

DEFERRED CESÀRO FIBONACCI STATISTICAL CONVERGENCE AND KOROVKIN-TYPE APPROXIMATION THEOREMS

H.M. SRIVASTAVA^{1,2,3,4,5,6,*}, BIDU BHUSAN JENA⁷, SUSANTA KUMAR PAIKRAY⁸, PRIYADARSINI
PARIDA⁹

¹*Department of Mathematics and Statistics, University of Victoria, Victoria, V8W 3R4, Canada*

²*Department of Medical Research, China Medical University Hospital, Taichung 40402, Taiwan*

³*Center for Converging Humanities, Kyung Hee University, Seoul 02447, Korea*

⁴*Department of Applied Mathematics, Chung Yuan Christian University, Taoyuan City 320314, Taiwan*

⁵*Department of Mathematics and Informatics, Azerbaijan University, Baku AZ1007, Azerbaijan*

⁶*Section of Mathematics, International Telematic University Uninettuno,*

39 Corso Vittorio Emanuele II, Rome I-00186, Italy

⁷*Section of Mathematics, Faculty of Science, Sri Sri University, Cuttack 754006, India*

⁸*Department of Mathematics, Veer Surendra Sai University of Technology, Burla 768018, India*

⁹*Department of Mathematics, Kuntala Kumari Sabat Women's College, Balasore 756003, India*

Abstract. In this paper, we introduce and investigate the notions of deferred Cesàro statistical convergence and deferred Cesàro Fibonacci statistical convergence within the framework of real sequence spaces, by establishing an inclusion relation between these two notions. Upon building on these foundations, we prove a new Korovkin-type approximation theorem for sequences of positive linear operators acting on trigonometric polynomials in $\mathcal{C}_{2\pi}(\mathbb{R})$, where convergence is considered in the sense of deferred Cesàro statistical convergence generated by Fibonacci sequences. The proposed theorem is shown to provide a genuine extension of several previously known Korovkin-type approximation results. Furthermore, we investigate the rate of deferred Cesàro Fibonacci statistical convergence for the same class of 2π -periodic functions. Finally, we show how the theoretical findings are supported by some illustrative examples that highlight the scope and applicability of the proposed approach in approximation theory.

Keywords. Deferred Cesàro Fibonacci statistical convergence; Fibonacci matrix; Statistical convergence; Korovkin-type approximation theorem; Measures the relative frequency; Rate of convergence.

1. INTRODUCTION AND MOTIVATION

Let E be a subset of the positive integers. For each $n \in \mathbb{N}$, consider the initial segment $[1, n]$, and select an integer uniformly at random from this set. The ratio $\frac{|E \cap [1, n]|}{n}$ measures the relative frequency of the elements of E among the first n positive integers. If this ratio converges to a limit as $n \rightarrow \infty$, the limit is called the asymptotic density of E . This quantity may be interpreted as the probability that a randomly chosen positive integer belongs to E .

*Corresponding author.

E-mail address: harimsri@uvic.ca (H.M. Srivastava).

Received 9 November 2025; Accepted 1 June 2026; Published online 10 July 2026.

Before proceeding to our main constructions, we recall some fundamental terminology related to asymptotic density. Let \mathbb{Z}^+ denote the set of all positive integers. For subsets $E, F \subseteq \mathbb{Z}^+$, the sets E and F are said to be asymptotically equivalent (denoted $E \sim F$) if their symmetric difference $E\Delta F$ is finite. In 1981, Freedman and Sember [8] further refined this framework by introducing the notions of lower asymptotic density and convergence in density, which have provided a fundamental basis for various extensions in summability and approximation theory.

Definition 1.1. (see [8]) Let the function g be defined on all subsets of the set of natural numbers, which take on values in the interval $[0, 1]$. The function g is called a *lower asymptotic density* if it satisfies the following conditions:

- (i) $g(E) = g(F)$ whenever $E \sim F$;
- (ii) If $E \cap F = \emptyset$, then $g(E) + g(F) \leq g(E \cup F)$;
- (iii) For all subsets $E, F \subseteq \mathbb{Z}^+$, $g(E) + g(F) \leq 1 + g(E \cap F)$;
- (iv) $g(\mathbb{Z}^+) = 1$.

Corresponding to the notion of lower asymptotic density, we define the following concept of upper asymptotic density.

Definition 1.2. Let the function \bar{g} be a function defined on all subsets of the set of natural numbers, which takes on values in the interval $[0, 1]$. The function \bar{g} is called an *upper asymptotic density* if it satisfies the following conditions:

- (i) $\bar{g}(E) = \bar{g}(F)$ whenever $E \sim F$;
- (ii) If $E \cap F = \emptyset$, then $\bar{g}(E) + \bar{g}(F) \geq \bar{g}(E \cup F)$;
- (iii) For all subsets $E, F \subseteq \mathbb{Z}^+$, $\bar{g}(E) + \bar{g}(F) \geq \bar{g}(E \cap F)$;
- (iv) $\bar{g}(\mathbb{Z}^+) = 1$.

The functions g and \bar{g} serve as, respectively, the lower and upper bounds for the asymptotic behavior of subsets of \mathbb{Z}^+ . A set $E \subseteq \mathbb{Z}^+$ is said to possess a *natural density* with respect to g if and only if $g(E) = \bar{g}(E)$. The term asymptotic density is commonly used for the function $d(E) = \lim_{n \rightarrow \infty} \inf \frac{E(n)}{n}$, where $E \subset \mathbb{N}$ and $E(n) = \sum_{a \leq n, a \in E} 1$. Equivalently, the natural density of E can be expressed as $d(E) = \lim_{n \rightarrow \infty} n^{-1} |E(n)|$, where $|E(n)|$ denotes the number of elements of E not exceeding n . The notion of statistical convergence was initiated by Fast [7] and further investigated by Schoenberg [31], who interpreted it as a summability method and established some of its basic properties. These works demonstrated that every bounded statistically convergent sequence is also Cesàro summable to the same limit. Statistical convergence can likewise be regarded as a particular form of convergence in density [4]. Zygmund [40] referred to this phenomenon as almost convergence and discussed its relation with strong summability. Since then, statistical convergence has been widely explored in several mathematical domains, including number theory [6], trigonometric series [40], summability theory [8], measure theory [25], and Hausdorff locally convex spaces [24]. Subsequently, Aktuglu [2] introduced the concept of $\alpha\beta$ -statistical convergence, which was later generalized by Karakaya and Karaisa [20] through the notions of weighted $\alpha\beta$ -statistical convergence of order γ , weighted $\alpha\beta$ -summability of order γ , and strong weighted $\alpha\beta$ -summability of order γ . Braha [3] later proposed weighted equi-statistical convergence and derived a corresponding Korovkin-type theorem. Further developments include the work of Parida et al. [28], who established a Korovkin-type theorem based on deferred Nörlund means, and the introduction

of statistical deferred weighted Riemann summability with fuzzy approximation theorems by Parida et al. [29]. In 2024, Jena et al. [16] studied statistical Riemann–Stieltjes integrability and deferred Cesàro summability, while Dutta et al. [5] analyzed statistical convergence criteria for martingales via deferred Cesàro means. More recently, Srivastava et al. [36, 39] proved Korovkin-type theorems for positive linear operators based on statistical derivatives of deferred Cesàro summability, including extensions to double sequences. In 2025, Jena et al. [17] introduced statistical gauge integrability and derived corresponding Korovkin approximation results. Most recently, Satapathy et al. [30] proved a new class of Korovkin-type approximation theorems using equi-statistical convergence of double sequences.

