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CHARNES-COOPER SCALARIZATION TO NON-SMOOTH SEMIVECTORIAL BILEVEL OPTIMIZATION PROBLEMS

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Abstract. Semivectorial bilevel optimization problems, particularly those where the lower-level problem is solved up to efficiency, have attracted significant attention in optimization theory. While previous research in this area has often focused on problems with continuously differentiable functions, many real-world applications of bilevel optimization feature non-smooth functions. Motivated by this observation, the current study aims to propose new necessary optimality conditions for semivectorial bilevel programs under weaker regularity assumptions that allow for non-differentiability. Specifically, we revisit the Charnes-Cooper scalarization technique and present tailored optimality results for problems where the data satisfy only local Lipschitz continuity. Through this generalization, our results provide a more flexible theoretical framework applicable to a broader class of non-smooth optimization models arising in practical semivectorial bilevel programming contexts.

Keywords. Charnes-Cooper scalarization; Efficient solution; Optimality conditions; Optimal value function; Semivectorial bilevel programming.

1. INTRODUCTION

The fundamental structure of semivectorial bilevel programming, which was originally established by Bonnel and Morgan [1], is expressed as

$$\min_{x,y} F(x,y) \ \text{ s.t. } x \in X,\ y \in \Phi(x). \tag{SVBOP}$$

The nonempty and closed set $X \subseteq \mathbb{R}^n$ denotes the upper-level feasible set, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ represents the objective function, and $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is the solution set mapping of the vector lower-level problem

$$\min_{y} \left(f_1(x,y), \dots, f_s(x,y)\right) \ \text{ s.t. } y \in \Upsilon(x), \tag{P[x]}$$

where $\Upsilon(x) \subseteq \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^s$ denote the feasible set and the vector-valued objective function of the lower-level problem, respectively.

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Semivectorial bilevel optimization problems arise naturally in hierarchical decision-making processes involving multiple conflicting criteria at the lower level. These models have numerous applications in areas such as transportation planning, engineering design, supply chain management, economics, and energy systems; see, for example, [2, 3, 4] for further details. In these situations, the lower-level decision-maker generally seeks Pareto efficient solutions rather than weakly efficient ones, since efficient solutions exclude decisions that can be improved in one criterion without deteriorating another. Thus the study of semivectorial bilevel programming with efficient lower-level solutions is not only mathematically challenging but also highly relevant from a practical viewpoint. Except for [5, 6], most of the existing literature has mainly focused on the case where Φ is the weakly efficient solution map associated with (P[x]). Earlier seminal contributions on this topic include [7, 8, 9, 10, 11], where various optimality conditions, reformulation procedures, and numerical algorithms were established. The motivation of the present work is threefold.

- Most of the available contributions deal with weakly efficient lower-level solutions, whereas the efficient case remains significantly less investigated despite its practical relevance. Indeed, weakly efficient solutions may correspond to decisions that are dominated in some criteria and therefore may have limited applicability in real-world hierarchical optimization problems. This motivates the study of semivectorial bilevel problems where the lower-level problem is solved up to efficiency.
- A common feature of the above references is the use of weighted-sum scalarization techniques in order to transform the lower-level multiobjective problem into a scalar optimization problem. Although this approach is applicable in several semivectorial optimization settings, particularly in the weakly efficient case, it suffers from important drawbacks. First, the weighting parameters are usually incorporated as additional upper-level variables, which enlarges the dimension of the bilevel problem and complicates the associated analysis. Second, in nonconvex multiobjective optimization, weighted-sum scalarization generally fails to characterize properly the efficient solution set. Consequently, this scalarization technique becomes inadequate when the lower-level problem is solved up to efficiency rather than weak efficiency. These limitations motivate the development of an alternative scalarization framework based on the Charnes–Cooper approach.
- Existing works employing the Charnes–Cooper method in semivectorial bilevel optimization, such as [6], rely essentially on differentiability assumptions. Nevertheless, many practical bilevel optimization models involve non-smooth data, as emphasized in [12]. This motivates the extension of the available theory to a non-smooth setting by using tools from variational and generalized differentiation analysis under locally Lipschitz assumptions.

Motivated by these points, we focus on the semivectorial bilevel program (SVBOP) where the lower-level problem (P[x]) is solved up to efficiency. Specifically, we re-examine the Charnes–Cooper method originally proposed in the seminal works [13, 14] and recently applied for convex vector optimization problems in [15]. In Section 3, the Charnes–Cooper approach is explored in more detail. Before exploring the non-smooth analysis of problem (SVBOP), it is essential to establish that the efficiency map of a completely linear parametric multiobjective optimization problem possesses a closed graph. This provides an additional criterion for establishing the existence of solutions to (SVBOP). Subsequently, by employing an enhanced Charnes–Cooper technique, we are able to transform the multiobjective bilevel programming problem into a bilevel programming problem that incorporates a scalar lower-level problem. This innovative scalarization approach proves to be effective in addressing significant classes of vectorial bilevel problems, while simultaneously overcoming the limitations of previous scalarization methods. Furthermore, we reformulate the bilevel optimization problem as a single-level optimization problem, ensuring global and local equivalence to the original problem. Notably, this reformulation is achieved without necessitating the scalar lower-level solution set derived through the utilization of Charnes–Cooper to be either inner semicompact or inner semicontinuous. Building upon these foundations, we establish the necessary optimality conditions for the problem under consideration.

In the next section, we introduce key concepts from multiobjective optimization and variational analysis. Section 3 first examines the graph-closedness of the associated efficiency mapping, outlines our scalarization technique, and explains the transformation process yielding a single-level optimization problem. Moving on to Section 4, we establish a value function estimation and provide sufficient conditions to ensure its local Lipschitz property. Subsequently, in Section 5, we use an appropriate weak Mangasarian–Fromowitz constraint qualification to establish the necessary optimality conditions. To enhance these conditions, we introduce a general partial calmness concept specifically tailored to the optimal value function reformulation. Moreover, we study the special case where the lower-level multiobjective problem is linear in the lower decision variable. Finally, we draw our conclusions in Section 6.

2. Preliminaries

In this section, we introduce some basic notations that are used throughout the remainder of this paper. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $\mathbb{R}_+$ be the set of non-negative real numbers. Given a subset $M \subseteq \mathbb{R}^n$, the notation $\operatorname{conv} M$ refers to the convex hull of M, and the distance of a point $\bar{x} \in \mathbb{R}^n$ to M is given by $d(\bar{x}, M) := \inf\{\|x-\bar{x}\| \ : \ x \in M\}$.

Consider the following multi-objective optimization problem:

$$\min_x \left(\omega_1(x), \dots, \omega_s(x)\right) \quad \text{s.t. } x \in M, \tag{2.1}$$

where $\omega_k : \mathbb{R}^n \to \mathbb{R}$, $k = 1, \dots, s$, and s is a positive integer.

Definition 2.1. Let $\bar{x} \in M$ be a feasible point of problem (2.1).

(1) We say that $\bar{x}$ is an efficient solution to problem (2.1) if there exists no other feasible point $x \in M$ such that $\omega_k(x) \le \omega_k(\bar{x})$ for all $k = 1, \dots, s$ and $\omega_k(x) < \omega_k(\bar{x})$ for some $k = 1, \dots, s$.
(2) We say that $\bar{x}$ is a weakly efficient solution to problem (2.1) if there exists no $x \in M$ such that $\omega_k(x) < \omega_k(\bar{x})$ for all $k = 1, \dots, s$.

Given a set-valued mapping $\Pi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we define:

- The graph of Π is given by:

$$\operatorname{gph} \Pi := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in \Pi(x)\};$$

- The Kuratowski-Painlevée upper limit of Π at a point $\bar{x}$ is given by

$$\limsup_{x \to \bar{x}} \Pi(x) := \{y \in \mathbb{R}^m : \ \exists x_k \to \bar{x}, \ y_k \to y \text{ with } y_k \in \Pi(x_k) \ \forall k \in \mathbb{N}\}.$$

Let Θ be a locally closed subset of $\mathbb{R}^n$, and $\bar{x} \in \Theta$. Then,

- The normal cone of Fréchet $\hat{N}(\bar{x},\Theta)$ is given by

$$\widehat{N}(\bar{x},\Theta) := \left\{ x^* \in \mathbb{R}^n : \limsup_{x \xrightarrow{\Theta} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq 0 \right\}.$$

- The normal cone of Mordukhovich $N(\bar{x},\Theta)$ to Θ in $\bar{x}$ is defined as

$$N(\bar{x},\Theta) := \limsup_{x \xrightarrow{\Theta} \bar{x}} \widehat{N}(x,\Theta),$$

where $x \xrightarrow{\Theta} \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Theta$.

Unless otherwise stated, $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is assumed to be a lower semi-continuous function. For $\bar{x} \in \text{dom}\ \varphi := \{x \in \mathbb{R}^n \ : \ \varphi(x) < +\infty\}$, the Fréchet subdifferential of φ at $\bar{x}$ is

$$\widehat{\partial}\varphi(\bar{x}) := \left\{ x^* \in \mathbb{R}^n \ : \ \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \geq 0 \right\},$$

and the Mordukhovich (limiting) subdifferential of φ at $\bar{x}$ is given as follows:

$$\partial\varphi(\bar{x}) := \left\{ x^* \in \mathbb{R}^n \ : \ \exists x_k \xrightarrow{\varphi} \bar{x}, \text{ and } x_k^* \in \widehat{\partial}\varphi(x_k)\, \forall k \text{ with } x_k^* \to x^* \right\},$$

and its singular subdifferential at $\bar{x}$ is

$$\partial^\infty\varphi(\bar{x}) := \left\{ x^* \in \mathbb{R}^n \ : \ \exists x_k \xrightarrow{\varphi} \bar{x}, t_k \searrow 0, \text{ and } x_k^* \in \widehat{\partial}\varphi(x_k)\, \forall k \text{ with } t_k x_k^* \to x^* \right\},$$

where $x \xrightarrow{\varphi} \bar{x}$ means that $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$. When φ is a local Lipschitz continuous function at $\bar{x}$, $\partial\varphi(\bar{x})$ is nonempty and compact. Moreover, its convex hull is the Clarke subdifferential, i.e.,

$$\partial_C\varphi(\bar{x}) := \text{conv}\ \partial\varphi(\bar{x}).$$

Given this connection between Mordukhovich and Clarke subdifferentials, we have the following convex hull property

$$\text{conv}\ \partial(-\varphi)(\bar{x}) = -\text{conv}\ \partial\varphi(\bar{x}). \tag{2.2}$$

The concept of semismoothness was applied to sets in [16] by means of the Euclidean distance function d_E. A set $\Theta \subseteq \mathbb{R}^n$ is called semismooth at $\bar{x} \in cl\ \Theta$ if, for any sequence $x_k \to \bar{x}$ with $x_k \in \Theta$ and $\|x_k - \bar{x}\|^{-1}(x_k - \bar{x}) \to \alpha$, $\langle x_k^*, \alpha\rangle \to 0$ for all selections $x_k^* \in \partial_C d_E(x_k,\Theta)$. Moreover, we say that the set Θ is regular at $\bar{x}$ if $N(\bar{x},\Theta) = \widehat{N}(\bar{x},\Theta)$.

