^\kappa} \right) [(1 + b^\varepsilon)h^\varepsilon \ell^\varepsilon + (1 - b^\varepsilon)m^\varepsilon g^\varepsilon] p^\varepsilon q^\varepsilon S_2^\varepsilon \\ &\quad - \frac{\kappa c_2^\varepsilon}{r^\varepsilon} p^\varepsilon q^\varepsilon g^\varepsilon h^\varepsilon S_2^\varepsilon + \varepsilon \frac{(c_1^\varepsilon)'}{2c_1^\varepsilon(r^\varepsilon)^\kappa} (p^\varepsilon \ell^\varepsilon + q^\varepsilon m^\varepsilon) R_2^\varepsilon, \end{aligned} \quad (3.34)$$

and

$$\begin{aligned} &\partial_X[(c_1^\varepsilon + c_2^\varepsilon)q^\varepsilon h^\varepsilon S_2^\varepsilon + \varepsilon c_1^\varepsilon S_2^\varepsilon] + \partial_Y[(c_1^\varepsilon - c_2^\varepsilon)p^\varepsilon g^\varepsilon S_2^\varepsilon + \varepsilon c_1^\varepsilon S_2^\varepsilon] \\ &= \left( \frac{(c_1^\varepsilon)'}{2c_1^\varepsilon(r^\varepsilon)^\kappa} + \frac{(c_2^\varepsilon)'}{4c_2^\varepsilon(r^\varepsilon)^\kappa} \right) [(1 + b^\varepsilon)h^\varepsilon \ell^\varepsilon + (1 - b^\varepsilon)m^\varepsilon g^\varepsilon] p^\varepsilon q^\varepsilon S_2^\varepsilon \\ &\quad - \left( \frac{(a^\varepsilon)'}{2a^\varepsilon(r^\varepsilon)^\kappa} + \frac{(c_2^\varepsilon)'}{4c_2^\varepsilon(r^\varepsilon)^\kappa} \right) [(1 - b^\varepsilon)h^\varepsilon \ell^\varepsilon + (1 + b^\varepsilon)m^\varepsilon g^\varepsilon] p^\varepsilon q^\varepsilon R_2^\varepsilon \\ &\quad + \frac{\kappa c_2^\varepsilon}{r^\varepsilon} p^\varepsilon q^\varepsilon g^\varepsilon h^\varepsilon R_2^\varepsilon + \varepsilon \frac{(c_1^\varepsilon)'}{2c_1^\varepsilon(r^\varepsilon)^\kappa} (p^\varepsilon \ell^\varepsilon + q^\varepsilon m^\varepsilon) S_2^\varepsilon. \end{aligned} \quad (3.35)$$

Combining (3.32)-(3.35) and making use of the div-curl lemma again, we have

$$\begin{aligned}\overline{(c_1 - c_2)qhR_2pg} &= (c_1 - c_2)\overline{qhR_2} \cdot \overline{pg}, \quad \overline{(c_1 + c_2)pgR_2qh} = (c_1 + c_2)\overline{pgR_2} \cdot \overline{qh}, \\ \overline{(c_1 + c_2)qhS_2pg} &= (c_1 + c_2)\overline{qhS_2} \cdot \overline{pg}, \quad \overline{(c_1 - c_2)pgS_2qh} = (c_1 - c_2)\overline{pgS_2} \cdot \overline{qh},\end{aligned}$$

which indicates that

$$\overline{qhR_2} \cdot \overline{pg} = \overline{pgR_2} \cdot \overline{qh} \quad \text{and} \quad \overline{qhS_2} \cdot \overline{pg} = \overline{pgS_2} \cdot \overline{qh}.$$

Thus one can define  $(\overline{R_2}, \overline{S_2})$  as follows:

$$\overline{R_2} := \frac{\overline{qhR_2}}{\overline{qh}} = \frac{\overline{pgR_2}}{\overline{pg}} \quad \text{and} \quad \overline{S_2} := \frac{\overline{qhS_2}}{\overline{qh}} = \frac{\overline{pgS_2}}{\overline{pg}}.$$

It is obvious by the estimates established in previous subsection that  $(\overline{R_2}, \overline{S_2}) \in L_{loc}^\infty(\Omega)$ .

**Lemma 3.1.** *For a.e.  $(X, Y) \in \Omega$ , there hold*

$$\begin{aligned}\overline{(pgR_2^2 - pg \cdot R_2^2)}(X, Y) &= 0, \quad \overline{(qhR_2^2 - qh \cdot R_2^2)}(X, Y) = 0, \\ \overline{(pgS_2^2 - pg \cdot S_2^2)}(X, Y) &= 0, \quad \overline{(qhS_2^2 - qh \cdot S_2^2)}(X, Y) = 0,\end{aligned} \quad (3.36)$$

and

$$\begin{aligned}(\overline{g^2} - \overline{g}^2)(X, Y) &= 0, \quad (\overline{h^2} - \overline{h}^2)(X, Y) = 0, \\ (\overline{p^2} - \overline{p}^2)(X, Y) &= 0, \quad (\overline{q^2} - \overline{q}^2)(X, Y) = 0, \\ (\overline{\ell^2} - \overline{\ell}^2)(X, Y) &= 0, \quad (\overline{m^2} - \overline{m}^2)(X, Y) = 0,\end{aligned} \quad (3.37)$$

*Proof.* We only consider the point  $(X, Y) \in \Omega_2$ . To show the terms of  $R_2$  in (3.36), it is necessary to derive the equations for  $\mathcal{T}_1 := \overline{(pgR_2^2 - pg \cdot R_2^2)}$  and  $\mathcal{T}_2 := \overline{(qhR_2^2 - qh \cdot R_2^2)}$ . We take  $\varepsilon \rightarrow 0^+$  in (3.34) and apply the div-curl lemma to obtain

$$\begin{aligned}& \partial_X [(c_1 - c_2)\overline{qh} \cdot \overline{R_2}] + \partial_Y [(c_1 + c_2)\overline{pg} \cdot \overline{R_2}] \\ &= \left( \frac{c'_1}{2c_1 r^\kappa} + \frac{c'_2}{4c_2 r^\kappa} \right) [(1-b)\overline{p\ell} \cdot \overline{qh} + (1+b)\overline{qm} \cdot \overline{pg}] \overline{R_2} \\ & \quad - \left( \frac{a'}{2ar^\kappa} + \frac{c'_2}{4c_2 r^\kappa} \right) [(1+b)\overline{p\ell} \cdot \overline{qh} + (1-b)\overline{qm} \cdot \overline{pg}] \overline{S_2} - \frac{\kappa c_2}{r} \overline{pg} \cdot \overline{qh} \cdot \overline{S_2}.\end{aligned} \quad (3.38)$$