Motivated by the aforementioned investigations, we introduce and study the notions of deferred Cesàro statistical convergence and its Fibonacci counterpart within the class of real sequence spaces, and we establish an inclusion relation these two concepts. Based on this analytical foundation, we establish a new Korovkin-type approximation theorem for sequences of positive linear operators acting on trigonometric polynomials in $\mathfrak{C}_{2\pi}(\mathbb{R})$, where convergence is considered in the sense of deferred Cesàro statistical convergence induced by Fibonacci sequences. We further demonstrate that this result provides a genuine extension of several previously known Korovkin-type approximation theorems. Moreover, we examine the rate of deferred Cesàro Fibonacci statistical convergence for the same family of 2π -periodic functions. Finally, we present illustrative examples that verify and clarify the applicability of the established approximation results.

2. A SET OF PRELIMINARIES

The numbers appearing in the lower rows of the well-known Fibonacci triangle are called the Fibonacci numbers. The resulting sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$ is referred to as the Fibonacci sequence [23].

Definition 2.1. The *Fibonacci sequence* (g_n) , for $n = 1, 2, 3, \dots$, is defined by the linear recurrence relation $g_n = g_{n-1} + g_{n-2}$, ≥ 2 , with the initial terms $(g_1) = 1$ and $(g_2) = 1$.

From the above recurrence relation, the initial terms of the Fibonacci sequence are typically defined either as $g_1 = g_2 = 1$ or, equivalently, $g_0 = 0$ and $g_1 = 1$. Every subsequent term in the sequence is obtained by summing the two preceding terms. The Fibonacci sequence was introduced to the Western world by Leonardo of Pisa, better known as Fibonacci, in his book *Liber Abaci*, published in 1202. Nevertheless, the concept itself existed long before Fibonacci's work. Similar numerical patterns appeared earlier in Indian mathematics, where they were known as the *Virahanka numbers* [11]. In *Liber Abaci*, the sequence begins with 1, although in contemporary notation it is usually initiated with either $g_0 = 0$ or $g_1 = 1$. Let g_n represent the n^{th} Fibonacci number for each $n \in \mathbb{N}$. The infinite matrix $\widehat{F} = (\widehat{g}_{nk})$ is then defined [18] as

$$\widehat{g}_{nk} = \begin{cases} -\frac{g_{n+1}}{g_n}, & \text{if } k = n - 1, \\ \frac{g_n}{g_{n+1}}, & \text{if } k = n, \\ 0, & \text{if } 0 \leq k < n - 1 \text{ or } k > n. \end{cases}$$

We use the convention $g_0 = 0, g_1 = 1$ so that $g_2 = 1, g_3 = 2, g_4 = 3, g_5 = 5, g_6 = 8, \dots$. The matrix $\widehat{F} = (\widehat{g}_{nk})$ is defined by

$$\widehat{g}_{nk} = \begin{cases} -\frac{g_{n+1}}{g_n}, & k = n - 1, \\ \frac{g_n}{g_{n+1}}, & k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for $n = 1, 2, 3, 4, 5$, we obtain the following truncated matrix:

$$\widehat{F} = \begin{pmatrix} \frac{g_1}{g_2} & 0 & 0 & 0 & 0 & \dots \\ -\frac{g_3}{g_2} & \frac{g_2}{g_3} & 0 & 0 & 0 & \dots \\ 0 & -\frac{g_4}{g_3} & \frac{g_3}{g_4} & 0 & 0 & \dots \\ 0 & 0 & -\frac{g_5}{g_4} & \frac{g_4}{g_5} & 0 & \dots \\ 0 & 0 & 0 & -\frac{g_6}{g_5} & \frac{g_5}{g_6} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Upon substituting the numerical Fibonacci values, we find that

$$\widehat{F} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ -2 & \frac{1}{2} & 0 & 0 & 0 & \dots \\ 0 & -\frac{3}{2} & \frac{2}{3} & 0 & 0 & \dots \\ 0 & 0 & -\frac{5}{3} & \frac{3}{5} & 0 & \dots \\ 0 & 0 & 0 & -\frac{8}{5} & \frac{5}{8} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

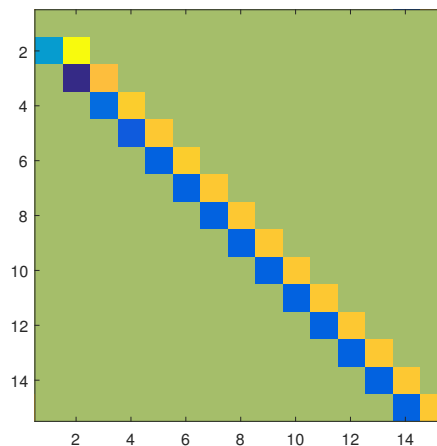


FIGURE 1. Fibonacci Matrix

In the visual representation of the matrix \widehat{F} (Figure 1), the diagonal entries $\frac{g_n}{g_{n+1}}$ are distinguished by a distinct colour. This distinction is meaningful for two reasons. First, these diagonal elements form a strictly decreasing sequence converging to the inverse of the golden ratio, φ^{-1} ,

thereby encoding the limiting behaviour of the transformation. Second, the diagonal constitutes the only nonzero component of \widehat{F} that lies along the main line of the matrix, whereas all other nonzero entries are located on the first subdiagonal. Emphasizing the diagonal thus visually distinguishes the two structurally different contributions of the operator.

We now introduce the general Fibonacci sequence space $Y(\widehat{F})$ as defined in [18]. Let Y be a sequence space and $k \in \mathbb{N}$. Then $Y(\widehat{F}) = \{y = (y_k) \in \omega : (\widehat{F}y_k) \in Y\}$. It is evident that if Y is a linear space, then $Y(\widehat{F})$ is also a linear space. Kara [18] further showed that if Y is a Banach space, then $Y(\widehat{F})$ inherits the Banach space structure with the norm $\|y\|_{Y(\widehat{F})} = \|\widehat{F}y\|_Y$.

Definition 2.2. A sequence $y = (y_k)$ is said to be *Fibonacci statistically convergent* over $Y(\widehat{F})$ if there exists a real number L such that, for every $\varepsilon > 0$, $K_\varepsilon(\widehat{F}) = \{k \leq n : |\widehat{F}y_k - L| \geq \varepsilon\}$ has natural density zero, that is, $d(K_\varepsilon(\widehat{F})) = 0$. Equivalently,

$$\frac{1}{n} |\{k \leq n : |\widehat{F}y_k - L| \geq \varepsilon\}| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In this case, we write $d(\widehat{F}) \lim y_k = L$ or, equivalently, $y_k \rightarrow L(S(\widehat{F}))$. The set of all \widehat{F} statistically convergent sequences is denoted by $S(\widehat{F})$.

Next, we recall the concept of the deferred Cesàro mean $D(a_n, b_n)$. Let (a_n) and (b_n) be sequences of non-negative integers satisfying the following conditions:

- (i) $a_n < b_n$ for all n ,
- (ii) $\lim_{n \rightarrow \infty} b_n = \infty$.

Then, the deferred Cesàro mean $D(a_n, b_n)$ of a sequence (y_n) is defined (see Agnew [1, p. 414]) by

$$D(a_n, b_n) = D(y_n) = \frac{y_{a_n+1} + y_{a_n+2} + \dots + y_{b_n}}{b_n - a_n} = \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} y_k.$$

It is well known that the deferred Cesàro mean $D(a_n, b_n)$ is regular under conditions (i) and (ii) (see Agnew [1]). We now recall the following definition.