Now, we define the coderivative $D^*\Pi(\bar{x},\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ for the set-valued mapping $\Pi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x},\bar{y}) \in \text{gph}\ \Pi$ by

$$D^*\Pi(\bar{x},\bar{y})(v) := \{u \in \mathbb{R}^n \mid (u,-v) \in N((\bar{x},\bar{y}), \text{gph}\,\Pi)\} \ \text{for} \quad v \in \mathbb{R}^m.$$

Moreover, if the function $\omega : \mathbb{R}^n \to \mathbb{R}^m$ is strictly differentiable at $\bar{x}$, then

$$D^*\omega(\bar{x})(y^*) = \left\{ \nabla\omega(\bar{x})^\top y^* \right\} \quad \text{for all } y^* \in \mathbb{R}^m.$$

For $(\bar{x},\bar{y}) \in \text{gph}\ \Pi$, the set-valued mapping Π is said to be

- inner semicontinuous at $(\bar{x},\bar{y})$ if, for any sequence $x_k \to \bar{x}$ there exists a sequence $y_k \in \Pi(x_k)$ which converges to $\bar{y}$ as $k \to \infty$.
- inner semicompact at $\bar{x}$, $\Pi(\bar{x}) \neq \emptyset$ if, for every sequence $x_k \to \bar{x}$ with $\Pi(x_k) \neq \emptyset$ there exists a sequence of $y_k \in \Pi(x_k)$ that contains a convergent subsequence as $k \to +\infty$.

- calm at $(\bar{x},\bar{y})$ if there exist neighborhoods U of $\bar{x}$, V of $\bar{y}$, and a constant $\lambda > 0$ such that
$$d\left(y,\Pi\left(\bar{x}\right)\right) \leq \lambda \left\|x-\bar{x}\right\| \quad \forall x \in U \text{ and } \forall y \in \Pi\left(x\right) \cap V.$$
- has Aubin property at $(\bar{x},\bar{y}) \in \operatorname{gph} \Pi$ if there exist neighborhoods U of $\bar{x}$, V of $\bar{y}$, and a scalar $\tau > 0$ such that for all $x,x^* \in U$ and $y \in \Pi(x) \cap V$ there is $y^* \in \Pi\left(x^*\right)$ such that
$$\left\|y-y^*\right\| \leq \tau \left\|x-x^*\right\|.$$

In general, the notion of inner semicontinuity is required to satisfy the Lipschitz property. For a closed graph mapping Π, the coderivative criterion $D^*\Pi\left(\bar{x},\bar{y}\right)(0) = \{0\}$ provides a sufficient condition for the inner semicontinuity of Π at $(\bar{x},\bar{y})$.

3. Charnes-Cooper scalarization to semivectorial bilevel programs

This paper primarily focuses on semivectorial optimistic bilevel optimization problem (SVBOP), where the set-valued mapping $\Phi = \Phi_{\text{eff}}$ represents the collection of efficient optimal solutions for the parametric multiobjective optimization problem (P[x]). Furthermore, throughout the paper, for clarity of exposition, the upper- and lower-level feasible sets are defined as:
$$X := \{x \in \mathbb{R}^n : G_i(x) \leq 0,\ i \in I\} \text{ and } \Upsilon(x) := \{y \in \mathbb{R}^m : g_j(x,y) \leq 0,\ j \in J\}$$
with $I = \{1,\ldots,q\}$, $J = \{1,\ldots,p\}$, $G_i : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. We assume that all involved functions at both levels are Lipschitz continuous.

We also assume that the graph of Φ_{eff} is closed. This closedness assumption on the graph of the efficiency mapping is necessary to ensure the existence of solutions to (SVBOP). However, it is worth noting that when solving the lower-level problem (P[x]) efficiently, the graph of the corresponding efficiency map Φ_{eff} is generally not closed. To illustrate, we consider the following lower-level problem:
$$\min_y \left(\lfloor y \rfloor; x^2 + \lfloor y+2 \rfloor\right) \quad \text{s.t. } y \in [2,3],$$
where $\lfloor\cdot\rfloor$ denotes the floor function. Here, the efficiency map is given by $\Phi_{\text{eff}}(x) = [2,3[$ for all $x \in \mathbb{R}$. Hence, $\operatorname{gph} \Phi_{\text{eff}} = \mathbb{R} \times [2,3[$, which is not closed.

The non-closedness of $\operatorname{gph} \Phi_{\text{eff}}$ poses challenges, for example in proving the existence of solutions to (SVBOP). Considering the lower-level problem above, the upper-level problem
$$\min_{x,y} |x| - y + 3 \ \text{ s.t. } x \leq 0 \ \text{ s.t. } (x,y) \in \operatorname{gph} \Phi_{\text{eff}}$$
clearly admits no solution.

In this section, we prove that, for a completely linear multiobjective parametric optimization problem, the efficiency map has a closed graph. This closedness property ensures the existence of solutions to (SVBOP), as will be shown subsequently.

Recall that a fully linear multiobjective parametric optimization problem takes the following form:
$$\min_y Ay \ \text{ s.t. } Bx + Cy \leq e, \tag{L[x]}$$
where $A \in \mathbb{R}^{s\times m}$, $B \in \mathbb{R}^{p\times n}$, $C \in \mathbb{R}^{p\times m}$, and $e \in \mathbb{R}^p$. We represent every matrix in terms of its row vectors as $A = (a_k)_{1\leq k\leq s}$, $B = \left(b_j\right)_{1\leq j\leq p}$ and $C = \left(c_j\right)_{1\leq j\leq p}$. For this problem, the feasible set $\Upsilon(x)$, with $x \in \mathbb{R}^n$, is given by
$$\Upsilon(x) = \{y \in \mathbb{R}^m \ : \ Bx + Cy \leq e\}.$$

The following proposition shows that the efficiency map of a fully linear multiobjective parametric optimization problem possesses a closed graph, proven here without any scalarization.

Proposition 3.1. *The mapping* Φ_{eff} *associated with problem* (L[x]) *possesses a closed graph.*

Proof. Suppose, on the contrary, that gph Φ_{eff} is not closed. Then, there exists a sequence $\{(x_t, y_t)\} \subseteq \text{gph } \Phi_{\text{eff}}$ converging to $(x, y) \in \text{gph } \Upsilon \setminus \text{gph } \Phi_{\text{eff}}$. Hence, there is $\bar{y} \in \Upsilon(x)$ such that $a_k\bar{y} \leq a_k y$ for all $k = 1, \ldots, s$ and $a_k\bar{y} < a_k y$ for some $k = 1, \ldots, s$. Hence $\bar{y} - y \in \Gamma$, where

$$\Gamma := \{y \in \mathbb{R}^m \ : \ a_k y \leq 0 \ \text{ for all } k = 1, \ldots, s, \text{ and } a_k y < 0 \text{ for some } k = 1, \ldots, s\}.$$

Let $\mathscr{J} \subseteq J$ such that $\mathscr{J} = \mathscr{J}(t) := \left\{j \in J \ : \ b_j x_t + c_j y_t = e_j\right\}$. Then, $b_j x + c_j y = e_j$ and $c_j\bar{y} \leq c_j y$ for all $j \in \mathscr{J}$.

We claim there exists $j \in J \setminus \mathscr{J}$ with $c_j\bar{y} > c_j y$. Suppose to the contrary that $c_j(\bar{y} - y) \leq 0$ for all $j \in J \setminus \mathscr{J}$. Henceforth, for a given $z \in \Phi_{\text{eff}}(x)$, we have $(x, z + (\bar{y} - y)) \in \text{gph}\Upsilon$ since $(x, z) \in \text{gph}\Upsilon$. However, $z + (\bar{y} - y) - z = \bar{y} - y \in \Gamma$, so $a_k(z + (\bar{y} - y)) \leq a_k z$ for all k and $a_k(z + (\bar{y} - y)) < a_k z$ for some k, contradicting z being efficient for (L[x]). Hence there exists the claimed j.

Let $\mathscr{J}_1, \mathscr{J}_2 \subseteq J$ such that

$$\mathscr{J}_1 := \left\{j \in J \ : \ c_j(\bar{y} - y) \leq 0\right\} \text{ and } \mathscr{J}_2 := \left\{j \in J \ : \ c_j(\bar{y} - y) > 0\right\}.$$

Observe that $\mathscr{J} \subseteq \mathscr{J}_1$, $\mathscr{J}_2 \neq \emptyset$ and $\mathscr{J}_1 \cup \mathscr{J}_2 = J$. Fix $t \in \mathbb{N}$, and define

$$\rho_t := \min_{j \in \mathscr{J}_2} \frac{e_j - \left(b_j x_t + c_j y_t\right)}{c_j(\bar{y} - y)} = \frac{e_l - (b_l x_t + c_l y_t)}{c_l(\bar{y} - y)} > 0,$$

with $l \in \mathscr{J}_2$ and taking into account the fact that $\mathscr{J} \nsubseteq \mathscr{J}_2$. Let

$$\tilde{y}_t := y_t + \rho_t(\bar{y} - y). \tag{3.1}$$

Since $\rho_t > 0$ and $\bar{y} \neq y$, then $\tilde{y}_t \neq y_t$. Two cases have to be considered.

- For $j \in \mathscr{J}_1$, we obtain

$$b_j x_t + c_j\tilde{y}_t = b_j x_t + c_j y_t + \rho_t c_j(\bar{y} - y) \leq b_j x_t + c_j y_t \leq e_j.$$

- For $j \in \mathscr{J}_2$, we obtain

$$\begin{aligned} b_j x_t + c_j\tilde{y}_t &= b_j x_t + c_j y_t + \rho_t c_j(\bar{y} - y) \\ &= b_j x_t + c_j y_t + \left(\rho_t \times \frac{c_j(\bar{y} - y)}{e_j - \left(b_j x_t + c_j y_t\right)} \times \left(e_j - \left(b_j x_t + c_j y_t\right)\right)\right) \\ &\leq b_j x_t + c_j y_t + \left(1 \times \left(e_j - \left(b_j x_t + c_j y_t\right)\right)\right) \\ &\leq e_j. \end{aligned}$$

Consequently, $(x_t, \tilde{y}_t) \in \text{gph } \Upsilon$. Moreover, we have from (3.1) that $a_k\tilde{y}_t \leq a_k y_t$ for all $k = 1, \ldots, s$ and $a_k\tilde{y}_t < a_k y_t$ for some $k = 1, \ldots, s$, which contradicts the fact that $y_t \in \Phi_{\text{eff}}(x_t)$. Therefore, gph Φ_{eff} is closed. □

Returning to the initial semivectorial bilevel problem (SVBOP), our goal is to transform it into a single optimistic bilevel programming problem using the Charnes-Cooper scalarization method. To initiate this transformation, let's define the set Ω of upper and lower-level constraints as follows:

$$\Omega = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : x \in X \text{ and } y \in \Upsilon(x)\}.$$

For $(x,y) \in \Omega$, the scalarization technique of the transformed problem (P[x]) is given by the following program:

$$\min_{w} \sum_{k=1}^{s} f_k(x,w) \text{ s.t. } f_k(x,w) \le f_k(x,y),\ k = 1,\dots,s,\ \ w \in \Upsilon(x). \tag{CCS[x,y]}$$

We refer to (CCS[x,y]) as a Charnes-Cooper scalarization problem. Its solution set is denoted as $S(x,y)$ and its feasible set is given by $\Xi(x,y) = \{w \in \Upsilon(x) : f_k(x,w) \le f_k(x,y), k = 1,\dots,s\}$. When considering efficient solutions with respect to $\mathbb{R}^s_+$ for lower-level problem (P[x]), we establish the relationship between (CCS[x,y]) and (P[x]). According to [6, Proposition 3.1], we obtain the following result:

Proposition 3.2. *For $(x,y) \in \Omega$, y is an efficient solution of* (P[x])*; that is, $y \in \Phi_{\text{eff}}(x)$, if and only if it is a solution of* (CCS[x,y]).

Let us point out that Proposition 3.2 is no longer true when considering weak efficient solutions for the lower-level problem (P[x]); see [6, Example 3.2]. Based on the aforementioned arguments and the preceding proposition, we now replace semivectorial optimization problem (SVBOP) with the following equivalent scalarized bilevel optimization problem:

$$\min_{x,y} F(x,y) \text{ s.t. } x \in X,\ y \in S(x,y). \tag{3.2}$$

One possible way to reformulate scalarized problem (3.2) as a single-level mathematical program is through the application of the optimal value function reformulation. This process involves replacing the lower-level solution set $S(x,y)$ with its characterization using the optimal value function.