By convolving (3.38) with the standard Friedrichs' mollifier  $j_\sigma$  and multiplying the resulting by  $j_\sigma * \overline{R_2}$  and then taking  $\sigma \rightarrow 0$ , we deduce

$$\begin{aligned}& \partial_X [(c_1 - c_2)\overline{qh} \cdot \overline{R_2^2}] + \partial_Y [(c_1 + c_2)\overline{pg} \cdot \overline{R_2^2}] \\ &= \left( \frac{c'_1}{2c_1 r^\kappa} (1-b) + \frac{c'_2}{2c_2 r^\kappa} \right) \overline{p\ell} \cdot \overline{qh} \cdot \overline{R_2^2} + \left( \frac{c'_1}{2c_1 r^\kappa} (1+b) + \frac{c'_2}{2c_2 r^\kappa} \right) \overline{qm} \cdot \overline{pg} \cdot \overline{R_2^2} \\ & \quad - \left( \frac{a'}{ar^\kappa} + \frac{c'_2}{2c_2 r^\kappa} \right) [(1+b)\overline{p\ell} \cdot \overline{qh} + (1-b)\overline{qm} \cdot \overline{pg}] \overline{S_2} \cdot \overline{S_2} - \frac{2\kappa c_2}{r} \overline{pg} \cdot \overline{qh} \cdot \overline{S_2} \cdot \overline{R_2}.\end{aligned} \quad (3.39)$$

On the other hand, one multiplies (3.34) by  $2R_2^\varepsilon$  and then taking  $\varepsilon \rightarrow 0^+$  in the resulting to gain by the div-curl lemma

$$\begin{aligned} & \partial_X[(c_1 - c_2)\overline{qhR_2^2}] + \partial_Y[(c_1 + c_2)\overline{pgR_2^2}] \\ &= \left(\frac{c'_1}{2c_1r^\kappa}(1 - b) + \frac{c'_2}{2c_2r^\kappa}\right)\overline{p\ell} \cdot \overline{qhR_2^2} + \left(\frac{c'_1}{2c_1r^\kappa}(1 + b) + \frac{c'_2}{2c_2r^\kappa}\right)\overline{qm} \cdot \overline{pgR_2^2} \\ & \quad - \left(\frac{a'}{ar^\kappa} + \frac{c'_2}{2c_2r^\kappa}\right)[(1 + b)\overline{p\ell qhR_2S_2} + (1 - b)\overline{qgqmR_2S_2}] - \frac{2\kappa c_2}{r}\overline{pg} \cdot \overline{qh} \cdot \overline{S_2} \cdot \overline{R_2}. \end{aligned} \quad (3.40)$$

We subtract (3.39) from (3.40) to achieve

$$\begin{aligned} & \partial_X[(c_1 - c_2)\mathcal{T}_2] + \partial_Y[(c_1 + c_2)\mathcal{T}_1] \\ &= \left(\frac{c'_1}{2c_1r^\kappa}(1 - b) + \frac{c'_2}{2c_2r^\kappa}\right)\overline{p\ell}\mathcal{T}_2 + \left(\frac{c'_1}{2c_1r^\kappa}(1 + b) + \frac{c'_2}{2c_2r^\kappa}\right)\overline{qm}\mathcal{T}_1 \\ & \quad - \left(\frac{a'}{ar^\kappa} + \frac{c'_2}{2c_2r^\kappa}\right)(1 + b)(\overline{p\ell hR_2S_2} - \overline{p\ell} \cdot \overline{qh} \cdot \overline{R_2} \cdot \overline{S_2}) \\ & \quad - \left(\frac{a'}{ar^\kappa} + \frac{c'_2}{2c_2r^\kappa}\right)(1 - b)(\overline{pqmgR_2S_2} - \overline{qm} \cdot \overline{pg} \cdot \overline{R_2} \cdot \overline{S_2}). \end{aligned} \quad (3.41)$$

Putting

$$\mathcal{T}_3 = (\overline{pgS_2^2} - \overline{pg} \cdot \overline{S_2^2}), \quad \mathcal{T}_4 = (\overline{qhS_2^2} - \overline{qh} \cdot \overline{S_2^2}),$$

we also have

$$\begin{aligned} & \partial_X[(c_1 + c_2)\mathcal{T}_4] + \partial_Y[(c_1 - c_2)\mathcal{T}_3] \\ &= \left(\frac{c'_1}{2c_1r^\kappa}(1 + b) + \frac{c'_2}{2c_2r^\kappa}\right)\overline{p\ell}\mathcal{T}_4 + \left(\frac{c'_1}{2c_1r^\kappa}(1 - b) + \frac{c'_2}{2c_2r^\kappa}\right)\overline{qm}\mathcal{T}_3 \\ & \quad - \left(\frac{a'}{ar^\kappa} + \frac{c'_2}{2c_2r^\kappa}\right)(1 - b)(\overline{p\ell hR_2S_2} - \overline{p\ell} \cdot \overline{qh} \cdot \overline{R_2} \cdot \overline{S_2}) \\ & \quad - \left(\frac{a'}{ar^\kappa} + \frac{c'_2}{2c_2r^\kappa}\right)(1 + b)(\overline{pqmgR_2S_2} - \overline{qm} \cdot \overline{pg} \cdot \overline{R_2} \cdot \overline{S_2}). \end{aligned} \quad (3.42)$$

Summing up (3.41) and (3.42) obtains

$$\begin{aligned} & \partial_X[(c_1 - c_2)\mathcal{T}_2 + (c_1 + c_2)\mathcal{T}_4] + \partial_Y[(c_1 + c_2)\mathcal{T}_1 + (c_1 - c_2)\mathcal{T}_3] \\ &= \left(\frac{c'_1}{2c_1r^\kappa} + \frac{c'_2}{2c_2r^\kappa}\right)\overline{p\ell}(\mathcal{T}_2 + \mathcal{T}_4) + \left(\frac{c'_1}{2c_1r^\kappa} + \frac{c'_2}{2c_2r^\kappa}\right)\overline{qm}(\mathcal{T}_1 + \mathcal{T}_3) \\ & \quad + \frac{c'_1b}{2c_1r^\kappa}[\overline{p\ell}(\mathcal{T}_4 - \mathcal{T}_2) + \overline{qm}(\mathcal{T}_1 - \mathcal{T}_3)] - \left(\frac{2a'}{ar^\kappa} + \frac{c'_2}{c_2r^\kappa}\right)(\overline{p\ell hR_2S_2} - \overline{p\ell} \cdot \overline{qh} \cdot \overline{R_2} \cdot \overline{S_2}) \\ & \quad - \left(\frac{2a'}{ar^\kappa} + \frac{c'_2}{c_2r^\kappa}\right)(\overline{pqmgR_2S_2} - \overline{qm} \cdot \overline{pg} \cdot \overline{R_2} \cdot \overline{S_2}). \end{aligned} \quad (3.43)$$