Definition 2.3. (see [15]) A sequence (y_n) is said to be *deferred Cesàro statistically convergent* to a real number L if, for every $\varepsilon > 0$, $\{k : a_n < k \leq b_n \text{ and } |y_k - L| \geq \varepsilon\}$ has deferred Cesàro density zero, that is,

$$\lim_{n \rightarrow \infty} \frac{1}{b_n - a_n} |\{k : a_n < k \leq b_n \text{ and } |y_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $\text{stat}_{DC} \lim_{n \rightarrow \infty} y_n = L$.

Next, we present the following definition introducing our proposed convergence criterion.

Definition 2.4. A sequence $y = (y_k)$ is said to be *deferred Cesàro Fibonacci statistically convergent* (or *deferred Cesàro $\widehat{F}y_k$ -statistically convergent*) over $Y(\widehat{F})$ if there exists a real number L such that, for every $\varepsilon > 0$, the set $K_\varepsilon(\widehat{F}) = \{k : a_n < k \leq b_n \text{ and } |\widehat{F}y_k - L| \geq \varepsilon\}$ has deferred Cesàro density zero. That is,

$$\lim_{n \rightarrow \infty} \frac{1}{b_n - a_n} |\{k : a_n < k \leq b_n \text{ and } |\widehat{F}y_k - L| \geq \varepsilon\}| = 0.$$

When this condition holds true, we denote it by $DC(\widehat{F}) \lim y_k = L$.

In view of the above definitions, we now state and prove that every deferred every deferred Cesàro statistically convergent sequence is also deferred Cesàro Fibonacci statistically convergent. However, the converse does not necessarily hold true.

Theorem 2.1. *Every deferred Cesàro statistically convergent sequence is also deferred Cesàro Fibonacci statistically convergent. That is, if $\text{stat}_{DC} \lim_{n \rightarrow \infty} y_n = L$, then $DC(\widehat{F}) \lim_{n \rightarrow \infty} y_n = L$. However, the converse does not necessarily hold true.*

Proof. Let (y_n) be deferred Cesàro statistically convergent to L . By definition, for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{b_n - a_n} |\{k : a_n < k \leq b_n, |y_k - L| \geq \varepsilon\}| = 0.$$

Consider now the Fibonacci transform $(\widehat{F}y)_n$ defined by

$$(\widehat{F}y)_n = -\frac{g_{n+1}}{g_n} x_{n-1} + \frac{g_n}{g_{n+1}} y_n,$$

where g_n is the n th Fibonacci number. Since $y_n \rightarrow L$ in the deferred Cesàro statistical sense, for sufficiently large n , we have $|y_n - L| < \varepsilon/2$ and $|y_{n-1} - L| < \varepsilon/2$ outside a set of deferred Cesàro density zero. Hence, we have

$$|(\widehat{F}y)_n - L| \leq \frac{g_{n+1}}{g_n} |y_{n-1} - L| + \frac{g_n}{g_{n+1}} |y_n - L|.$$

For large n , we know that $\frac{g_{n+1}}{g_n} \rightarrow \varphi$ and $\frac{g_n}{g_{n+1}} \rightarrow \varphi^{-1}$, where φ is the golden ratio. Thus, clearly, there exists a constant $C > 0$ such that

$$|(\widehat{F}y)_n - L| \leq C \max\{|y_{n-1} - L|, |y_n - L|\}.$$

Since the set of indices for which $|y_n - L| \geq \varepsilon/C$ has deferred Cesàro density zero, it follows that $\{k : a_n < k \leq b_n, |(\widehat{F}y)_k - L| \geq \varepsilon\}$ also has deferred Cesàro density zero. Hence, $(\widehat{F}y)_n$ is deferred Cesàro Fibonacci statistically convergent to L . This completes the proof of Theorem 2.1. □

In view of the converse part of Theorem 2.1, we construct an illustrative example below.

Example 2.1. Consider the Fibonacci numbers $\{g_n\}$ defined by $g_0 = 0, g_1 = 1$, and $g_n = g_{n-1} + g_{n-2}$ for $n \geq 2$. Define the sequence $y = (y_k)$ by $y_k = (-1)^k \frac{g_{k+1}}{g_k}, k \geq 2$. Since $\frac{g_{k+1}}{g_k} \rightarrow \varphi \approx 1.618$, where φ is the golden ratio, the sequence (y_k) oscillates between approximately $\pm\varphi$ and hence is not deferred Cesàro statistically convergent.

Now, consider its Fibonacci transform

$$(\widehat{F}y)_n = -\frac{g_{n+1}}{g_n} y_{n-1} + \frac{g_n}{g_{n+1}} y_n.$$

Substituting the definition of y_k yields

$$(\widehat{F}y)_n = -\frac{g_{n+1}}{g_n} \left((-1)^{n-1} \frac{g_n}{g_{n-1}} \right) + \frac{g_n}{g_{n+1}} \left((-1)^n \frac{g_{n+1}}{g_n} \right) = (-1)^n \left(\frac{g_{n+1}}{g_{n-1}} - 1 \right).$$

Since $\frac{g_{n+1}}{g_{n-1}} \rightarrow \varphi^2 \approx 2.618$, the terms of $(\widehat{F}y)_n$ approach a constant magnitude and alternate in sign. Consequently, the absolute difference $|(\widehat{F}y)_n - 0|$ is bounded, and the set

$$\{k : a_n < k \leq b_n, |(\widehat{F}y)_k - 0| \geq \varepsilon\}$$

has deferred Cesàro density zero for any $\varepsilon > 0$. Thus, the sequence $y = (y_k)$ is deferred Cesàro Fibonacci statistically convergent to 0, but it is not deferred Cesàro statistically convergent in the ordinary sense.

3. KOROVKIN-TYPE APPROXIMATION THEOREMS

Korovkin-type approximation theorems provide a powerful criterion for verifying when a sequence of positive linear operators $(\mathfrak{A}_n)_{n \geq 1}$ on $\mathfrak{C}[a, b]$ defines an approximation process. Korovkin-type approximation theorems provide a powerful criterion for determining when a sequence of positive linear operators $(\mathfrak{A}_n)_{n \geq 1}$ on $\mathfrak{C}[a, b]$ defines an approximation process. These theorems simplify the verification of convergence on the entire function space by reducing it to checking convergence on a small set of test functions. The classical result of Korovkin [22] established that the test functions $1, y, y^2$ suffice for $\mathfrak{C}[a, b]$, and that $1, \cos y, \sin y$ serve as test functions for the space of 2π -periodic continuous functions.

The connection between approximation processes and statistical convergence was first investigated by Gadjiev and Orhan [10], who proved Korovkin-type theorems via statistical convergence. Since then, numerous extensions and refinements have appeared, including results involving statistical lacunary summability [26], weighted statistical convergence [27], deferred Nörlund means [37], weighted trigonometric Korovkin theorems [38], statistical probability convergence [12], and several other variants (see [13, 14, 33, 34, 35]). In parallel, several notions of density have been introduced in number theory to generalize asymptotic density, especially in cases where the classical density fails to exist. Asymptotic density captures the intuitive idea that some infinite subsets of \mathbb{N} are sparser than others, even though both are countably infinite. Within the study of sequence spaces, Fibonacci numbers were first employed by Kara and Basarir [19], and subsequently, Kara [18] introduced the Fibonacci difference matrix \widehat{F} , leading to new matrix domain spaces. This direction of research has since been widely developed (see [18, 19, 21, 23]). Motivated by the two lines of development, statistical convergence in approximation theory and the use of Fibonacci structures in sequence spaces, we introduce in this paper a new mode of convergence, which we call the deferred Cesàro Fibonacci statistical convergence.