Let $\upsilon : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ denote the optimal value function associated with the lower-level problem (CCS[x,y]), defined as:

$$\upsilon(x,y) = \inf_{w \in \Xi(x,y)} \sum_{k=1}^{s} f_k(x,w), \quad \text{for all } (x,y) \in \mathbb{R}^n \times \mathbb{R}^m. \tag{3.3}$$

Then, the lower-level solution set $S(x,y)$ can be expressed as:

$$\begin{aligned} S(x,y) &= \arg\min_{w} \left\{ \sum_{k=1}^{s} f_k(x,w), w \in \Xi(x,y) \right\} \\ &= \left\{ w \in \Upsilon(x) : \sum_{k=1}^{s} f_k(x,w) \le \upsilon(x,y),\ f_k(x,w) \le f_k(x,y),\ k = 1,\dots,s \right\}. \end{aligned}$$

This optimal value reformulation allows casting problem (3.2) into a single-level form without requiring additional assumptions. It transforms (3.2) into the equivalent single-level problem:

$$\min_{(x,y)\in\Omega} F(x,y) \text{ s.t. } \sum_{k=1}^{s} f_k(x,y) \le \upsilon(x,y). \tag{3.4}$$

Remark 3.1. (i) The constraints $f_k(x,w) \le f_k(x,y)$ for $k = 1,\dots,s$ are not explicitly included in the single-level formulation (3.4) as they hold true for $w = y$.

(ii) When solving multi-objective optimization problems numerically, efficient solutions tend to be computed more easily than weakly efficient solutions using typical algorithms. Additionally, our reformulation method offers two advantages over prior works [1, 5, 7, 8, 10]. First, we do not require the scalarized problem solution set to be closed. Second, we avoid introducing an extra variable at the upper optimization level - the single-level program (3.4) only involves the original decision variables x and y, maintaining a more compact formulation.

4. Subdifferential Estimate of Charnes-Cooper Value Function

This section investigates the local sensitivity and stability properties of the optimal value function υ defined in (3.3). We derive an upper estimate of the Mordukhovich subdifferential of υ at a reference point, providing insight into its point-based sensitivity. Additionally, various conditions are established for the local Lipschitz continuity of υ around the reference point.

To facilitate the analysis, some key definitions and notation are introduced. Let $(x,y,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$, and set:

$$\begin{aligned} f_{0k}(x,y,w) &:= f_k(x,w), \text{ for all } k=1,\ldots,s; \\ f_0(x,y,w) &:= \sum_{k=1}^{s} f_{0k}(x,y,w); \\ g_0(x,y,w) &:= g(x,w). \end{aligned}$$

We impose the calmness property as a significant constraint qualification at the reference points of the following set-valued mappings:

$$\Psi_{\Upsilon_0}(\mu) := \left\{(x,y,w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \;:\; g_0(x,y,w) \in \mu - \mathbb{R}^p_+\right\}, \text{ for all } \mu \in \mathbb{R}^p$$

$$\Psi_{\Xi}(\kappa) := \left\{(x,y,w) \in \operatorname{gph} \Upsilon_0 \;:\; m(x,y,w) \in \kappa - \mathbb{R}^s_+\right\}, \text{ for all } \kappa \in \mathbb{R}^s,$$

where $\Upsilon_0 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is the set-valued mapping given by $\Upsilon_0(x,y) = \Upsilon(x)$ and $m(x,y,w) = \Big(f_1(x,w) - f_1(x,y), \ldots, f_s(x,w) - f_s(x,y)\Big)$.

Theorem 4.1. *Let $(\bar{x},\bar{y})$ be a feasible point of problem (SVBOP). Assume that Ξ possesses the Aubin property at $(\bar{x},\bar{y},w)$ for all $w \in S(\bar{x},\bar{y})$. Moreover, suppose that the set-valued mapping S is inner semicompact at $(\bar{x},\bar{y})$ and that the set-valued mapping Ψ_{Ξ} (respectively, Ψ_{Υ_0}) is calm at $(0_{\mathbb{R}^s},\bar{x},\bar{y},w)$ (respectively, at $(0_{\mathbb{R}^p},\bar{x},\bar{y},w)$) for all $w \in S(\bar{x},\bar{y})$. Then, the optimal value function υ is Lipschitz continuous near $(\bar{x},\bar{y})$, and*

$$\begin{aligned} \partial \upsilon(\bar{x},\bar{y}) \subseteq \bigcup_{w\in S(\bar{x},\bar{y})} \bigcup_{\delta \geq 0} \bigcup_{\vartheta \geq 0} \Bigg\{ (\bar{x}^*,\bar{y}^*) \;:\; (\bar{x}^*,\bar{y}^*,0_{\mathbb{R}^m}) \in \sum_{k=1}^{s} \partial f_k(\bar{x},w) \\ + \sum_{k=1}^{s} \delta_k\left(\partial f_k(\bar{x},w) - conv\, \partial f_k(\bar{x},\bar{y})\right) + \sum_{j=1}^{p} \vartheta_j \partial g_j(\bar{x},w) \Bigg\}. \end{aligned}$$

Proof. Due to the inner semicompactness of S at $(\bar{x},\bar{y})$ and the fact that f_{0k}, $k=1,\ldots,s$, is locally Lipschitz continuous around $(\bar{x},\bar{y},w)$ for all $w \in S(\bar{x},\bar{y})$, we apply [17, Theorem 7(ii)]. This result yields a variational representation of both the singular and limiting subdifferentials of the optimal value function in terms of coderivatives of the multifunction Ξ. More precisely, we obtain:

- the singular subdifferential satisfies

$$\partial^\infty \upsilon(\bar{x},\bar{y}) \subseteq \bigcup_{w\in S(\bar{x},\bar{y})} D^*\Xi(\bar{x},\bar{y},w)(0),$$

- and the limiting subdifferential satisfies

$$\partial \upsilon(\bar{x},\bar{y}) \subseteq \bigcup_{w\in S(\bar{x},\bar{y})} \Big\{(\eta,0)^\top + D^*\Xi(\bar{x},\bar{y},w)(\sigma):\ (\eta,0,\sigma)\in\partial f_0(\bar{x},\bar{y},w)\Big\}.$$

By the Aubin property of Ξ at $(\bar{x},\bar{y},w)$ for all $w\in S(\bar{x},\bar{y})$, the coderivative satisfies the stability condition

$$D^*\Xi(\bar{x},\bar{y},w)(0)\subseteq\{0\},$$

which implies that no nontrivial singular directions exist. Hence $\partial^\infty\upsilon(\bar{x},\bar{y})=\{0\}$. This property ensures that the value function υ is locally Lipschitz continuous near $(\bar{x},\bar{y})$. Furthermore, for any $(\bar{x}^*,\bar{y}^*)\in\partial\upsilon(\bar{x},\bar{y})$, there exists $\overline{w}\in S(\bar{x},\bar{y})$ such that, for some $(\eta,0,\sigma)\in\partial f_0(\bar{x},\bar{y},\overline{w})$,

$$\begin{pmatrix}\bar{x}^*\\ \bar{y}^*\end{pmatrix}\in\begin{pmatrix}\eta\\ 0\end{pmatrix}+D^*\Xi(\bar{x},\bar{y},\overline{w})(\sigma).$$

Exploiting the coderivative definition together with the subdifferential sum rule in the last inclusion, we obtain

$$(\bar{x}^*-\eta,\bar{y}^*,-\sigma)\in N\Big((\bar{x},\bar{y},\overline{w}),\mathrm{gph}\ \Xi\Big).$$

Combining this with the calmness of Ψ_Ξ while observing that $\Psi_\Xi(0)=\mathrm{gph}\ \Xi$, we conclude from [18, Theorem 4.1] that there exists $\delta\in N\left(m(\bar{x},\bar{y},\overline{w}),-\mathbb{R}^s_+\right)$ such that

$$(\bar{x}^*-\eta,\bar{y}^*,-\sigma)\in\partial\langle\delta,m\rangle(\bar{x},\bar{y},\overline{w})+N\Big((\bar{x},\bar{y},\overline{w}),\mathrm{gph}\ \Upsilon_0\Big).$$

Invoking the structure of the functions f_{0k} and utilizing the fact that $N\left(m(\bar{x},\bar{y},\overline{w}),-\mathbb{R}^s_+\right)\subseteq\mathbb{R}^s_+$, it follows that there exists $(\eta_k,0,\sigma_k)\in\partial f_{0k}(\bar{x},\bar{y},\overline{w})$, with $k=1,\ldots,s$, such that $(\eta,0,\sigma)=\sum_{k=1}^{s}(\eta_k,0,\sigma_k)$. Additionally, there exist positive scalars δ_k, with $k=1,\ldots,s$, such that

$$(\bar{x}^*-\eta,\bar{y}^*,-\sigma)\in\sum_{k=1}^{s}\delta_k\left(\partial f_k(\bar{x},\overline{w})-\mathrm{conv}\ \partial f_k(\bar{x},\bar{y})\right)+N\Big((\bar{x},\bar{y},\overline{w}),\mathrm{gph}\ \Upsilon_0\Big). \quad (4.1)$$

This follows by applying the calculus rules for generalized derivatives along with the convexity property stated in (2.2).

Let us now compute the normal cone $N\Big((\bar{x},\bar{y},\overline{w}),\mathrm{gph}\ \Upsilon_0\Big)$. Taking into consideration that $\mathrm{gph}\ \Upsilon_0=\Psi_{\Upsilon_0}(0)$ and that Ψ_{Υ_0} is calm at $(0_{\mathbb{R}^p},\bar{x},\bar{y},\overline{w})$, we derive from [18, Theorem 4.1] that, for any (a,b,c) in this normal cone ,there exists $\vartheta\in\mathbb{R}^p_+$ such that

$$(a,b,c)\in\sum_{j=1}^{p}\vartheta_j\partial g_j(\bar{x},\overline{w}). \quad (4.2)$$

The combination of (4.1) and (4.2) yields

$$(\bar{x}^* - \eta, \bar{y}^*, -\sigma) \in \sum_{k=1}^{s} \delta_k \left(\partial f_k(\bar{x}, \bar{w}) - \text{conv}\, \partial f_k(\bar{x}, \bar{y})\right) + \sum_{j=1}^{p} \vartheta_j \partial g_j(\bar{x}, \bar{w}).$$

This proves the claimed estimate for $\partial \upsilon(\bar{x}, \bar{y})$. □

Remark 4.1. (1) Let $j \in J$. One see that $N\left(g_{0j}(\bar{x}, \bar{y}, \bar{w}), -\mathbb{R}_+\right) = \mathbb{R}_+$ when $g_{0j}(\bar{x}, \bar{y}, \bar{w}) = 0$; and $N\left(g_{0j}(\bar{x}, \bar{y}, \bar{w}), -\mathbb{R}_+\right) = \{0\}$ when $g_{0j}(\bar{x}, \bar{y}, \bar{w}) < 0$. Hence, we deduce from the proof of Theorem 4.1 that $\vartheta_j \geq 0$ and $\vartheta_j g_{0j}(\bar{x}, \bar{y}, \bar{w}) = 0$ for all $j \in J$.

(2) It bears noting that the Aubin property imposed on the set-valued mapping Ξ is a sufficient but not necessary condition. As an alternative, the following constraint qualification could be imposed for each $w \in S(\bar{x}, \bar{y})$:

$$\left[\begin{array}{l} (x^*, y^*, 0) \in \partial \left\langle \delta_g, g_0 \right\rangle (\bar{x}, \bar{y}, w) + \partial \left\langle \delta_m, m \right\rangle (\bar{x}, \bar{y}, w) \\ \\ \text{with } \delta_g \in N\left(g_0(\bar{x}, \bar{y}, w), -\mathbb{R}^p_+\right) \text{ and } \delta_m \in \mathbb{R}^s_+ \end{array}\right] \Rightarrow \quad (x^*, y^*) = (0, 0).$$

This point-based constraint qualification establishes a weaker assumption that could still ensure the conclusions of Theorem 4.1 are valid, namely the claimed subdifferential estimate and Lipschitz continuity of υ.