Thanks to the definition of weak  $*$  limit, one can derive

$$\begin{aligned}
 \int_{\Omega} \overline{pg(R_2 - \overline{R_2})^2} \chi \, dXdY &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} p^\varepsilon g^\varepsilon (R_2^\varepsilon - \overline{R_2})^2 \chi \, dXdY \\
 &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} [p^\varepsilon g^\varepsilon R_2^{2\varepsilon} - 2p^\varepsilon g^\varepsilon R_2^\varepsilon \overline{R_2} + p^\varepsilon g^\varepsilon \overline{R_2}^2] \chi \, dXdY \\
 &= \int_{\Omega} [\overline{pgR_2^2} - 2\overline{pgR_2} \cdot \overline{R_2} + \overline{pg} \cdot \overline{R_2}^2] \chi \, dXdY \\
 &= \int_{\Omega} [\overline{pgR_2^2} - \overline{pg} \cdot \overline{R_2}^2] \chi \, dXdY = \int_{\Omega} \mathcal{T}_1 \chi \, dXdY, \tag{3.44}
 \end{aligned}$$

for any test function  $\chi \in C_0^\infty(\Omega)$ , which means that  $T_1 = \overline{pg(R_2 - \overline{R_2})^2} \geq 0$ . Performing similar arguments can yield

$$\begin{aligned}
 T_1 &= \overline{pg(R_2 - \overline{R_2})^2} \geq 0, & T_3 &= \overline{pg(S_2 - \overline{S_2})^2} \geq 0, \\
 T_3 &= \overline{qh(R_2 - \overline{R_2})^2} \geq 0, & T_4 &= \overline{qh(S_2 - \overline{S_2})^2} \geq 0, \\
 \overline{pqh\ell R_2 S_2} - \overline{p\ell} \cdot \overline{qh} \cdot \overline{R_2} \cdot \overline{S_2} &= \overline{pqh\ell(R_2 - \overline{R_2})(S_2 - \overline{S_2})}, \\
 \overline{pqgmR_2 S_2} - \overline{pg} \cdot \overline{qm} \cdot \overline{R_2} \cdot \overline{S_2} &= \overline{pqgm(R_2 - \overline{R_2})(S_2 - \overline{S_2})}.
 \end{aligned} \tag{3.45}$$

Moreover, we see that

$$\begin{aligned}
 |\overline{pqgm(R_2 - \overline{R_2})(S_2 - \overline{S_2})}| &\leq \widehat{K} [\overline{pg(R_2 - \overline{R_2})^2}]^{\frac{1}{2}} \cdot [\overline{pg(S_2 - \overline{S_2})^2}]^{\frac{1}{2}} \leq \widehat{K} (\mathcal{T}_1 + \mathcal{T}_3), \\
 |\overline{pqh\ell(R_2 - \overline{R_2})(S_2 - \overline{S_2})}| &\leq \widehat{K} [\overline{qh(R_2 - \overline{R_2})^2}]^{\frac{1}{2}} \cdot [\overline{qh(S_2 - \overline{S_2})^2}]^{\frac{1}{2}} \leq \widehat{K} (\mathcal{T}_2 + \mathcal{T}_4),
 \end{aligned} \tag{3.46}$$

hold in the domain  $\widehat{\Omega}$  for some positive constant  $\widehat{K}$  depending only on  $(\widehat{X}, \widehat{Y})$ , where  $(\widehat{X}, \widehat{Y})$  is any point in  $\Omega_2$  and  $\widehat{\Omega}$  is the domain bounded by  $\Gamma_0, \Gamma_b, X = \widehat{X}$  and  $Y = \widehat{Y}$ . Let  $(X, Y)$  be any point in  $\widehat{\Omega} \cap \Omega_2$ . We now integrate (3.43) over the domain  $\{(X', Y') \mid Y' \leq X' \leq X, \varphi(X) \leq Y' \leq Y\}$  and utilize (3.45) and (3.46) to conclude that

$$\begin{aligned}
 &\int_{\varphi(X)}^Y (\mathcal{T}_2 + \mathcal{T}_4)(X, Y') \, dY' + \int_Y^X (\mathcal{T}_1 + \mathcal{T}_3)(X', Y) \, dX' \\
 &\leq \widehat{K} \left\{ \int_Y^X \int_{\varphi(X')}^Y + \int_0^Y \int_{\varphi(X')}^{X'} \right\} (\mathcal{T}_2 + \mathcal{T}_4)(X', Y') \, dY' dX' \\
 &\quad + \widehat{K} \left\{ \int_{\varphi(X)}^0 \int_{\varphi^{-1}(Y')}^X + \int_0^Y \int_{Y'}^X \right\} (\mathcal{T}_1 + \mathcal{T}_3)(X', Y') \, dX' dY', \tag{3.47}
 \end{aligned}$$

for some positive constant  $\widehat{K}$  depending only on  $(\widehat{X}, \widehat{Y})$ , where  $\varphi^{-1}$  represents the inverse of  $\varphi$ . Here we used the facts that  $\mathcal{T}_i = 0 (i = 1, \dots, 4)$  a.e. on  $\Gamma_0$  by (2.22) and  $\mathcal{T}_2 = \mathcal{T}_3, \mathcal{T}_1 = \mathcal{T}_4$  a.e. on  $\Gamma_0$  by (2.26). Based on Lemma 3.3 in Zhang and Zheng [37], we obtain by (3.47)

$$\int_{\varphi(X)}^Y (\mathcal{T}_2 + \mathcal{T}_4)(X, Y') \, dY' = \int_Y^X (\mathcal{T}_1 + \mathcal{T}_3)(X', Y) \, dX' = 0, \quad \forall (X, Y) \in \widehat{\Omega} \cap \Omega_2, \tag{3.48}$$

which implies that

$$\mathcal{T}_1(X, Y) = \mathcal{T}_2(X, Y) = \mathcal{T}_3(X, Y) = \mathcal{T}_4(X, Y) = 0, \quad \text{a.e. } (X, Y) \in \widehat{\Omega} \cap \Omega_2. \tag{3.49}$$

Thus it follows by (3.49) and the arbitrariness of  $(\hat{X}, \hat{Y}) \in \Omega_2$  that

$$\begin{aligned} (\overline{pgR_2^2} - \overline{p\bar{g} \cdot R_2^2})(X, Y) &= 0, \quad (\overline{pgS_2^2} - \overline{p\bar{g} \cdot S_2^2})(X, Y) = 0, \\ (\overline{qhR_2^2} - \overline{q\bar{h} \cdot R_2^2})(X, Y) &= 0, \quad (\overline{qhS_2^2} - \overline{q\bar{h} \cdot S_2^2})(X, Y) = 0, \end{aligned} \quad \text{a.e. } (X, Y) \in \Omega_2. \quad (3.50)$$

