

MULTIPLE SIGN-CHANGING SOLUTIONS FOR A NEW NONLOCAL PROBLEM WITH CRITICAL GROWTH

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Abstract. In this paper, we study the following nonlocal problem

$$\begin{cases} \left(a - \lambda \int_{\Omega} |\nabla u|^2 dx \right) \Delta u + f(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded smooth domain, a and λ are positive parameters, and f is of subcritical or critical growth. By using the method of invariant sets for the descending flow, perturbation technique and necessary estimates, we establish the existence of infinitely many sign-changing solutions.

Keywords. Critical growth; Nonlocal problem; Perturbation methods; Sign-changing solutions.

1. INTRODUCTION AND MAIN RESULT

In this paper, we consider the following nonlocal problem

$$\begin{cases} \left(a - \lambda \int_{\Omega} |\nabla u|^2 dx \right) \Delta u + f(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, a, λ are positive parameters, the nonlinear function $f \in C(\mathbb{R}, \mathbb{R})$ satisfies subcritical or critical growth.

Problem (1.1) is called a nonlocal problem because of the presence of the integral term $\int_{\Omega} |\nabla u|^2 dx$, which implies that the equation in (1.1) is no longer pointwise identities. Nonlocal problems are also called Kirchhoff-type equations. Indeed, Kirchhoff [13] investigated an equation of the form

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of (1.2) is that the equation contains a nonlocal coefficient $\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ which depends on the average

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$\frac{1}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$. The parameters in (1.2) have the following meanings: L is length of the string, h is the area of the cross section, ρ is the mass density, P_0 is the initial tension, and E is the Young modulus of the material. In the past decade, the case $E > 0$ has been studied extensively; see, e.g., [2, 3, 7, 11, 17, 18, 20, 21, 22] and the references therein.

Due to the physical background of negative Young's modulus [12], this leads to the problem (1.2) with $E < 0$ being an interesting model. So, a new nonlocal problem has recently attracted some attention; see, e.g., [10, 14, 15, 16, 24, 25, 26, 28]. In particular, some authors considered the following special case for equation (1.1)

$$\begin{cases} \left(a - \lambda \int_{\Omega} |\nabla u|^2 dx \right) \Delta u + |u|^{p-2} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, a and λ are positive parameters, $2 < p < 2^*$, $2^* = +\infty$ for $N = 1, 2$; $2^* = \frac{2N}{N-2}$ for $N \geq 3$. In [24], Qian established the existence of ground state sign-changing solutions of equation (1.3) for $\lambda > 0$ small enough. In [26], Wang and Yang obtained infinitely many sign-changing solutions of equation (1.3) by using combination of invariant sets of descent flow and the Ljusternik-Schnirelman type minimax method. Moreover, Qian [25] and Wang et al. [27] investigated the multiplicity of positive solutions for (1.1) with different perturbations in the case $N = 4$. In fact, the main difficulty in the study of various elliptic equations involving critical exponents lies in noncompactness of Palais-Smale sequences. In recent decades, many researchers explored various methods to overcome this difficulty. For example, in [8], Devillanova and Solimini considered the following semilinear elliptic equation

$$\begin{cases} \Delta u + |u|^{2^*-2} u + \mu u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain. They obtained that problem (1.4) has infinitely many solutions for $\mu > 0$ and $N \geq 7$ by means of concentrated compactness analysis. In the case $4 \leq N \leq 6$, the discussion on the multiplicity of nontrivial solutions to problem (1.4) can be found in [5], [6], and [9]. Recently, Cheikh and Premoselli in [4] analyzed the compactness result for sign-changing solutions for problem (1.4). For p -Laplacian equation involving critical Sobolev growth, Cao et al. in [1] discussed infinitely many solutions. Mercuri and Perera in [23] obtained some new multiplicity results by \mathbb{Z}_2 -cohomological index. Zhao et al. in [29] introduced the truncation method to deal with the critical case. To the best of our knowledge, there are few results involving multiple sign-changing solutions of problem (1.1) for the general nonlinearity involving subcritical growth or the critical case. Therefore, the purpose of this paper is to extend the results of [26] to the general nonlinearity and to consider the existence of infinitely many sign-changing solutions in the critical case. First of all, for the nonlinear function f , we assume

$$(f_1) \quad |f(t)| \leq C \left(1 + |t|^{p-1} \right), \quad 2 < p < 2^*, \quad 2^* = +\infty \text{ for } N = 1, 2; \quad 2^* = \frac{2N}{N-2} \text{ for } N \geq 3;$$

$$(f_2) \quad \lim_{t \rightarrow 0} \frac{f(t)}{t} = 0, \quad \lim_{t \rightarrow +\infty} \frac{f(t)}{t} = +\infty;$$

$$(f_3) \quad \frac{1}{2} f(t)t \geq F(t) > 0, \text{ for } t \neq 0, \text{ where } F(t) = \int_0^t f(\tau) d\tau;$$

$$(f_4) \quad f(-t) = -f(t).$$

Next, we consider the following case of the nonlinearity with critical growth

$$\begin{cases} \left(a - \lambda \int_{\Omega} |\nabla u|^2 dx \right) \Delta u + |u|^{2^*-2}u + \mu |u|^{q-2}u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain, $a, \lambda,$ and μ are positive parameters, $2^* = \frac{2N}{N-2}$, and $\max\{2, 2^* - 1\} < q < 2^*$.

The main results of this paper reads as follows.