(3) In fact, the Aubin property appears to be strong, making it difficult to satisfy; however, many studies in the literature have a large number of properties and sufficient conditions to ensure its validity; see, e.g.g, [18, 19, 20].

In Theorem 4.1, it is certainly important to simplify further the upper estimate of the subdifferential for the value function υ. One way to achieve this is by applying the inner semicontinuity property of the solution map S rather than its inner semicompactness.

Theorem 4.2. *Let $(\bar{x}, \bar{y})$ be a feasible point of problem (SVBOP). Assume that Ξ possesses the Aubin property at $(\bar{x}, \bar{y}, \bar{w})$ while S is inner semicontinuous at this point. Suppose that the set-valued mapping Ψ_Ξ (respectively, Ψ_{Υ_0}) is calm at $(0_{\mathbb{R}^s}, \bar{x}, \bar{y}, \bar{w})$ (respectively, at $(0_{\mathbb{R}^p}, \bar{x}, \bar{y}, \bar{w})$). Then, the optimal value function υ is Lipschitz continuous near $(\bar{x}, \bar{y})$, and*

$$\begin{aligned} \partial \upsilon(\bar{x}, \bar{y}) \quad \subseteq \quad & \bigcup_{\delta \geq 0} \bigcup_{\vartheta \geq 0} \Bigg\{ (\bar{x}^*, \bar{y}^*) \ : \ (\bar{x}^*, \bar{y}^*, 0_{\mathbb{R}^m}) \in \sum_{k=1}^{s} \partial f_k(\bar{x}, \bar{w}) \\ & + \sum_{k=1}^{s} \delta_k \left(\partial f_k(\bar{x}, \bar{w}) - \textit{conv}\, \partial f_k(\bar{x}, \bar{y})\right) + \sum_{j=1}^{p} \vartheta_j \partial g_j(\bar{x}, \bar{w}) \Bigg\}. \end{aligned}$$

Proof. We have that S is inner semicontinuous at $(\bar{x}, \bar{y}, \bar{w})$. As stated previously, [17, Theorem 7(i)] ensures the optimal value function υ is Lipschitz near $(\bar{x}, \bar{y})$, owing to the validity of the Aubin property of Ξ at $(\bar{x}, \bar{y}, \bar{w})$. Again from [17, Theorem 7(i)], by the inner semicontinuity of S at $(\bar{x}, \bar{y}, \bar{w})$ and the local Lipschitz continuity of f_{0k}, $k = 1, \ldots, s$, at this point, for any $(\bar{x}, \bar{y}) \in \partial \upsilon(\bar{x}, \bar{y})$ there exists $(\eta, 0, \sigma) \in \partial f_0(\bar{x}, \bar{y}, \bar{w})$ such that

$$\begin{pmatrix} \bar{x}^* \\ \bar{y}^* \end{pmatrix} \in \begin{pmatrix} \eta \\ 0 \end{pmatrix} + D^* \Xi(\bar{x}, \bar{y}, \bar{w})(\sigma).$$

As shown in the proof of Theorem 4.1, there exist $(\eta_k, 0, \sigma_k) \in \partial f_{0k}(\bar{x}, \bar{y}, \bar{w})$, $k = 1, \ldots, s$, such that

$$(\eta, 0, \sigma) = \sum_{k=1}^{s} (\eta_k, 0, \sigma_k), \;\; \delta \in N\left(m(\bar{x}, \bar{y}, \bar{w}), -\mathbb{R}^s_+\right) \subseteq \mathbb{R}^s_+ \text{ and } \vartheta \in N\left(g_0(\bar{x}, \bar{y}, \bar{w}), -\mathbb{R}^p_+\right) \subseteq \mathbb{R}^p_+$$

fulfill

$$(\bar{x}^* - \eta, \bar{y}^*, -\sigma) \in \sum_{k=1}^{s} \delta_k \left(\partial f_k(\bar{x}, \bar{w}) - \operatorname{conv} \partial f_k(\bar{x}, \bar{y})\right) + \sum_{j=1}^{p} \vartheta_j \partial g_j(\bar{x}, \bar{w}).$$

In conclusion, this establishes the upper estimate of $\partial \upsilon(\bar{x}, \bar{y})$ that the theorem claims. □

Remark 4.2. It should be noted that the Aubin property of the set-valued mapping Ξ is not required for the upper estimates of the fundamental subdifferential of υ given in Theorems 4.1 and 4.2. The Aubin property is exclusively needed to obtain the Lipschitz continuity of the already defined lower-level value function.

Remark 4.3. Assume the functions involved in (SVBOP) are continuously differentiable around the reference point.

- (a) Under the inner semicontinuity of S, the upper estimate of the optimal value function in Theorem 4.2 reduces to that achieved in [6, Theorem 4.2].
- (b) Under the inner semicompactness of S, the optimal value function estimation in Theorem 4.1 reduces to the following estimation:

$$\begin{aligned}\partial \upsilon(\bar{x}, \bar{y}) \;\subseteq\; & \bigcup_{w \in S(\bar{x}, \bar{y})} \bigcup_{\delta \geq 0} \bigcup_{\vartheta \geq 0} \Bigg\{ (\bar{x}^*, \bar{y}^*) \;:\; (\bar{x}^*, \bar{y}^*, 0_{\mathbb{R}^m}) \in \sum_{k=1}^{s} \nabla f_k(\bar{x}, w) \\ & + \sum_{k=1}^{s} \delta_k \left(\nabla f_k(\bar{x}, w) - \nabla f_k(\bar{x}, \bar{y})\right) + \sum_{j=1}^{p} \vartheta_j \nabla g_j(\bar{x}, w) \Bigg\}.\end{aligned}$$

  This estimation extends the one stated in [6, Theorem 4.2], and provides new necessary optimality conditions in smooth settings.

5. Necessary Optimality Conditions

In this section, we determine necessary optimality conditions for the auxiliary optimistic bilevel programming problem (3.4) by using the lower-level value function (3.3) and sensitivity results from Section 4.

First, we define

$$\psi(x, y) = \sum_{k=1}^{s} f_k(x, y) - \upsilon(x, y),$$
$$\Sigma = \{(x, y) \in \Omega : \; \psi(x, y) \leq 0\}.$$

We also require additional constraint qualifications to establish necessary optimality conditions. The following conditions are non-smooth counterparts of regularity assumptions at the lower and upper levels:

- **•:** We say that the weak Mangasarian-Fromowitz constraint qualification is satisfied at $(\bar{x}, \bar{y})$ if

$$\partial \psi(\bar{x}, \bar{y}) \cap -\operatorname{bd} N((\bar{x}, \bar{y}), \Omega) = \emptyset. \tag{WMFCQ}$$

•: The upper-level regularity of (SVBOP) at $\bar{x}$ is given by

$$\left.\begin{array}{r} 0 \in \sum_{i=1}^{q} \zeta_i \partial G_i(\bar{x}) \\ \zeta_i \geq 0,\ \zeta_i G_i(\bar{x}) = 0,\ i \in I \end{array}\right\} \implies \zeta_i = 0,\ i = 1, \ldots, q. \tag{ULR}$$

•: The lower-level regularity of (P[x]) at $(\bar{x}, \bar{y})$ is given by

$$\left.\begin{array}{r} 0 \in \sum_{j=1}^{p} \gamma_j \partial_y g_j(\bar{x}, \bar{y}) \\ \gamma_j \geq 0,\ \gamma_j g_j(\bar{x}, \bar{y}) = 0, \quad j \in J, \end{array}\right\} \implies \gamma_j = 0,\ j = 1, \ldots, p. \tag{LLR}$$

The next result gives us the normal cone formula for Ω at each point $(\bar{x}, \bar{y}) \in \Omega$.

Lemma 5.1. *Let $(\bar{x}, \bar{y}) \in \Omega$. Assume the upper-level and lower-level regularities hold at $\bar{x}$ and $(\bar{x}, \bar{y})$, respectively. Then*

$$N((\bar{x}, \bar{y}), \Omega) \subseteq$$
$$\left\{ \begin{pmatrix} \sum_{i=1}^{q} \zeta_i \partial G_i(\bar{x}) + \sum_{j=1}^{p} \gamma_j \partial_x g_j(\bar{x}, \bar{y}) \\ \sum_{j=1}^{p} \gamma_j \partial_y g_j(\bar{x}, \bar{y}) \end{pmatrix} : \begin{array}{ll} \zeta_i \geq 0, & \zeta_i G_i(\bar{x}) = 0, i \in I \\ \gamma_j \geq 0, & \gamma_j g_j(\bar{x}, \bar{y}) = 0, j \in J. \end{array} \right\}$$

Proof. Setting $\chi(x,y) = [G_i(x),\ i = 1, \ldots, q,\ g_j(x,y), j = 1, \ldots, p]^\top$, the set Ω can be rewritten as $\Omega = \{(x,y) : \chi(x,y) \leq 0\}$. Then, fulfillment of the upper-level and lower-level regularities at $\bar{x}$ and $(\bar{x}, \bar{y})$ respectively implies

$$\left.\begin{array}{r} 0 \in \sum_{i=1}^{q} \zeta_i \partial \chi_i(\bar{x}, \bar{y}) + \sum_{j=1}^{p} \gamma_i \partial \chi_{q+j}(\bar{x}, \bar{y}) \\ \zeta_i \geq 0,\ \zeta_i \chi_i(\bar{x}, \bar{y}) = 0,\ i \in I \\ \gamma_j \geq 0,\ \gamma_j \chi_{q+j}(\bar{x}, \bar{y}) = 0,\ j \in J \end{array}\right\} \implies \zeta_1, \ldots, \zeta_q, \gamma_{q+1}, \ldots, \gamma_{q+p} = 0.$$

Consequently, from [21, Theorem 6.14], we obtain the desired result. □

Subsequently, we are going to provide one of the primary results of this study, which gives the necessary optimality conditions for auxiliary problem (3.4). We initially focus on the situation where the weak Mangasarian-Fromowitz constraint qualification holds.