Let $\mathfrak{F}(\mathbb{R})$ denote the linear space of real-valued functions on \mathbb{R} . Let $\mathfrak{C}(\mathbb{R})$ be the space of all real-valued continuous functions g on \mathbb{R} . It is known that $\mathfrak{C}(\mathbb{R})$ is a Banach space with the norm $\|g\|_\infty = \sup_{y \in \mathbb{R}} |g(y)|$, $g \in \mathfrak{C}(\mathbb{R})$. We denote by $\mathfrak{C}_{2\pi}(\mathbb{R})$ the space of all 2π -periodic functions $g \in \mathfrak{C}(\mathbb{R})$, which is a Banach space with the norm $\|g\|_{2\pi} = \sup_{y \in \mathbb{R}} |g(y)|$, $g \in \mathfrak{C}_{2\pi}(\mathbb{R})$. We say that \mathfrak{A} is a positive operator if, for every nonnegative g and $y \in I$, $\mathfrak{A}(g, y) \geq 0$, where I is any given interval on the real semi-axis.

The first and second classical Korovkin approximation theorems state the following (see [22]).

Theorem 3.1. (see [22]) *Let (\mathfrak{A}_k) be a sequence of positive linear operators from $\mathfrak{C}[0, 1]$ into $\mathfrak{F}[0, 1]$. Then*

$$\lim_{k \rightarrow \infty} \|\mathfrak{A}_k(g, y) - g(y)\|_{\mathfrak{C}[a, b]} = 0 \iff \lim_{k \rightarrow \infty} \|\mathfrak{A}_k(e_i, y) - e_i\|_{\mathfrak{C}[a, b]} = 0,$$

where $e_i = y^i$ for $i = 0, 1, 2$.

Theorem 3.2. (see [22]) *Let $\{\mathfrak{T}_k\}$ be a sequence of positive linear operators from $\mathfrak{C}_{2\pi}(\mathbb{R})$ into $\mathfrak{F}(\mathbb{R})$. Then*

$$\lim_{k \rightarrow \infty} \|\mathfrak{T}_k(g, y) - g(y)\|_{2\pi} = 0 \iff \lim_{k \rightarrow \infty} \|\mathfrak{T}_k(g_i, y) - g_i\|_{2\pi} = 0 \quad (i = 0, 1, 2),$$

where $g_0 = 1$, $g_1 = \sin y$ and $g_2 = \cos y$.

In 2018, Jena et al. [15] stated and proved the Korovkin-type approximation theorem based on deferred Cesàro statistical convergence.

Theorem 3.3. (see [15]) *Let $\mathfrak{L}_k : \mathfrak{C}[0, \infty) \rightarrow \mathfrak{C}[0, \infty)$ be a sequence of positive linear operators. Then, for all $g \in \mathfrak{C}[0, \infty)$, $\text{stat}_{\text{DC}} \lim_{k \rightarrow \infty} \|\mathfrak{L}_k(g; y) - g(y)\|_{\infty} = 0$ if and only if*

$$\text{stat}_{\text{DC}} \lim_{k \rightarrow \infty} \|\mathfrak{L}_k(1; y) - 1\|_{\infty} = 0, \quad \text{stat}_{\text{DC}} \lim_{k \rightarrow \infty} \|\mathfrak{L}_k(e^{-t}; y) - e^{-y}\|_{\infty} = 0,$$

and $\text{stat}_{\text{DC}} \lim_{k \rightarrow \infty} \|\mathfrak{L}_k(e^{-2t}; y) - e^{-2y}\|_{\infty} = 0$.

Next, we establish a Korovkin-type approximation theorem based on deferred Cesàro Fibonacci statistical convergence and support it with an illustrative example.

Theorem 3.4. *Let $\{\mathfrak{L}_k\}$ be a sequence of positive linear operators from $\mathfrak{C}_{2\pi}(\mathbb{R})$ into $\mathfrak{C}_{2\pi}(\mathbb{R})$. Then, for every $g \in \mathfrak{C}_{2\pi}(\mathbb{R})$,*

$$DC(\widehat{F}) \lim_{k \rightarrow \infty} \|\mathfrak{L}_k(g, y) - g(y)\|_{2\pi} = 0 \tag{3.1}$$

if and only if

$$DC(\widehat{F}) \lim_{k \rightarrow \infty} \|\mathfrak{L}_k(g_i, y) - g_i(y)\|_{2\pi} = 0 \quad (i = 0, 1, 2), \tag{3.2}$$

where $g_0(y) = 1$, $g_1(y) = \sin y$, and $g_2(y) = \cos y$.

Proof. As $1, \sin y, \cos y \in \mathfrak{C}_{2\pi}(\mathbb{R})$, conditions (3.2) follow immediately from (3.1). Let conditions (3.2) hold true and let $I_1 = (a, a + 2\pi)$ be any subinterval of length 2π in \mathbb{R} . Fix $y \in I_1$. By the continuity of g , it follows that, for given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|g(y) - g(t)| < \varepsilon, \quad \text{whenever } |t - y| < \delta. \tag{3.3}$$

If $|t - y| \geq \delta$, we assume that $t \in (y + \delta, 2\pi + y + \delta)$. Then

$$|g(y) - g(t)| \leq 2\|g\|_{2\pi} \leq \frac{2\|g\|_{2\pi}}{\sin^2(\delta/2)} \psi(t), \tag{3.4}$$

where $\psi(t) = \sin^2(\frac{t-y}{2})$. Combining (3.3) and (3.4), we obtain $|g(y) - g(t)| < \varepsilon + \frac{2\|g\|_{2\pi}}{\sin^2(\delta/2)} \psi(t)$, which implies that

$$-\varepsilon - \frac{2\|g\|_{2\pi}}{\sin^2(\delta/2)} \psi(t) < g(y) - g(t) < \varepsilon + \frac{2\|g\|_{2\pi}}{\sin^2(\delta/2)} \psi(t).$$

By positivity and linearity of $\{\mathfrak{L}_k\}$, we have

$$\begin{aligned} \mathfrak{L}_k(1, y) \left(-\varepsilon - \frac{2\|g\|_{2\pi}}{\sin^2(\delta/2)} \psi(t) \right) &< \mathfrak{L}_k(1, y)(g(y) - g(t)) \\ &< \mathfrak{L}_k(1, y) \left(\varepsilon + \frac{2\|g\|_{2\pi}}{\sin^2(\delta/2)} \psi(t) \right), \end{aligned}$$

so that

$$\begin{aligned}
 -\varepsilon \mathfrak{L}_k(1, y) - \frac{2\|g\|_{2\pi}}{\sin^2(\delta/2)} \mathfrak{L}_k(\psi(t), y) &< \mathfrak{L}_k(g, y) - g(y) \mathfrak{L}_k(1, y) \\
 &< \varepsilon \mathfrak{L}_k(1, y) + \frac{2\|g\|_{2\pi}}{\sin^2(\delta/2)} \mathfrak{L}_k(\psi(t), y). \tag{3.5}
 \end{aligned}$$

Furthermore, we have

$$\begin{aligned}
 \mathfrak{L}_k(g, y) - g(y) &= \mathfrak{L}_k(g, y) - g(y) \mathfrak{L}_k(1, y) + f(y) \mathfrak{L}_k(1, y) - g(y) \\
 &= \mathfrak{L}_k(g, y) - g(y) \mathfrak{L}_k(1, y) + g(y) (\mathfrak{L}_k(1, y) - 1). \tag{3.6}
 \end{aligned}$$

From (3.5) and (3.6), we obtain

$$\mathfrak{L}_k(g, y) - g(y) < \varepsilon \mathfrak{L}_k(1, y) + \frac{2\|g\|_{2\pi}}{\sin^2(\delta/2)} \mathfrak{L}_k(\psi(t), y) + g(y) + g(y) (\mathfrak{L}_k(1, y) - 1). \tag{3.7}$$