To show the equalities in (3.37), we perform the same process as above to derive their equations. Let's take the term  $(\bar{\ell}^2 - \overline{\ell^2})$  as an example. Taking  $\varepsilon \rightarrow 0^+$  in the equation of  $\ell^\varepsilon$  in (3.2), we apply the div-curl lemma and (3.50) to arrive at

$$\begin{aligned} \partial_Y \bar{\ell} &= \frac{c'_1}{8c_1^2 r^\kappa} [\bar{q} \cdot \overline{g(1-2g)} - \overline{q\bar{h}(1-2\bar{g})}] + \frac{c'_2}{8c_1 c_2 r^\kappa} \overline{q\bar{h} \cdot g(1-2g)} (\overline{R_2^2} + \overline{S_2^2}) \\ &\quad - \left( \frac{c'_2}{4c_1 c_2 r^\kappa} + \frac{a'}{2ac_1 r^\kappa} \right) \overline{q\bar{h} \cdot g(1-2g)} \cdot \overline{R_2 \cdot S_2} + \frac{\kappa}{2r} \overline{q\bar{m} \cdot g(1-2g)}, \end{aligned}$$

from which one has

$$\begin{aligned} \partial_Y \bar{\ell}^2 &= \frac{c'_1}{4c_1^2 r^\kappa} [\bar{q} \cdot \overline{g(1-2g)} - \overline{q\bar{h}(1-2\bar{g})}] \bar{\ell} + \frac{c'_2}{4c_1 c_2 r^\kappa} (\overline{R_2^2} + \overline{S_2^2}) \overline{q\bar{h} \cdot g(1-2g)} \cdot \bar{\ell} \\ &\quad - \left( \frac{c'_2}{2c_1 c_2 r^\kappa} + \frac{a'}{ac_1 r^\kappa} \right) \overline{R_2 \cdot S_2} \cdot \overline{q\bar{h} \cdot g(1-2g)} \cdot \bar{\ell} + \frac{\kappa}{r} \overline{q\bar{m} \cdot g(1-2g)} \cdot \bar{\ell}. \end{aligned} \quad (3.51)$$

On the other hand, we multiply the equation of  $\ell^\varepsilon$  in (3.2) by  $2\ell^\varepsilon$  and take  $\varepsilon \rightarrow 0$  to obtain by the div-curl lemma

$$\begin{aligned} \partial_Y \bar{\ell}^2 &= \frac{c'_1}{4c_1^2 r^\kappa} [\bar{q} \cdot \overline{g(1-2g)\ell} - \overline{q\bar{h} \cdot (1-2\bar{g})\ell}] + \frac{c'_2}{4c_1 c_2 r^\kappa} (\overline{R_2^2} + \overline{S_2^2}) \overline{q\bar{h} \cdot g(1-2g)\ell} \\ &\quad - \left( \frac{c'_2}{2c_1 c_2 r^\kappa} + \frac{a'}{ac_1 r^\kappa} \right) \overline{R_2 \cdot S_2} \cdot \overline{q\bar{h} \cdot g(1-2g)\ell} + \frac{\kappa}{r} \overline{q\bar{m} \cdot g(1-2g)\ell}. \end{aligned} \quad (3.52)$$

Combining (3.51) and (3.52) gives

$$\begin{aligned} \partial_Y \overline{(\ell - \bar{\ell})^2} &= \frac{c'_1}{2c_1^2 r^\kappa} \overline{q\bar{h}(g - \bar{g})(\ell - \bar{\ell})} - \left\{ \frac{c'_1}{4c_1^2 r^\kappa} \bar{q} + \frac{c'_2}{4c_1 c_2 r^\kappa} (\overline{R_2^2} + \overline{S_2^2}) \overline{q\bar{h}} + \frac{\kappa}{r} \overline{q\bar{m}} \right. \\ &\quad \left. - \left( \frac{c'_2}{2c_1 c_2 r^\kappa} + \frac{a'}{ac_1 r^\kappa} \right) \overline{q\bar{h} \cdot R_2 \cdot S_2} \right\} \left[ \overline{2(g + \bar{g})(g - \bar{g})(\ell - \bar{\ell})} - \overline{(g - \bar{g})(\ell - \bar{\ell})} \right]. \end{aligned} \quad (3.53)$$

Here we used the following relations

$$\begin{aligned} \bar{\ell}^2 - \overline{\ell^2} &= \overline{(\ell - \bar{\ell})^2}, \quad \overline{g\ell} - \overline{g \cdot \bar{\ell}} = \overline{(g - \bar{g})(\ell - \bar{\ell})}, \\ \overline{g(2g - 1)\ell} - \overline{g(2g - 1) \cdot \bar{\ell}} &= \overline{2(g + \bar{g})(g - \bar{g})(\ell - \bar{\ell})} - \overline{(g - \bar{g})(\ell - \bar{\ell})}, \\ \overline{g\ell^2} - \overline{g\bar{\ell} \cdot \bar{\ell}} &= \overline{(\ell + \bar{\ell})(\ell - \bar{\ell})(g - \bar{g})} - \overline{\bar{\ell} \cdot (g - \bar{g})(\ell - \bar{\ell})} + \overline{g \cdot (\ell - \bar{\ell})^2}, \end{aligned}$$

which can be verified similar to (3.44).

By the same arguments, the corresponding equations for other terms in (3.37) can be derived analogously. We add up the obtained equations, integrate the resulting and then use the

boundary conditions to conclude that

$$\begin{aligned} (\overline{g^2} - \overline{g}^2)(X, Y) &= 0, & (\overline{h^2} - \overline{h}^2)(X, Y) &= 0, \\ (\overline{p^2} - \overline{p}^2)(X, Y) &= 0, & (\overline{q^2} - \overline{q}^2)(X, Y) &= 0, & \text{a.e. } (X, Y) \in \widehat{\Omega} \cap \Omega_2. \\ (\overline{\ell^2} - \overline{\ell}^2)(X, Y) &= 0, & (\overline{m^2} - \overline{m}^2)(X, Y) &= 0, \end{aligned} \quad (3.54)$$

The proofs of the above equalities are similar to those in Zhang and Zheng [37] and Hu [26], so we omit them here. Due to the arbitrariness of  $(\hat{X}, \hat{Y})$ , (3.54) is valid for a.e.  $(X, Y) \in \Omega_2$ . The proof of the lemma is complete.  $\square$

**Proof Theorem 3.2.** Based on Lemma 3.1, we know that there exists a subsequence of  $\{g^\varepsilon, h^\varepsilon, p^\varepsilon, q^\varepsilon, \ell^\varepsilon, m^\varepsilon\}$ , still denoted by  $\{g^\varepsilon, h^\varepsilon, p^\varepsilon, q^\varepsilon, \ell^\varepsilon, m^\varepsilon\}$ , such that

$$\{g^\varepsilon, h^\varepsilon, p^\varepsilon, q^\varepsilon, \ell^\varepsilon, m^\varepsilon\} \rightarrow (\overline{g}, \overline{h}, \overline{p}, \overline{q}, \overline{\ell}, \overline{m}) \quad \text{strongly in } L_{\text{loc}}^\theta(\Omega), \quad \forall \theta < \infty. \quad (3.55)$$

Thus there hold by (3.50) and (3.55)

$$\begin{aligned} (\overline{pgR_2^2} - \overline{p} \cdot \overline{g} \cdot \overline{R_2^2})(X, Y) &= 0, & (\overline{pgS_2^2} - \overline{p} \cdot \overline{g} \cdot \overline{S_2^2})(X, Y) &= 0, \\ (\overline{qhR_2^2} - \overline{q} \cdot \overline{h} \cdot \overline{R_2^2})(X, Y) &= 0, & (\overline{qhS_2^2} - \overline{q} \cdot \overline{h} \cdot \overline{S_2^2})(X, Y) &= 0, \end{aligned} \quad \text{a.e. } (X, Y) \in \Omega. \quad (3.56)$$

Moreover, in view of (3.45), (3.46) and (3.55) that

$$\overline{pqhlR_2S_2} = \overline{p} \cdot \overline{q} \cdot \overline{h} \cdot \overline{\ell} \cdot \overline{R_2} \cdot \overline{S_2}, \quad \overline{pqmgR_2S_2} = \overline{p} \cdot \overline{q} \cdot \overline{m} \cdot \overline{g} \cdot \overline{R_2} \cdot \overline{S_2}, \quad (3.57)$$

for almost every  $(X, Y) \in \Omega$ . Now we take  $\varepsilon \rightarrow 0^+$  in system (3.1)-(3.5) and apply (3.55)-(3.57) to see that  $(\overline{g}, \overline{h}, \overline{p}, \overline{q}, \overline{\ell}, \overline{m}, \overline{R_2}, \overline{S_2}, u, v, r)$  thus obtained is indeed a global weak solution of system (2.15)-(2.19) with boundary conditions (2.22), (2.26). Furthermore, taking  $\varepsilon \rightarrow 0^+$  in (3.6)-(3.7) obtains (3.28)-(3.30), which complete the proof of Theorem 3.2. In addition, the continuous dependence of solutions on the initial data follows directly from the compactness and convergence.

