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EXISTENCE RESULTS FOR MIXED P-LAPLACIAN AND FRACTIONAL P-LAPLACIAN EQUATIONS WITH SOBOLEV CRITICAL EXPONENTS

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Abstract. By developing a method of Lieb's compactness Theorem with refined ernergy estimation, we overcome the lack of translation invariance and establish existence of ground states. Our unified approach handles both subcritical and doubly critical cases under nonsymmetric perturbations.

Keywords. Critical exponent; Fractional p-Laplacian equations; Ground state; Local and nonlocal operators; Scaling invariance.

1. INTRODUCTION

In this paper, we investigate the following problem

$$-\Delta_p u + (-\Delta)_p^s u = F(x)|u|^{r-2}u, \; x \in \mathbb{R}^N, \tag{$\mathscr{P}$}$$

where $N > p > 1$, $0 < s < 1$, $p_s^* < r < p^*$, with $p_s^* = \frac{pN}{N-ps}$, $p^* = \frac{pN}{N-p}$ denoting the fractional Sobolev critical exponent and the classical Sobolev critical exponent, respectively. The perturbation function is nonsymmetric and satisfies

(F_1) $1 \not\equiv F \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$1 = \lim_{|x| \to +\infty} F(x) = \inf_{x \in \mathbb{R}^N} F(x).$$

To analyze equation ($\mathscr{P}$), we introduce some standard notation: the p-Laplacian is defined as $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, and the fractional p-Laplacian $(-\Delta)_p^s$, up to a normalization factor, is given for $u \in C_0^\infty(\mathbb{R}^N)$ by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy, \quad x \in \mathbb{R}^N,$$

where $B_\varepsilon(x)$ denotes the open ball of radius ε centered at x.

In recent years, there has been growing interest in nonlinear problems involving the mixed type operators $-\Delta_p + (-\Delta)_p^s$, particularly in relation to optimal animal foraging patterns (see

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[13]). It is worth emphasizing that significant progress has been made in several key areas, including the local behavior of mixed local and nonlocal problems [11], global gradient estimates [4], the Brezis–Oswald approach for mixed operators [5], and regularity properties of mixed local and nonlocal elliptic equations [29]. In [27], the authors dealt with the following equation involving local and nonlocal operators

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = \lambda_p u^q + u^r, & \text{in } \Omega, \\ u(x) > 0, & \text{in } \Omega, \\ u(x) = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $s \in (0,1), 2 \leqslant p < \infty, 0 < q(p) < p-1 < r(p) < \infty$ and $0 < \lambda_p < \infty$. They obtained existence and global uniform explicit boundedness results for weak solutions. Silva, Fiscella, and Viloria [28] extended the results of [27] to the critical case. Specifically, they considered the problem

$$\begin{cases} -\Delta_p u + (-\Delta)_p^s u = \lambda u^{q-2} u + u^{p^*-2} u, & \text{in } \Omega, \\ u(x) = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary, $N \geqslant 2, s \in (0,1), 1 \leqslant p < N, q \in (1, p^*), \lambda > 0$, and $p^* = \frac{Np}{N-p}$. By combining variational methods with topological techniques such as Krasnoselskii genus and Lusternik–Schnirelman category theory, they analyzed three different scenarios depending on the exponent q. From a mathematical perspective, this operator poses a particularly striking challenge because of the interplay between its nonlocal nature and the lack of scaling invariance. When $s = 0, p = 2$, equation ($\mathscr{P}$) reduces to

$$-\Delta u + u = F(x)|u|^{r-2} u, \ x \in \mathbb{R}^N, \tag{$\mathscr{S}$}$$

where $N > 2$ and $2 < r < 2^* = \frac{2N}{N-2}$. Prior research on this equation includes contributions from Berestycki-Lions [6], Bahri-Lions [3], and Mederski [19]. A significant challenge in this problem arises from the unboundedness of $\mathbb{R}^N$, which prevents the compact embedding of $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$. Weinstein [30] explored the existence of a positive ground state solution for equation ($\mathscr{S}$) with $F(x) \equiv 1$. When $F(x)$ is radially symmetric, the problem can be restricted to the radial Sobolev space, where the compact embedding $H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ holds. In this setting, Nehari [21], Strauss [26], and Berestycki-Lions [7] proved that equation ($\mathscr{S}$) possesses infinitely many solutions. When $F(x)$ is G-invariant under some group action, Mederski [19] introduced a refined version of Lions' lemma and established the existence of solutions. However, these methods do not directly apply when $F(x)$ is neither radial nor symmetry-invariant. To address such cases, Lions [18] developed the celebrated concentration-compactness principle, which was subsequently used to prove the existence of a positive solution for equation ($\mathscr{S}$) under the following condition:

(F_2) $1 = \lim\limits_{|x| \to +\infty} F(x) = \inf_{x \in \mathbb{R}^N} F(x)$.

Ding and Ni [12] and Willem [31, Section 1.8] explored the existence of solutions for equation ($\mathscr{S}$) under condition (F_1). Zhu [32] further established the existence of sign-changing solutions for $N \geqslant 5$, assuming that F satisfies (F_2) together with an additional condition. For further related results, we refer to [1, 9, 10, 14, 20, 23, 24, 25].

Despite significant advances in the study of mixed-operator equations [27, 28], several key challenges remain unresolved. The seminal works of [27] and [28] established existence results for mixed operators, but their methods rely heavily on bounded domains or subcritical nonlinearities, leaving the case of double critical exponents—where fractional Sobolev and classical Sobolev criticalities coexist—untouched. Although Ding-Ni [12] and Zhu [32] made progress in dealing with nonsymmetric perturbations of local operators, their techniques have not yet been extended to the mixed-operators framework, owing to the inherent tension between local and nonlocal scaling behaviors.

Prior works leave three fundamental questions unresolved: (i) Can existence results be extended to the doubly critical case? (ii) What techniques can handle the fractioanl Sobolev and classical Sobolev critical exponents? (iii) How does a nonsymmetric perturbation affect the solution structure?

The present work provides positive answers to all three questions. We highlight the main challenges of our problem as follows. Although r lies in the subcritical range, the energy functional fails to satisfy the $(PS)_c$ condition due to the nonsymmetric perturbation F. Moreover, the simultaneous presence of local and nonlocal operators further complicates the compactness analysis. Having identified the key challenges posed by nonsymmetric perturbations, we now present the first existence result in the subcritical regime.

Theorem 1.1. *Under the assumptions* $N > p > 1$, $0 < s < 1$, $p_s^* < r < p^*$, *and* (F_1), *equation* $(\mathscr{P})$ *admits a ground state solution.*

Remark 1.1. The nonlocal nature of the fractional p-Laplacian brings considerable complexity to our analysis. Unlike its classical counterpart, this operator requires sophisticated technical treatment due to the interplay between nonlinearity and nonlocality, which presents both challenges and opportunities in our investigation. Moreover, under condition (F_1), translation invariance is lost. To overcome this difficulty, we combine Lieb's compactness theorem with Lemma 3.6 to recover $(PS)_c$ condition.

Based on the existence result established in Theorem 1.1 for the single-nonlinearity case, we now turn to the more complex scenario involving combined nonlinearities.

$$-\Delta_p u + (-\Delta)_p^s u = F(x)|u|^{r-2}u + F(x)|u|^{q-2}u,\ x \in \mathbb{R}^N, \tag{$D\mathscr{P}$}$$

where $N > p > 1$, $0 < s < 1$, and $p_s^* < r < q < p^*$.

Our second main result is as follows.

Theorem 1.2. *Under the assumptions* $N > p > 1$, $0 < s < 1$, $p_s^* < r < q < p^*$, *and* (F_1), *equation* $(D\mathscr{P})$ *admits a ground state solution.*

We extend our method to the more challenging critical case and focus on the following problem involving both fractional Sobolev and classical Sobolev critical exponents:

$$-\Delta_p u + (-\Delta)_p^s u = F(x)|u|^{p_s^*-2}u + F(x)|u|^{p^*-2}u + \lambda F(x)|u|^{r-2}u,\ x \in \mathbb{R}^N, \tag{$\mathscr{C}$}$$

where $\lambda > 0$ is a parameter.

This leads to our final existence result, which extends the previous theorems to the critical case while addressing the structural challenges posed by the double critical terms and the lack of symmetry.

Theorem 1.3. *Under the assumptions* $N > p > 1$, $0 < s < 1$, $p_s^* < r < p^*$, *and* (F_1), *there exists* $\tilde{\lambda} \in (0, +\infty)$ *such that, for every* $\lambda > \tilde{\lambda}$, *equation* $(\mathscr{C})$ *admits a ground state solution.*

Remark 1.2. To the best of our knowledge, ground state solutions for mixed p-Laplacian and fractional p-Laplacian equations involving double critical Sobolev exponents remain largely unexplored. The presence of double critical exponents renders this problem particularly challenging. We introduce an innovative strategy for verifying the $(PS)_c$ condition. This method combines the generalized Lieb compactness theorem (Lemma 5.1) with analytical techniques (Lemma 5.4).

Remark 1.3. In response to the three questions posed in Section 1, our work provides: (i) We establish the existence of ground state solutions for problems involving both the fractional Sobolev critical exponent p_s^* and the classical Sobolev critical exponent p^*, thereby extending previous results that were limited to subcritical or single-critial nonlinearities. (ii) To handle the coexistence of two critical exponents and the doubly nonlocal nature of the operator, we introduce a generalized Lieb compactness theorem (Lemma 5.1) and new scaling functions adapted to the fractional setting. These tools allow us to recover compactness and verify the $(PS)_c$ condition below a sharp threshold. (iii) The perturbation F affects the solution structure mainly through the energy threshold, but the mountain pass geometry and Nehari manifold method remain effective.

This paper is organized as follows: In Section 2, we give some preliminaries and key inequalities. Section 3 proves Lieb's compactness Theorem via two different approaches and establishes Theorem 1.1 concerning the existence of a ground state solution to $(\mathscr{P})$. Section 4 is devoted to Theorem 1.2, and Section 5 completes the proof of Theorem 1.3. The Appendix contains supplementary proofs.

2. PRELIMINARIES

The homogeneous fractional Sobolev space $D^{s,p}(\mathbb{R}^N)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the seminorm

$$\|u\|_{D^{s,p}(\mathbb{R}^N)}^p := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy.$$

The homogeneous Sobolev space $D^{1,p}(\mathbb{R}^N)$ is defined with the seminorm

$$\|u\|_{D^{1,p}(\mathbb{R}^N)}^p := \int_{\mathbb{R}^N} |\nabla u|^p\,dx.$$

The mixed Sobolev space $\mathbb{X}$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ under the seminorm

$$\|u\|_{\mathbb{X}}^p := \int_{\mathbb{R}^N} |\nabla u|^p\,dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy.$$

The weak solutions of $(\mathscr{P})$ corresponds to critical points of the associated energy functional $I : \mathbb{X} \to \mathbb{R}$ defined by

$$I(u) := \frac{1}{p}\left(\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right) - \frac{1}{r}\int_{\mathbb{R}^N} F(x)|u|^r\,dx.$$

The Fréchet derivative $I'(u)$ is given by

$$\langle I'(u),\varphi\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy$$
$$+\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx-\int_{\mathbb{R}^N}F(x)|u|^{r-2}u\varphi\,dx,$$

where $\varphi\in\mathbb{X}$.