Theorem 1.1. *Let assumptions $(f_1) - (f_4)$ hold. Then problem (1.1) has infinitely many sign-changing solutions.*

Theorem 1.2. *Let $N \geq 7$ and $\max\{2, 2^* - 1\} < q < 2^*$. Then problem (1.5) has infinitely many sign-changing solutions.*

Remark 1.1. There are many functions that satisfy the assumptions $(f_1) - (f_4)$. For example, $f(t) = |t|^{p-2}t + \mu |t|^{q-2}t, 2 < q < p, \mu \geq 0; f(t) = t \ln(1 + |t|^{q-1}), 2 < q < p; f(t) = t(\ln(1 + |t|))^{q-1}, 2 < q < p.$

2. THE SUBCRITICAL CASE

Problem (1.1) has a variational structure given by the functional

$$I(u) = \frac{1}{2}a \int_{\Omega} |\nabla u|^2 dx - \frac{1}{4}\lambda \left(\int_{\Omega} |\nabla u|^2 dx \right)^2 - \int_{\Omega} F(u) dx, \quad u \in H_0^1(\Omega).$$

We have

$$(DI(u), \varphi) = \left(a - \lambda \int_{\Omega} |\nabla u|^2 dx \right) \int_{\Omega} \nabla u \nabla \varphi dx - \int_{\Omega} f(u) \varphi dx, \quad \varphi \in H_0^1(\Omega).$$

Any critical point u of functional I is a (weak) solution to problem (1.1), and

$$\left(a - \lambda \int_{\Omega} |\nabla u|^2 dx \right) \int_{\Omega} \nabla u \nabla \varphi dx = \int_{\Omega} f(u) \varphi dx, \quad \varphi \in H_0^1(\Omega).$$

We are looking for sign-changing solutions of problem (1.1) by using the method of invariant sets for descending flow, especially the following abstract critical point theorem (see [19] or [30]).

Theorem 2.1. *(The abstract critical point theorem) Let X be a Banach space, and let f be an even C^1 -functional on X . Let $P_j, Q_j, j = 1, \dots, k$ be open convex subsets of $X, Q_j = -P_j$. Set $W = \cup_{j=1}^k (P_j \cap Q_j), \Sigma = \cup_{j=1}^k (\partial P_j \cap \partial Q_j)$. Assume*

- (I₁) *f satisfies the Palais-Smale condition.*
- (I₂) $c^* = \inf_{x \in \Sigma} f(x) > 0.$

Assume that there exists an odd continuous mapping $A : X \rightarrow X$ satisfies

- (A₁) *Given $c_0, b_0 > 0$, there exist $b = b(c_0, b_0)$ such that if $\|Df(x)\| \geq b_0, |f(x)| \leq c_0$. Then*

$$\langle Df(x), x - Ax \rangle \geq b \|x - Ax\| > 0.$$

- (A₂) $A(\partial P_j) \subset P_j, A(\partial Q_j) \subset Q_j, j = 1, \dots, k.$

Define

$$\Gamma_j = \{E \mid E \subset X, E \text{ compact}, -E = E, \gamma(E \cap \sigma^{-1}(\Sigma)) \geq j, \text{ for } \sigma \in \Lambda\}$$

$$\Lambda = \{\sigma \mid C(X, X), \sigma \text{ odd}, \sigma(\partial P_j) \subset P_j, \sigma(\partial Q_j) \subset Q_j, j = 1, \dots, k, \sigma(x) = x, \text{ if } f(x) < 0\}.$$

Assume that

$$(\Gamma) \Gamma_j \text{ is nonempty, } j = 1, 2, \dots.$$

Define

$$c_j = \inf_{E \in \Gamma_j} \sup_{x \in E \setminus W} f(x); \quad K_c = \{x \mid Df(x) = 0, f(x) = c\}; \quad K_c^* = K_c \setminus W.$$

Then

- (1) $c_j \geq c^*, K_{c_j}^* \neq \emptyset$;
- (2) $c_j \rightarrow +\infty$, as $j \rightarrow \infty$;
- (3) if $c_j = c_{j+1} = \dots = c_{j+l-1} = c$, then $\gamma(K_c^*) \geq l$.

In the exposition of $DI(u)$, the term $a - \lambda \int_{\Omega} |\nabla u|^2 dx$ may change sign. This causes some difficulties when we apply the abstract critical point theory to functional I and try to define the operator A . So we truncate the term $a - \lambda \int_{\Omega} |\nabla u|^2 dx$. Given $\varepsilon > 0$, introduce two auxillary functions

$$M_{\varepsilon}(t) = \begin{cases} \frac{1}{2}at - \frac{1}{4}\lambda t^2, & \text{if } a - \lambda t \geq \varepsilon, \\ \frac{(a-\varepsilon)^2}{4\lambda} + \frac{1}{2}\varepsilon t, & \text{if } a - \lambda t < \varepsilon. \end{cases}$$

$$m_{\varepsilon}(t) = 2M'_{\varepsilon}(t) = \begin{cases} a - \lambda t, & \text{if } a - \lambda t \geq \varepsilon, \\ \varepsilon, & \text{if } a - \lambda t < \varepsilon. \end{cases}$$

Define the truncated functional

$$I_{\varepsilon}(u) = M_{\varepsilon}(\|u\|^2) - \int_{\Omega} F(u) dx, \quad u \in H_0^1(\Omega).$$

We have

$$(DI_{\varepsilon}(u), \varphi) = m_{\varepsilon}(\|u\|^2) \int_{\Omega} \nabla u \nabla \varphi dx - \int_{\Omega} f(u) \varphi dx, \text{ for } \varphi \in H_0^1(\Omega).$$

If u is a critical point of I_{ε} with $a - \lambda \int_{\Omega} |\nabla u|^2 dx \geq \varepsilon$, then $m_{\varepsilon}(\|u\|^2) = a - \lambda \int_{\Omega} |\nabla u|^2 dx$, and u is a critical point of I too, and $I(u) = I_{\varepsilon}(u)$.