Theorem 5.1. *Let $(\bar{x}, \bar{y})$ be a local optimal solution of problem (3.4). Assume Ω is regular and semismooth at $(\bar{x}, \bar{y})$ and (WMFCQ) holds at this point. Moreover, suppose that the upper-level and lower-level regularities are satisfied at $\bar{x}$ and $(\bar{x}, \bar{y})$, respectively. Then the following statements are valid:*

*(a)**:** Under the hypotheses of Theorem 4.1, there exist $\nu \geqslant 0$, $\xi \in \mathbb{R}_+^{n+m+1}$, $w_r \in S(\bar{x}, \bar{y})$, $\delta_r \in \mathbb{R}_+^s$, $\vartheta_r \in \mathbb{R}_+^p$, $v_r^k \in \mathbb{R}_+^{n+m+1}$, $(\zeta, \gamma) \in \mathbb{R}^q \times \mathbb{R}^p$, and*

$$(x_F^*, y_F^*) \in \partial F(\bar{x}, \bar{y}), \qquad x_{G_i}^* \in \partial G_i(\bar{x}), \qquad \left(x_{f_k}^*, y_{f_k}^*\right) \in \partial f_k(\bar{x}, \bar{y}),$$
$$\left(x_{g_j}^*, y_{g_j}^*\right) \in \partial g(\bar{x}, \bar{y}), \qquad \left(x_{f_k}^{*r}, w_{f_k}^{*r}\right) \in \partial f_k(\bar{x}, w_r), \qquad \left(\tilde{x}_{f_k}^r, \tilde{w}_{f_k}^r\right) \in \partial f_k(\bar{x}, w_r),$$
$$\left(\hat{x}_{f_k}^{r,l}, \hat{y}_{f_k}^{r,l}\right) \in \partial f_k(\bar{x}, \bar{y}), \qquad \left(\underline{x}_{g_j}^r, \underline{w}_{g_j}^r\right) \in \partial g_j(\bar{x}, w_r),$$

with $i \in I$, $j \in J$, $k = 1, \ldots, s$, $r = 1, \ldots, n+m+1$, $l = 1, \ldots, n+m+1$, $\sum_{r=1}^{n+m+1} \xi_r = 1$ *and* $\sum_{l=1}^{n+m+1} \nu_{r,l}^k = 1$, $r = 1, \ldots, n+m+1$, $k = 1, \ldots, s$, *such that*

$$0 = x_F^* + \sum_{i=1}^{q} \zeta_i x_{G_i}^* + \nu \sum_{k=1}^{s} x_{f_k}^* + \sum_{j=1}^{p} \gamma_j x_{g_j}^* - \nu \sum_{r=1}^{n+m+1} \xi_r \left(\sum_{k=1}^{s} x_{f_k}^{*r} + \sum_{k=1}^{s} \delta_{r,k} \left(\tilde{x}_{f_k}^r - \sum_{l=1}^{n+m+1} \nu_{r,l}^k \hat{x}_{f_k}^{r,l} \right) + \sum_{j=1}^{p} \vartheta_{r,j} \underline{x}_{g_j}^r \right),$$

$$0 = y_F^* + \nu \sum_{k=1}^{s} y_{f_k}^* + \sum_{j=1}^{p} \gamma_j y_{g_j}^* + \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} \sum_{l=1}^{n+m+1} \nu_{r,l}^k \hat{y}_{f_k}^{r,l},$$

$$0 = \sum_{k=1}^{s} w_{f_k}^{*r} + \sum_{k=1}^{s} \delta_{r,k} \tilde{w}_{f_k}^r + \sum_{j=1}^{p} \vartheta_{r,j} \underline{w}_{g_j}^r, \quad \forall r = 1, \ldots, n+m+1,$$

$$\zeta_i \geqslant 0,\ \zeta_i G_i(\bar{x}) = 0, \quad \forall i = 1, \ldots, q,$$
$$\gamma_j \geqslant 0,\ \gamma_j g_j(\bar{x}, \bar{y}) = 0, \quad \forall j = 1, \ldots, p$$
$$\vartheta_{r,j} \geqslant 0,\ \vartheta_{r,j} g_j(\bar{x}, w_r) = 0, \quad \forall j = 1, \ldots, p, \quad \forall r = 1, \ldots, n+m+1.$$

(b)**:** *Under the hypotheses of Theorem 4.2, there exist* $\nu \geqslant 0$, $\xi \in \mathbb{R}_+^{n+m+1}$, $\delta_r \in \mathbb{R}_+^s$, $\vartheta_r \in \mathbb{R}_+^p$, $\nu_r^k \in \mathbb{R}_+^{n+m+1}$, $(\zeta, \gamma) \in \mathbb{R}^q \times \mathbb{R}^p$, *and*

$$(x_F^*, y_F^*) \in \partial F(\bar{x}, \bar{y}), \qquad x_{G_i}^* \in \partial G_i(\bar{x}), \qquad \left(x_{f_k}^*, y_{f_k}^*\right) \in \partial f_k(\bar{x}, \bar{y}),$$
$$\left(x_{g_j}^*, y_{g_j}^*\right) \in \partial g(\bar{x}, \bar{y}), \qquad \left(x_{f_k}^{*r}, y_{f_k}^{*r}\right) \in \partial f_k(\bar{x}, \bar{y}), \qquad \left(\tilde{x}_{f_k}^r, \tilde{y}_{f_k}^r\right) \in \partial f_k(\bar{x}, \bar{y}),$$
$$\left(\hat{x}_{f_k}^{r,l}, \hat{y}_{f_k}^{r,l}\right) \in \partial f_k(\bar{x}, \bar{y}), \qquad \left(\underline{x}_{g_j}^r, \underline{y}_{g_j}^r\right) \in \partial g_j(\bar{x}, \bar{y}),$$

with $i \in I$, $j \in J$, $k = 1, \ldots, s$, $r = 1, \ldots, n+m+1$, $l = 1, \ldots, n+m+1$, $\sum_{r=1}^{n+m+1} \xi_r = 1$ *and* $\sum_{l=1}^{n+m+1} \nu_{r,l}^k = 1$, $r = 1, \ldots, n+m+1$, $k = 1, \ldots, s$, *such that*

$$0 = x_F^* + \sum_{i=1}^{q} \zeta_i x_{G_i}^* + \nu \sum_{k=1}^{s} x_{f_k}^* + \sum_{j=1}^{p} \gamma_j x_{g_j}^* - \nu \sum_{r=1}^{n+m+1} \xi_r \left(\sum_{k=1}^{s} x_{f_k}^{*r} + \sum_{k=1}^{s} \delta_{r,k} \left(\tilde{x}_{f_k}^r - \sum_{l=1}^{n+m+1} \nu_{r,l}^k \hat{x}_{f_k}^{r,l} \right) + \sum_{j=1}^{p} \vartheta_{r,j} \underline{x}_{g_j}^r \right),$$

$$0 = y_F^* + \nu \sum_{k=1}^{s} y_{f_k}^* + \sum_{j=1}^{p} \gamma_j y_{g_j}^* + \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} \sum_{l=1}^{n+m+1} \nu_{r,l}^k \hat{y}_{f_k}^{r,l},$$

$$0 = \sum_{k=1}^{s} y_{f_k}^{*r} + \sum_{k=1}^{s} \delta_{r,k} \tilde{y}_{f_k}^r + \sum_{j=1}^{p} \vartheta_{r,j} \underline{y}_{g_j}^r, \quad \forall r = 1, \ldots, n+m+1,$$
$$\zeta_i \geqslant 0,\ \zeta_i G_i(\bar{x}) = 0, \quad \forall i = 1, \ldots, q,$$
$$\gamma_j \geqslant 0,\ \gamma_j g_j(\bar{x}, \bar{y}) = 0, \quad \forall j = 1, \ldots, p,$$
$$\vartheta_{r,j} \geqslant 0,\ \vartheta_{r,j} g_j(\bar{x}, \bar{y}) = 0, \quad \forall j = 1, \ldots, p.$$

Proof. Let $(\bar{x}, \bar{y})$ be a local optimal solution of (3.4). Regarding the hypotheses of (a) or (b), the lower-level value function υ is Lipschitz continuous near $(\bar{x}, \bar{y})$, as shown in Theorems 4.1

and 4.2. Considering the closedness of the set Σ and Lipschitz property of all functions, we obtain by applying the necessary optimality conditions from [20, Proposition 5.3] to problem (3.4) that

$$0 \in \partial F(\bar{x},\bar{y}) + N((\bar{x},\bar{y}),\Sigma).$$

Then, applying [22, Lemma 3.3] together with (WMFCQ), there exists $\nu \geq 0$ such that

$$0 \in \partial F(\bar{x},\bar{y}) + \nu \sum_{k=1}^{s} \partial f_k(\bar{x},\bar{y}) + \nu \partial(-\upsilon)(\bar{x},\bar{y}) + N((\bar{x},\bar{y}),\Omega)$$

while considering the regularity and semi-smoothness of Ω. Using the convex hull properties, $\text{conv}\,\partial(-\upsilon)(\bar{x},\bar{y}) = -\text{conv}\,\partial\upsilon(\bar{x},\bar{y})$, we get

$$0 \in \partial F(\bar{x},\bar{y}) + \nu \sum_{k=1}^{s} \partial f_k(\bar{x},\bar{y}) - \nu\,\text{conv}\,\partial\upsilon(\bar{x},\bar{y}) + N((\bar{x},\bar{y}),\Omega).$$

Observing this expression along with Lemma 5.1 and the upper estimate for υ from Theorems 4.1 and 4.2, and Remark 4.1, we can deduce assertions (a) and (b) of the theorem. Specifically, we apply Carathéodory's theorem to compute an element of $\text{conv}\,\partial\upsilon(\bar{x},\bar{y})$. □

Example 5.1. Consider the following semivectorial optimistic bilevel optimization problem

$$\min_{x,y} F(x,y) = |x| + x + y^2 \ \text{ s.t. } G(x) = x^2 - 2x \leq 0, \ \ y \in \Phi_{\text{eff}}(x), \tag{5.1}$$

where $\Phi_{\text{eff}}(x)$ represents the solution set of the lower-level multiobjective optimization problem

$$\min_{(x,y)} \left(x^2 + xy + x \, , \, y^2 + xy + y\right) \ \text{ s.t. } g(x,y) = -y + x \leq 0. \tag{5.2}$$

For this example, we have

$$X = [0,2], \ \ \Upsilon(x) = [x,+\infty[, \ \text{ and } \ \Omega = \left\{(x,y) \in \mathbb{R}^2 : \ (x,y) \in [0,2] \times [x,+\infty[\right\}.$$

Using the Charnes-Cooper scalarization technique, we can conclude that the lower-level multiobjective problem (5.2) is equivalent to the following single-objective program:

$$\begin{cases} \min\limits_{w} w^2 + w + x^2 + 2xw + x \\ (w-y)(w+y+x+1) \leq 0, \\ xw - xy \leq 0, \\ x \leq w. \end{cases} \tag{5.3}$$

Letting $(x,y) \in \Omega$, we have $\Xi(x,y) = [x,y]$. The optimal value function of problem (5.3) is given by $\upsilon(x,y) = 4x^2 + 2x$. Therefore, the optimal value reformulation of the original problem (5.1) is given by

$$\begin{cases} \min\limits_{x,y} F(x,y) = |x| + x + y^2 \\ y^2 - 3x^2 + 2xy + y - x \leq 0, \\ x \leq y, \\ 0 \leq x \leq 2. \end{cases} \tag{5.4}$$

Observe first that $0 \in \Phi_{\text{eff}}(0)$. Then $(\bar{x},\bar{y}) := (0,0)$ is an optimal solution to problem (5.4). The normal cone to Ω at $(\bar{x},\bar{y})$, and the partial of ψ at $(\bar{x},\bar{y})$ are given by

$$N((\bar{x},\bar{y}),\Omega) = \mathbb{R} \times \mathbb{R}^-, \ \ \partial\psi(\bar{x},\bar{y}) = \{(-1,1)\}.$$

Then, $\partial\psi(\bar{x},\bar{y}) \cap -\text{bd}\,N((\bar{x},\bar{y}),\Omega) = \emptyset$.

- The constraint qualification (WMFCQ) is satisfied at $(\bar{x},\bar{y})$.
- The set-valued mapping S given by $S(x,y)=\{x\}$, for all $(x,y)\in\Omega$, is inner semicompact and inner semicontinuous at $(\bar{x},\bar{y})$.
- For $\mu\in\mathbb{R}$, $\Psi_{\Upsilon_0}(\mu)=\{(x,y,w)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\,:\,-w+x\leq\mu\}$. Hence, the set-valued mapping Ψ_{Υ_0} is calm at $(0,\bar{x},\bar{y},w)$ for $w\in S(\bar{x},\bar{y})$.
- For $\kappa\in\mathbb{R}$, $\Psi_{\Xi}(\kappa)=\{(x,y,w)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\,:\,-w+x\leq 0,\ xw-xy\leq\kappa\}$. Hence, the set-valued mapping Ψ_{Ξ} is calm at $(0,\bar{x},\bar{y},w)$ for $w\in S(\bar{x},\bar{y})$.
- The upper-level and lower-level regularities are satisfied at $\bar{x}$ and $(\bar{x},\bar{y})$, respectively.