We now compute

$$\begin{aligned}
 \mathfrak{L}_k(\psi(t), y) &= \mathfrak{L}_k\left(\sin^2\left(\frac{y-t}{2}\right), y\right) \\
 &= \frac{1}{2} \left(\mathfrak{L}_k(1, y) - \cos y \mathfrak{L}_k(\cos t, y) - \sin y \mathfrak{L}_k(\sin t, y) \right) \\
 &= \frac{1}{2} \left\{ \mathfrak{L}_k(1, y) - \cos y [\mathfrak{L}_k(\cos t, y) - \cos y] - \sin y [\mathfrak{L}_k(\sin t, y) - \sin y] \right\}.
 \end{aligned}$$

Upon substituting this into (3.7) and estimating absolute values, we have

$$\begin{aligned}
 |\mathfrak{L}_k(g, y) - g(y)| &\leq \varepsilon + \left(\varepsilon + |g(y)| + \frac{\|g\|_{2\pi}}{\sin^2(\delta/2)} \right) |\mathfrak{L}_k(1, y) - 1| \\
 &\quad + \frac{\|g\|_{2\pi}}{\sin^2(\delta/2)} (|\mathfrak{L}_k(\cos t, y) - \cos y| + |\mathfrak{L}_k(\sin t, y) - \sin y|).
 \end{aligned}$$

Since ε is arbitrary, we obtain

$$\begin{aligned}
 \|\mathfrak{L}_k(g, y) - g(y)\|_{2\pi} &\leq \varepsilon + R \left(\|\mathfrak{L}_k(1, y) - 1\|_{2\pi} + \|\mathfrak{L}_k(\cos t, y) - \cos t\|_{2\pi} \right. \\
 &\quad \left. + \|\mathfrak{L}_k(\sin t, y) - \sin y\|_{2\pi} \right),
 \end{aligned}$$

where

$$R = \max \left(\varepsilon + \|g\|_{2\pi} + \frac{\|g\|_{2\pi}}{\sin^2(\delta/2)}, \frac{\|g\|_{2\pi}}{\sin^2(\delta/2)} \right).$$

Finally, replacing $\mathfrak{L}_k(\cdot, y)$ by $\mathfrak{T}_k(\cdot, y) = \widehat{F} \mathfrak{L}_k(\cdot, y)$ and, for $\varepsilon' > 0$, we write

$$\mathfrak{A} := \left\{ k : a_k \leq b_k : \|\mathfrak{T}_k(1, y) - 1\|_{2\pi} + \|\mathfrak{T}_k(\sin t, y) - \sin y\|_{2\pi} + \|\mathfrak{T}_k(\cos t, y) - \cos y\|_{2\pi} \geq \frac{\varepsilon'}{R} \right\},$$

$$\mathfrak{A}_1 := \left\{ k : a_k \leq b_k : \|\mathfrak{T}_k(1, y) - 1\|_{2\pi} \geq \frac{\varepsilon'}{3R} \right\},$$

$$\mathfrak{A}_2 := \left\{ k : a_k \leq b_k : \|\mathfrak{T}_k(\sin t, y) - \sin y\|_{2\pi} \geq \frac{\varepsilon'}{3R} \right\}$$

and

$$\mathfrak{A}_3 := \left\{ k : a_k \leq b_k : \|\mathfrak{T}_k(\text{cost}, y) - \text{cos } y\|_{2\pi} \geq \frac{\varepsilon'}{3R} \right\}.$$

Then $\mathfrak{A} \subset \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3$, so $DC(\mathfrak{A}) \leq DC(\mathfrak{A}_1) + DC(\mathfrak{A}_2) + DC(\mathfrak{A}_3)$. In view of (3.1) and (3.2), we conclude that

$$DC(\widehat{F}) \lim_{k \rightarrow \infty} \|\mathfrak{L}_k(g, y) - g(y)\|_{2\pi} = 0,$$

which completes the proof of Theorem 3.4. □

We remark that our Theorem 3.4 is stronger than Theorem 2.1 (see [15, p. 2311]) as well as the Theorem of Gadjiev and Orhan [10]. To demonstrate this, we present the following example.

Example 3.1. For $n \in \mathbb{N}$, let $\mathfrak{S}_n(g)$ denote the n th partial sum of the Fourier series of g , that is,

$$\mathfrak{S}_n(g, y) = \frac{1}{2}a_0(g) + \sum_{k=0}^n (a_k(g) \cos ky + b_k(g) \sin ky).$$

For each $n \in \mathbb{N}$, we define $\mathfrak{F}_n(g, y) = \frac{1}{n+1} \sum_{k=a_{n+1}}^{b_n} \mathfrak{S}_k(g, y)$. A standard computation shows that, for every $t \in \mathbb{R}$,

$$\mathfrak{F}_n(g, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \varphi_n(y-t) dt,$$

where

$$\varphi_n(y) = \begin{cases} \frac{\sin^2\left(\frac{(n+1)(y-t)}{2}\right)}{(n+1) \sin^2\left(\frac{y-t}{2}\right)} & \text{if } y \text{ is not a multiple of } 2\pi, \\ n+1 & \text{if } y \text{ is a multiple of } 2\pi. \end{cases}$$

The sequence $(\varphi_n)_{n \in \mathbb{N}}$ forms a positive kernel, known as the Fejér kernel, and the corresponding operators \mathfrak{F}_n for $n \geq 1$ are called Fejér convolution operators.

We define the sequence of linear operators $K_n : \mathfrak{C}_{2\pi}(\mathbb{R}) \rightarrow \mathfrak{C}_{2\pi}(\mathbb{R})$ by

$$K_n(g, y) = (1 + y_n)\mathfrak{F}_n(g, y), \tag{3.8}$$

where $y = (y_n)$ is defined in Example 2.1. Then

$$K_n(1, y) = 1 + (-1)^n \frac{g_{n+1}}{g_n},$$

$$K_n(\sin t, y) = \left(1 + (-1)^n \frac{g_{n+1}}{g_n} \right) \frac{n}{n+1} \sin y,$$

$$K_n(\text{cost}, y) = \left(1 + (-1)^n \frac{g_{n+1}}{g_n} \right) \frac{n}{n+1} \text{cos } y.$$

Hence, (K_n) satisfies the conditions (3.1) and (3.2). Therefore,

$$DC(\widehat{F}) \lim_{k \rightarrow \infty} \|K_k(g, y) - g(y)\|_{2\pi} = 0.$$

It is clearly observed that the sequence of operators (K_n) defined in (3.8) does not satisfy Theorem 2.1 (see [15, p. 2311]) or the theorem of Gadjiev and Orhan [10]. This is because the sequence (y_n) is deferred Cesàro \widehat{F} -statistically convergent to 0, while it is neither convergent nor deferred Cesàro statistically convergent in the usual sense.

Next, we carry out a geometric analysis (see Figures 2, 3 and 4) of the operators defined in (3.8) for the test functions 1, $\sin t$, and $\cos t$, and discuss their convergence behaviour.