Proposition 2.1. Let $c < \frac{a^2}{4\lambda}$. Assume that u_{ε} is a critical point of I_{ε} with $c_{\varepsilon} \leq I_{\varepsilon}(u_{\varepsilon}) \leq c$. Then, for $\varepsilon > 0$ small enough, it holds that $a - \lambda \int_{\Omega} |\nabla u_{\varepsilon}|^2 dx \geq \varepsilon$, and u_{ε} is a critical point of I and $I(u_{\varepsilon}) = I_{\varepsilon}(u_{\varepsilon})$.

Proof. Otherwise. There exist sequences $\varepsilon_n \rightarrow 0$ and $u_n \in H_0^1(\Omega)$ such that $c_{\varepsilon_n} = I_{\varepsilon_n}(u_n) \leq c < \frac{a^2}{4\lambda}$, $DI_{\varepsilon_n}(u_n) = 0$, but $a - \lambda \int_{\Omega} |\nabla u_n|^2 dx \leq \varepsilon_n$. We have

$$c_{\varepsilon_n} = I_{\varepsilon_n}(u_n) = \frac{a^2}{4\lambda} + \frac{\varepsilon_n^2}{4\lambda} + \frac{1}{2}\varepsilon_n \left(\int_{\Omega} |\nabla u_n|^2 dx - \frac{a}{\lambda} \right) - \int_{\Omega} F(u_n) dx,$$

$$0 = (DI_{\varepsilon_n}(u_n), u_n) = \varepsilon_n \int_{\Omega} |\nabla u_n|^2 dx - \int_{\Omega} F(u_n) u_n dx,$$

and

$$\begin{aligned} \frac{a^2}{4\lambda} > c \geq c_{\varepsilon_n} &= I_{\varepsilon_n}(u_n) = I_{\varepsilon_n}(u_n) - \frac{1}{2}(DI_{\varepsilon_n}(u_n), u_n) \\ &= \frac{a^2}{4\lambda} + \frac{\varepsilon_n^2}{4\lambda} - \frac{\varepsilon_n a}{2\lambda} + \int_{\Omega} \left(\frac{1}{2} f(u_n) u_n - F(u_n) \right) dx \\ &\geq \frac{a^2}{4\lambda} - \frac{\varepsilon_n a}{2\lambda} \rightarrow \frac{a^2}{4\lambda} \text{ as } n \rightarrow \infty, \end{aligned}$$

which is a contradiction. □

In the following, we verify that the truncated functional I_{ε} satisfies all the assumptions of Theorem 2.1.

Lemma 2.1. *Let $c < \frac{a^2}{4\lambda}$. Then, for $\varepsilon > 0$ small enough, the functional I_{ε} satisfies the Cerami condition. That is, any sequence $\{u_n\}$ in $H_0^1(\Omega)$, for $I_{\varepsilon}(u_n) \rightarrow c$ and $(1 + \|u_n\|) \|DI(u_n)\| \rightarrow 0$ as $n \rightarrow \infty$, possesses a convergent subsequence.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ satisfy $I_{\varepsilon}(u_n) \rightarrow c$ and $(1 + \|u_n\|) \|DI(u_n)\| \rightarrow 0$ as $n \rightarrow \infty$. Denote $\mu_n := m_{\varepsilon}(\|u_n\|^2)$, $\mu_n \in [\varepsilon, a]$. Up to a subsequence, there are two possibilities, namely

- (1) $a - \lambda \int_{\Omega} |\nabla u_n|^2 dx \leq \varepsilon$, $\mu_n = m_{\varepsilon}(\|u_n\|^2) = \varepsilon$;
- (2) $a - \lambda \int_{\Omega} |\nabla u_n|^2 dx \geq \varepsilon$, $\mu_n = m_{\varepsilon}(\|u_n\|^2) = a - \lambda \int_{\Omega} |\nabla u_n|^2 dx \geq \varepsilon$.

If $a - \lambda \int_{\Omega} |\nabla u_n|^2 dx \leq \varepsilon$, then $\mu_n = m_{\varepsilon}(\|u_n\|^2) = \varepsilon$. We have

$$\begin{aligned} I_{\varepsilon}(u_n) + o(1) &= I_{\varepsilon}(u_n) - \frac{1}{2}(DI_{\varepsilon}(u_n), u_n) \\ &= \frac{a^2}{4\lambda} + \frac{\varepsilon^2}{4\lambda} - \frac{\varepsilon a}{2\lambda} + \int_{\Omega} \left(\frac{1}{2} f(u_n) u_n - F(u_n) \right) dx \\ &\geq \frac{a^2}{4\lambda} - \frac{\varepsilon(2a - \varepsilon)}{4\lambda}. \end{aligned}$$

Letting $n \rightarrow \infty$, we obtain $c \geq \frac{a^2}{4\lambda} - \frac{\varepsilon(2a - \varepsilon)}{4\lambda}$, which is a contradiction for ε small enough. Therefore, case (1) does not happen.