We also consider the minimization problem (see [17, 18])

$$S_r := \inf_{u\in\mathbb{X}\setminus\{0\}}\frac{\|u\|^p_{\mathbb{X}}}{\left(\int_{\mathbb{R}^N}|u|^r\,dx\right)^{\frac{p}{r}}}. \tag{2.1}$$

Next, we present the inequalities that are crucial in the proof.

Lemma 2.1. (see [16, Refined Sobolev inequality with the Morrey norm]) *For $N>p>1$, there exists $C>0$ such that for any ι, ϑ satisfying*

$$\frac{2}{p^*}\leqslant\iota<1,\quad 1\leqslant\vartheta<p^*,$$

the following inequality holds for every $u\in D^{1,p}(\mathbb{R}^N)$:

$$\left(\int_{\mathbb{R}^N}|u|^{p^*}\,dx\right)^{\frac{1}{p^*}}\leqslant C\|u\|^{\iota}_{D^{1,p}(\mathbb{R}^N)}\|u\|^{1-\iota}_{\mathscr{M}^{\vartheta,\frac{N-p}{p}\vartheta}(\mathbb{R}^N)}.$$

Lemma 2.2. (see [22]) *Let $s\in(0,1]$ and $N>ps$. Then there exists a constant $S_s>0$ such that, for any $u\in D^{s,p}(\mathbb{R}^N)$, $\|u\|^p_{L^{p_s^*}(\mathbb{R}^N)}\leqslant S_s^{-1}\|u\|^p_{D^{s,p}(\mathbb{R}^N)}$.*

3. SUBCRITICAL CASE

In this section, we prove the existence of a ground state solution for equation $(\mathscr{P})$ under the assumptions of Theorem 1.1. The proof is divided into several steps.

3.1. **Lieb's compactness theorem.**

Lemma 3.1. (Lieb's compactness Theorem) *Let $\tau\in(p_s^*,p^*)$ and let $\{u_n\}$ be a bounded sequence in $\mathbb{X}$. If $\lim\limits_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^\tau\,dx>0$, then there exists $\{z_n\}\subset\mathbb{R}^N$ such that $\bar{u}_n:=u_n(x+z_n)\to\bar{u}\not\equiv0$, in $L^\tau_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, if $\{z_n\}$ is bounded, then $u_n\to u\not\equiv0$,* in $L^\tau_{\mathrm{loc}}(\mathbb{R}^N)$.

Our proof is based on the following refined Sobolev type inequality.

Lemma 3.2. *Let $\tau\in(p_s^*,p^*)$. Then, for all $u\in\mathbb{X}$,*

$$\int_{\mathbb{R}^N}|u|^\tau\,dx\leqslant C\left(\sup_{z\in\mathbb{R}^N}\int_{B(z,1)}|u|^\tau\,dx\right)^{\frac{\tau-p}{\tau}}\|u\|^p_{\mathbb{X}},$$

where C is a positive constant independent of u.

Proof. While related to [31, Lemma 1.21], our method introduces key modifications. For $u\in\mathbb{X}$ and $\tau\in(p_s^*,p^*)$, the Sobolev embedding theorem yields

$$\int_{B(z,1)}|u|^\tau\,dx\leqslant C\left(\int_{B(z,1)}|\nabla u|^p\,dx+\int_{B(z,1)}\int_{B(z,1)}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)^{\frac{\tau}{p}}.$$

Using the above inequality, we obtain

$$\begin{aligned}&\int_{B(z,1)}|u|^\tau dx\\&=\left(\int_{B(z,1)}|u|^\tau dx\right)^{\frac{p}{\tau}}\left(\int_{B(z,1)}|u|^\tau dx\right)^{\frac{\tau-p}{\tau}}\\&\leqslant C\left(\int_{B(z,1)}|\nabla u|^p dx+\int_{B(z,1)}\int_{B(z,1)}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)\left(\int_{B(z,1)}|u|^\tau dx\right)^{\frac{\tau-p}{\tau}}.\end{aligned}$$

Using a covering of $\mathbb{R}^N$ with unit balls such that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we derive

$$\int_{\mathbb{R}^N}|u|^\tau dx\leqslant C\left(\sup_{z\in\mathbb{R}^N}\int_{B(z,1)}|u|^\tau dx\right)^{\frac{\tau-p}{\tau}}\|u\|_{\mathbb{X}}^p.$$

□

Proof of Lemma 3.1. The boundedness of $\{u_n\}$ in $\mathbb{X}$ allows us to select a subsequence such that

$$u_n\rightharpoonup u \text{ in } \mathbb{X},\ u_n\to u \text{ a.e. in } \mathbb{R}^N, u_n\to u \text{ in } L^\tau_{\rm loc}(\mathbb{R}^N),$$

for any $\tau\in(p_s^*,p^*)$. Because $\lim\limits_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^\tau dx>0$, Lemma 3.2 implies that for sufficiently large n, there exists $C>0$ such that $\sup_{z\in\mathbb{R}^N}\int_{B(z,1)}|u_n|^\tau dx\geqslant C>0$. Since $\{u_n\}$ is bounded in $\mathbb{X}$ and $\mathbb{X}\hookrightarrow L^\tau(\mathbb{R}^N)$, we observe that

$$\sup_{z\in\mathbb{R}^N}\int_{B(z,1)}|u_n|^\tau dx\leqslant\int_{\mathbb{R}^N}|u_n|^\tau dx\leqslant C.$$

Therefore, there exists C_0 such that $C_0\leqslant\sup_{z\in\mathbb{R}^N}\int_{B(z,1)}|u_n|^\tau dx\leqslant C_0^{-1}$. According to above inequality, there exists $\{z_n\}\subset\mathbb{R}^N$ such that

$$\int_{B(z_n,1)}|u_n|^\tau dx\geqslant\sup_{z\in\mathbb{R}^N}\int_{B(z,1)}|u_n|^\tau dx-\frac{C}{2n}\geqslant C_1>0.$$

Set $\bar{u}_n:=u_n(x+z_n)$. Then $\|\bar{u}_n\|_{\mathbb{X}}=\|u_n\|_{\mathbb{X}}$ and $\int_{B(0,1)}|\bar{u}_n|^\tau dx\geqslant C_1>0$. Up to a subsequence, there exists $\bar{u}$ such that $\bar{u}_n\rightharpoonup\bar{u}$ in $\mathbb{X}$, $\bar{u}_n\to\bar{u}$ a.e. in $\mathbb{R}^N$. Using the embedding $\mathbb{X}\hookrightarrow L^\tau_{\rm loc}(\mathbb{R}^N)$ is compact, we derive that $\bar{u}\not\equiv 0$. If $\{z_n\}$ is bounded, there exists $C>0$ such that $B(z_n,1)\subset B(0,C)$ and

$$\int_{B(0,C)}|u_n|^\tau dx\geqslant\int_{B(z_n,1)}|u_n|^\tau dx\geqslant C_1>0.$$

This implies that $u\not\equiv 0$. □

3.2. $(PS)_c$ **condition.** We set

$$c:=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}I(\gamma(t))>0 \text{ and } \Gamma:=\{\gamma\in C([0,1],\mathbb{X})\,|\,\gamma(0)=0,I(\gamma(1))<0\}.$$

The Nehari manifold for $(\mathscr{P})$ is defined as $\mathcal{N}:=\{u\in\mathbb{X}\backslash\{0\}|\langle I'(u),u\rangle=0\}$, and we denote

$$\bar{c}:=\inf_{u\in\mathcal{N}}I(u),\quad \bar{\bar{c}}:=\inf_{u\in\mathbb{X}\backslash\{0\}}\sup_{t\geqslant0}I(tu).$$

Lemma 3.3. *Under the hypotheses of Theorem 1.1, the following statements hold.*

(1) The energy functional I satisfies the mountain pass geometry.

(2) *For every* $u \in \mathbb{X} \setminus \{0\}$, *there exists a unique* $t_u > 0$ *such that* $t_u u \in \mathscr{N}$ *and* $I(t_u u) = \max_{t>0} I(tu)$.

(3) $\bar{c} > 0$.

(4) $c = \bar{c} = \bar{\bar{c}}$.

(5) *If* $u \in \mathscr{N}$ *and* $I(u) = c$, *then* u *is a ground state solution.*

Proof. The proof follows standard arguments. Full details are provided in the Appendix for completeness. ☐

It follows from the Mountain Pass Theorem [2] and Lemma 3.3 that there exists a $(PS)_c$ sequence with

$$I(u_n) \to c, \text{ and } \langle I'(u_n), \varphi \rangle \to 0, \text{ as } n \to +\infty,$$

for every $\varphi \in \mathbb{X}$.

Lemma 3.4. *Let* $\{u_n\}$ *be a* $(PS)_c$ *sequence at level* $c \in (0, c^*)$. *Then,* $\{u_n\}$ *is bounded in* $\mathbb{X}$.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence at level $c \in (0, c^*)$. Then,

$$c + 1 + o(1) \geqslant I(u_n) - \frac{1}{r}\langle I'(u_n), u_n \rangle = \left(\frac{1}{p} - \frac{1}{r}\right) \|u\|_{\mathbb{X}}^p,$$

which shows that $\{u_n\}$ is bounded in $\mathbb{X}$. ☐

Lemma 3.5. *Under the hypotheses of Theorem 1.1, it holds that* $c < c^* := \left(\frac{1}{p} - \frac{1}{r}\right) S_r^{\frac{r}{r-p}}$.