Finally, for $(x_F^*,y_F^*):=(2,0)$, $x_G^*:=-2$, $\left(x^*_{f_1},y^*_{f_1}\right)=\left(x^{*r}_{f_1},w^{*r}_{f_1}\right)=\left(\tilde{x}^r_{f_1},\tilde{w}^r_{f_1}\right)=\left(\hat{x}^{r,l}_{f_1},\hat{y}^{r,l}_{f_1}\right):=(1,0)$, $\left(x^*_{f_2},y^*_{f_2}\right)=\left(x^{*r}_{f_2},w^{*r}_{f_2}\right)=\left(\tilde{x}^r_{f_2},\tilde{w}^r_{f_2}\right)=\left(\hat{x}^{r,l}_{f_2},\hat{y}^{r,l}_{f_2}\right):=(0,1)$, $\left(x^*_g,y^*_g\right)=\left(\underline{x}^r_g,\underline{w}^r_g\right):=(1,-1)$, $\nu:=1$, $\xi=\left(\frac{1}{6},\frac{1}{3},\frac{1}{2}\right)^\top$, $w_r:=0\in S(\bar{x},\bar{y})$, $\delta_{r,1}:=3$, $\delta_{r,2}:=1$, $\vartheta_r:=2$, $v_r^k:=\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)^\top$, $\zeta:=1$, and $\gamma:=2$, $r=1,2,3$, $l=1,2,3$, $k=1,2$, the optimality conditions (a) and (b) stated in Theorem 5.1 are satisfied.

As stated in [22], the weak basic constraint qualification (WMFCQ) has been demonstrated to work, particularly for simple bilevel programming problems. Following that, we look at another constraint qualification, the partial calmness condition, which was presented in [23] and has lately been extensively studied and used to deduce necessary optimality conditions for a traditional optimistic bilevel program via its optimal value reformulation.

Definition 5.1. [23] Let $(\bar{x},\bar{y})$ be a local solution of (3.4). The bilevel programming problem (3.4) is said to be partially calm at $(\bar{x},\bar{y})$ if there exist $\nu>0$ and a neighborhood U of $(\bar{x},\bar{y},0)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}$ such that $F(x,y)-F(\bar{x},\bar{y})+\nu|u|\geq 0$, for all $(x,y,u)\in U$ also satisfying $x\in X$, $y\in\Upsilon(x)$, and $\psi(x,y)+u=0$.

Proposition 5.1. [23] *Let $(\bar{x},\bar{y})$ be a local solution of (3.4). This problem is partially calm at $(\bar{x},\bar{y})$ if and only if there exists $\nu>0$ such that $(\bar{x},\bar{y})$ is a local optimal solution of the following penalized problem:*

$$\min_{x,y}\ F(x,y)+\nu\psi(x,y)\ \ s.t.\ x\in X,\ y\in\Upsilon(x). \tag{5.5}$$

Building upon this proposition, we now derive necessary optimality conditions for problem (3.4) from the feasible set using partial penalization, which eliminates the value function constraint responsible for the failure of well-known constraint qualifications.

Theorem 5.2. *Let $(\bar{x},\bar{y})$ be a local optimal solution of problem (3.4). Assume that the problem (3.4) is partially calm at $(\bar{x},\bar{y})$, and that the upper-level and lower-level regularities are satisfied at $\bar{x}$ and $(\bar{x},\bar{y})$, respectively. Then the following statements are valid*

(a)**:** *Based on the assumptions of Theorem 4.1, there exist $\nu>0$, $\xi\in\mathbb{R}^{n+m+1}_+$, $w_r\in S(\bar{x},\bar{y})$, $\delta_r\in\mathbb{R}^s_+$, $\vartheta_r\in\mathbb{R}^p_+$, $v_r^k\in\mathbb{R}^{n+m+1}_+$, $(\zeta,\gamma)\in\mathbb{R}^q\times\mathbb{R}^p$, and*

$$(x_F^*,y_F^*)\in\partial F(\bar{x},\bar{y}),\qquad x^*_{G_i}\in\partial G_i(\bar{x}),\qquad \left(x^*_{f_k},y^*_{f_k}\right)\in\partial f_k(\bar{x},\bar{y}),$$
$$\left(x^*_{g_j},y^*_{g_j}\right)\in\partial g(\bar{x},\bar{y}),\qquad \left(x^{*r}_{f_k},w^{*r}_{f_k}\right)\in\partial f_k(\bar{x},w_r),\qquad \left(\tilde{x}^r_{f_k},\tilde{w}^r_{f_k}\right)\in\partial f_k(\bar{x},w_r),$$
$$\left(\hat{x}^{r,l}_{f_k},\hat{y}^{r,l}_{f_k}\right)\in\partial f_k(\bar{x},\bar{y}),\qquad \left(\underline{x}^r_{g_j},\underline{w}^r_{g_j}\right)\in\partial g_j(\bar{x},w_r),$$

with $i \in I$, $j \in J$, $k = 1, \ldots, s$, $r = 1, \ldots, n+m+1$, $l = 1, \ldots, n+m+1$, $\sum_{r=1}^{n+m+1} \xi_r = 1$ *and* $\sum_{l=1}^{n+m+1} v_{r,l}^k = 1$, $r = 1, \ldots, n+m+1$, $k = 1, \ldots, s$, *such that*

$$\begin{aligned}
0 &= x_F^* + \sum_{i=1}^{q} \zeta_i x_{G_i}^* + \nu \sum_{k=1}^{s} x_{f_k}^* + \sum_{j=1}^{p} \gamma_j x_{g_j}^* - \nu \sum_{r=1}^{n+m+1} \xi_r \Bigg(\sum_{k=1}^{s} x_{f_k}^{*r} \\
&\quad + \sum_{k=1}^{s} \delta_{r,k} \left(\tilde{x}_{f_k}^r - \sum_{l=1}^{n+m+1} v_{r,l}^k \hat{x}_{f_k}^{r,l} \right) + \sum_{j=1}^{p} \vartheta_{r,j} \underline{x}_{g_j}^r \Bigg), \\
0 &= y_F^* + \nu \sum_{k=1}^{s} y_{f_k}^* + \sum_{j=1}^{p} \gamma_j y_{g_j}^* + \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} \sum_{l=1}^{n+m+1} v_{r,l}^k \hat{y}_{f_k}^{r,l}, \\
0 &= \sum_{k=1}^{s} w_{f_k}^{*r} + \sum_{k=1}^{s} \delta_{r,k} \tilde{w}_{f_k}^r + \sum_{j=1}^{p} \vartheta_{r,j} \underline{w}_{g_j}^r, \quad \forall r = 1, \ldots, n+m+1, \\
&\zeta_i \geqslant 0,\ \zeta_i G_i(\bar{x}) = 0, \quad \forall i = 1, \ldots, q, \\
&\gamma_j \geqslant 0,\ \gamma_j g_j(\bar{x}, \bar{y}) = 0, \quad \forall j = 1, \ldots, p \\
&\vartheta_{r,j} \geqslant 0,\ \vartheta_{r,j} g_j(\bar{x}, w_r) = 0, \quad \forall j = 1, \ldots, p, \quad \forall r = 1, \ldots, n+m+1.
\end{aligned}$$

(b)**:** *Based on the assumptions of Theorem 4.2, there exist* $\nu > 0$, $\xi \in \mathbb{R}_+^{n+m+1}$, $\delta_r \in \mathbb{R}_+^s$, $\vartheta_r \in \mathbb{R}_+^p$, $v_r^k \in \mathbb{R}_+^{n+m+1}$, $(\zeta, \gamma) \in \mathbb{R}^q \times \mathbb{R}^p$, *and*

$$\begin{array}{lll}
(x_F^*, y_F^*) \in \partial F(\bar{x}, \bar{y}), & x_{G_i}^* \in \partial G_i(\bar{x}), & \left(x_{f_k}^*, y_{f_k}^*\right) \in \partial f_k(\bar{x}, \bar{y}), \\
\left(x_{g_j}^*, y_{g_j}^*\right) \in \partial g(\bar{x}, \bar{y}), & \left(x_{f_k}^{*r}, y_{f_k}^{*r}\right) \in \partial f_k(\bar{x}, \bar{y}), & \left(\tilde{x}_{f_k}^r, \tilde{y}_{f_k}^r\right) \in \partial f_k(\bar{x}, \bar{y}), \\
\left(\hat{x}_{f_k}^{r,l}, \hat{y}_{f_k}^{r,l}\right) \in \partial f_k(\bar{x}, \bar{y}), & \left(\underline{x}_{g_j}^r, \underline{y}_{g_j}^r\right) \in \partial g_j(\bar{x}, \bar{y}), &
\end{array}$$

with $i \in I$, $j \in J$, $k = 1, \ldots, s$, $r = 1, \ldots, n+m+1$, $l = 1, \ldots, n+m+1$, $\sum\limits_{r=1}^{n+m+1} \xi_r = 1$ *and* $\sum_{l=1}^{n+m+1} v_{r,l}^k = 1$, $r = 1, \ldots, n+m+1$, $k = 1, \ldots, s$, *such that*

$$\begin{aligned}
0 &= x_F^* + \sum_{i=1}^{q} \zeta_i x_{G_i}^* + \nu \sum_{k=1}^{s} x_{f_k}^* + \sum_{j=1}^{p} \gamma_j x_{g_j}^* - \nu \sum_{r=1}^{n+m+1} \xi_r \Bigg(\sum_{k=1}^{s} x_{f_k}^{*r} \\
&\quad + \sum_{k=1}^{s} \delta_{r,k} \left(\tilde{x}_{f_k}^r - \sum_{l=1}^{n+m+1} v_{r,l}^k \hat{x}_{f_k}^{r,l} \right) + \sum_{j=1}^{p} \vartheta_{r,j} \underline{x}_{g_j}^r \Bigg), \\
0 &= y_F^* + \nu \sum_{k=1}^{s} y_{f_k}^* + \sum_{j=1}^{p} \gamma_j y_{g_j}^* + \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} \sum_{l=1}^{n+m+1} v_{r,l}^k \hat{y}_{f_k}^{r,l}, \\
0 &= \sum_{k=1}^{s} y_{f_k}^{*r} + \sum_{k=1}^{s} \delta_{r,k} \tilde{y}_{f_k}^r + \sum_{j=1}^{p} \vartheta_{r,j} \underline{y}_{g_j}^r, \quad \forall r = 1, \ldots, n+m+1, \\
&\zeta_i \geqslant 0,\ \zeta_i G_i(\bar{x}) = 0, \quad \forall i = 1, \ldots, q, \\
&\gamma_j \geqslant 0,\ \gamma_j g_j(\bar{x}, \bar{y}) = 0, \quad \forall j = 1, \ldots, p, \\
&\vartheta_{r,j} \geqslant 0,\ \vartheta_{r,j} g_j(\bar{x}, \bar{y}) = 0, \quad \forall j = 1, \ldots, p.
\end{aligned}$$

Proof. Let $(\bar{x},\bar{y})$ be a local optimal solution of (3.4). By either assumptions of Theorem 4.1 or Theorem 4.2, the lower-level optimal value function υ is locally Lipscitz at this point. Since problem (3.4) is partially calm at $(\bar{x},\bar{y})$, we deduce from Proposition 5.1 that there exists a constant $\nu > 0$ such that $(\bar{x},\bar{y})$ is a local optimal solution to penalized problem (5.5).