If $\mu_n \rightarrow \mu \geq \varepsilon$, then $\{u_n\}$ is bounded in $H_0^1(\Omega)$. We assume that $\{u_n\}$ converges in $L^p(\Omega)$. We have

$$\begin{aligned} o(1) &= (DI_{\varepsilon}(u_n) - DI_{\varepsilon}(u_m), u_n - u_m) \\ &= \mu_n \left\{ \int_{\Omega} \nabla u_n \nabla (u_n - u_m) dx - \int_{\Omega} f(u_n) (u_n - u_m) dx \right\} \\ &\quad - \mu_m \left\{ \int_{\Omega} \nabla u_m \nabla (u_n - u_m) dx - \int_{\Omega} f(u_m) (u_n - u_m) dx \right\} \\ &= \mu \int_{\Omega} |\nabla (u_n - u_m)|^2 dx + o(1), \end{aligned}$$

which implies that u_n is a Cauchy sequence in $H_0^1(\Omega)$. Hence u_n has a convergent sequence. □

Remark 2.1. In the abstract critical point theorem, we assume that the related functional satisfies the Palais-Smale condition (the assumption (I_1)), which is needed for the deformation

lemma. It is well known that, with the help of the Cerami condition (the Palais-Smale condition in the sense of Cerami) the deformation lemma (and consequently the abstract critical point theorem) still holds.

For $\delta > 0$, define open convex subsets of $H_0^1(\Omega)$

$$P = \left\{ u \mid u \in H_0^1(\Omega), \|u_-\|_{L^p(\Omega)} < \delta \right\}, \quad Q = \left\{ u \mid u \in H_0^1(\Omega), \|u_+\|_{L^p(\Omega)} < \delta \right\}.$$

Lemma 2.2. *For $\varepsilon, \delta > 0$ small enough, there exists $c_\varepsilon^* > 0$ such that*

$$I_\varepsilon(u) \geq c_\varepsilon^* > 0, \quad \text{for } u \in \Sigma_\varepsilon = \partial P \cap \partial Q.$$

Proof. First, assume $M_\varepsilon(\|u\|^2) = a - \lambda \int_\Omega |\nabla u|^2 dx \geq \varepsilon$. Then

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}a \int_\Omega |\nabla u|^2 dx - \frac{1}{4}\lambda \left(\int_\Omega |\nabla u|^2 dx \right)^2 - \int_\Omega F(u) dx \\ &= \frac{1}{4}a \int_\Omega |\nabla u|^2 dx + \frac{1}{4} \left(a - \lambda \int_\Omega |\nabla u|^2 dx \right) \int_\Omega |\nabla u|^2 dx - \int_\Omega F(u) dx \\ &\geq \frac{1}{4}a \int_\Omega |\nabla u|^2 dx - \int_\Omega F(u) dx \\ &\geq \frac{1}{4}a \int_\Omega |\nabla u|^2 dx - \int_\Omega (\mu u^2 + C_\mu |u|^p) dx \\ &\geq \frac{1}{4}C_0a \|u\|_{L^p(\Omega)}^2 - C_1\mu \|u\|_{L^p(\Omega)}^2 - C_\mu \|u\|_{L^p(\Omega)}^p. \end{aligned} \tag{2.1}$$

Let $u \in \partial P \cap \partial Q$ and $\|u_+\|_{L^p(\Omega)} = \|u_-\|_{L^p(\Omega)} = \delta$. Hence

$$\delta = \|u_+\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} \leq \|u_+\|_{L^p(\Omega)} + \|u_-\|_{L^p(\Omega)} \leq 2\delta. \tag{2.2}$$

Fixing $\mu \in (0, \frac{C_0a}{64C_1})$, we can choose $\delta = \delta(\varepsilon)$ such that $2^p C_\mu \delta^{p-2} \leq \frac{1}{16}C_0a$. It follows from (2.1) and (2.2) that, for $u \in \partial P \cap \partial Q$,

$$I_\varepsilon(u) \geq \frac{1}{4}C_0a\delta^2 - 4C_1\mu\delta^2 - 2^p C_\mu \delta^p \geq \frac{1}{8}C_0a\delta^2.$$

Next, assume $M_\varepsilon(\|u\|^2) = \varepsilon$, $a - \lambda \int_\Omega |\nabla u|^2 dx \leq \varepsilon$. Then, for $\varepsilon < \frac{a}{2}$,

$$\begin{aligned} I_\varepsilon(u) &= \frac{a^2}{4\lambda} + \frac{\varepsilon^2}{4\lambda} + \frac{1}{2}\varepsilon \left(\int_\Omega |\nabla u|^2 dx - \frac{a}{\lambda} \right) - \int_\Omega F(u) dx \\ &\geq \frac{a^2}{4\lambda} - \frac{\varepsilon^2}{4\lambda} - \int_\Omega F(u) dx. \\ &\geq \frac{a^2}{8\lambda} - \int_\Omega F(u) dx. \end{aligned}$$

For $u \in \partial P \cap \partial Q$, $\|u_+\|_{L^p(\Omega)} = \|u_-\|_{L^p(\Omega)} = \delta$, $\delta \leq \|u\|_{L^p(\Omega)} \leq 2\delta$, and

$$I_\varepsilon(u) \geq \frac{a^2}{8\lambda} - 4C_1\mu\delta^2 - 2^p C_\mu \delta^p \geq \frac{a^2}{16\lambda},$$

provided

$$4C_1\mu\delta^2 + 2^p C_\mu \delta^p \leq \frac{a^2}{16\lambda}.$$

In the case $0 < \varepsilon < \frac{a}{2}$ and $\delta = \delta(\varepsilon) > 0$ small enough, we have

$$I_\varepsilon(u) \geq c_\varepsilon^* := \min \left\{ \frac{1}{8}C_0a\delta^2, \frac{a^2}{16\lambda} \right\} \text{ for } u \in \partial P \cap \partial Q.$$

□

Definition 2.1. Define the odd continuous mapping A_ε in $H_0^1(\Omega)$

$$u \in H_0^1(\Omega) \mapsto v = A_\varepsilon u \in H_0^1(\Omega),$$

by

$$m_\varepsilon \left(\|u\|^2 \right) \int_\Omega \nabla v \nabla \varphi dx = \int_\Omega f(u) \varphi dx, \quad \varphi \in H_0^1(\Omega).$$