Proof. Let v be a minimizer of S_r defined in (2.1). From condition (F_1), we deduce that

$$\int_{\mathbb{R}^N} F(x)|v|^r \, dx > \int_{\mathbb{R}^N} |v|^r \, dx.$$

Define the function

$$h_r(t) = \frac{t^p}{p}\|v\|_{\mathbb{X}}^p - \frac{t^r}{r}\int_{\mathbb{R}^N} F(x)|v|^r \, dx, \qquad t \geqslant 0.$$

Direct calculation shows

$$h_r'(t) = t^{p-1}\|v\|_{\mathbb{X}}^p - t^{r-1}\int_{\mathbb{R}^N} F(x)|v|^r \, dx.$$

Hence there exists a unique positive critical point

$$t_r = \left[\frac{\|v\|_{\mathbb{X}}^p}{\int_{\mathbb{R}^N} F(x)|v|^r \, dx}\right]^{\frac{1}{r-p}},$$

and h_r attains its maximum at $t = t_r$ with

$$h_r(t_r) = \left(\frac{1}{p} - \frac{1}{r}\right)\left(\frac{\|v\|_{\mathbb{X}}^p}{\left(\int_{\mathbb{R}^N} F(x)|v|^r \, dx\right)^{\frac{p}{r}}}\right)^{\frac{r}{r-p}}.$$

According to (2.1), we conclude that

$$c \leqslant \max_{t>0} I(tv) \leqslant \max_{t>0} h_r(t) = h_r(t_r) < \left(\frac{1}{p} - \frac{1}{r}\right)\left[\frac{\|v\|_{\mathbb{X}}^p}{\left(\int_{\mathbb{R}^N} |v|^r \, dx\right)^{\frac{p}{r}}}\right]^{\frac{r}{r-p}}$$

$$= \left(\frac{1}{p} - \frac{1}{r}\right) S_r^{\frac{r}{r-p}} = c^*.$$

Therefore $c < c^*$. $\square$

Lemma 3.6. *The functional I satisfies the $(PS)_c$ condition if $c \in (0, c^*)$.*

Proof. Let $\{u_n\} \subset \mathbb{X}$ be a $(PS)_c$ sequence, i.e., $I(u_n) \to c$, $I'(u_n) \to 0$ in $\mathbb{X}^*$. By Lemma 3.4, one sees that $\{u_n\}$ is bounded in $\mathbb{X}$. Moreover, from

$$c + o(1) = I(u_n) - \frac{1}{r}\langle I'(u_n), u_n\rangle = \left(\frac{1}{p} - \frac{1}{r}\right)\|u_n\|_{\mathbb{X}}^p,$$

we have $\|u_n\|_{\mathbb{X}}^p \to \frac{rc}{r-p} > 0$. Hence $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} F(x)|u_n|^r dx > 0$. By $F \in L^\infty(\mathbb{R}^N)$, we deduce that

$$0 < \lim_{n\to\infty} \int_{\mathbb{R}^N} F(x)|u_n|^r dx \leqslant \|F\|_{L^\infty(\mathbb{R}^N)} \liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^r dx,$$

which yields $\liminf\limits_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^r dx > 0$. It follows from Lemma 3.1 that there exists $\{z_n\} \subset \mathbb{R}^N$ and a function $\bar{u} \in \mathbb{X}$ such that, after passing to a subsequence,

$$\bar{u}_n := u_n(x + z_n) \to \bar{u} \not\equiv 0, \text{ in } L^r_{\text{loc}}(\mathbb{R}^N).$$

and

$$\int_{B(0,1)} |\bar{u}_n|^r \, dx = \int_{B(z_n,1)} |u_n|^r \, dx \geqslant C > 0, \text{ for all } n.$$

Define the functional

$$\bar{I}(\bar{u}_n) = \frac{1}{p}\|\bar{u}_n\|_{\mathbb{X}}^p - \frac{1}{r}\int_{\mathbb{R}^N} F(x + z_n)|\bar{u}_n|^r \, dx.$$

Then, as $n \to +\infty$, $\bar{I}(\bar{u}_n) = I(u_n) \to c$. For all $\varphi \in \mathbb{X}$, set $\varphi_n = \varphi(x - z_n)$. Then, $\|\varphi_n\|_{\mathbb{X}} = \|\varphi\|_{\mathbb{X}}$, and

$$|\langle \bar{I}'(\bar{u}_n), \varphi\rangle| = |\langle I'(u_n), \varphi_n\rangle| \leqslant \|I'(u_n)\|_{\mathbb{X}^{-1}} \|\varphi_n\|_{\mathbb{X}} = o_n(1)\|\varphi\|_{\mathbb{X}}.$$

This yields

$$\bar{I}'(\bar{u}_n) \to 0. \tag{3.1}$$

This indicates that $\{\bar{u}_n\}$ is a $(PS)_c$ sequence of $\bar{I}$. In order to show that $\{z_n\}$ is bounded in $\mathbb{R}^N$, we suppose on the contrary that $|z_n| \to +\infty$ as $n \to +\infty$. Obviously, for any $\varphi \in \mathbb{X}$,

$$\int_{\mathbb{R}^N} F(x + z_n)|\bar{u}_n|^{r-2}\bar{u}_n\varphi \, dx = \int_{\mathbb{R}^N} |\bar{u}_n|^{r-2}\bar{u}_n\varphi \, dx + o_n(1), \text{ as } n \to +\infty, \tag{3.2}$$

since

$$\begin{aligned}
&\int_{\mathbb{R}^N} [F(x + z_n) - 1]|\bar{u}_n|^{r-1}|\varphi| \, dx \\
\leqslant &\left(\int_{\mathbb{R}^N} [F(x + z_n) - 1]|\bar{u}_n|^r \, dx\right)^{\frac{r-1}{r}} \left(\int_{\mathbb{R}^N} [F(x + z_n) - 1]|\varphi|^r \, dx\right)^{\frac{1}{r}} \\
\leqslant &C\left(\int_{\mathbb{R}^N} [F(x + z_n) - 1]|\varphi|^r \, dx\right)^{\frac{1}{r}} \to 0, \text{ as } n \to +\infty.
\end{aligned}$$

It follows from $\bar{u}_n \rightharpoonup \bar{u}$ weakly in $\mathbb{X}$, (3.1)-(3.2) that $\langle \bar{I}'(\bar{u}_n), \varphi\rangle = \langle I'_\infty(\bar{u}), \varphi\rangle = 0$, which implies that $\bar{u}$ is a nontrivial weak solution of the limit equation

$$-\Delta_p u + (-\Delta)_p^s u = |u|^{r-2}u, \; x \in \mathbb{R}^N, \tag{S_∞}$$

and the functional is

$$I_\infty(\bar{u}) = \frac{1}{p}\|\bar{u}\|_{\mathbb{X}}^p - \frac{1}{r}\int_{\mathbb{R}^N} |\bar{u}|^r\, dx.$$

In particular, $\bar{u} \in \mathcal{N}_\infty$ (the Nehari manifold for I_∞) and $I_\infty(\bar{u}) > 0$.

Now, we compute the energy level. From $\bar{I}(\bar{u}_n) \to c$ and $\langle \bar{I}'(\bar{u}_n), \bar{u}_n\rangle \to 0$, we have

$$c = \lim_{n\to\infty}\left(\bar{I}(\bar{u}_n) - \frac{1}{r}\langle \bar{I}'(\bar{u}_n), \bar{u}_n\rangle\right) = \left(\frac{1}{p} - \frac{1}{r}\right)\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x+z_n)|\bar{u}_n|^r dx.$$

Since $F \geq 1$, it follows that

$$c \geq \left(\frac{1}{p} - \frac{1}{r}\right)\liminf_{n\to\infty}\int_{\mathbb{R}^N} |\bar{u}_n|^r dx.$$

Because $\bar{u}_n \to \bar{u}$ in $L^r_{\text{loc}}(\mathbb{R}^N)$, by Fatou's lemma,

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N} |\bar{u}_n|^r dx \geq \int_{\mathbb{R}^N} |\bar{u}|^r dx.$$

Thus

$$c \geq \left(\frac{1}{p} - \frac{1}{r}\right)\int_{\mathbb{R}^N} |\bar{u}|^r dx = I_\infty(\bar{u}) \geq c_\infty,$$

where c_∞ denotes the mountain pass level (or the least energy) for I_∞. A standard computation (similar to Lemma 3.5 with $F \equiv 1$) gives

$$c_\infty = \left(\frac{1}{p} - \frac{1}{r}\right) S_r^{\frac{r}{r-p}} = c^*.$$

But Lemma 3.5 tells us that $c < c^*$. Hence we obtain $c \geq c^*$, a contradiction. Therefore $\{z_n\}$ must be bounded. By using Lemma 3.1 again, up to a subsequence, we have $u_n(x) \to u \not\equiv 0$, in $L^r_{\text{loc}}(\mathbb{R}^N)$. We prove $u_n \to u$ in $\mathbb{X}$ by demostrating $I(u) = c$. From $u_n \rightharpoonup u \not\equiv 0$ in $\mathbb{X}$, we know

$$\langle I'(u_n), \varphi\rangle = \langle I'(u), \varphi\rangle + o(1)$$

Then $u \in \mathcal{N}$. From Brezis-Lieb Lemma [8], we infer that

$$\begin{aligned} c \leqslant I(u) =& I(u) - \frac{1}{r}\langle I'(u), u\rangle \\ =& \left(\frac{1}{p} - \frac{1}{r}\right)\|u\|_{\mathbb{X}}^p \leqslant \left(\frac{1}{p} - \frac{1}{r}\right)\lim_{n\to\infty}\|u_n\|_{\mathbb{X}}^p \\ =& \lim_{n\to\infty} I(u_n) - \frac{1}{r}\lim_{n\to\infty}\langle I'(u_n), u_n\rangle \\ =& c, \end{aligned}$$

This shows $I(u) = c$ and $\lim\limits_{n\to\infty}\|u_n\|_{\mathbb{X}}^p = \|u\|_{\mathbb{X}}^p$. Consequently, we obtain the required conclusion. □

Proof of Theorem 1.1. By Lemmas 3.3-3.6, we obtain the desired result. The proof is complete. □

4. COMBINED NONLINEARITIES

This section establishes Theorem 1.2 regarding the existence of a ground state to ($D\mathscr{P}$). The corresponding energy functional $E : \mathbb{X} \to \mathbb{R}$ is defined by

$$\begin{aligned} E(u) :=& \frac{1}{p}\left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \, dx\, dy\right) \\ & -\frac{1}{r}\int_{\mathbb{R}^N} F(x)|u|^r \, dx - \frac{1}{q}\int_{\mathbb{R}^N} F(x)|u|^q \, dx. \end{aligned}$$

The Fréchet derivative $E'(u)$ is given by

$$\begin{aligned} \langle E'(u), \varphi\rangle =& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} \, dx\, dy \\ & + \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx - \int_{\mathbb{R}^N} F(x)|u|^{r-2}u\varphi \, dx - \int_{\mathbb{R}^N} F(x)|u|^{q-2}u\varphi \, dx, \end{aligned}$$

for any $\varphi \in \mathbb{X}$.