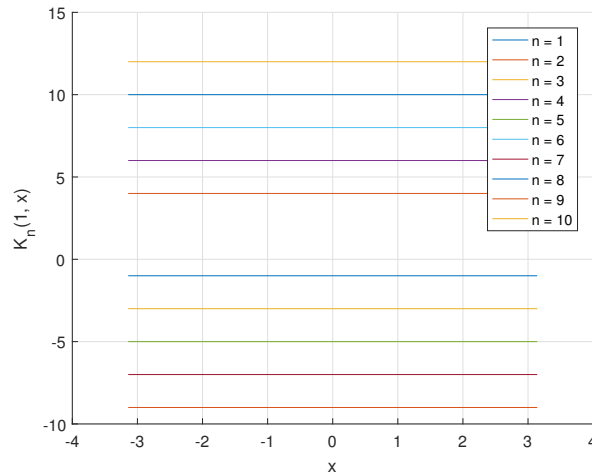


FIGURE 2. Convergence behaviour of $K_n(1; y)$

The operator given by

$$K_n(1, y) = 1 + (-1)^n \frac{g_{n+1}}{g_n}$$

reflects the normalization property of the operator sequence. As $n \rightarrow \infty$, the ratio $\frac{g_{n+1}}{g_n}$ tends to the golden ratio $\phi \approx 1.618$, inducing oscillations with decreasing magnitude due to the alternating sign factor $(-1)^n$. This indicates that the sequence $\{K_n\}$ statistically preserves constants in the Fibonacci deferred Cesàro sense, that is, convergence occurs on a subsequence indexed by Fibonacci numbers whose natural density tends to 1.

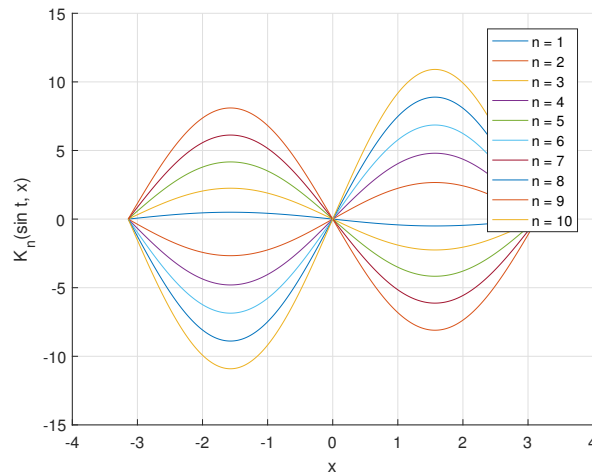


FIGURE 3. Convergence behaviour of $K_n(\sin t; y)$

For the sine function, the operator takes the following form:

$$K_n(\sin t, y) = \left(1 + (-1)^n \frac{g_{n+1}}{g_n} \right) \frac{n}{n+1} \sin y.$$

The multiplicative factors $\frac{n}{n+1}$ and the Fibonacci-weighted term together regulate the oscillations, ensuring that deviations from $\sin y$ vanish statistically. Hence, the sequence $\{K_n(\sin t, y)\}$ converges to $\sin y$ in the deferred Cesàro Fibonacci statistical convergence sense, implying that the approximation improves for almost all indices n in a density sense rather than in the classical point-wise sense.

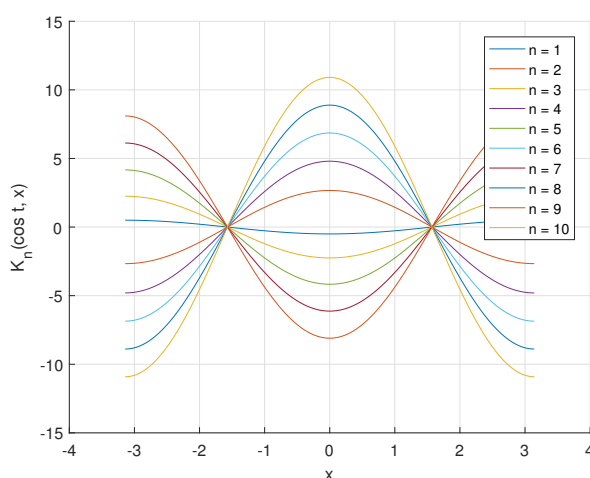


FIGURE 4. Convergence behaviour of $K_n(\cos t; y)$

Similarly, for the cosine function, we have

$$K_n(\cos t, y) = \left(1 + (-1)^n \frac{g_{n+1}}{g_n} \right) \frac{n}{n+1} \cos y,$$

which demonstrates that the operator approximates $\cos y$ within the framework of deferred Cesàro Fibonacci statistical convergence. This ensures that the trigonometric structure is statistically preserved, and the convergence is stabilized through Fibonacci weighting. Taken together, these operators constitute a Korovkin-type sequence under the deferred Cesàro Fibonacci statistical convergence framework in the space $\mathcal{C}_{2\pi}(\mathbb{R})$. The classical Fejér summability process is thereby extended via Fibonacci scaling, inducing statistical convergence along subsequences determined by Fibonacci indices. Graphically, the oscillations in these operators diminish as n increases, illustrating the inherent smoothing and stabilizing effect of the deferred Cesàro Fibonacci means.

4. RATE OF DEFERRED CESÀRO \widehat{F} -STATISTICAL CONVERGENCE

In this section, we study the rate of deferred Cesàro \widehat{F} -statistical convergence for a sequence of positive linear operators mapping $\mathcal{C}_{2\pi}(\mathbb{R})$ into itself. We begin with the following definition.

Definition 4.1. Let (u_n) be a positive, non-increasing sequence. A sequence $y = (y_k)$ is said to be *deferred Cesàro \widehat{F} -statistically convergent* to a limit L with rate $o(u_n)$ if, for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{\|\Omega_n(y, \varepsilon)\|_\infty}{u_n(b_n - a_n)} = 0,$$

where

$$\Omega_n(y, \varepsilon) = \left| \left\{ k : a_n < k \leq b_n \text{ and } |\widehat{F}y_k - L| \geq \varepsilon \right\} \right|.$$

In this case, we write

$$y_k - L = DC(\widehat{F}) o(u_n).$$

Next, we establish a useful auxiliary result which is stated as Lemma 4.1 below.

Lemma 4.1. *Let (u_n) and (v_n) be two positive and non-increasing sequences. Suppose that $y = (y_n)$ and $z = (z_n)$ are sequences such that $y_k - L_1 = DC(\widehat{F})o(u_n)$ and $z_k - L_2 = DC(\widehat{F})o(v_n)$. Then the following properties hold true:*

- (i) $\alpha(y_n - L_1) = DC(\widehat{F}) o(u_n)$ for any scalar α ,
- (ii) $(y_n - L_1) \pm (z_n - L_2) = DC(\widehat{F}) o(s_n)$, where $s_n = \max\{u_n, v_n\}$,
- (iii) $(y_n - L_1)(z_n - L_2) = DC(\widehat{F}) o(u_n v_n)$.

Proof. To prove the assertion (i) of Lemma 4.1, we consider the following sets for $\varepsilon > 0, s \in I$, and sequences $y = (y_k)$ and $z = (z_k)$:

$$\begin{aligned} A_n(s, \varepsilon) &= \left| \left\{ k : a_n < k \leq b_n \text{ and } |(\widehat{F}y_k - L_1) \pm (\widehat{F}z_k - L_2)| \geq \varepsilon \right\} \right|, \\ A_{0,n}(s, \varepsilon) &= \left| \left\{ k : a_n < k \leq b_n \text{ and } |\widehat{F}y_k - L_1| \geq \frac{\varepsilon}{2} \right\} \right|, \\ A_{1,n}(s, \varepsilon) &= \left| \left\{ k : a_n < k \leq b_n \text{ and } |\widehat{F}z_k - L_2| \geq \frac{\varepsilon}{2} \right\} \right|. \end{aligned}$$

Clearly, we have

$$A_n(s, \varepsilon) \subseteq A_{0,n}(s, \varepsilon) \cup A_{1,n}(s, \varepsilon).$$