#### 4. SOLUTIONS IN THE ORIGINAL VARIABLES

By returning the functions  $(u, v)(X, Y)$  constructed in Section 3 to the original  $(t, r)$  plane, we show in this section the global existence of conservative weak solutions to the initial-boundary value problem (1.7)-(1.9) on the domain  $D$ .

**4.1. Inverse transformation.** We first observe that there hold on bounded domain of the  $X$ - $Y$  plane

- The functions  $p, g$  and  $\ell$  are Lipschitz continuous w.r.t.  $Y$ , measurable w.r.t.  $X$ .
- The functions  $q, h$  and  $m$  are Lipschitz continuous w.r.t.  $X$ , measurable w.r.t.  $Y$ .
- The functions  $R_2$  and  $S_2$  are bounded and measurable in both  $X$  and  $Y$ .
- The functions  $u, v$  and  $r$  are Lipschitz continuous w.r.t. both  $X$  and  $Y$ .

Along with the solving process in Section 3, we obtained the function  $r = r(X, Y)$ . Thus we need to acquire the function  $t = t(X, Y)$  to construct the inverse functions  $X = X(t, r)$ ,  $Y = Y(t, r)$ . Owing to (3.29), it suggests that

$$t_{XY} = t_{YX},$$

which indicates that we can integrate one of the equations  $t_X$  and  $t_Y$  to gain the function  $t = t(X, Y)$ . Hence we have the functions  $(t, r) = (t(X, Y), r(X, Y))$ . It is pointed out that the map

$(X, Y) \mapsto (t, r)$  may not be one-to-one mapping. However, we assert that if  $t(X_1, Y_1) = t(X_2, Y_2)$  and  $r(X_1, Y_1) = r(X_2, Y_2)$  for two points  $(X_1, Y_1)$  and  $(X_2, Y_2)$  in  $\Omega$ , there hold

$$\begin{aligned} u(t(X_1, Y_1), r(X_1, Y_1)) &= u(t(X_2, Y_2), r(X_2, Y_2)), \\ v(t(X_1, Y_1), r(X_1, Y_1)) &= v(t(X_2, Y_2), r(X_2, Y_2)), \end{aligned} \tag{4.1}$$

which mean that the values of  $u$  and  $v$  do not depend on the choice of  $(X, Y)$ . Thus one can choose an arbitrary point  $(X, Y)$  satisfying  $t(X, Y) = t, r(X, Y) = r$  and then define  $u(t, r) := u(X, Y)$  and  $v(t, r) := v(X, Y)$ . To show the above assertion, we divide the analysis into two cases: Case I.  $X_1 \leq X_2, Y_1 \leq Y_2$  and Case II.  $X_1 \leq X_2, Y_1 \geq Y_2$ . We just only discuss Case I, and the Case II can be checked similarly. For Case I, if  $r(X_1, Y_1) = r(X_2, Y_2) = r^* > 1$ , we consider the set  $D_{r^*} := \{(X, Y) : r(X, Y) \leq r^*\}$  and denote its boundary by  $\partial D_{r^*}$  which is a Lipschitz continuous curve in  $\Omega$ . We then construct a Lipschitz continuous curve  $\gamma_1$  connecting points  $(X_1, Y_1)$  and  $(X_2, Y_2)$ , which consists a horizontal segment  $Y \equiv Y_1, \partial D_{r^*}$  and a vertical segment  $X \equiv X_2$ . Due to the monotonicity of  $r$  and  $t$  by (2.19)-(2.20), one knows that there hold  $x(X, Y) \equiv x(X_1, Y_1)$  and  $t(X, Y) \equiv t(X_1, Y_1)$  on  $\gamma_1$ . Thus it suggests that  $pgdX = qhdY = 0$  and then  $p\ell dX = qmdY = 0$  along  $\Gamma_1$ . Then we calculate

$$u(t(X_2, Y_2), r(X_2, Y_2)) - u(t(X_1, Y_1), r(X_1, Y_1)) = \int_{\gamma_1} \frac{p\ell}{2c_1 r^\kappa} dX + \frac{qm}{2c_1 r^\kappa} dY = 0,$$

and

$$\begin{aligned} &v(t(X_2, Y_2), r(X_2, Y_2)) - v(t(X_1, Y_1), r(X_1, Y_1)) \\ &= \int_{\gamma_1} \frac{pg}{4ac_2 r^\kappa} [(1+b)R_2 - (1-b)S_2] dX + \frac{qh}{4ac_2 r^\kappa} [(1+b)S_2 - (1-b)R_2] dY = 0, \end{aligned}$$

by the boundedness of  $R_2$  and  $S_2$  on the bounded domain. If  $r(X_1, Y_1) = r(X_2, Y_2) = 1$ , we acquire  $pgdX = 0$  and then  $p\ell dX = 0$  on  $Y = Y_1$  ( $Y_1 \leq X \leq X_1$ ),  $qhdY = 0$  and then  $qmdY = 0$  on  $X = X_2$  ( $Y_2 \leq Y \leq X$ ). By a direct calculation, one applies the boundary conditions on  $\Gamma_b$  to obtain

$$\begin{aligned} u(t(X_1, Y_1), r(X_1, Y_1)) &= u(t(X_1, Y_1), r(X_1, Y_1)) - u(t(Y_1, Y_1), r(Y_1, Y_1)) \\ &= \int_{Y_1}^{X_1} \frac{p\ell}{2c_1 r^\kappa}(X, Y_1) dX = 0, \end{aligned}$$

and

$$\begin{aligned} u(t(X_2, Y_2), r(X_2, Y_2)) &= u(t(X_2, Y_2), r(X_2, Y_2)) - u(t(X_2, X_2), r(X_2, X_2)) \\ &= \int_{X_2}^{Y_2} \frac{qm}{2c_1 r^\kappa}(X_2, Y) dY = 0. \end{aligned}$$

Hence we still have  $u(t(X_1, Y_1), r(X_1, Y_1)) = u(t(X_2, Y_2), r(X_2, Y_2))$ . The equality of  $v$  can be arrived similarly.