Define the mountain pass level $c_F := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} E(\gamma(t))$, where

$$\Gamma := \left\{\gamma \in C([0,1],\mathbb{X}) \mid \gamma(0) = 0,\ E(\gamma(1)) < 0\right\}.$$

The Nehari manifold for ($D\mathscr{P}$) is defined as

$$\mathscr{N}_F := \left\{u \in \mathbb{X}\setminus\{0\} \mid \langle E'(u), u\rangle = 0\right\},$$

and we set

$$\bar{c}_F := \inf_{u\in\mathscr{N}_F} E(u), \qquad \bar{\bar{c}}_F := \inf_{u\in\mathbb{X}\setminus\{0\}} \sup_{t\geq 0} E(tu).$$

Lemma 4.1. *Let the assumptions in Theorem 1.2 hold. Then*

1. *The functional E satisfies mountain pass geometry.*
2. *For any $u \in \mathbb{X}\setminus\{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathscr{N}_F$ and $E(t_u u) = \max_{t>0} E(tu)$.*
3. $\bar{c}_F > 0$.
4. $c_F = \bar{c}_F = \bar{\bar{c}}_F$.
5. *If $u \in \mathscr{N}_F$ and $E(u) = c_F$, then u is a ground state solution.*

Proof. The proof follows analogous arguments to those in Lemma 3.3. □

Lemma 4.2. *Let $\{u_n\} \subset X$ be a $(PS)_{c_F}$ sequence at level $c_F > 0$. Then, $\{u_n\}$ is bounded in $\mathbb{X}$.*

Proof. Let $\{u_n\}$ be a $(PS)_{c_F}$ sequence at level $c_F > 0$. It is easy to see that

$$\begin{aligned} c_F + 1 + o(1) \geqslant& E(u_n) - \frac{1}{r}\langle E'(u_n), u_n\rangle \\ =& \left(\frac{1}{p}-\frac{1}{r}\right)\left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \, dx\, dy\right) \\ & + \left(\frac{1}{r}-\frac{1}{q}\right)\int_{\mathbb{R}^N} F(x)|u_n|^q \, dx \\ \geqslant& \left(\frac{1}{p}-\frac{1}{r}\right)\|u\|_{\mathbb{X}}^p, \end{aligned}$$

which shows that $\{u_n\}$ is bounded in $\mathbb{X}$. □

Lemma 4.3. *The functional E satisfies $(PS)_{c_F}$ condition under $c_F > 0$.*

Proof. **Step 1.** We claim that if $\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^q\,dx = 0$, then $\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^r\,dx = 0$. From Hölder's inequality and $p_s^* < r < q < p^*$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} F(x)|u_n|^r\,dx &\leqslant \left(\int_{\mathbb{R}^N} F(x)|u_n|^{p_s^*}\,dx\right)^{\frac{q-r}{q-p_s^*}}\left(\int_{\mathbb{R}^N} F(x)|u_n|^q\,dx\right)^{\frac{r-p_s^*}{q-p_s^*}}\\ &\leqslant C\left(\int_{\mathbb{R}^N} F(x)|u_n|^q\,dx\right)^{\frac{r-p_s^*}{q-p_s^*}} = o_n(1).\end{aligned}$$

Similarly, if $\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^r\,dx = 0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} F(x)|u_n|^q\,dx &\leqslant \left(\int_{\mathbb{R}^N} F(x)|u_n|^r\,dx\right)^{\frac{q-r}{p^*-r}}\left(\int_{\mathbb{R}^N} F(x)|u_n|^{p^*}\,dx\right)^{\frac{p^*-q}{p^*-r}}\\ &\leqslant C\left(\int_{\mathbb{R}^N} F(x)|u_n|^r\,dx\right)^{\frac{q-r}{p^*-r}} = o_n(1).\end{aligned}$$

Step 2. We prove $\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^r\,dx > 0$.

If $\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^r\,dx = 0$, one sees that $c_F + o(1) = \frac{1}{p}\|u_n\|_{\mathbb{X}}^p$, and $o(1) = \|u_n\|_{\mathbb{X}}^p$, which gives $c_F = 0$. This is contradictory to $c_F > 0$. Hence, $\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^r\,dx > 0$. Similarly, we derive that $\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^q\,dx > 0$. Due to $F \in L^\infty(\mathbb{R}^N)$, we infer that

$$0 < \lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^r\,dx \leqslant \|F\|_{L^\infty(\mathbb{R}^N)}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^r\,dx$$

and

$$0 < \lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^q\,dx \leqslant \|F\|_{L^\infty(\mathbb{R}^N)}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^q\,dx.$$

Therefore,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^r\,dx > 0,\ \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^q\,dx > 0.$$

Step 3. From Lemma 3.1, there exists $\{z_n\} \subset \mathbb{R}^N$ such that $\bar{u}_n := u_n(x+z_n) \to \bar{u} \not\equiv 0$, in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$\int_{B(0,1)}|\bar{u}_n|^r\,dx = \int_{B(z_n,1)}|u_n|^r\,dx \geqslant C > 0.$$

Note that the functional E lacks translation invariance. We verify that, as $n \to +\infty$, $\bar{E}(\bar{u}_n) = E(u_n) \to c_F$ and $\bar{E}'(\bar{u}_n) \to 0$, where

$$\bar{E}(\bar{u}_n) = \frac{1}{p}\|\bar{u}_n\|_{\mathbb{X}}^p - \frac{1}{r}\int_{\mathbb{R}^N} F(x+z_n)|\bar{u}_n|^r\,dx - \frac{1}{q}\int_{\mathbb{R}^N} F(x+z_n)|\bar{u}_n|^q\,dx. \tag{4.1}$$

For all $\varphi \in \mathbb{X}$, we derive that

$$|\langle \bar{E}'(\bar{u}_n),\varphi\rangle| = |\langle E'(u_n),\varphi_n\rangle| \leqslant \|E'(u_n)\|_{\mathbb{X}^{-1}}\|\varphi_n\|_{\mathbb{X}} = o_n(1)\|\varphi\|_{\mathbb{X}},$$

where $\varphi_n = \varphi(x-z_n)$ and $\|\varphi_n\|_{\mathbb{X}} = \|\varphi\|_{\mathbb{X}}$, which leads to $\bar{E}'(\bar{u}_n) \to 0$, and $\{\bar{u}_n\}$ is a $(PS)_{c_F}$ sequence of $\bar{E}$.

Step 4. We claim that $\{z_n\}$ is bounded in $\mathbb{R}^N$.

Suppose on the contrary that $|z_n| \to +\infty$ as $n \to +\infty$. Consequently,

$$\int_{\mathbb{R}^N} F(x+z_n)|\bar{u}_n|^{r-2}\bar{u}_n\varphi\,dx = \int_{\mathbb{R}^N}|\bar{u}_n|^{r-2}\bar{u}_n\varphi\,dx,\ \text{as } n \to +\infty, \tag{4.2}$$

and

$$\int_{\mathbb{R}^N} F(x+z_n)|\bar{u}_n|^{q-2}\bar{u}_n\varphi\,dx = \int_{\mathbb{R}^N} |\bar{u}_n|^{q-2}\bar{u}_n\varphi\,dx, \text{ as } n\to+\infty, \tag{4.3}$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}^N}[F(x+z_n)-1]|\bar{u}_n|^{r-1}|\varphi|\,dx\\
\leqslant&\left(\int_{\mathbb{R}^N}[F(x+z_n)-1]|\bar{u}_n|^{r}\,dx\right)^{\frac{r-1}{r}}\left(\int_{\mathbb{R}^N}[F(x+z_n)-1]|\varphi|^{r}\,dx\right)^{\frac{1}{r}}\\
\leqslant& C\left(\int_{\mathbb{R}^N}[F(x+z_n)-1]|\varphi|^{r}\,dx\right)^{\frac{1}{r}}\to 0, \text{ as } n\to+\infty.
\end{aligned}$$

It follows from $\bar{u}_n \rightharpoonup \bar{u}$ weakly in $\mathbb{X}$, (4.2) and (4.3) that $\langle \bar{E}'(\bar{u}_n),\varphi\rangle = \langle E'_\infty(\bar{u}),\varphi\rangle = 0$, which implies that $\bar{u}$ is a weak solution to the following equation

$$-\Delta_p\bar{u}+(-\Delta)^s_p\bar{u} = |\bar{u}|^{r-2}\bar{u}+|\bar{u}|^{q-2}\bar{u},\ x\in\mathbb{R}^N, \tag{S_∞}$$

and the corresponding energy functional is

$$E_\infty(\bar{u}) = \frac{1}{p}\|\bar{u}\|_X^p - \frac{1}{r}\int_{\mathbb{R}^N}|\bar{u}|^r\,dx - \frac{1}{q}\int_{\mathbb{R}^N}|\bar{u}|^q\,dx.$$

Let $\tilde{c}_\infty$ be the ground state energy of E_∞. It follows from (4.1), $F(x)\geqslant 1$, Brezis-Lieb Lemma [8] and $\langle E'_\infty(\bar{u}),\varphi\rangle = 0$ that

$$\begin{aligned}
\tilde{c}_\infty > c_F =&\bar{E}(\bar{u}_n) - \frac{1}{p}\langle \bar{E}'(\bar{u}_n),\bar{u}_n\rangle\\
=&\left(\frac{1}{p}-\frac{1}{r}\right)\lim_{n\to\infty}\int_{\mathbb{R}^N}F(x+z_n)|\bar{u}_n|^r\,dx + \left(\frac{1}{p}-\frac{1}{q}\right)\lim_{n\to\infty}\int_{\mathbb{R}^N}F(x+z_n)|\bar{u}_n|^q\,dx\\
\geqslant&\left(\frac{1}{p}-\frac{1}{r}\right)\lim_{n\to\infty}\int_{\mathbb{R}^N}|\bar{u}_n|^r\,dx + \left(\frac{1}{p}-\frac{1}{q}\right)\lim_{n\to\infty}\int_{\mathbb{R}^N}|\bar{u}_n|^q\,dx\\
\geqslant&\left(\frac{1}{p}-\frac{1}{r}\right)\int_{\mathbb{R}^N}|\bar{u}|^r\,dx + \left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^N}|\bar{u}|^q\,dx\\
=&E_\infty(\bar{u}) - \frac{1}{p}\langle E'_\infty(\bar{u}),\bar{u}\rangle\\
=&E_\infty(\bar{u}) \geqslant \tilde{c}_\infty,
\end{aligned}$$

which yields a contradiction. Hence $\{z_n\}$ is bounded in $\mathbb{R}^N$. By using Lemma 3.1 again, up to a subsequence, we conclude that $u_n(x)\to u\not\equiv 0$, in $L^r_{\text{loc}}(\mathbb{R}^N)$.

Step 5. We show that $u_n\to u$ in $\mathbb{X}$.

From $u_n \rightharpoonup u \not\equiv 0$ in $\mathbb{X}$, it follows that $\langle E'(u_n),\varphi\rangle = \langle E'(u),\varphi\rangle + o(1)$, which yields $u \in \mathscr{N}_F$. According to Brezis-Lieb Lemma [8], we infer that

$$\begin{aligned} c_F \leqslant E(u) =& E(u) - \frac{1}{r}\langle E'(u),u\rangle \\ =& \left(\frac{1}{p}-\frac{1}{r}\right)\|u\|_{\mathbb{X}}^p + \left(\frac{1}{r}-\frac{1}{q}\right)\int_{\mathbb{R}^N} F(x)|u|^q\,dx \\ \leqslant& \left(\frac{1}{p}-\frac{1}{r}\right)\lim_{n\to\infty}\|u_n\|_{\mathbb{X}}^p + \left(\frac{1}{r}-\frac{1}{q}\right)\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^q\,dx \\ =& \lim_{n\to\infty} E(u_n) - \frac{1}{r}\lim_{n\to\infty}\langle E'(u_n),u_n\rangle \\ =& c_F, \end{aligned}$$

which leads to $E(u) = c_F$ and $\lim_{n\to\infty}\|u_n\|_{\mathbb{X}}^p = \|u\|_{\mathbb{X}}^p$. Then, the proof is complete. ☐

Proof of Theorem 1.2. We can obtain the desired result by Lemmas 4.1-4.3. The proof is complete. ☐

5. CRITICAL CASE

Theorem 1.3 is proved in this section by combining the generalized Lieb compactness theorem (Lemma 5.1) with an analytical method (Lemma 5.4).