Now, taking into account the local Lipschitz continuity of F and ψ at $(\bar{x},\bar{y})$, we have from [20, Proposition 5.3] that

$$0 \in \partial F(\bar{x},\bar{y}) + \nu\, \partial \psi(\bar{x},\bar{y}) + N((\bar{x},\bar{y}),\Omega). \tag{5.6}$$

From the structure of ψ, the basic subdifferential is given by

$$\partial \psi(\bar{x},\bar{y}) = \partial\left(\sum_{k=1}^{s} f_k(\bar{x},\bar{y}) - \upsilon(\bar{x},\bar{y})\right).$$

Thus $\partial \psi(\bar{x},\bar{y}) \subseteq \sum_{k=1}^{s} \partial f_k(\bar{x},\bar{y}) + \partial(-\upsilon)(\bar{x},\bar{y})$. Since $\partial(-\upsilon)(\bar{x},\bar{y}) \subseteq -\operatorname{conv}(\partial \upsilon(\bar{x},\bar{y}))$, one sees that

$$\partial \psi(\bar{x},\bar{y}) \subseteq \sum_{k=1}^{s} \partial f_k(\bar{x},\bar{y}) - \operatorname{conv}(\partial \upsilon(\bar{x},\bar{y}))). \tag{5.7}$$

By combining inclusions (5.6), (5.7) with Lemma 5.1, Theorems 4.1 and 4.2, and Remark 4.1, we obtain results (a) and (b) by utilizing the Carathéodory theorem to compute an element of $\operatorname{conv} \partial \upsilon(\bar{x},\bar{y})$. □

Example 5.2. We apply the necessary optimality conditions derived in Theorem 5.2 to verify that they are satisfied at the given candidate optimal solution. Define the problem data for (SVBOP). For $(x,y) \in \mathbb{R}\times\mathbb{R}$, set

$$F(x,y) = |x| + (y+2)^2, \quad G(x) = -x,$$
$$f_1(x,y) = x + (y+2)^2, \quad f_2(x,y) = x^2 + y + 2, \quad g(x,y) = |y| - 2.$$

Then, $\Omega = \mathbb{R}^+ \times [-2,2]$, and $\Xi(x,y) = [-2,y]$ for all $(x,y)\in\Omega$. Moreover, we have

$$S(x,y) = \{y\}, \quad \upsilon(x,y) = x + x^2.$$

The point $(\bar{x},\bar{y}) := (0,-2)$ is an optimal solution of the considered problem. It is then shown that

- The functions F, g, f_1, and f_2 are locally Lipschitz continuous.
- The open Cartesian cube $U := (-1,1)\times(-3,-1)\times(-1,1)$ is a neighborhood of $(\bar{x},\bar{y},0)$. For $(x,y,u)\in U$ such that $x\in X$, $y \in \Upsilon(x)$, and $\psi(x,y)+u = 0$, we have

  $$F(x,y) - F(\bar{x},\bar{y}) + \nu|u| = F(x,y) + \nu|u| \geq 0.$$

  Therefore, problem (3.4) is partially calm at $(\bar{x},\bar{y})$.
- The set-valued map Ξ possesses the Aubin property at $(\bar{x},\bar{y},w)$ with $w\in S(\bar{x},\bar{y})$.
- By construction, the set-valued mapping S is inner semicompact and inner semicontinuous at $(\bar{x},\bar{y})$.
- For $\mu\in\mathbb{R}$, we have $\Psi_{\Upsilon_0}(\mu) = \{(x,y,w)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R} \,:\, |w|-2\leq\mu\}$. Hence, the set-valued mapping Ψ_{Υ_0} is calm at $(0,\bar{x},\bar{y},w)$ with $w\in S(\bar{x},\bar{y})$.
- For $\kappa\in\mathbb{R}$, we have $\Psi_{\Xi}(\kappa) = \{(x,y,w)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R} \,:\, |w|-2\leq 0,\ w-y\leq\kappa\}$. Hence, the set-valued mapping Ψ_{Ξ} is calm at $(0,\bar{x},\bar{y},w)$ with $w\in S(\bar{x},\bar{y})$.

- The upper-level and lower-level regularities are satisfied at $\bar{x}$ and $(\bar{x},\bar{y})$ respectively.

By explicitly constructing the multipliers and constants, the optimality conditions (a) and (b) in Theorem 5.2 are shown to hold at the optimal solution $(\bar{x},\bar{y}) = (0,-2)$. The multipliers and constants are constructed as follows: $(x_F^*, y_F^*) := (1,0)$, $x_G^* := -1$, $\left(x_{f_1}^*, y_{f_1}^*\right) = \left(x_{f_1}^{*r}, w_{f_1}^{*r}\right) = \left(\tilde{x}_{f_1}^r, \tilde{w}_{f_1}^r\right) = \left(\hat{x}_{f_1}^{r,l}, \hat{y}_{f_1}^{r,l}\right) := (1,0)$, $\left(x_{f_2}^*, y_{f_2}^*\right) = \left(x_{f_2}^{*r}, w_{f_2}^{*r}\right) = \left(\tilde{x}_{f_2}^r, \tilde{w}_{f_2}^r\right) = \left(\hat{x}_{f_2}^{r,l}, \hat{y}_{f_2}^{r,l}\right) := (0,1)$, $\left(x_g^*, y_g^*\right) = \left(\underline{x}_g^r, \underline{w}_g^r\right) := (0,-1)$, $\nu := 1$, $\xi = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)^\top$, $w_r := -2 \in S(\bar{x},\bar{y})$, $\delta_{r,1} := 3$, $\delta_{r,2} := 1$, $\vartheta_r := 2$, $v_r^k := \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)^\top$, $\zeta := 1$, and $\gamma := 2$, $r = 1,2,3$, $l = 1,2,3$, $k = 1,2$.

Example 5.3. Consider the semivectorial bilevel optimization problem

$$\begin{cases} \min\limits_{x,y} & F(x,y) = x - y \\ \text{s.t.} & x \in X, \\ & y \in \Phi_{\text{eff}}(x), \end{cases}$$

where $X = \{x \in \mathbb{R} \;:\; G_1(x) = -x \leq 0 \;\text{ and }\; G_2(x) = x - 1 \leq 0\}$, and $\Phi_{\text{eff}}(x)$ denotes the efficient solution set of the lower-level problem

$$\begin{cases} \min\limits_{y} & \left(f_1(x,y) = |y|, f_2(x,y) = |y-1| + \frac{1}{2}y\right) \\ \text{s.t.} & y \in \Upsilon(x) = \{y \in \mathbb{R} \;:\; g_1(x,y) = x - y \leq 0,\; g_2(x,y) = y - 2 \leq 0\} = [x,2]. \end{cases}$$

One can see that $\Phi_{\text{eff}}(x) = [x,1]$. Therefore, the unique optimal solution of the considered problem is $(\bar{x},\bar{y}) = (0,1)$. For $(x,y) \in \Omega = \left\{(x,y) \in \mathbb{R}^2 \;:\; x \in X \text{ and } y \in \Upsilon(x)\right\}$, the Charnes-Cooper scalarized lower-level problem is

$$\begin{cases} \min\limits_{w} & |w| + |w-1| + \frac{1}{2}w \\ \text{s.t.} & x - w \leq 0, \\ & w - 2 \leq 0, \\ & |w| \leq |y|, \\ & |w-1| + \frac{1}{2}w \leq |y-1| + \frac{1}{2}y. \end{cases}$$

The feasible set of the above problem is given by:

$$\Xi(x,y) = \begin{cases} \{y\}, & x \leq y \leq 1, \\ [\max\{x, 4-3y\}, y], & 1 \leq y \leq 2. \end{cases}$$

One can see that the optimal solution set is

$$S(x,y) = \begin{cases} \{y\}, & x \leq y \leq 1, \\ \{\max\{x, 4-3y\}\}, & 1 \leq y \leq 2, \end{cases}$$

while the optimal value function associated with the Charnes-Cooper scalarized lower-level problem is given by:

$$\upsilon(x,y)=\begin{cases}1+\dfrac{1}{2}y, & x\le y\le 1,\\ 1+\dfrac{1}{2}\max\{x,4-3y\}, & 1\le y\le 2.\end{cases}$$

Thus

$$\psi(x,y)=\begin{cases}0, & x\le y\le 1,\\ \dfrac{5}{2}y-2-\dfrac{1}{2}\max\{x,4-3y\}, & 1\le y\le 2.\end{cases}$$

We now verify partial calmness at $(\bar{x},\bar{y})$. Let (x,y,u) be sufficiently close to $(0,1,0)$ such that $x\in X$, $y\in\Upsilon(x)$, and $\psi(x,y)+u=0$. Moreover, we have

$$F(x,y)-F(\bar{x},\bar{y})+\nu|u|=x-y+1+\nu\psi(x,y).$$

For $x\le y\le 1$, we have $\psi(x,y)=0$ and $x-y+1\ge 0$. For $1\le y\le 2$ near $(\bar{x},\bar{y})$, we have $\max\{x,4-3y\}=4-3y$, so $\psi(x,y)=4(y-1)$, and thus

$$F(x,y)-F(\bar{x},\bar{y})+\nu|u|=x+(4\nu-1)(y-1)\ge 0$$

if and only if $\nu\ge\dfrac{1}{4}$. Thus, for any $\nu\ge\dfrac{1}{4}$, we have $F(x,y)-F(\bar{x},\bar{y})+\nu|u|\ge 0$. Hence the partial calmness condition holds at $(\bar{x},\bar{y})$.

- The upper-level and lower-level regularities are satisfied at $\bar{x}$ and $(\bar{x},\bar{y})$ respectively.
- The multifunction Ξ satisfies the Aubin property at $(\bar{x},\bar{y},w)$ for all $w\in S(\bar{x},\bar{y})$.
- The set-valued mapping S is inner semicompact and inner semicontinuous at $(\bar{x},\bar{y})$.
- For $\mu\in\mathbb{R}^2$, we have $\Psi_{\Upsilon_0}(\mu)=\left\{(x,y,w)\in\mathbb{R}^3 : x-w\le\mu_1,\ w-2\le\mu_2\right\}$. Hence, Ψ_{Υ_0} is calm at $(0,\bar{x},\bar{y},w)$ $w\in S(\bar{x},\bar{y})$.
- For $\kappa\in\mathbb{R}^2$,

$$\Psi_{\Xi}(\kappa)=\left\{(x,y,w)\in\mathbb{R}^3 : x\le w\le 2,\ |w|-|y|\le\kappa_1,\ |w-1|-|y-1|+\frac{1}{2}(w-y)\le\kappa_2\right\},$$

Hence, the set-valued mapping Ψ_{Ξ} is calm at $(0,\bar{x},\bar{y},w)$ with $w\in S(\bar{x},\bar{y})$.

The multipliers and constants are constructed as follows: $(x_F^*,y_F^*):=(1,-1)$, $x_{G_1}^*:=-1$, $x_{G_2}^*:=1$, $\left(x_{f_1}^*,y_{f_1}^*\right)=\left(x_{f_1}^{*r},w_{f_1}^{*r}\right)=\left(\tilde{x}_{f_1}^r,\tilde{w}_{f_1}^r\right)=\left(\hat{x}_{f_1}^{r,l},\hat{y}_{f_1}^{r,l}\right):=(0,1)$, $\left(x_{f_2}^*,y_{f_2}^*\right)=\left(0,\dfrac{3}{2}\right)$, $\left(\hat{x}_{f_2}^{r,l},\hat{y}_{f_2}^{r,l}\right)=\left(0,\dfrac{1}{2}\right)$, $\left(x_{f_2}^{*r},w_{f_2}^{*r}\right)=\left(\tilde{x}_{f_2}^r,\tilde{w}_{f_2}^r\right):=\left(0,-\dfrac{1}{2}\right)$, $\left(x_{g_1}^*,y_{g_1}^*\right)=\left(\underline{x}_{g_1}^r,\underline{w}_{g_1}^r\right):=(1,-1)$, $\left(x_{g_2}^*,y_{g_2}^*\right)=\left(\underline{x}_{g_2}^r,\underline{w}_{g_2}^r\right):=(0,1)$, $\nu:=\dfrac{1}{3}$, $\xi=(1,0,0)^\top$, $w_r:=1\in S(\bar{x},\bar{y})$, $\delta_{r,1}:=0$, $\delta_{r,2}:=1$, $\vartheta_r:=(0,0)^\top$, $v_r^k:=(1,0,0)^\top$, $\zeta:=(1,0)^\top$, and $\gamma:=(0,0)^\top$, $r=1,2,3$, $l=1,2,3$, $k=1,2$. This confirms the validity of the theoretical results in a non-smooth and nonconvex bilevel framework with coupled lower-level objectives.