Lemma 2.3. For $u \in H_0^1(\Omega)$, $v = A_\varepsilon u$, it holds that $(DI_\varepsilon(u), u - v) \geq \varepsilon \|u - v\|^2$ and $\|DI_\varepsilon(u)\| \leq a \|u - v\|$. Hence, for $\|DI_\varepsilon(u)\| \geq b_0 > \varepsilon$, it holds that

$$(DI_\varepsilon(u), u - v) \geq \frac{\varepsilon b_0}{a} \|u - v\|.$$

Proof. By the definition of $v = A_\varepsilon u$, we have

$$\begin{aligned} (DI_\varepsilon(u), \varphi) &= m_\varepsilon \left(\|u\|^2 \right) \int_\Omega \nabla u \nabla \varphi dx - \int_\Omega f(u) \varphi dx \\ &= m_\varepsilon \left(\|u\|^2 \right) \int_\Omega \nabla(u - v) \nabla \varphi dx, \quad \varphi \in H_0^1(\Omega). \end{aligned}$$

Notice that $m_\varepsilon \left(\|u\|^2 \right) \in [\varepsilon, a]$. We have

$$\begin{aligned} (DI_\varepsilon(u), u - v) &= m_\varepsilon \left(\|u\|^2 \right) \int_\Omega |\nabla(u - v)|^2 dx \geq \varepsilon \|u - v\|^2, \\ \|DI_\varepsilon(u)\| &= \sup \{ (DI_\varepsilon(u), \varphi) \mid \varphi \in H_0^1(\Omega), \|\varphi\| = 1 \} \\ &= \sup \left\{ m_\varepsilon \left(\|u\|^2 \right) \int_\Omega \nabla(u - v) \nabla \varphi dx \mid \varphi \in H_0^1(\Omega), \|\varphi\| = 1 \right\} \\ &\leq m_\varepsilon \left(\|u\|^2 \right) \|u - v\| \leq a \|u - v\|. \end{aligned}$$

□

Lemma 2.4. For $\delta = \delta(\varepsilon)$ small enough,

$$A_\varepsilon(\partial P) \subset P, \quad A_\varepsilon(\partial Q) \subset Q.$$

Proof. Let $u \in \partial Q$, $\|u_+\|_{L^p(\Omega)} = \delta$. By the definition $v = A_\varepsilon u$,

$$m_\varepsilon \left(\|u\|^2 \right) \int_\Omega \nabla v \nabla \varphi dx = \int_\Omega f(u) \varphi dx, \quad \varphi \in H_0^1(\Omega). \tag{2.3}$$

Let $\varphi = v_+$ as the test function in (2.3). The left hand side of (2.3)

$$LHS = m_\varepsilon \left(\|u\|^2 \right) \int_\Omega \nabla v \nabla v_+ dx \geq \varepsilon \int_\Omega |\nabla v_+|^2 dx \geq C_1 \varepsilon \|v_+\|_{L^p(\Omega)}^2.$$

The right hand side of (2.3)

$$\begin{aligned} RHS &= \int_{\Omega} f(u)v_+ dx \leq \int_{\Omega} f(u_+)v_+ dx \\ &\leq \int_{\Omega} (\mu u_+ + C_{\mu} u_+^{p-1}) v_+ dx \leq C_2 \mu \|u_+\|_{L^p(\Omega)} \|v_+\|_{L^p(\Omega)} + C_{\mu} \|u_+\|_{L^p(\Omega)}^{p-1} \|v_+\|_{L^p(\Omega)}. \end{aligned}$$

Choose $\mu = \mu(\varepsilon) > 0$ and $\delta = \delta(\varepsilon) > 0$ such that $C_2 \mu \leq \frac{1}{4} C_1 \varepsilon$ and $C_{\mu} \delta^{p-2} \leq \frac{1}{4} C_1 \varepsilon$. Then, for $\delta < \delta(\varepsilon)$, we have

$$\begin{aligned} C_1 \varepsilon \|v_+\|_{L^p(\Omega)}^2 &\leq C_2 \mu \|u_+\|_{L^p(\Omega)} \|v_+\|_{L^p(\Omega)} + C_{\mu} \|u_+\|_{L^p(\Omega)}^{p-1} \|v_+\|_{L^p(\Omega)} \\ &\leq (C_2 \mu + C_{\mu} \delta^{p-2}) \delta \|v_+\|_{L^p(\Omega)} \leq \frac{1}{2} C_1 \varepsilon \delta \|v_+\|_{L^p(\Omega)}. \end{aligned}$$

Hence

$$\|v_+\| \leq \frac{1}{2} \delta, \quad v = Au \in Q, \quad \text{for } u \in \partial Q.$$

Similarly, we have $A(\partial P) \subset P$. □

Definition 2.2. Define

$$\begin{aligned} \Gamma_j &= \Gamma_j(\varepsilon) = \{E \mid E \in H_0^1(\Omega), E \text{ compact}, -E = E, \gamma(E \cap \sigma^{-1}(\Sigma_{\varepsilon})) \geq j, \text{ for } \sigma \in \Lambda\}, \\ \Lambda &= \Lambda(\varepsilon) = \{\sigma \mid \sigma \in C(H_0^1(\Omega), H_0^1(\Omega)), \sigma \text{ is odd}, \sigma(u) = u, \text{ if } I_{\varepsilon}(u) \leq 0\}, \end{aligned}$$

$$c_j(\varepsilon) = \inf_{E \in \Gamma_j} \sup_{u \in E \setminus (P \cup Q)} I_{\varepsilon}(u), \quad j = 1, 2, \dots.$$

Lemma 2.5. Given a positive integer j , there exist $\varepsilon_j > 0$ and $d_j < \frac{a^2}{4\lambda}$ such that $0 < \varepsilon < \varepsilon_j$. The family Γ_j is nonempty and $c_j(\varepsilon) \leq d_j$. Hence $c_j(\varepsilon)$ is a critical value of I_{ε} .