5.1. **Generalization of Lieb's compactness Theorem.** A measurable function $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Morrey space $\mathscr{M}^{p,\varpi}(\mathbb{R}^N)$ with norm $\|u\|_{\mathscr{M}^{p,\varpi}(\mathbb{R}^N)}$, where $p \in [1,\infty)$ and $\varpi \in (0,N]$, if and only if

$$\|u\|^p_{\mathscr{M}^{p,\varpi}(\mathbb{R}^N)} := \sup_{R>0,x\in\mathbb{R}^N} R^{\varpi-N}\int_{B(x,R)} |u(y)|^p\,dy < \infty,$$

as introduced in [15].

The choice $\tau = p^*$ or $\tau = p_s^*$ is not admissible in Lemma 2.1. This motivates the following generalized version of Lieb's compactness theorem.

Lemma 5.1. *Let $\tau \in (p_s^*, p^*)$, and let $\{u_n\}$ be a bounded sequence in $\mathbb{X}$. Suppose that*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p_s^*}\,dx > 0 \text{ and } \lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p^*}\,dx > 0.$$

Then there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ such that $\bar{u}_n(x) := u_n(x+z_n) \to \bar{u} \not\equiv 0$ in $L^\tau_{\mathrm{loc}}(\mathbb{R}^N)$. Furthermore, if $\{z_n\}$ is bounded, then $u_n \to u \not\equiv 0$ in $L^\tau_{\mathrm{loc}}(\mathbb{R}^N)$.

Proof. We divide this proof into three steps.

Step 1. Since $\{u_n\} \subset \mathbb{X}$ is a bounded sequence, up to a subsequence, we assume

$$u_n \rightharpoonup u \text{ in } \mathbb{X},\ \ u_n \to u \text{ a.e. in } \mathbb{R}^N,\ \ u_n \to u \text{ in } L^\tau_{\mathrm{loc}}(\mathbb{R}^N),$$

for any $\tau \in (p_s^*, p^*)$. It follows from Lemma 2.1 and $\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p^*}\,dx > 0$ that there exists $C > 0$ such that $\|u_n\|_{\mathscr{M}^{p,N-p}(\mathbb{R}^N)} \geqslant C > 0$. We have the continuous embeddings

$$\mathbb{X} \hookrightarrow D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N) \hookrightarrow \mathscr{M}^{p,N-p}(\mathbb{R}^N).$$

Consequently, $\|u_n\|_{\mathscr{M}^{p,N-p}(\mathbb{R}^N)} \leqslant C$. Thus there exists a constant $C_0 > 0$, independent of n, such that $C_0 \leqslant \|u_n\|_{\mathscr{M}^{p,N-p}(\mathbb{R}^N)} \leqslant C_0^{-1}$. Due to this inequality, we infer that there exist $\sigma_n > 0$ and $z_n \in \mathbb{R}^N$ such that

$$\sigma_n^{-p} \int_{B(z_n,\sigma_n)} |u_n(y)|^p \, dy \geqslant \|u_n\|^p_{\mathscr{M}^{p,N-p}(\mathbb{R}^N)} - \frac{1}{n} \geqslant \frac{C_0}{2} > 0, \tag{5.1}$$

where the last inequality holds for sufficiently large n.

Step 2. We first show that $\{\sigma_n\}$ is bounded. Suppose, for contradiction, that $\sigma_n \to \infty$ (up to a subsequence). By the boundedness of $\{u_n\}$, one has $0 < \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p_s^*} dy \le C$. Since $N > p > 1$ and $s \in (0,1)$, we derive that

$$-p + N\left(1 - \frac{p}{p_s^*}\right) < 0. \tag{5.2}$$

Using (5.1), Hölder's inequality, and (5.2), we get

$$\begin{aligned} 0 < \frac{C_0}{2} \leqslant & \sigma_n^{-p} \int_{B(z_n,\sigma_n)} |u_n(y)|^p \, dy \\ \leqslant & \sigma_n^{-p} \left(\frac{\omega_{N-1}}{N} \sigma_n^N\right)^{\frac{p_s^*-p}{p_s^*}} \left(\int_{B(z_n,\sigma_n)} |u_n(y)|^{p_s^*} dy\right)^{\frac{p}{p_s^*}} \\ \leqslant & C\sigma_n^{-p+N\left(1-\frac{p}{p_s^*}\right)} \to 0, \text{ as } n \to \infty. \end{aligned}$$

This yields a contradiction. By the Bolzano–Weierstrass theorem, up to a subsequence, still denoted by $\{\sigma_n\}$, there exists $\bar{\sigma} \in [0,\infty)$ such that $\lim_{n\to\infty} \sigma_n = \bar{\sigma}$.

We next show that $\bar{\sigma} > 0$. Suppose on the contrary that $\lim_{n\to\infty} \sigma_n = \bar{\sigma} = 0$. We will use a covering argument to derive a contradiction with the hypothesis $\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p_s^*} dx > 0$. Using the boundedness of $\{u_n\}$, we have

$$C \lim_{n\to\infty} \left(\int_{\mathbb{R}^N} |u_n|^{p_s^*} dy\right)^{\frac{p}{p_s^*}} \leqslant \lim_{n\to\infty} \|u_n\|^p_{\mathbb{X}} \leqslant \bar{C}.$$

For any $z \in \mathbb{R}^N$, from Hölder's and Sobolev's inequalities,

$$\begin{aligned} \int_{B(z,\sigma_n)} |u_n|^{p_s^*} dy \leqslant & \left(\int_{B(z,\sigma_n)} dy\right)^{\frac{p^*-p_s^*}{p^*}} \left(\int_{B(z,\sigma_n)} |u_n|^{p^*} dy\right)^{\frac{p_s^*}{p^*}} \\ \leqslant & S_1^{-\frac{p_s^*}{p}} \left(\int_{B(z,\sigma_n)} dy\right)^{\frac{p^*-p_s^*}{p^*}} \left(\int_{B(z,\sigma_n)} |\nabla u_n|^p \mathrm{dy}\right)^{\frac{p_s^*}{p}} \\ \leqslant & S_1^{-\frac{p_s^*}{p}} \left(\int_{B(z,\sigma_n)} dy\right)^{\frac{p^*-p_s^*}{p^*}} \|u_n\|^{p_s^*-p}_{D^{1,p}(\mathbb{R}^N)} \int_{B(z,\sigma_n)} |\nabla u_n|^p \mathrm{dy} \\ \leqslant & CS_1^{-\frac{p_s^*}{p}} \left(\int_{B(z,\sigma_n)} dy\right)^{\frac{p^*-p_s^*}{p^*}} \int_{B(z,\sigma_n)} |\nabla u_n|^p \mathrm{dy}. \end{aligned}$$

Covering $\mathbb{R}^N$ by balls of radius σ_n, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, one gets

$$\begin{aligned}\int_{\mathbb{R}^N}|u_n|^{p_s^*}\,dy \leqslant& C(N+1)S_1^{-\frac{p_s^*}{p}}\left(\int_{B(0,\sigma_n)}dy\right)^{\frac{p^*-p_s^*}{p^*}}\int_{\mathbb{R}^N}|\nabla u_n|^p \mathrm{d}y\\ \leqslant& C(N+1)S_1^{-\frac{p_s^*}{p}}\left(\int_{B(0,\sigma_n)}dy\right)^{\frac{p^*-p_s^*}{p^*}}\\ \leqslant& C(N+1)S_1^{-\frac{p_s^*}{p}}\sigma_n^{N(1-\frac{p_s^*}{p^*})}.\end{aligned}$$

Using $\lim_{n\to\infty}\sigma_n=0$, one leads to

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p_s^*}\,dy \leqslant C(N+1)S_1^{-\frac{p_s^*}{p}}\lim_{n\to\infty}\sigma_n^{N(1-\frac{p_s^*}{p^*})}=0,$$

which contradicts $\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p_s^*}\,dx>0$. Therefore, $\bar\sigma>0$.

Step 3. Since $\lim_{n\to\infty}\sigma_n=\bar\sigma>0$, for sufficiently large n, we have $\{\sigma_n\}\subset(\bar\sigma/2,2\bar\sigma)$. From (5.1),

$$\int_{B(z_n,2\bar\sigma)}|u_n(y)|^p\,dy \geqslant \frac{C_0\bar\sigma^p}{2^{1-p}}>0. \tag{5.3}$$

Define $\bar u_n:=u_n(x+z_n)$. Then $\{\bar u_n\}\subset\mathbb{X}$ is a bounded sequence satisfying

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\bar u_n|^{p_s^*}\,dx>0 \text{ and } \lim_{n\to\infty}\int_{\mathbb{R}^N}|\bar u_n|^{p^*}\,dx>0.$$

Thanks to (5.3), we derive that

$$\int_{B(0,2\bar\sigma)}|\bar u_n(y)|^p\,dy \geqslant \frac{C_1\bar\sigma^p}{2^{1-p}}>0. \tag{5.4}$$

Applying the embedding $\mathbb{X}\hookrightarrow D^{1,p}(\mathbb{R}^N)\hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and (5.4), we obtain $\bar u_n\rightharpoonup\bar u\not\equiv 0$.

Step 4. If $\{z_n\}$ is bounded, there exists $C>0$ such that $B(z_n,2\bar\sigma)\subset B(0,C)$ and

$$\int_{B(0,C)}|u_n|^p\,dx\geqslant\int_{B(z_n,2\bar\sigma)}|u_n|^p\,dx\geqslant C_1>0.$$

This implies that $u\not\equiv 0$. □

5.2. $(PS)_c$ **condition.** We introduce the associated energy functional $J:\mathbb{X}\to\mathbb{R}$ corresponding to ($\mathscr{C}$) as

$$\begin{aligned}J(u):=&\frac{1}{p}\left(\int_{\mathbb{R}^N}|\nabla u|^p\,dx+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\right)-\frac{1}{p_s^*}\int_{\mathbb{R}^N}F(x)|u|^{p_s^*}\,dx\\ &-\frac{1}{p^*}\int_{\mathbb{R}^N}F(x)|u|^{p^*}\,dx-\frac{\lambda}{r}\int_{\mathbb{R}^N}F(x)|u|^r\,dx.\end{aligned}$$

The Fréchet derivative of $J'(u)$ is given by

$$\begin{aligned}\langle J'(u),\varphi\rangle = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy\\ &+\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx-\int_{\mathbb{R}^N}F(x)|u|^{p_s^*-2}u\varphi\,dx\\ &-\int_{\mathbb{R}^N}F(x)|u|^{p^*-2}u\varphi\,dx-\lambda\int_{\mathbb{R}^N}F(x)|u|^{r-2}u\varphi\,dx,\end{aligned}$$

for any $\varphi\in\mathbb{X}$. Define the mountain pass level

$$c_\lambda := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} J(\gamma(t))$$

where

$$\Gamma := \{\gamma\in C([0,1],\mathbb{X})\,|\,\gamma(0)=0, J(\gamma(1))<0\}.$$

Under the assumptions of Theorem 1.3, one has $c_\lambda>0$. The Nehari manifold for ($\mathscr{C}$) is defined as follows

$$\mathscr{N}_\lambda := \{u\in\mathbb{X}\backslash\{0\}\,|\,\langle J'(u),u\rangle=0\},$$

and we set

$$\bar{c}_\lambda := \inf_{u\in\mathscr{N}_\lambda} J(u),\ \bar{\bar{c}}_\lambda := \inf_{u\in\mathbb{X}\backslash\{0\}}\sup_{t\geqslant 0} J(tu).$$

Lemma 5.2. *Under the assumptions of Theorem 1.3, the following statements hold.*

(1) The functional J satisfies the mountain pass geometry.