Remark 5.1. Obviously, the necessary optimality conditions obtained in Theorems 5.1 and 5.2 are almost the same; the only difference between them is that the scalar ν is just a nonnegative scalar in Theorem 5.1, and in Theorem 5.2 it is a positive scalar.

To conclude this section, we demonstrate a case where the partial calmness property holds for problem (3.4) at every local optimum. Specifically, consider the semi-vectorial optimization problem (SVBOP) with the parametric linear multiobjective problem (L[x]). The scalarized formulation is the linear program:

$$\min_{w} \sum_{k=1}^{s} a_k w \text{ s.t. } a_k w - a_k y \leq 0,\ k = 1,\dots,s,\ \ b_j x + c_j y \leq e_j,\ j = 1,\dots,p. \tag{5.8}$$

The feasible set and the solution set of problem (5.8), respectively, are given by:

$$\Xi(x,y) = \left\{ w \in \mathbb{R}^m \ :\ a_k w - a_k y \leq 0,\ k = 1,\dots,s;\ \ b_j x + c_j y \leq e_j,\ j = 1,\dots,p \right\},$$

$$S(x,y) = \arg\min_{w} \left\{ \sum_{k=1}^{s} a_k w \ :\ w \in \Xi(x,y) \right\}.$$

The optimal value function $\upsilon : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ associated with (5.8) is given by:

$$\upsilon(x,y) = \inf_{w \in \Xi(x,y)} \sum_{k=1}^{s} a_k w, \ \text{ for all } (x,y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Then, problem (SVBOP) is equivalent to the following single-level optimization problem:

$$\begin{cases} \min_{x,y} F(x,y) \\ G_i(x) \leq 0,\ i = 1\dots,q \\ b_j x + c_j y \leq e_j,\ j = 1,\dots,p \\ \sum_{k=1}^{s} a_k y - \upsilon(x,y) \leq 0. \end{cases} \tag{5.9}$$

Theorem 5.3. *Let $(\bar{x},\bar{y})$ be a local optimal solution of problem (5.9). Assume that the upper-level and lower-level regularities are satisfied at $\bar{x}$ and $(\bar{x},\bar{y})$, respectively. Then the following statements are valid:*

(a)**:** *Under the hypotheses of Theorem 4.1, there exist $\nu > 0$, $\xi \in \mathbb{R}^{n+m+1}$, $w_r \in S(\bar{x},\bar{y})$, $\delta_r \in \mathbb{R}^s_+$, $\vartheta_r \in \mathbb{R}^p$, $(\zeta,\gamma) \in \mathbb{R}^q \times \mathbb{R}^p$, and $(x_F^*, y_F^*) \in \partial F(\bar{x},\bar{y})$, $x_{G_i}^* \in \partial G_i(\bar{x})$, with $i \in I$, $r = 1,\dots,n+m+1$ and $\sum_{r=1}^{n+m+1} \xi_r = 1$ such that:*

$$0 = x_F^* + \sum_{i=1}^{q} \zeta_i x_{G_i}^* + \sum_{j=1}^{p} \gamma_j b_j - \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{j=1}^{p} \vartheta_{r,j} b_j,$$

$$0 = y_F^* + \nu \sum_{k=1}^{s} a_k + \sum_{j=1}^{p} \gamma_j c_j + \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} a_k,$$

$$0 = \sum_{k=1}^{s} a_k + \sum_{k=1}^{s} \delta_{r,k} a_k + \sum_{j=1}^{p} \vartheta_{r,j} c_j, \quad \forall r = 1,\dots,n+m+1,$$

$$\zeta_i \geqslant 0,\ \zeta_i G_i(\bar{x}) = 0, \quad \forall i = 1,\dots,q,$$

$$\gamma_j \geqslant 0,\ \gamma_j \left(b_j \bar{x} + c_j \bar{y} - e_j \right) = 0, \quad \forall j = 1,\dots,p$$

$$\vartheta_{r,j} \geqslant 0,\ \vartheta_{r,j} \left(b_j \bar{x} + c_j w_r - e_j \right) = 0, \quad \forall j = 1,\dots,p, \quad \forall r = 1,\dots,n+m+1.$$

(b)**:** *Based on the assumptions of Theorem 4.2, there exist* $\nu > 0$, $\xi \in \mathbb{R}^{n+m+1}$, $\delta_r \in \mathbb{R}^s_+$, $\vartheta_r \in \mathbb{R}^p$, $(\zeta, \gamma) \in \mathbb{R}^q \times \mathbb{R}^p$, *and* $(x^*_F, y^*_F) \in \partial F(\bar{x}, \bar{y})$, $x^*_{G_i} \in \partial G_i(\bar{x})$, $i \in I$, $r = 1 \ldots, n+m+1$ *and* $\sum_{r=1}^{n+m+1} \xi_r = 1$ *such that:*

$$0 = x^*_F + \sum_{i=1}^{q} \zeta_i x^*_{G_i} + \sum_{j=1}^{p} \gamma_j b_j - \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{j=1}^{p} \vartheta_{r,j} b_j,$$

$$0 = y^*_F + \nu \sum_{k=1}^{s} a_k + \sum_{j=1}^{p} \gamma_j c_j + \nu \sum_{r=1}^{n+m+1} \xi_r \sum_{k=1}^{s} \delta_{r,k} a_k,$$

$$0 = \sum_{k=1}^{s} a_k + \sum_{k=1}^{s} \delta_{r,k} a_k + \sum_{j=1}^{p} \vartheta_{r,j} c_j, \quad \forall r = 1, \ldots, n+m+1,$$

$$\zeta_i \geqslant 0,\ \zeta_i G_i(\bar{x}) = 0, \quad \forall i = 1, \ldots, q,$$

$$\gamma_j \geqslant 0,\ \gamma_j \left(b_j \bar{x} + c_j \bar{y} - e_j\right) = 0, \quad \forall j = 1, \ldots, p,$$

$$\vartheta_{r,j} \geqslant 0,\ \vartheta_{r,j} \left(b_j \bar{x} + c_j \bar{y} - e_j\right) = 0, \quad \forall j = 1, \ldots, p, \quad \forall r = 1, \ldots, n+m+1.$$

Proof. Let $(\bar{x}, \bar{y})$ be a local optimal solution to single-level linear program (5.9). According to [23, Proposition 4.1], we see that parametric linear multiobjective problem (L[x]) satisfies partial calmness at $(\bar{x}, \bar{y})$. By Proposition 5.1, there exists a scalar $\nu > 0$ such that $(\bar{x}, \bar{y})$ is a local solution to penalized problem (5.5). The remainder of the proof follows analogously to the proof of Theorem 5.2, where necessary optimality conditions were derived under partial calmness. □

6. Conclusion

This paper demonstrated how the Charnes-Cooper transformation could be applied to derive necessary optimality conditions for semivectorial optimization problems with an efficiently solvable lower-level program (SVBOP). By reformulating the problem as a classical bilevel program, we were able to utilize tools from parametric analysis and sensitivity estimation of the lower-level value function. Specifically, analyzing Lipschitz continuity and subdifferential properties allowed obtaining optimality conditions without differentiability assumptions on the value function. A promising avenue for future work is algorithmic implementation of this value function reformulation approach to solve semivectorial optimization problems numerically. Prior literature had validated this methodology algorithmically when the lower-level involves scalar optimization; see the survey in [24]. With the increasing relevance of multi-objective decision making, extending these computational techniques to problems where the lower-level involves vector optimization presents an opportunity for developing practical solution methods. Overall, the value function perspective adopted here laid theoretical groundwork for addressing semivectorial programs while retaining their intrinsic multiobjective structure.

References

[1] H. Bonnel, J. Morgan, Semivectorial bilevel optimization problem: penalty approach, J. Optim. Theory Appl. 131 (2006), 365-382.

[2] M. J. Alves, C. H. Antunes, A semivectorial bilevel programming approach to optimize electricity dynamic time-of-use retail pricing, Comput. Oper. Res. 92 (2018), 130–144.

[3] S. Dempe, A. Zemkoho, Bilevel optimization, Springer Optimization and Its Applications, Cham, Springer, 2020. doi: 10.1007/978-3-030-52119-6
[4] Y. El-Yahyaoui, M. Jennane, E. M. Kalmoun, L. Lahoussine, Optimality conditions for a class of nonsmooth semidefinite bilevel optimization problems via convexifications applied to bilevel supply chain under uncertainty, J. Comput. Appl. Math. 471 (2026), 116734.
[5] H. Bonnel, Optimality conditions for the semivectorial bilevel optimization problem, Pacific J. Optim. 2 (2006),447-467.
[6] Y. El-Yahyaoui, L. Lafhim, Necessary optimality conditions for a semivectorial bilevel optimization problem via a revisited Charnes-Cooper scalarization, Optimization 74 (2025), 165-188.
[7] S. Dempe, P. Mehlitz, Semivectorial bilevel programming versus scalar bilevel programming, Optimization. 69 (2019), 657-679.
[8] S. Dempe, G. Nazih, A.B. Zemkoho, New optimality conditions for the semivectorial bilevel optimization problem, J. Optim. Theory Appl. 157 (2013), 54-74.
[9] G. Eichfelder, Methods for multiobjective bilevel optimization, In: S. Dempe, A.B. Zemkoho, (ed.) Bilevel optimization: Advances and next challenges, pp. 423-449, Springer, 2020.
[10] L. Lafhim, New optimality conditions and a scalarization approach for a nonconvex semi-vectorial bilevel optimization problem, Positivity 24 (2020), 395-413.
[11] A. Sinha, P. Malo, K. Deb, A review on bilevel optimization: from classical to evolutionary approaches and applications, IEEE Trans. Evolutionary Computn. 22 (2017), 276-295.
[12] S. Dempe, A.B. Zemkoho, KKT reformulation and necessary conditions for optimality in non-smooth bilevel optimization, SIAM J. Optim. 24 (2014), 1639-1669.
[13] A. Charnes, W.W. Cooper, Programming with linear fractional functionals, Naval Research Logistics Quarterly, 8 (1961), 181-186.
[14] A. Charnes, W.W. Cooper, Management Models and Industrial Applications of Linear Programming. Vol. I and II. Wiley, New York, 1962.
[15] P. Kesarwani, J. Dutta, Charnes-Cooper scalarization and convex vector optimization, Optim. Lett. 15 (2021), 833-846.
[16] R. Henrion, J.V. Outrata, Subdifferential condition for calmness of multifunctions, J. Math. Anal. Appl. 258 (2001), 110-130.
[17] B.S. Mordukhovich, N.M. Nam, N.D. Yen, Subgradients of marginal functions in parametric mathematical programming, Math. Program. 116 (2009), 369-396.
[18] R. Henrion, A. Jourani, J.V. Outrata, On the calmness of a class of multifunctions, SIAM J. Optim. 13 (2002), 603-618.
[19] R. Henrion, J.V. Outrata, Calmness of constraint systems with applications, Math. Program. 104 (2005), 437-464.
[20] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, Berlin-Heidelberg, Springer-Verlag, 2006.
[21] R.T. Rockafellar, R.J.B. Wets, Variational Analysis, Berlin, Springer, 1998.
[22] S. Dempe, A.B. Zemkoho, The generalized Mangasarian-Fromowitz constraint qualification and optimality conditions for bilevel programs, J. Optim. Theory Appl. 148 (2011), 46-68.
[23] J.J. Ye, D.L. Zhu, Optimality conditions for bilevel programming problems, Optimization. 33 (1995), 9-27.
[24] S. Dempe, Bilevel optimization: theory, algorithms, applications and a bibliography, In: S. Dempe, A.B. Zemkoho, (ed.) Bilevel Optimization: Advances and Next Challenges, pp. 581-672, Springer, 2020.