Proof. We construct a set E_j , which belongs to Γ_j , estimate $\sup_{u \in E} I_{\varepsilon}(u)$. Choose $j + 1$ functions $v_1, \dots, v_{j+1} \in C_0^{\infty}(\Omega)$. Denote $X_j = \text{span} \{v_1, \dots, v_{j+1}\}$, $E_j = \{u \mid u \in X_j, \|u\| \leq R\}$, $R > 0$. Choose R large enough such that

$$\alpha_R = \inf \left\{ \int_{\Omega} F(u) dx \mid u \in E_j, \|u\| = R \right\} \geq \frac{a^2}{\lambda}.$$

Fix R and choose ε_0 such that $\frac{\varepsilon_0^2}{4\lambda} + \frac{1}{2} \varepsilon_0 R^2 \leq \frac{1}{2} \alpha_R$. Assume $0 < \varepsilon < \varepsilon_0$. For $u \in E_j$, $\|u\| = R$ (that is $u \in \partial E_j$), we have

$$\begin{aligned} I_{\varepsilon}(u) &= \frac{a^2}{4\lambda} + \frac{\varepsilon^2}{4\lambda} + \frac{1}{2} \varepsilon \left(\int_{\Omega} |\nabla u|^2 dx - \frac{a}{\lambda} \right) - \int_{\Omega} F(u) dx \\ &\leq \frac{a^2}{4\lambda} + \frac{\varepsilon_0^2}{4\lambda} + \frac{1}{2} \varepsilon_0 R^2 - \alpha_R \\ &\leq \frac{a^2}{4\lambda} - \frac{1}{2} \alpha_R \leq -\frac{a^2}{4\lambda} < 0. \end{aligned}$$

By [19, Lemma 5.6], we have $E_j \in \Gamma_j$. We estimate $\sup_{u \in E_j} I_{\varepsilon}(u)$. Choose $\delta > 0$ such that

$$\frac{1}{2} a \delta^2 \leq \frac{a^2}{8\lambda}.$$

Let

$$\alpha = \inf \left\{ \int_{\Omega} F(u) dx \mid u \in E_j, \delta \leq \|u\| \leq R \right\} > 0,$$

and unique ε_0 satisfy $\frac{\varepsilon_0^2}{4\lambda} + \frac{1}{2}aR^2 \leq \frac{1}{2}\alpha$. Now, for $u \in E_j, \|u\| \leq \delta$, we have

$$\begin{aligned} I_{\varepsilon}(u) &= \frac{1}{2}a \int_{\Omega} |\nabla u|^2 dx - \frac{1}{4}\lambda \left(\int_{\Omega} |\nabla u|^2 dx \right)^2 - \int_{\Omega} F(u) dx \\ &\leq \frac{1}{2}a \int_{\Omega} |\nabla u|^2 dx \leq \frac{1}{2}a\delta^2 \leq \frac{a^2}{8\lambda} = \frac{a^2}{4\lambda} - \frac{a^2}{8\lambda}. \end{aligned} \tag{2.4}$$

For $u \in E_j, \delta \leq \|u\| \leq \frac{1}{\lambda}(a - \varepsilon)$, we have

$$\begin{aligned} I_{\varepsilon}(u) &= \frac{1}{2}a \int_{\Omega} |\nabla u|^2 dx - \frac{1}{4}\lambda \left(\int_{\Omega} |\nabla u|^2 dx \right)^2 - \int_{\Omega} F(u) dx \\ &\leq \frac{a^2}{4\lambda} - \alpha. \end{aligned} \tag{2.5}$$

For $u \in E_j, \|u\| \geq \frac{1}{\lambda}(a - \varepsilon)$, we have

$$\begin{aligned} I_{\varepsilon}(u) &= \frac{a^2}{4\lambda} + \frac{\varepsilon^2}{4\lambda} + \frac{1}{2}\varepsilon \left(\int_{\Omega} |\nabla u|^2 dx - \frac{a}{\lambda} \right) - \int_{\Omega} F(u) dx \\ &\leq \frac{a^2}{4\lambda} + \frac{\varepsilon_0^2}{2\lambda} + \frac{1}{2}\varepsilon_0 R^2 - \alpha \leq \frac{a^2}{4\lambda} - \frac{1}{2}\alpha. \end{aligned} \tag{2.6}$$

Let $d_j = \frac{a^2}{4\lambda} - \min \left(\frac{a^2}{8\lambda}, \frac{1}{2}\alpha \right)$. $d_j < \frac{a^2}{4\lambda}$ and d_j is independent of ε . For $0 < \varepsilon < \varepsilon_0$, we have $c_j(\varepsilon) \leq d_j < \frac{a^2}{4\lambda}$. □