(2) For any $u\in\mathbb{X}\setminus\{0\}$, there exists a unique $t_u>0$ such that $t_u u\in\mathscr{N}_\lambda$ and $J(t_u u)=\max_{t>0} J(tu)$.

(3) $\bar{c}_\lambda>0$.

(4) $c_\lambda=\bar{c}_\lambda=\bar{\bar{c}}_\lambda$.

(5) If $u\in\mathscr{N}_\lambda$ and $J(u)=c_\lambda$, then u is a ground state solution.

Proof. The proof follows by the same arguments as in Lemma 3.3. □

Lemma 5.3. *Let $\{u_n\}\subset\mathbb{X}$ be a $(PS)_{c_\lambda}$ sequence for the functional J at level $c_\lambda\in(0,c_\lambda^*)$, i.e.,*

$$J(u_n)\to c_\lambda,\ \text{and}\ \langle J'(u_n),\varphi\rangle\to 0,\quad \textit{for all } \varphi\in\mathbb{X},\ \text{as } n\to+\infty.$$

Then $\{u_n\}$ is bounded in $\mathbb{X}$.

Proof. Let $\{u_n\}$ be a $(PS)_{c_\lambda}$ sequence at level $c_\lambda\in(0,c_\lambda^*)$. Therefore,

$$\begin{aligned}c_\lambda = J(u_n) =&J(u_n)-\frac{1}{p_s^*}\langle J'(u_n),u_n\rangle\\ &\geqslant\left(\frac{1}{p}-\frac{1}{p_s^*}\right)\|u_n\|_{\mathbb{X}}^p+\lambda\left(\frac{1}{p_s^*}-\frac{1}{r}\right)\int_{\mathbb{R}^N}F(x)|u_n|^r\,dx\\ &+\left(\frac{1}{p_s^*}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N}F(x)|u_n|^{p^*}\,dx,\end{aligned}$$

which implies that $\{u_n\}$ is bounded in $\mathbb{X}$. □

Lemma 5.4. *Under the assumptions of Theorem 1.3, there exists* $\bar{\lambda} \in (0,+\infty)$ *such that, for any* $\lambda > \bar{\lambda}$, $c_\lambda \in (0, c_\lambda^*)$, *where*

$$c_\lambda^* := \min\left\{ \left(\frac{1}{p} - \frac{1}{p_s^*}\right) S_s^{\frac{p_s^*}{p_s^*-p}} \|F\|_{L^\infty(\mathbb{R}^N)}^{-\frac{p}{p_s^*-p}}, \left(\frac{1}{p} - \frac{1}{p^*}\right) S_1^{\frac{p^*}{p^*-p}} \|F\|_{L^\infty(\mathbb{R}^N)}^{-\frac{p}{p^*-p}} \right\},$$

and S_s, S_1 *are the constants from Lemma 2.2.*

Proof. Choose a function $u \in \mathbb{X}$ such that $\|u\|_{\mathbb{X}} = 1$ and $\int_{\mathbb{R}^N} F(x)|u|^r\,dx > 0$. From Lemma 5.2, one has $\lim_{t\to+\infty} J(tu) = -\infty$, and there exists a unique $t_{u,\lambda} > 0$ such that $t_{u,\lambda}u \in \mathcal{N}_\lambda$ and $\sup_{t\geqslant 0} J(tu) = J(t_{u,\lambda}u)$. The parameter $t_{u,\lambda}$ satisfies the following Euler–Lagrange equation:

$$t_{u,\lambda}^p \|u\|_{\mathbb{X}}^p = t_{u,\lambda}^{p^*} \int_{\mathbb{R}^N} F(x)|u|^{p^*}\,dx + \lambda t_{u,\lambda}^r \int_{\mathbb{R}^N} F(x)|u|^r\,dx + t_{u,\lambda}^{p_s^*} \int_{\mathbb{R}^N} F(x)|u|^{p_s^*}\,dx. \tag{5.5}$$

Furthermore,

$$t_{u,\lambda}^p \|u\|_{\mathbb{X}}^p \geqslant t_{u,\lambda}^{p^*} \int_{\mathbb{R}^N} F(x)|u|^{p^*}\,dx.$$

This gives that $\{t_{u,\lambda}\}_\lambda$ is bounded.

We now claim that $t_{u,\lambda} \to 0$ as $\lambda \to +\infty$. Suppose, for contradiction, that there exists a sequence $\{\lambda_n\} \to +\infty$ such that $t_{u,\lambda_n} \to t_1 > 0$ (up to a subsequence). One has

$$\lambda_n t_{u,\lambda_n}^r \int_{\mathbb{R}^N} F(x)|u|^r\,dx \to +\infty, \text{ as } n \to +\infty.$$

Putting this into (5.5), we see that $t_1^p \|u\|_{\mathbb{X}}^p = +\infty$. This is a contradiction to $\|u\|_{\mathbb{X}} = 1$. Hence $t_{u,\lambda} \to 0$ as $\lambda \to +\infty$. Consequently,

$$\lim_{\lambda\to+\infty} \sup_{t\geqslant 0} J(tu) = \lim_{\lambda\to+\infty} J(t_{u,\lambda}u) = 0.$$

Therefore, there exists $\bar{\lambda} \in (0,+\infty)$ such that, for every $\lambda > \bar{\lambda}$, $\sup_{t\geqslant 0} J(tu) < c_\lambda^*$. For any $\lambda > \bar{\lambda}$, construct a mountain pass path as follows. Choose $T > 0$ sufficiently large so that $J(Tu) < 0$. Define $\gamma(t) = t(Tu)$ for $t \in [0,1]$. Then $\gamma \in \Gamma$ and

$$c_\lambda \leqslant \max_{t\in[0,1]} J(\gamma(t)) \leqslant \sup_{t\geqslant 0} J(tu) < c_\lambda^*.$$

Since $c_\lambda > 0$ by the mountain pass geometry, we conclude that $c_\lambda \in (0, c_\lambda^*)$ for all $\lambda > \bar{\lambda}$. □

Let us consider the following limit equation

$$(-\Delta)_p u + (-\Delta)_p^s u = |u|^{p_s^*-2}u + \lambda |u|^{r-2}u + |u|^{p^*-2}u, \ x \in \mathbb{R}^N. \tag{$S_{\lambda,\infty}$}$$

Lemma 5.5. *Let* $N > p > 1$, $0 < s < 1$, *and* $p_s^* < r < p^*$. *Then there exists* $\bar{\bar{\lambda}} \in (0,+\infty)$ *such that for every* $\lambda > \bar{\bar{\lambda}}$, *equation* ($S_{\lambda,\infty}$) *admits a ground state solution.*

Proof. We first recall the limit functional $J_{\lambda,\infty} : \mathbb{X} \to \mathbb{R}$ defined by

$$J_{\lambda,\infty}(u) = \frac{1}{p}\|u\|_{\mathbb{X}}^p - \frac{1}{p_s^*}\int_{\mathbb{R}^N} |u|^{p_s^*}\,dx - \frac{\lambda}{r}\int_{\mathbb{R}^N} |u|^r\,dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*}\,dx.$$

The mountain pass level and Nehari manifold for $J_{\lambda,\infty}$ are defined analogously to those in Lemma 5.2. Specifically, set

$$c_{\lambda,\infty} := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J_{\lambda,\infty}(\gamma(t)) > 0$$

where

$$\Gamma := \{\gamma \in C([0,1],\mathbb{X}) \,|\, \gamma(0)=0, J_{\lambda,\infty}(\gamma(1))<0\}.$$

The Nehari manifold for $(S_{\lambda,\infty})$ is $\mathcal{N}_{\lambda,\infty} := \{u \in \mathbb{X}\backslash\{0\} | \langle J'_{\lambda,\infty}(u),u\rangle = 0\}$, and we denote

$$\bar{c}_{\lambda,\infty} := \inf_{u\in\mathcal{N}_{\lambda,\infty}} J_{\lambda,\infty}(u), \qquad \bar{\bar{c}}_{\lambda,\infty} := \inf_{u\in\mathbb{X}\backslash\{0\}} \sup_{t\geqslant 0} J_{\lambda,\infty}(tu).$$

From Lemma 5.4, there exists $\bar{\bar{\lambda}} \in (0,+\infty)$ such that, for all $\lambda > \bar{\bar{\lambda}}$, $c_{\lambda,\infty} \in (0, c^*_{\lambda,\infty})$, where

$$c^*_{\lambda,\infty} := \min\left\{ \left(\frac{1}{p}-\frac{1}{p_s^*}\right) S_s^{\frac{p_s^*}{p_s^*-p}}, \left(\frac{1}{p}-\frac{1}{p^*}\right) S_1^{\frac{p^*}{p^*-p}} \right\},$$

and S_s, S_1 are given in Lemma 2.2. From Lemmas 5.2-5.3, there exists a bounded $(PS)_{c_{\lambda,\infty}}$ sequence $\{u_n\} \subset \mathcal{N}_{\lambda,\infty}$ at level $c_{\lambda,\infty}$. In particular,

$$J_{\lambda,\infty}(u_n) \to c_{\lambda,\infty}, \text{ and } \langle J'_{\lambda,\infty}(u_n),u_n\rangle \to 0, \text{ as } n\to+\infty.$$

We now show that both $\lim_{n\to\infty}\int |u_n|^{p^*}\,dx > 0$ and $\lim_{n\to\infty}\int |u_n|^{p_s^*}\,dx > 0$ must hold.

Case 1. Suppose $\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p^*}\,dx = 0$. Then,

$$c_{\lambda,\infty} = J_{\lambda,\infty}(u_n) = \frac{1}{p}\|u_n\|_{\mathbb{X}}^p - \frac{1}{p_s^*}\int_{\mathbb{R}^N} |u_n|^{p_s^*}\,dx + o_n(1),$$

and

$$0 = \langle J'_{\lambda,\infty}(u_n),u_n\rangle = \|u_n\|_{\mathbb{X}}^p - \int_{\mathbb{R}^N} |u_n|^{p_s^*}\,dx, \tag{5.6}$$

which gives

$$c_{\lambda,\infty} + o_n(1) = \left(\frac{1}{p}-\frac{1}{p_s^*}\right)\|u_n\|_{\mathbb{X}}^p. \tag{5.7}$$

It follows from (5.6) and Lemma 2.2 that

$$\|u_n\|_{\mathbb{X}}^p = \int_{\mathbb{R}^N} |u|^{p_s^*}\,dx \leqslant S_s^{-p_s^*/p}\|u_n\|_{D^{s,p}(\mathbb{R}^N)}^{p_s^*} \leqslant S_s^{-p_s^*/p}\|u_n\|_{\mathbb{X}}^{p_s^*},$$

Thus $\|u_n\|_{\mathbb{X}}^{p_s^*-p} \geq S_s^{p_s^*/p}$, i.e., $\|u_n\|_{\mathbb{X}}^p \geq S_s^{p_s^*/(p_s^*-p)}$. Consequently,

$$c_{\lambda,\infty} \geq \left(\frac{1}{p}-\frac{1}{p_s^*}\right) S_s^{\frac{p_s^*}{p_s^*-p}},$$

which contradicts the fact that $c_{\lambda,\infty} < c^*_{\lambda,\infty}$. Hence $\lim_{n\to\infty}\int |u_n|^{p^*}\,dx > 0$.