Let $s_n = \max\{u_n, v_n\}$. Then, by applying assertion (3.2) of Theorem 3.4, we obtain

$$\frac{\|A_n(s, \varepsilon)\|_{2\pi}}{s_n(b_n - a_n)} \leq \frac{\|A_{0,n}(s, \varepsilon)\|_{2\pi}}{u_n(b_n - a_n)} + \frac{\|A_{1,n}(s, \varepsilon)\|_{2\pi}}{v_n(b_n - a_n)}. \tag{4.1}$$

Since both terms on the right-hand side tend to zero as $n \rightarrow \infty$, the left-hand side also tends to zero. This establishes assertion (i) of Lemma 4.1. The remaining assertions (ii) and (iii) follow by analogous arguments. This completes the proof of Lemma 4.1 is complete. \square

To estimate the rate of deferred Cesàro \widehat{F} -statistical convergence for sequences of positive linear operators, we make use of the modulus of continuity, which provides a quantitative measure of the uniform continuity of a function. We recall that the modulus of continuity of a function $g \in \mathfrak{C}_{2\pi}(\mathbb{R})$ is defined by

$$\omega(g, \delta) = \sup_{\substack{y, z \in Y \\ |z - y| \leq \delta}} |g(z) - g(y)| \quad (\delta > 0). \tag{4.2}$$

It is known that, for any $g \in \mathfrak{C}_{2\pi}(\mathbb{R})$, $\lim_{\delta \rightarrow 0^+} \omega(g, \delta) = 0$. Moreover, for all $y, z \in \mathbb{R}$,

$$|g(y) - g(z)| \leq \omega(g, \delta) \left(\frac{|g(y) - g(z)|}{\delta} + 1 \right). \tag{4.3}$$

We now establish Theorem 4.1 that connects the modulus of continuity with the rate of deferred Cesàro \widehat{F} -statistical convergence of a sequence of positive linear operators.

Theorem 4.1. *Let (\mathfrak{L}_k) be a sequence of positive linear operators from $\mathfrak{C}_{2\pi}(\mathbb{R})$ into $\mathfrak{C}_{2\pi}(\mathbb{R})$. Suppose that the following conditions hold true:*

- (i) $\|\mathfrak{L}_k(1, y) - y\|_{2\pi} = DC(\widehat{F})o(u_n)$,
- (ii) $\omega(g, \theta_k) = DC(\widehat{F})o(v_n)$, where $\theta_k = \sqrt{\mathfrak{L}_k(\sin^2 \frac{t-y}{2}, y)}$.

Then, for all $g \in \mathfrak{C}_{2\pi}(\mathbb{R})$, $\|\mathfrak{L}_k(g, y) - g(y)\|_{2\pi} = DC(\widehat{F}) o(w_n)$, $w_n = \max\{u_n, v_n\}$.

Proof. Let $g \in \mathfrak{C}_{2\pi}(\mathbb{R})$ and $y \in [-\pi, \pi]$. From (3.6) and (4.3), we can write

$$|\mathfrak{L}_k(g, y) - g(y)| \leq \mathfrak{L}_k(|g(t) - g(y)|; y) + |g(y)| |\mathfrak{L}_k(1, y) - 1|.$$

Using inequality (4.3), we obtain

$$|\mathfrak{L}_k(g, y) - g(y)| \leq \mathfrak{L}_k\left(\frac{|t-y|}{\delta} + 1; y\right) \omega(g, \delta) + |g(y)| |\mathfrak{L}_k(1, y) - 1|.$$

Since $|t - y| \leq \pi$, it follows that

$$|\mathfrak{L}_k(g, y) - g(y)| \leq \mathfrak{L}_k\left(\frac{\pi^2}{\delta^2} \sin^2 \frac{t-y}{2} + 1; y\right) \omega(g, \delta) + |g(y)| |\mathfrak{L}_k(1, y) - 1|.$$

That is,

$$|\mathfrak{L}_k(g, y) - g(y)| \leq \left\{ \mathfrak{L}_k(1, y) + \frac{\pi^2}{\delta^2} \mathfrak{L}_k\left(\sin^2 \frac{t-y}{2}; y\right) \right\} \omega(g, \delta) + |g(y)| |\mathfrak{L}_k(1, y) - 1|.$$

Now, by choosing $\sqrt{\theta_k} = \delta$ so that $\theta_k = \mathfrak{L}_k(\sin^2 \frac{t-y}{2}, y)$, we find that

$$\|\mathfrak{L}_k(g, y) - g(y)\|_{2\pi} \leq \|g\|_{2\pi} \|\mathfrak{L}_k(1, y) - y\|_{2\pi} + 2\omega(g, \theta_k) + \omega(g, \theta_k) \|\mathfrak{L}_k(1, y) - y\|_{2\pi}.$$

Consequently, we have

$$\|\mathfrak{L}_k(g, y) - g(y)\|_{2\pi} \leq K \left(\|\mathfrak{L}_k(1, y) - y\|_{2\pi} + \omega(g, \theta_k) + \omega(g, \theta_k) \|\mathfrak{L}_k(1, x) - y\|_{2\pi} \right),$$

where $K = \max\{2, \|g\|_{2\pi}\}$. From Definition 4.1 and the assumptions (i) and (ii), we have

$$\|\mathfrak{L}_k(1, y) - y\|_{2\pi} = d(\widehat{F}) o(u_n), \quad \omega(g, \theta_k) = DC(\widehat{F}) o(v_n).$$

Therefore, by Lemma 4.1, it follows that

$$\|\mathfrak{L}_k(g, y) - g(y)\|_{2\pi} = DC(\widehat{F}) o(w_n), \quad w_n = \max\{u_n, v_n\}.$$

This completes the proof of Theorem 4.1. □

5. CONCLUDING REMARKS AND FURTHER DISCUSSION

The Fibonacci sequence remains one of the most celebrated and extensively studied numerical sequences in mathematics, serving as a foundational construct that continues to enrich mathematical exploration across diverse research domains. In this study, we have introduced and systematically examined deferred Cesàro Fibonacci statistical convergence, a novel generalization that unites classical summability theory with the intrinsic properties of Fibonacci sequences. By establishing an inclusion relation between deferred Cesàro statistical convergence and its Fibonacci counterpart, we have shown that the Fibonacci weighting generates a broader class of convergence behaviors, particularly suitable for sequences that fail to converge

in the ordinary or deferred Cesàro statistical sense. Our main contribution, a Korovkin-type approximation theorem for positive linear operators on $\mathcal{C}_{2\pi}(\mathbb{R})$, offers a powerful criterion for verifying convergence across the entire function space by testing only on the fundamental set $\{1, \sin y, \cos y\}$, thereby extending several classical results in approximation theory.

The theoretical framework developed herein opens new perspectives for practical applications. In signal processing and communication systems, Fejér-type convolution operators governed by Fibonacci statistical convergence can be utilized to design adaptive filters capable of effectively handling noisy or irregularly sampled data-situations in which traditional convergence criteria may fail. In numerical analysis and computational mathematics, the deferred Cesàro Fibonacci means offer enhanced stability for iterative algorithms, particularly in scenarios involving oscillatory or slowly converging sequences. Moreover, within approximation theory and data-driven methodologies, the Korovkin-type results enable efficient assessment of operator sequences for function approximation, interpolation, and machine learning models relying on trigonometric basis expansions.

Future research directions include extending the present framework to multidimensional settings via higher-order Fibonacci matrices, exploring its applicability to fuzzy and probabilistic convergence structures, and developing computational algorithms that exploit Fibonacci-scaled summability for optimization problems arising in engineering and applied sciences, and the mathematics of finance.