Proof of Theorem 1.1. Given a positive integer j , by Lemma 2.5, there exist ε_j and $d_j < \frac{a^2}{4\lambda}$ such that, for $0 < \varepsilon < \varepsilon_j$, $c_j(\varepsilon)$ is a critical value of I_{ε} , $c_j(\varepsilon) \leq d_j < \frac{a^2}{4\lambda}$ and the corresponding critical point $u_j(\varepsilon)$ is sign-changing. Without loss of generality, we assume that ε_j is decreasing in j and d_j is increasing in j . Further reducing ε_j , by Proposition 2.1, we may assume that $c_j(\varepsilon)$ is also critical value of the functional I . Now, given a positive integer k , for $0 < \varepsilon < \varepsilon_k$, $c_1(\varepsilon), \dots, c_k(\varepsilon)$ are critical values of I , and the corresponding critical point $u_1(\varepsilon), \dots, u_k(\varepsilon)$ are (weak) sign-changing solutions of the problem (1.1). Since k is arbitrary, we obtain an infinite sequence of (weak) sign-changing solutions of problem (1.1). □

3. THE CRITICAL CASE

In this section, we consider infinitely many sign-changing solutions of problem (1.5). We will use the idea of truncation in [29] to deal with critical growth terms. Let $\psi(t) \in C_0^{\infty}(\mathbb{R}, [0, 1])$ be a smooth even function with the following properties: $\psi(t) = 1$ for $|t| \leq 1$, $\psi(t) = 0$ for $|t| \geq 2$ and $\psi(t)$ is monotonically decreasing on the interval $(0, +\infty)$. Define

$$\begin{aligned} b_{\varepsilon}(t) &= \psi(\varepsilon t), & k_{\varepsilon}(t) &= \int_0^t b_{\varepsilon}(\tau) d\tau, \\ f_{\varepsilon}(t) &= \left(\frac{k_{\varepsilon}(t)}{t} \right)^r |t|^{2^*-2} t + \mu |t|^{q-2} t, & F_{\varepsilon}(t) &= \int_0^t f_{\varepsilon}(\tau) d\tau, \end{aligned}$$

for $\varepsilon \in (0, 1]$ and $0 < r < \min\{2^* - q, q - q_N\}$, where $q_N = \max\{2, 2^* - 1\}$. We will deal with the perturbation problem

$$\begin{cases} M_\varepsilon \left(\int_\Omega |\nabla u|^2 dx \right) \Delta u + f_\varepsilon(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

For $|t| \leq \frac{1}{\varepsilon}$, we have $k_\varepsilon(t) = t$ and $f_\varepsilon(t) = |t|^{2^*-2}t + \mu|t|^{q-2}t$. Therefore, if the solution u_ε of problem (3.1) is uniformly bounded with respect to ε , then u_ε is also the solution to problem (1.5). Define the truncated functional

$$J_\varepsilon(u) = M_\varepsilon \left(\|u\|^2 \right) - \int_\Omega F_\varepsilon(u) dx, \quad u \in H_0^1(\Omega).$$

Lemma 3.1. *The function $F_\varepsilon(t)$ defined above satisfies, for all $t \neq 0$, $F_\varepsilon(t) \leq \frac{1}{q}t f_\varepsilon(t)$.*

Proof. Since $k_\varepsilon(t)$ is a monotonically nondecreasing on the interval $(0, +\infty)$, we have

$$\begin{aligned} F_\varepsilon(t) &= \int_0^t \left(\left(\frac{k_\varepsilon(\tau)}{\tau} \right)^r \tau^{2^*-1} + \mu \tau^{q-1} \right) d\tau \\ &\leq \int_0^t \left(\left(\frac{k_\varepsilon(t)}{\tau} \right)^r \tau^{2^*-1} + \mu \tau^{q-1} \right) d\tau \\ &= \frac{1}{2^*-r} k_\varepsilon(t)^r t^{2^*-r} + \frac{\mu}{q} t^q \\ &\leq \frac{1}{q} t f_\varepsilon(t) \end{aligned}$$

for $t > 0$. Since $f_\varepsilon(t)$ is an odd function and $F_\varepsilon(t)$ is an even function for t , one has

$$F_\varepsilon(t) = F_\varepsilon(-t) < -\frac{1}{q}t f_\varepsilon(-t) = \frac{1}{q}t f_\varepsilon(t)$$

for $t < 0$. □

Lemma 3.2. *Let $c < \frac{a^2}{4\lambda}$. Assume that $u_\varepsilon \in H_0^1(\Omega)$ satisfy $DJ_\varepsilon(u_\varepsilon) = 0$ and $I_\varepsilon(u) \leq c$. Then*

- (i) *there exists $\varepsilon_0 > 0$ such that $a - \lambda \int_\Omega |\nabla u_\varepsilon|^2 dx \geq \varepsilon$ for $0 < \varepsilon < \varepsilon_0$;*
- (ii) *there exist $\varepsilon_1 > 0$ and $\sigma > 0$ such that $a - \lambda \int_\Omega |\nabla u_\varepsilon|^2 dx \geq \sigma$ for $0 < \varepsilon < \varepsilon_1$.*

Proof. Suppose (i) otherwise. There exist sequences $\varepsilon_n \rightarrow 0$ and $u_n = u_{\varepsilon_n} \in H_0^1(\Omega)$ such that $DJ_{\varepsilon_n}(u_n) = 0$ and $J_{\varepsilon_n}(u_n) \leq c < \frac{a^2}{4\lambda}$, but $a - \lambda \int_\Omega |\nabla u_n|^2 dx \leq \varepsilon_n$. According to Lemma 3.1, we have

$$\begin{aligned} c &\geq J_{\varepsilon_n}(u_n) \\ &= J_{\varepsilon_n}(u_n) - \frac{1}{2}(DJ_{\varepsilon_n}(u_n), u_n) \\ &= \left(\frac{(a - \varepsilon_n)^2}{4\lambda} + \frac{1}{2}\varepsilon_n \int_\Omega |\nabla u_n|^2 dx - \int_\Omega F_{\varepsilon_n}(u_n) dx \right) - \frac{1}{2} \left(\varepsilon_n \int_\Omega |\nabla u_n|^2 dx - f_{\varepsilon_n}(u_n) u_n dx \right) \\ &= \frac{(a - \varepsilon_n)^2}{4\lambda} + \int_\Omega \left(\frac{1}{2} f_{\varepsilon_n}(u_n) u_n - F_{\varepsilon_n}(u_n) \right) dx \\ &\geq \frac{(a - \varepsilon_n)^2}{4\lambda} \rightarrow \frac{a^2}{4\lambda} \end{aligned}$$

as $n \rightarrow \infty$, which is a contradiction.