Finally, we point out that the functions  $(u(t, r), v(t, r))$  constructed above are Hölder continuous with exponent  $1/2$  on bounded sets and all characteristic curves are  $C^1$  with Hölder continuous derivative. These conclusions can be directly gained by the boundedness of the following terms

$$\int_0^\tau [r^\kappa (u_t \pm c_1(u)u_r)]^2 dt \leq C_\tau, \quad \int_0^\tau [r^\kappa a(u)(v_t \pm c_2(u)v_r)]^2 dt \leq C_\tau, \tag{4.2}$$

for some constant  $C_\tau$  depending only on  $\tau$ . Moreover, it suggests by (4.2) that the functions  $R_1, R_2, S_1$  and  $S_2$  at (2.1) are square integrable on bounded subsets of the  $t$ - $r$  plane. In addition, we also have

$$r^\kappa [u_t + c_1(u)u_r] = r^\kappa \cdot 2c_1 X_r u_X = r^\kappa \cdot 2c_1 \cdot \frac{1}{pg} \cdot \frac{p\ell}{2c_1 r^\kappa} = \frac{\ell}{g} = R_1,$$

$$r^\kappa [u_t - c_1(u)u_r] = r^\kappa \cdot (-2c_1 Y_r u_Y) = r^\kappa \cdot 2c_1 \cdot \frac{1}{qh} \cdot \frac{qm}{2c_1 r^\kappa} = \frac{m}{h} = S_1,$$

which mean that the functions  $R_1$  and  $S_1$  at (2.1) are indeed the same as recovered from (2.11).

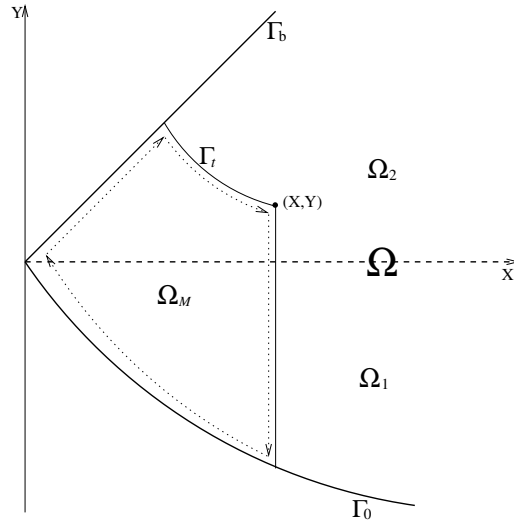


FIGURE 2. The region  $\Omega_M$ .

4.2. **The energy estimate.** In this subsection, we shall derive the energy estimate (1.14), that is

$$\mathcal{E}(t) = \frac{1}{2} \int_1^\infty \left( r^{2\kappa} [u_t^2(t, r) + c_1^2 u_r^2(t, r)] + r^{2\kappa} a^2 [v_t^2(t, r) + c_2^2 v_r^2(t, r)] \right) dr \leq \mathcal{E}_0. \tag{4.3}$$

Let  $t > 0$  be any fixed time and  $M \gg 1$  be a constant. We use  $\Gamma_t \subset \Omega$  to express the transformation of the horizontal segment of  $t$  with  $r \in [1, \infty)$  in the  $t$ - $r$  plane. Denote  $(X_b, Y_b)$   $(X_M, Y_M)$  are the corresponding points of the points  $(t, 1)$  and  $(t, M)$ , respectively. Let  $\Omega_M$  be the domain bounded by  $\Gamma_0, \Gamma_b, \Gamma_t$  and  $X = X_M$ . See Figure 2 for the illustration. We integrate the energy equation (3.30) on  $\Omega_M$  and employ Green's theorem to acquire

$$\begin{aligned} & \int_{\Gamma_t \cap \{X \leq X_M\}} e_1 dX - e_2 dY + \int_{\varphi(X_M)}^{Y_M} e_2 dY + \int_{\Gamma_b \cap \{X \leq X_M\}} e_1 dX - e_2 dY \\ &= \int_{\Gamma_0 \cap \{X \leq X_M\}} e_1 dX - e_2 dY, \end{aligned} \tag{4.4}$$

where

$$e_1 = p + \frac{1}{2} pg [(1+b)R_2^2 + (1-b)S_2^2], \quad e_2 = q + \frac{1}{2} qh [(1-b)R_2^2 + (1+b)S_2^2].$$

According to the boundary conditions in (2.26), we see that  $e_1 = e_2$  and  $dX = dY$  on  $\Gamma_b$ , from which and (4.4) we obtain

$$\begin{aligned} & \int_{\Gamma_t \cap \{X \leq X_M\}} e_1 dX - e_2 dY \leq \int_{\Gamma_0 \cap \{X \leq X_M\}} e_1 dX - e_2 dY \\ &= \frac{1}{2} \int_1^{r_M} \left( r^{2\kappa} [u_1^2(r) + c_1^2(u_0(r))u_{0r}^2] + r^{2\kappa} a^2(u_0(r)) [v_1^2(r) + c_2^2(u_0(r))v_{0r}^2(r)] \right) dr \\ &\leq \mathcal{E}_0, \end{aligned} \tag{4.5}$$

where  $r_M = r(X_M, \varphi(X_M))$ . On the other hand, we apply the following formulas

$$dt = \frac{pg}{2c_1} dX + \frac{qh}{2c_1} dY, \quad dr = \frac{pg}{2} dX - \frac{qh}{2} dY,$$

to compute

$$\begin{aligned} & \frac{1}{2} \int_1^M \left( r^{2\kappa} [u_t^2(t, r) + c_1^2 u_r^2(t, r)] + r^{2\kappa} a^2 [v_t^2(t, r) + c_2^2 v_r^2(t, r)] \right) dr \\ &= \int_{\Gamma_t \cap \{g \neq 0, X \leq X_M\}} e_1 dX - \int_{\Gamma_t \cap \{h \neq 0, X \leq X_M\}} e_2 dY \\ &\leq \int_{\Gamma_t \cap \{X \leq X_M\}} e_1 dX - e_2 dY, \end{aligned} \tag{4.6}$$

which together with (4.5) to obtain

$$\frac{1}{2} \int_1^M \left( r^{2\kappa} [u_t^2(t, r) + c_1^2 u_r^2(t, r)] + r^{2\kappa} a^2 [v_t^2(t, r) + c_2^2 v_r^2(t, r)] \right) dr \leq \mathcal{E}_0. \tag{4.7}$$

One lets  $M \rightarrow \infty$  in (4.7) to arrive at (4.3).

Based on the energy estimate (4.3), we can verify the Lipschitz continuity with respect to the  $L^2$  distance for  $(u, v)(t, r)$ , that is (1.12) holds. For any  $t, s \in [0, \infty)$ , one first has

$$u(t, r) - u(s, r) = (t - s) \int_0^1 u_t(s + \xi(t - s), r) d\xi,$$

for some number  $\xi$ , from which and (4.3) we acquire for any  $M > 1$

$$\begin{aligned} \|u(t, r) - u(s, r)\|_{L^2([1, \infty))} &\leq |t - s| \int_0^1 \|u_t(s + \xi(t - s), \cdot)\|_{L^2([1, \infty))} d\xi \\ &\leq \sqrt{2\mathcal{E}_0} |t - s|. \end{aligned} \tag{4.8}$$

The check for the function  $v$  is similar.