We conclude this paper by further remarking about the potential for developing some of the currently ongoing research areas which we have considered herein. As we pointed out in Section 1, many authors have contributed to the initial Fast-Zygmund notion of *statistical convergence*. In fact, by calling it *almost convergence*, Zygmund went on to discuss its relationship with *strong summability*. We choose to refer the reader to the closely-related and frequently-cited work by Fridy [9]. The currently popular research areas are not only limited to the notion of statistical convergence, but also include the related notion of statistical summability as well as their corresponding families of statistical *weighted* convergence and statistical *weighted* summability (see Srivastava et al. [32] for the related details).

REFERENCES

- [1] R.P. Agnew, On deferred Cesàro means, *Ann. Math.* 33 (1932), 413–421.
- [2] H. Aktuğlu, Korovkin-type approximation theorems proved via $\alpha\beta$ -statistical convergence, *J. Comput. Appl. Math.* 259 (2014), 174–181.
- [3] N.L. Braha, Some weighted equi-statistical convergence and Korovkin-type theorem, *Results Math.* 70 (2016), 433–446.
- [4] R.C. Buck, Generalized asymptotic density, *Amer. J. Math.* 75 (1953), 335–346.
- [5] H. Dutta, B.B. Jena, N.P. Pahari, S.K. Paikray, On certain statistical convergence criteria for martingale sequence via deferred Cesàro mean with some application, *J. Nepal Math. Soc.* 7 (2024), 40–60.
- [6] R. Erdős and G. Tenenbaum, Sur les densités de certaines suites d’entiers. *Proc. London Math. Soc. (Ser. 2)* 59 (1989), 417–438.
- [7] H. Fast, Sur la convergence statistique, *Colloq. Math.* 2 (1951), 241–244.
- [8] A.R. Freedman, J.J. Sember, Densities and summability, *Pacific J. Math.* 95 (1981), 293–305.
- [9] J.A. Fridy, On statistical convergence, *Analysis* 5 (1985), 301–313.
- [10] A.D. Gadjev, C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.* 32 (2002), 129–138.
- [11] S. Goonatilake, *Toward a Global Science: Mining Civilizational Knowledge*, p. 126, Indiana University Press, Bloomington, Indiana, 1998.

- [12] B.B. Jena, S.K. Paikray, Product of statistical probability convergence and its applications to Korovkin-type theorem, *Miskolc Math. Notes* 20 (2019), 969–984.
- [13] B.B. Jena, S.K. Paikray, H. Dutta, On various new concepts of statistical convergence for sequences of random variables via deferred Cesàro mean, *J. Math. Anal. Appl.* 487 (2020), Article ID 123950, 1–18.
- [14] B.B. Jena, S.K. Paikray, H. Dutta, Statistically Riemann integrable and summable sequence of functions via deferred Cesàro mean, *Bull. Iran. Math. Soc.* 48 (2022), 1293–1309.
- [15] B.B. Jena, S.K. Paikray, U.K. Misra, Statistical deferred Cesàro summability and its applications to approximation theorems, *Filomat* 32 (2018), 2307–2319.
- [16] B.B. Jena, S.K. Paikray, M. Mursaleen, On statistical Riemann-Stieltjes integrability and deferred Cesàro summability, *Contemp. Math.* 5 (2024), 1–16.
- [17] B.B. Jena, S.K. Paikray, M. Mursaleen, Statistical gauge integrable functions and Korovkin-type approximation theorems, *Bull. Iran. Math. Soc.* 51 (2025), Article ID 47, 1–21.
- [18] E. E. Kara, Some topological and geometrical properties of new Banach sequence spaces, *J. Inequal. Appl.* 2013 (2013), Article ID 38, 1–15.
- [19] E. E. Kara, M. Basarir, An application of Fibonacci numbers into infinite Toeplitz matrices, *Caspian J. Math. Sci.* 1 (2012), 43–47.
- [20] V. Karakaya, A. Karaisa, Korovkin-type approximation theorems for weighted $\alpha\beta$ -statistical convergence, *Bull. Math. Sci.* 5 (2015), 159–169.
- [21] M. Kirişci, A. Karaisa, *Fibonacci statistical convergence and Korovkin type approximation theorems*, *J. Inequal. Appl.* 2017 (2017), Article ID 229, 1–15.
- [22] P.P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publishing Corporation, New Delhi, 1960.
- [23] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 2001.
- [24] I.J. Maddox, Statistical convergence in a locally convex sequence space, *Math. Proc. Cambridge Philos. Soc.* 104 (1988), 141–145.
- [25] H.I. Miller, A measure theoretical subsequence characterization of statistical convergence, *Trans. Amer. Math. Soc.* 347 (1995), 1811–1819.
- [26] M. Mursaleen, C. Belen, On statistical lacunary summability of double sequences, *Filomat* 28 (2014), 231–239.
- [27] M. Mursaleen, V. Karakaya, M. Ertürk, F. Gürsoy, Weighted statistical convergence and its application to Korovkin-type approximation theorem, *Appl. Math. Comput.* 218 (2012), 9132–9137.
- [28] P. Parida, B.B. Jena, S.K. Paikray, A new approach to Korovkin-type theorems based on deferred Nörlund summability mean, *Ann. Univ. Craiova-Mat.* 50 (2023), 1–11.
- [29] P. Parida, S.K. Paikray, B.B. Jena, Statistical deferred weighted Riemann summability and fuzzy approximation theorems, *Sahand Commun. Math. Anal.* 21 (2024), 327–345.
- [30] A. Satapathy, B.B. Jena, S.K. Paikray, A new class of Korovkin-type approximation theorems based on equi-statistical convergence of double sequences, *Filomat* 39 (2025), 6453–6471.
- [31] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* 66 (1959), 361–375.
- [32] H.M. Srivastava, E. Aljimi, B. Hazarika, Statistical weighted $(N_{\lambda, p, q})(E_{\lambda}, 1)$ A -summability with application to Korovkin’s type approximation theorem, *Bull. Sci. Math.* 178 (2022), 103146.
- [33] H.M. Srivastava, B.B. Jena, S.K. Paikray, A certain class of statistical probability convergence and its applications to approximation theorems, *Appl. Anal. Discrete Math.* 14 (2020), 579–598.
- [34] H.M. Srivastava, B.B. Jena, S.K. Paikray, Statistical probability convergence via the deferred Nörlund mean and its applications to approximation theorems, *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM)* 114 (2020), Article ID 144, 1–14.
- [35] H.M. Srivastava, B.B. Jena, S.K. Paikray, Some Korovkin-type approximation theorems associated with a certain deferred weighted statistical Riemann integrable sequence of functions, *Axioms* 11 (2022), Article ID 128, 1–11.
- [36] H.M. Srivastava, B.B. Jena, S.K. Paikray, A new class of Korovkin-type theorems on double sequences, *Iran. J. Sci.* 49 (2025), 1103–1114.

- [37] H.M. Srivastava, B.B. Jena, S.K. Paikray, U.K. Misra, Generalized equi-statistical convergence of the deferred Nörlund summability and its applications to associated approximation theorems, *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM)* **112** (2018), 1487–1501.
- [38] H.M. Srivastava, B.B. Jena, S.K. Paikray, U.K. Misra, A certain class of weighted statistical convergence and associated Korovkin-type approximation theorems involving trigonometric functions, *Math. Methods Appl. Sci.* **41** (2018), 671–683.
- [39] H.M. Srivastava, B.B. Jena, S.K. Paikray, U.K. Misra, Korovkin-type theorems for positive linear operators based on the statistical derivative of deferred Cesàro summability, *Algorithms* **18** (2025), Article ID 218, 1–21.
- [40] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, London and New York, 1979.