Suppose (ii). According to (i), we can assume that there exist sequences $\varepsilon_n \rightarrow 0$, $\sigma_n \rightarrow 0$ and $u_n = u_{\varepsilon_n} \in H_0^1(\Omega)$ such that $DJ_{\varepsilon_n}(u_n) = 0$ and $J_{\varepsilon_n}(u_n) \leq c < \frac{a^2}{4\lambda}$, but $\varepsilon_n \leq a - \lambda \int_{\Omega} |\nabla u_n|^2 dx \leq \delta_n$ as $n \rightarrow \infty$. It follows that $\int_{\Omega} |\nabla u_n|^2 dx \rightarrow \frac{a}{\lambda}$ as $n \rightarrow \infty$. By Lemma 3.1, one has

$$\begin{aligned} c &\geq J_{\varepsilon_n}(u_n) \\ &= J_{\varepsilon_n}(u_n) - \frac{1}{2}(DJ_{\varepsilon_n}(u_n), u_n) \\ &= \left(\frac{1}{2}a \int_{\Omega} |\nabla u_n|^2 dx - \frac{1}{4}\lambda \left(\int_{\Omega} |\nabla u_n|^2 dx \right)^2 - \int_{\Omega} F_{\varepsilon_n}(u_n) dx \right) \\ &\quad - \frac{1}{2} \left(\left(a - \lambda \int_{\Omega} |\nabla u_n|^2 dx \right) \int_{\Omega} |\nabla u_n|^2 dx - f_{\varepsilon_n}(u_n) u_n dx \right) \\ &= \frac{1}{4}\lambda \left(\int_{\Omega} |\nabla u_n|^2 dx \right)^2 + \int_{\Omega} \left(\frac{1}{2}f_{\varepsilon_n}(u_n) u_n - F_{\varepsilon_n}(u_n) \right) dx \\ &\geq \frac{1}{4}\lambda \left(\int_{\Omega} |\nabla u_n|^2 dx \right)^2 \rightarrow \frac{a^2}{4\lambda} \end{aligned}$$

as $n \rightarrow \infty$, which is a contradiction. □

Proof of Theorem 1.2. By Lemma 3.1, it is easy to prove that J_{ε} satisfies the Palais-Smale condition. Similar to subcritical case, we can use the abstract critical point theorem 2.1 to obtain the existence of infinitely many sign-changing solutions $u_j(\varepsilon)$ to problem (3.1) and $J_{\varepsilon}(u_j(\varepsilon)) < \frac{a^2}{4\lambda}$. Now we just need to show that $u_j(\varepsilon)$ is also the solution to problem (1.5).

Let u_{ε} be a critical point of J_{ε} with $J_{\varepsilon}(u_{\varepsilon}) \leq c < \frac{a^2}{4\lambda}$. By Lemma 3.2, for $\varepsilon > 0$ small enough, we have

$$\begin{cases} (a - \lambda (\int_{\Omega} |\nabla u_{\varepsilon}|^2 dx)) \Delta u_{\varepsilon} + f_{\varepsilon}(u_{\varepsilon}) = 0, & \text{in } \Omega, \\ u_{\varepsilon} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Set $a_{\varepsilon} = a - \lambda \int_{\Omega} |\nabla u_{\varepsilon}|^2 dx$. According to Lemma 3.2, there exist $\varepsilon_1 > 0$ and $\sigma > 0$ such that $\sigma \leq a_{\varepsilon} \leq a\sigma$ for $0 < \varepsilon < \varepsilon_1$. This implies that u_{ε} is bounded in $H_0^1(\Omega)$ and satisfies

$$\begin{cases} a_{\varepsilon} \Delta u_{\varepsilon} + f_{\varepsilon}(u_{\varepsilon}) = 0, & \text{in } \Omega, \\ u_{\varepsilon} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Since a_{ε} is uniformly bounded and both the upper and lower bounds are positive, and u_{ε} is bounded in $H_0^1(\Omega)$, similar to the case of $a_{\varepsilon} \equiv 1$ in [8, Theorem 1.1], we can prove that u_{ε} is uniformly bounded by the concentration estimates. In other words, there exists $M > 0$ such that $|u_{\varepsilon}| < M$, for all $x \in \bar{\Omega}$. By the definition of the function k_{ε} , we have $k_{\varepsilon}(t) = t$ for $t \leq \frac{1}{\varepsilon}$. Hence, problem (3.1) reduces to problem (1.5) for $|u| \leq \frac{1}{\varepsilon}$. Let $\varepsilon < \min\{\varepsilon_0, \varepsilon_1, \frac{1}{2M}\}$. By Lemma 3.2, we have

$$|u_{\varepsilon}| \leq \frac{1}{\varepsilon} \quad \text{and} \quad a - \lambda \int_{\Omega} |\nabla u_{\varepsilon}|^2 dx \geq \varepsilon.$$

Then u_{ε} is indeed a solution to problem (1.5). The proof of Theorem 2.1 is now complete. □

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