**4.3. The proof of (1.13).** This subsection is devoted to showing that the functions  $(u, v)(t, r)$  constructed in Subsection 4.1 satisfies system (1.7) in the distributional sense, that is (1.13) holds.

Let  $\phi$  be an any function in  $\mathcal{F}$ . For the equation  $u$ , we first have by (2.2), (2.10), (2.11) and (2.14)

$$\begin{aligned} \phi_t u_t - (\phi c_1)_r c_1 u_r &= (c_1 \phi_X X_r - c_1 \phi_Y Y_r) \frac{R_1 + R_2}{2r^\kappa} \\ &\quad - (c_1 \phi_X X_r + c_1 \phi_Y Y_r + \phi c'_1 u_X X_r + \phi c'_1 u_Y Y_r) \frac{R_1 - S_1}{2r^\kappa} \\ &= \frac{c_1 S_1}{r^\kappa} X_r \phi_X - \frac{c_1 R_1}{r^\kappa} Y_r \phi_Y - \phi c'_1 (u_X X_r + u_Y Y_r) \frac{R_1 - S_1}{2r^\kappa} \\ &= \frac{c_1 m}{r^\kappa pgh} \phi_X + \frac{c_1 \ell}{r^\kappa qgh} \phi_Y - \phi \frac{c'_1}{4c_1 r^{2\kappa}} \frac{g + h - 2(gh + \ell m)}{gh}, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &\frac{2\kappa c_1^2(u)u_r}{r} + a(u)a'(u)[v_t^2 - c_2^2(u)v_r^2] - a^2(u)c_2(u)c'_2(u)v_r^2 \\ &= \frac{\kappa c_1}{r^{\kappa+1}} \frac{\ell h - gm}{gh} - \frac{Q}{r^{2\kappa}}. \end{aligned} \tag{4.10}$$

Combining (4.9)-(4.10) and applying the Jacobian

$$\frac{\partial(r,t)}{\partial(X,Y)} = \frac{pqgh}{2c_1}, \tag{4.11}$$

one obtains

$$\begin{aligned} &\int_0^\infty \int_1^\infty \left\{ \phi_t u_t - (\phi c_1)_r c_1 u_r \right. \\ &\quad \left. + \phi \left( \frac{2\kappa c_1^2(u)u_r}{r} + a(u)a'(u)[v_t^2 - c_2^2(u)v_r^2] - a^2(u)c_2(u)c'_2(u)v_r^2 \right) \right\} dr dt \\ &= \iint_\Omega \left\{ \frac{qm}{2r^\kappa} \phi_X + \frac{p\ell}{2r^\kappa} \phi_Y \right. \\ &\quad \left. + \phi pq \left( \frac{\kappa}{2r^{\kappa+1}} (\ell h - gm) - \frac{Q}{2c_1 r^{2\kappa}} gh - \frac{c'_1}{8c_1^2 r^{2\kappa}} [g + h - 2(gh + \ell m)] \right) \right\} dXdY. \end{aligned} \tag{4.12}$$

Note by (2.16)-(2.17) that

$$\begin{aligned} &\frac{qm}{2r^\kappa} \phi_X + \frac{p\ell}{2r^\kappa} \phi_Y = \left( \frac{qm}{2r^\kappa} \phi \right)_X + \left( \frac{p\ell}{2r^\kappa} \phi \right)_Y - \phi \left\{ \left( \frac{qm}{2r^\kappa} \right)_X + \left( \frac{p\ell}{2r^\kappa} \right)_Y \right\} \\ &= \left( \frac{qm}{2r^\kappa} \phi \right)_X + \left( \frac{p\ell}{2r^\kappa} \phi \right)_Y - \phi \left\{ \frac{\kappa pq(\ell h - gm)}{4r^{\kappa+1}} + \frac{(qm)_X + (p\ell)_Y}{2r^\kappa} \right\}, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} (qm)_X + (p\ell)_Y &= \frac{pq}{c_1} \left\{ \frac{c'_1}{8c_1 r^\kappa} [-h - g + 2(gh + \ell m)] + \frac{\kappa c_1}{2r} h\ell - \frac{Q}{2r^\kappa} gh \right\} \\ &\quad + \frac{pq}{c_1} \left\{ \frac{c'_1}{8c_1 r^\kappa} [-h - g + 2(gh + \ell m)] - \frac{\kappa c_1}{2r} gm - \frac{Q}{2r^\kappa} gh \right\} \\ &= \frac{pq}{c_1} \left\{ \frac{c'_1}{4c_1 r^\kappa} [-h - g + 2(gh + \ell m)] + \frac{\kappa c_1}{2r} (h\ell - gm) - \frac{Q}{r^\kappa} gh \right\}. \end{aligned} \tag{4.14}$$

Putting (4.13) and (4.14) into (4.12) and utilizing Green's theorem and the boundary conditions in (2.26) yields

$$\begin{aligned}
 & \int_0^\infty \int_1^\infty \left\{ \phi_t u_t - (\phi c_1)_r c_1 u_r \right. \\
 & \quad \left. + \phi \left( \frac{2\kappa c_1^2(u) u_r}{r} + a(u) a'(u) [v_t^2 - c_2^2(u) v_r^2] - a^2(u) c_2(u) c_2'(u) v_r^2 \right) \right\} dr dt \\
 &= \iint_\Omega \left( \frac{qm}{2r^\kappa} \phi \right)_X + \left( \frac{p\ell}{2r^\kappa} \phi \right)_Y dXdY = \int_{\Gamma_b} \phi \frac{p\ell}{2r^\kappa} dX - \phi \frac{qm}{2r^\kappa} dY \\
 &= \int_{\Gamma_b} \phi \frac{\ell h - mg}{4r^\kappa gh} pg dX + \phi \frac{\ell h - mg}{2r^\kappa gh} qh dY = \int_0^\infty \phi c_1^2 u_r(t, 1) dt. \tag{4.15}
 \end{aligned}$$

Thus the equation of  $u$  in (1.13) is proved.

For the equation of  $v$ , we deduce

$$\begin{aligned}
 & \phi_t a^2 v_t - \phi_r a^2 c_2^2 v_r \\
 &= (c_1 X_r \phi_X - c_1 Y_r \phi_Y) a^2 \frac{R_2 + S_2}{2r^\kappa a} - (X_r \phi_X + Y_r \phi_Y) a^2 c_2^2 \frac{R_2 - S_2}{2r^\kappa a c_2} \\
 &= \frac{a}{2r^\kappa pg} [(c_1 - c_2)R_2 + (c_1 + c_2)S_2] \phi_X + \frac{a}{2r^\kappa qh} [(c_1 + c_2)R_2 + (c_1 - c_2)S_2] \phi_Y,
 \end{aligned}$$

from which and (4.11) one has

$$\begin{aligned}
 & \int_0^\infty \int_1^\infty \left\{ \phi_t a^2(u) v_t - \phi_r a^2(u) c_2^2(u) v_r + \phi \frac{2\kappa a^2(u) c_2^2(u) v_r}{r} \right\} dr dt \\
 &= \iint_\Omega \left\{ \frac{a}{4c_1 r^\kappa} [(c_1 - c_2)R_2 + (c_1 + c_2)S_2] qh \phi_X \right. \\
 & \quad \left. + \frac{a}{4c_1 r^\kappa} [(c_1 + c_2)R_2 + (c_1 - c_2)S_2] pg \phi_Y + \phi \frac{\kappa a c_2}{4c_1 r^{\kappa+1}} pqgh(R_2 - S_2) \right\} dXdY. \tag{4.16}
 \end{aligned}$$