Case 2. Similarly, if $\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p_s^*}\,dx = 0$, then by analogous reasoning we obtain

$$c_{\lambda,\infty} \geqslant \left(\frac{1}{p}-\frac{1}{p^*}\right) S_1^{\frac{p^*}{p^*-p}},$$

again contradicting $c_{\lambda,\infty} < c^*_{\lambda,\infty}$. Therefore $\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p_s^*}\,dx > 0$. From Lemma 5.1, there exist a sequence $\{z_n\} \subset \mathbb{R}^N$ and $\bar{u} \not\equiv 0$ such that, after passing to a subsequence,

$$\bar{u}_n := u_n(x+z_n) \to \bar{u}, \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N),$$

and

$$\int_{B(0,2\bar{\sigma})} |\bar{u}_n|^p\,dx = \int_{B(z_n,2\bar{\sigma})} |u_n|^p\,dx \geqslant C > 0.$$

Moreover, $\bar{u}_n \rightharpoonup \bar{u}$ in $\mathbb{X}$.

We now prove that $\bar{u}$ is a ground state solution. Using Brezis-Lieb Lemma [8], one deduces

$$\begin{aligned}
c_{\lambda,\infty} \leqslant & J_{\lambda,\infty}(\bar{u}) \\
= & J_{\lambda,\infty}(\bar{u}) - \frac{1}{p_s^*}\langle J'_{\lambda,\infty}(\bar{u}), \bar{u}\rangle \\
= & \frac{1}{p}\|\bar{u}\|_{\mathbb{X}}^p - \frac{1}{p_s^*}\int_{\mathbb{R}^N}|\bar{u}|^{p_s^*}\,dx - \frac{\lambda}{r}\int_{\mathbb{R}^N}|\bar{u}|^r\,dx - \frac{1}{p^*}\int_{\mathbb{R}^N}|\bar{u}|^{p^*}\,dx \\
& - \frac{1}{p_s^*}\left(\|\bar{u}\|_{\mathbb{X}}^p - \int_{\mathbb{R}^N}|\bar{u}|^{p_s^*}\,dx - \int_{\mathbb{R}^N}|\bar{u}|^{p^*}\,dx - \lambda\int_{\mathbb{R}^N}|\bar{u}|^r\,dx\right) \\
= & \left(\frac{1}{p} - \frac{1}{p_s^*}\right)\|\bar{u}\|_{\mathbb{X}}^p + \left(\frac{1}{p_s^*} - \frac{1}{r}\right)\lambda\int_{\mathbb{R}^N}|\bar{u}|^r\,dx + \left(\frac{1}{p_s^*} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|\bar{u}|^{p^*}\,dx \\
\leqslant & \lim_{n\to\infty}\left[\left(\frac{1}{p} - \frac{1}{p_s^*}\right)\|u_n\|_{\mathbb{X}}^p + \left(\frac{1}{p_s^*} - \frac{1}{r}\right)\lambda\int_{\mathbb{R}^N}|\bar{u}_n|^r\,dx + \left(\frac{1}{p_s^*} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|\bar{u}_n|^{p^*}\,dx\right] \\
= & \lim_{n\to\infty} J_{\lambda,\infty}(\bar{u}_n) - \frac{1}{p_s^*}\lim_{n\to\infty}\langle J'_{\lambda,\infty}(\bar{u}_n), \bar{u}_n\rangle \\
= & \lim_{n\to\infty} J_{\lambda,\infty}(\bar{u}_n) \\
= & c_{\lambda,\infty},
\end{aligned}$$

which gives $\lim_{n\to\infty}\|\bar{u}_n\|_{\mathbb{X}}^p = \|\bar{u}\|_{\mathbb{X}}^p$ and $J_{\lambda,\infty}(\bar{u}) = c_{\lambda,\infty}$. Hence $\bar{u}$ is a nontrivial critical point of $J_{\lambda,\infty}$ at the mountain pass level, i.e., a ground state solution. ☐

Lemma 5.6. *Let $N > p > 1$, $0 < s < 1$ and let $\{u_n\} \subset \mathcal{N}_\lambda$ be a bounded $(PS)_{c_\lambda}$ sequence for the functional J. Then $u_n \to u \not\equiv 0$ strongly in $\mathbb{X}$. Moreover, $J(u) = c_\lambda$.*

Proof. From Lemmas 5.2- 5.4, there exists a bounded $(PS)_{c_\lambda}$ sequence $\{u_n\} \subset \mathcal{N}_\lambda$ at level $c_\lambda \in (0, c_\lambda^*)$.

Step 1. Non-vanishing of the critical terms. We first show that

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^{p^*}\,dx > 0, \qquad \liminf_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^{p_s^*}\,dx > 0.$$

Case 1. Suppose $\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^{p^*}\,dx = 0$, then

$$c_\lambda = J(u_n) = \frac{1}{p}\|u_n\|_{\mathbb{X}}^p - \frac{1}{p_s^*}\int_{\mathbb{R}^N} F(x)|u_n|^{p_s^*}\,dx,$$

and

$$0 = \langle J'(u_n), u_n\rangle = \|u_n\|_{\mathbb{X}}^p - \int_{\mathbb{R}^N} F(x)|u_n|^{p_s^*}\,dx, \tag{5.8}$$

which gives

$$c_\lambda = \left(\frac{1}{p} - \frac{1}{p_s^*}\right)\|u_n\|_{\mathbb{X}}^p. \tag{5.9}$$

It follows from (5.8) and Lemma 2.2 that

$$\|u_n\|_{\mathbb{X}}^p = \int_{\mathbb{R}^N} F(x)|u|^{p_s^*}\,dx \leqslant S_s^{-\frac{p_s^*}{p}}\|F\|_{L^\infty(\mathbb{R}^N)}\|u_n\|_{D^{s,p}(\mathbb{R}^N)}^{p_s^*} \leqslant S_s^{-\frac{p_s^*}{p}}\|F\|_{L^\infty(\mathbb{R}^N)}\|u_n\|_{\mathbb{X}}^{p_s^*},$$

which shows

$$S_s^{\frac{p_s^*}{p}}\|F\|_{L^\infty(\mathbb{R}^N)}^{-1} \leqslant \|u_n\|_{\mathbb{X}}^{p_s^*-p} \Rightarrow \|u_n\|_{\mathbb{X}}^p \geqslant S_s^{\frac{p_s^*}{p_s^*-p}}\|F\|_{L^\infty(\mathbb{R}^N)}^{-\frac{p}{p_s^*-p}}. \tag{5.10}$$

Combining (5.9) and (5.10), we can derive that

$$c_\lambda \geqslant \left(\frac{1}{p}-\frac{1}{p_s^*}\right) S_s^{\frac{p_s^*}{p_s^*-p}} \|F\|_{L^\infty(\mathbb{R}^N)}^{-\frac{p}{p_s^*-p}}.$$

contradicting $c_\lambda < c_\lambda^*$. Therefore, we get $\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x)|u_n|^{p^*}\,dx > 0$.

Case 2. The proof that $\liminf_{n\to\infty}\int F(x)|u_n|^{p_s^*}\,dx > 0$ is completely analogous. Using Lemma 2.2 leads to a contradiction with the definition of c_λ^*.

Step 2. Translation to recover compactness. From Lemma 5.1, there exists $\{z_n\}\subset\mathbb{R}^N$ such that $\bar{u}_n := u_n(x+z_n) \to \bar{u} \not\equiv 0$, in $L^p_{\text{loc}}(\mathbb{R}^N)$, and

$$\int_{B(0,2\bar{\sigma})} |\bar{u}_n|^p\,dx = \int_{B(z_n,2\bar{\sigma})} |u_n|^p\,dx \geqslant C > 0.$$

Since J is not translation invariant, we define the functional

$$\begin{aligned}\bar{J}(\bar{u}_n) =& \frac{1}{p}\|\bar{u}_n\|_{\mathbb{X}}^p - \frac{1}{p_s^*}\int_{\mathbb{R}^N} F(x+z_n)|\bar{u}_n|^{p_s^*}\,dx \\ & - \frac{\lambda}{r}\int_{\mathbb{R}^N} F(x+z_n)|\bar{u}_n|^r\,dx - \frac{1}{p^*}\int_{\mathbb{R}^N} F(x+z_n)|\bar{u}_n|^{p^*}\,dx.\end{aligned} \tag{5.11}$$

Then, $\bar{J}(\bar{u}_n) = J(u_n) \to c_\lambda$. For any $\varphi\in\mathbb{X}$, set $\varphi_n(x) = \varphi(x-z_n)$. Then $\|\varphi_n\|_{\mathbb{X}} = \|\varphi\|_{\mathbb{X}}$ and

$$\langle \bar{J}'(\bar{u}_n),\varphi\rangle = \langle J'(u_n),\varphi_n\rangle = o_n(1)\|\varphi_n\|_{\mathbb{X}} = o_n(1)\|\varphi\|_{\mathbb{X}}.$$

Thus $\bar{J}'(\bar{u}_n)\to 0$ in $\mathbb{X}^*$, i.e. $\{\bar{u}_n\}$ is a $(PS)_{c_\lambda}$ sequence for $\bar{J}_n$.

Step 3. Boundedness of the translation sequence. We claim that $\{z_n\}$ is bounded in $\mathbb{R}^N$. If not; then $|z_n|\to\infty$ (up to a subsequence). For any $\varphi\in C_c^\infty(\mathbb{R}^N)$, using Hölder's inequality and the fact that $F(x+z_n)\to 1$ in $L^1_{\text{loc}}(\mathbb{R}^N)$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}[F(x+z_n)-1]|\bar{u}_n|^{p_s^*-1}|\varphi|\,dx \\ \leqslant& \left(\int_{\mathbb{R}^N}[F(x+z_n)-1]|\bar{u}_n|^{p_s^*}\,dx\right)^{\frac{p_s^*-1}{p_s^*}} \left(\int_{\mathbb{R}^N}[F(x+z_n)-1]|\varphi|^{p_s^*}\,dx\right)^{\frac{1}{p_s^*}} \\ \leqslant& C\left(\int_{\mathbb{R}^N}[F(x+z_n)-1]|\varphi|^{p_s^*}\,dx\right)^{\frac{1}{p_s^*}} \to 0, \text{ as } n\to+\infty.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N} F(x+z_n)|\bar{u}_n|^{p_s^*-2}\bar{u}_n\varphi\,dx = \int_{\mathbb{R}^N}|\bar{u}_n|^{p_s^*-2}\bar{u}_n\varphi\,dx + o_n(1), \text{ as } n\to+\infty, \tag{5.12}$$

and similarly for the p^* and r terms. Consequently,

$$\int_{\mathbb{R}^N} F(x+z_n)|\bar{u}_n|^{p^*-2}\bar{u}_n\varphi\,dx = \int_{\mathbb{R}^N}|\bar{u}_n|^{p^*-2}\bar{u}_n\varphi\,dx + o_n(1), \text{ as } n\to+\infty. \tag{5.13}$$

It follows from $\bar{u}_n \rightharpoonup \bar{u}$ weakly in $\mathbb{X}$, (4.2), (5.12) and (5.13) that $\langle \bar{J}'(\bar{u}_n),\varphi\rangle = \langle J'_{\lambda,\infty}(\bar{u}),\varphi\rangle = 0$, which implies that $\bar{u}$ is a weak solution to equation $(S_{\lambda,\infty})$. It follows from (5.11), $F(x)\geqslant 1$, Brezis-Lieb Lemma [8] and $\langle J'_{\lambda,\infty}(\bar{u}),\varphi\rangle = 0$ that

$$\begin{aligned} c_{\lambda,\infty} > c_\lambda = &\lim_{n\to\infty}\left(\bar{J}(\bar{u}_n) - \frac{1}{p_s^*}\langle \bar{J}'(\bar{u}_n),\bar{u}_n\rangle\right)\\ \geqslant& J_{\lambda,\infty}(\bar{u}) - \frac{1}{p_s^*}\langle J'_{\lambda,\infty}(\bar{u}),\bar{u}\rangle\\ =&J_{\lambda,\infty}(\bar{u}) \geqslant c_{\lambda,\infty}, \end{aligned}$$

which yields a contradiction. Hence $\{z_n\}$ is bounded in $\mathbb{R}^N$. By using Lemma 3.1 again, up to a subsequence, we conclude that $u_n(x)\to u\not\equiv 0$, in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$.

Step 4. Strong convergence. Now we prove that $u_n\to u$ strongly in $\mathbb{X}$ and $J(u)=c_\lambda$. Using Brezis-Lieb Lemma [8], one deduces

$$\begin{aligned} &c_\lambda \leqslant J(u)\\ =&J(u) - \frac{1}{p_s^*}\langle J'(u),u\rangle\\ =&\frac{1}{p}\|u\|_{\mathbb{X}}^p - \frac{1}{p_s^*}\int_{\mathbb{R}^N}F(x)|u|^{p_s^*}\,dx - \frac{\lambda}{r}\int_{\mathbb{R}^N}F(x)|u|^r\,dx - \frac{1}{p^*}\int_{\mathbb{R}^N}F(x)|u|^{p^*}\,dx\\ &-\frac{1}{p_s^*}\left(\|u\|_{\mathbb{X}}^p - \int_{\mathbb{R}^N}F(x)|u|^{p_s^*}\,dx - \int_{\mathbb{R}^N}F(x)|u|^{p^*}\,dx - \lambda\int_{\mathbb{R}^N}F(x)|u|^r\,dx\right)\\ =&\left(\frac{1}{p}-\frac{1}{p_s^*}\right)\|u\|_{\mathbb{X}}^p + \left(\frac{1}{p_s^*}-\frac{1}{r}\right)\lambda\int_{\mathbb{R}^N}F(x)|u|^r\,dx + \left(\frac{1}{p_s^*}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N}F(x)|u|^{p^*}\,dx\\ \leqslant&\lim_{n\to\infty}\left[\left(\frac{1}{p}-\frac{1}{p_s^*}\right)\|u_n\|_{\mathbb{X}}^p + \left(\frac{1}{p_s^*}-\frac{1}{r}\right)\lambda\int_{\mathbb{R}^N}F(x)|u_n|^r\,dx + \left(\frac{1}{p_s^*}-\frac{1}{p^*}\right)\int_{\mathbb{R}^N}F(x)|u_n|^{p^*}\,dx\right]\\ =&\lim_{n\to\infty}J(u_n) - \frac{1}{p_s^*}\lim_{n\to\infty}\langle J'(u_n),u_n\rangle\\ =&\lim_{n\to\infty}J(u_n)\\ =&c_\lambda, \end{aligned}$$

which gives $\lim_{n\to\infty}\|u_n\|_{\mathbb{X}}^p = \|u\|_{\mathbb{X}}^p$ and $J(u)=c_\lambda$. ☐

Proof of Theorem 1.3. Setting $\tilde{\lambda} = \max\{\bar{\lambda},\bar{\bar{\lambda}}\}$, we get the desired result via Lemmas 5.2 - 5.6. The proof is complete. ☐

A. PROOF OF LEMMA 3.3

Lemma A.1. *The functional I satisfies the mountain pass geometry.*

Proof. It is easy to see that

$$I(u) \geqslant \frac{1}{p}\|u\|_{\mathbb{X}}^p - C\|F\|_{L^\infty(\mathbb{R}^N)}\|u\|_{\mathbb{X}}^r,$$

where $C>0$. From $p<r<p^*$, we know that there exists a sufficiently small positive number b such that $\varsigma := \inf_{\|u\|_{\mathbb{X}}=b} I(u) > I(0) \geqslant 0$. For $u\in\mathbb{X}\setminus\{0\}$, we have

$$I(tu) = \frac{t^p}{p}\|u\|_{\mathbb{X}}^p - \frac{t^r}{r}\int_{\mathbb{R}^N}F(x)|u|^r\,dx.$$

From $p_s^* < r < p^*$, it follows that $I(tu) < 0$ for t large enough. We can choose $t_u > 0$ corresponding to u such that $I(t_u u) < 0$ for $t > t_u$ and $\|t_u u\|_{\mathbb{X}} > b$. ☐

Lemma A.2. *For any $u \in \mathbb{X} \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathscr{N}$ and $I(t_u u) = \max_{t>0} I(tu)$.*

Proof. For any $u \in \mathbb{X} \setminus \{0\}$ and $t \in (0, \infty)$, we define

$$f_1(t) = I(tu) = \frac{t^p}{p}\|u\|_{\mathbb{X}}^p - \frac{t^r}{r}\int_{\mathbb{R}^N} F(x)|u|^r\,dx.$$

Then,

$$f_1'(t) = t^{p-1}\|u\|_{\mathbb{X}}^p - t^{r-1}\int_{\mathbb{R}^N} F(x)|u|^r\,dx.$$

We know that $f_1'(\cdot) = 0$ iff

$$\|u\|_{\mathbb{X}}^p = t^{r-p}\int_{\mathbb{R}^N} F(x)|u|^r\,dx.$$

Therefore, there must exist a unique value $0 < t_u < \infty$ such that $t_u u \in \mathscr{N}$. ☐

Lemma A.3. *It holds that $\bar{c} = \inf_{u \in \mathscr{N}} I(u) > 0$.*

Proof. By applying $\langle I'(u), u\rangle = 0$, we know

$$0 = \langle I'(u), u\rangle \geqslant \|u\|_{\mathbb{X}}^p - C\|F\|_{L^\infty(\mathbb{R}^N)}\|u\|_{\mathbb{X}}^r,$$

which gives $C\|F\|_{L^\infty(\mathbb{R}^N)}\|u\|_{\mathbb{X}}^{r-p} \geqslant 1$ and $\|u\|_{\mathbb{X}}^p \geqslant C$. For $u \in \mathscr{N}$, we have

$$I(u) = I(u) - \frac{1}{r}\langle I'(u), u\rangle \geqslant \left(\frac{1}{p} - \frac{1}{r}\right)\|u\|_{\mathbb{X}}^p \geqslant C.$$

We know that the functional I is bounded from below on $\mathscr{N}$. And $\bar{c} > 0$. ☐

Lemma A.4. *It holds $c = \bar{c} = \bar{\bar{c}}$, where $\bar{\bar{c}} := \inf_{u \in \mathbb{X}\setminus\{0\}} \sup_{t \geqslant 0} I(tu)$.*

Proof. By using Lemma 3.2, we can directly obtain $\bar{c} = \bar{\bar{c}}$. For any $u \in \mathbb{X} \setminus \{0\}$, there exists some $\tilde{t} > 0$ that is sufficiently large such that $I(\tilde{t}u) < 0$. We can construct a path $\gamma : [0,1] \to \mathbb{X}$ by setting $\gamma(t) = t\tilde{t}u$. It is clear that $\gamma \in \Gamma$ and that $c \leqslant \bar{\bar{c}}$. For every path $\gamma \in \Gamma$, we define $g(t) = \langle I'(\gamma(t)), \gamma(t)\rangle$. It is evident that $g(t) > 0$ for small values of t. We have

$$I(\gamma(1)) - \frac{1}{r}\langle I'(\gamma(1)), \gamma(1)\rangle \geqslant \left(\frac{1}{p} - \frac{1}{r}\right)\|\gamma(1)\|_X^p \geqslant 0,$$

which shows $\langle I'(\gamma(1)), \gamma(1)\rangle \leqslant pI(\gamma(1)) = pI(\tilde{t}u) < 0$. Thus, there exists $\tilde{\tilde{t}} \in (0,1)$ such that $g(\tilde{\tilde{t}}) = 0$, i.e. $\gamma(\tilde{\tilde{t}}) \in \mathscr{N}$ and $c \geqslant \bar{c}$. This deduces $c = \bar{c} = \bar{\bar{c}}$. ☐

Lemma A.5. *For $u \in \mathscr{N}$, $\Phi'(u) \neq 0$, where $\Phi(u) = \langle I'(u), u\rangle$ and*

$$\langle \Phi'(u), u\rangle = p\|u\|_{D^{1,p}(\mathbb{R}^N)}^p + p\|u\|_{D^{s,p}(\mathbb{R}^N)}^p - r\int_{\mathbb{R}^N} F(x)|u|^r\,dx. \tag{A.1}$$

Moreover, if $u \in \mathscr{N}$ and $I(u) = c$, then u is a ground state solution.

Proof. For $u \in \mathscr{N}$, it follows from (A.1) that

$$\begin{aligned}\langle \Phi'(u), u\rangle =& \langle \Phi'(u), u\rangle - p\Phi(u)\\ =& p\|u\|_{\mathbb{X}}^p - r\int_{\mathbb{R}^N} F(x)|u|^r\,dx - p\|u\|_{\mathbb{X}}^p + p\int_{\mathbb{R}^N} F(x)|u|^r\,dx\\ =& (p-r)\|u\|_{\mathbb{X}}^p < 0.\end{aligned}$$

Thus, $\Phi'(u) \neq 0$ for $u \in \mathscr{N}$. Suppose $u \in \mathscr{N}$ and $I(u) = \bar{c}$, where $\bar{c}$ is the minimum of I on $\mathscr{N}$. By using the Lagrange multiplier theorem, we can conclude that there exists a scalar $\lambda \in \mathbb{R}$ such that $I'(u) = \lambda\Phi'(u)$. Thus $\lambda\langle \Phi'(u), u\rangle = \langle I'(u), u\rangle = \Phi(u) = 0$. This implies $\lambda = 0$ and $I'(u) = 0$. ☐

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