Moreover, it follows by (2.18) that

$$\begin{aligned}
 & \left( [(c_1 - c_2)R_2 + (c_1 + c_2)S_2] qh \right)_X + \left( [(c_1 + c_2)R_2 + (c_1 - c_2)S_2] pg \right)_Y \\
 &= \left( \frac{c_1'}{2c_1 r^\kappa} - \frac{a'}{2ar^\kappa} \right) pq \{ [(1 - b)h\ell + (1 + b)mg]R_2 + [(1 + b)h\ell + (1 - b)mg]S_2 \} \\
 & \quad + \frac{\kappa c_2}{r} pqgh(R_2 - S_2), \tag{4.17}
 \end{aligned}$$

from which we gain

$$\begin{aligned}
& \left( \frac{a}{4c_1 r^\kappa} [(c_1 - c_2)R_2 + (c_1 + c_2)S_2]qh \right)_X + \left( \frac{a}{4c_1 r^\kappa} [(c_1 + c_2)R_2 + (c_1 - c_2)S_2]pg \right)_Y \\
&= \frac{a}{4c_1 r^\kappa} \left( \frac{c'_1}{2c_1 r^\kappa} - \frac{a'}{2ar^\kappa} \right) pq \{ [(1-b)hl + (1+b)mg]R_2 + [(1+b)hl + (1-b)mg]S_2 \} \\
& \quad + \frac{a}{4c_1 r^\kappa} \frac{\kappa c_2}{r} pqgh(R_2 - S_2) \\
& \quad + [(c_1 - c_2)R_2 + (c_1 + c_2)S_2]qh \left( \frac{a'}{4c_1 r^\kappa} \frac{p\ell}{2c_1 r^\kappa} - \frac{ac'_1}{4c_1^2 r^\kappa} \frac{p\ell}{2c_1 r^\kappa} - \frac{\kappa a}{4c_1 r^{\kappa+1}} \frac{pg}{2} \right) \\
& \quad + [(c_1 + c_2)R_2 + (c_1 - c_2)S_2]pg \left( \frac{a'}{4c_1 r^\kappa} \frac{qm}{2c_1 r^\kappa} - \frac{ac'_1}{4c_1^2 r^\kappa} \frac{qm}{2c_1 r^\kappa} + \frac{\kappa a}{4c_1 r^{\kappa+1}} \frac{qh}{2} \right) \\
&= \frac{\kappa ac_2}{4c_1 r^{\kappa+1}} pqgh(R_2 - S_2). \tag{4.18}
\end{aligned}$$

Thus there holds by (4.18)

$$\begin{aligned}
& \frac{a}{4c_1 r^\kappa} [(c_1 - c_2)R_2 + (c_1 + c_2)S_2]qh\phi_X + \frac{a}{4c_1 r^\kappa} [(c_1 + c_2)R_2 + (c_1 - c_2)S_2]pg\phi_Y \\
&= \left( \frac{a}{4c_1 r^\kappa} [(c_1 - c_2)R_2 + (c_1 + c_2)S_2]qh\phi \right)_X \\
& \quad + \left( \frac{a}{4c_1 r^\kappa} [(c_1 + c_2)R_2 + (c_1 - c_2)S_2]pg\phi \right)_Y - \phi \frac{\kappa ac_2}{4c_1 r^{\kappa+1}} pqgh(R_2 - S_2). \tag{4.19}
\end{aligned}$$

Inserting (4.19) into (4.16), we find by Green's theorem and the boundary conditions in (2.26) that

$$\begin{aligned}
& \int_0^\infty \int_1^\infty \left\{ \phi_t a^2(u) v_t - \phi_r a^2(u) c_2^2(u) v_r + \phi \frac{2\kappa a^2(u) c_2^2(u) v_r}{r} \right\} dr dt \\
&= \iint_\Omega \left\{ \left( \frac{a}{4c_1 r^\kappa} [(c_1 - c_2)R_2 + (c_1 + c_2)S_2]qh\phi \right)_X \right. \\
& \quad \left. + \left( \frac{a}{4c_1 r^\kappa} [(c_1 + c_2)R_2 + (c_1 - c_2)S_2]pg\phi \right)_Y \right\} dX dY \\
&= \int_{\Gamma_b} \frac{a}{4c_1 r^\kappa} [(c_1 + c_2)R_2 + (c_1 - c_2)S_2]pg\phi dX \\
& \quad - \frac{a}{4c_1 r^\kappa} [(c_1 - c_2)R_2 + (c_1 + c_2)S_2]qh\phi dY \\
&= \int_{\Gamma_b} \frac{\phi ac_2}{2r^\kappa} (R_2 - S_2) \left( \frac{pg}{2c_1} dX + \frac{qh}{2c_1} dY \right) = \int_0^\infty \phi a^2 c_2^2 v_r(t, 1) dt, \tag{4.20}
\end{aligned}$$

which is the equation of  $v$  in (1.13).

Finally, we can verify that the functions  $t \mapsto (u, v)(t, \cdot)$  are continuously differentiable as maps with values in  $L^\theta$ , for all  $1 \leq \theta < 2$ . Moreover, for any fixed time  $t$ , we define

$$\mu_t([1, \infty)) = \mu_t^-([1, \infty)) + \mu_t^+([1, \infty)),$$

where

$$\begin{aligned} \mu_t^-([1, \infty)) &= \int_{\Gamma_t} p + \frac{1}{2}pg[(1+b)R_2^2 + (1-b)S_2^2] \, dX, \\ \mu_t^+([1, \infty)) &= - \int_{\Gamma_t} q + \frac{1}{2}qh[(1-b)R_2^2 + (1+b)S_2^2] \, dY. \end{aligned}$$

One can show by the energy equation in (2.21) that  $\mu_t([1, \infty)) = \mathcal{E}_0$  at every time  $t$ . Moreover, for each  $t$ , the absolutely continuous part of  $\mu_t$  has density

$$\frac{1}{2}r^{2\kappa}[u_t^2(t, r) + c_1^2u_r^2(t, r)] + \frac{1}{2}r^{2\kappa}a^2[v_t^2(t, r) + c_2^2v_r^2(t, r)]$$

with respect to the Lebesgue measure, while for almost every  $t$ , the singular part of  $\mu_t$  is concentrated on the set where  $c'_1 = 0$ . This means that the total energy characterized by the measure  $\mu$  is conserved in time and the energy may concentrate on a set of times of zero measure or at points where  $c'_1 = 0$ . The proofs of the above properties of the solution are identical to that in Bressan and Zheng [8], Zhang and Zheng [36, 37] and Hu [23, 25], so we omit them here.

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