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SOLUTIONS OF AN ATTRACTION-REPULSION CHEMOTAXIS SYSTEM WITH SINGULAR SENSITIVITIES

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Abstract. This paper investigates the following chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (\frac{u}{v} \nabla v) + \chi_2 \nabla \cdot (\frac{u}{w} \nabla w) + \lambda u - \mu u^2, & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ 0 = \Delta w - w + u, & x \in \Omega, t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions, where $\tau \in \{0,1\}$. In the case of $w \not\equiv 0$, namely, singular attraction-repulsion mechanism, we prove the global boundedness of global classical solution to the system. For $\tau = 1$, if the parameters satisfy $\chi_1\chi_2 \in (0, \frac{1}{2})$, $\chi_2 \in (0, \frac{2}{9})$, and λ is sufficiently large, the system admits a unique global uniformly bounded solution. For $\tau = 0$, the system admits a unique global uniformly bounded solution with $\chi_1 \in (0, \frac{1}{2})$, which also indicates that no blow-up of solutions occurs over time. We complete our proof by using heat semigroup estimates, a priori estimates, parabolic-elliptic regularity theory, and the Moser iteration technique.

Keywords. Attraction-repulsion mechanism; Global solutions; Neumann boundary conditions; Singular sensitivity.

1. INTRODUCTION

In this paper, we consider a chemotaxis system featuring an attraction-repulsion mechanism and singular sensitivity

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (\frac{u}{v} \nabla v) + \chi_2 \nabla \cdot (\frac{u}{w} \nabla w) + \lambda u - \mu u^2, & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ 0 = \Delta w - w + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ (u, \tau v)(x,0) = (u_0(x), \tau v_0(x)), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, denotes a bounded domain with smooth boundary $\partial\Omega$, the parameter $\lambda, \mu, \chi_1, \chi_2 > 0$ and $\tau \in \{0,1\}$, and (u_0, v_0) fulfills

$$u_0 \in C^0(\bar{\Omega}), u_0 \geq 0, u_0 \not\equiv 0 \text{ in } \bar{\Omega}; \quad \tau v_0 \in W^{1,\infty}(\Omega), \tau v_0 > 0 \text{ in } \bar{\Omega}. \tag{1.2}$$

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In recent decades, chemotaxis systems have been extensively studied due to their crucial roles in elucidating fundamental biological processes, such as microbial aggregation, immune responses, and tumor metastasis. The pioneering research on these systems can be traced back to mathematical description of the aggregation of Dictyostelium discoideum by Keller and Segel [11]. From a biological point of view, system (1.1) characterizes the dynamic behavior of a cell population u under the influence of dual chemical signal gradients, where the chemoattractant v induces directional migration, while the chemorepellent w generates a repulsive effect. The singular chemotactic terms $-\nabla\cdot(u\chi_1(v)\nabla v)$ and $+\nabla\cdot(u\chi_2(w)\nabla w)$ represent the migration of cells towards regions with higher concentrations of the attractant signal and away from regions with higher concentrations of the repellent signal, respectively. The singular sensitivity functions $\chi_1(v)=\frac{\chi_1}{v}$ and $\chi_2(w)=\frac{\chi_2}{w}$ follow the Weber-Fechner law, indicating that the cell's perception of chemical stimuli decreases as the signal concentration increases. The logistic source $\lambda u-\mu u^2$ describes how cells proliferate and die. For $\tau=1$, the diffusion and degradation processes of signaling molecules are slow with their concentration gradually changing over time, whereas, for $\tau=0$, the diffusion rate of chemoattractants is much higher than that of cells, leading to v reaching a steady state instantaneously. For more comprehensive discussions on the biological mechanisms and mathematical analysis of systems like (1.1), we refer to [1, 3, 4, 11, 15] and the references therein.

Recently, there has been increasing attention to the global existence and boundedness of solutions for the following chemotaxis systems with singular sensitivity

$$\begin{cases} u_t=\Delta u-\nabla\cdot(u\chi_1(v)\nabla v)+g(u), & x\in\Omega, t>0,\\ \tau v_t=\Delta v-v+u, & x\in\Omega, t>0, \end{cases} \tag{1.3}$$

where $\tau\in\{0,1\}$. In the case of $\tau=1$, when $\chi_1(v)\equiv\chi_1>0$ and $g(u)=\mu_0u-\mu_1u^2$, Winkler [22] demonstrated that, in any spatial dimension, system (1.3) admits a unique globally bounded classical solution; when $\chi_1(v)=\chi_1v^{-1}$ and $g(u)=\mu_0u-\mu_1u^k$, Zhao [28] showed that global boundedness holds in higher dimensional ($N\geq 2$) provided χ_1 is sufficiently small. In the case of $\tau=0$, when $\chi_1(v)=v^{-1}$ and $g(u)=u(a(\cdot,t)-b(\cdot,t)u)$, Kurt and Shen [12] proved that the solution of (1.3) is uniformly bounded under certain conditions; when $\chi_1(v)=v^{-k}$, $k\in(0,1)$, by employing a recursive argument on k and Moser iteration technique, Zhao [29] proved that, when $\mu_1>0$ is sufficiently large, system (1.3) admits globally bounded solutions in two-dimensional space for $g(u)=\mu_0u-\mu_1u^k$. Subsequently, for $g(u)=ru-\frac{\mu u^2}{\ln^l(u+e)}$ ($\mu>0$, $l\in(0,1)$), Le [14] derived analogous results by using the $L\ln L$ estimate without imposing any constraints on μ. For more chemotaxis systems with singular sensitivity like system (1.3), we refer to [2, 13] and the references therein.

Another topic that has drawn increasing attention in recent years is the study of chemotaxis systems involving attraction-repulsion mechanisms

$$\begin{cases} u_t=\Delta u-\nabla\cdot(u\chi_1(v)\nabla v)+\nabla\cdot(u\chi_2(w)\nabla w)+g(u), & x\in\Omega, t>0,\\ \tau_1v_t=\Delta v-v+u, & x\in\Omega, t>0,\\ \tau_2w_t=\Delta w-w+u, & x\in\Omega, t>0, \end{cases} \tag{1.4}$$

where $\tau_1,\tau_2\in\{0,1\}$. In the case of $\chi_1(v)\equiv\chi_1>0$ and $\chi_2(w)\equiv\chi_2>0$, $\tau_1=0$, $\tau_2=1$ and $g(u)=\mu u(1-u^k)$, through a priori estimates, constructing a Lyapunov functional, and applying Moser iteration technique, Liu and Li [16] showed that system (1.4) has a unique globally bounded solution and analyzed asymptotic behavior, if $N = 2$, $k \geq 1$ or $N \geq 3$, $k > 1$, or $N \geq 3$, $k = 1$, $\mu > c(N)\chi_2 + \chi_1$. When $\tau_1 = \tau_2 = 0$, by utilizing methods such as elliptic regularity theory and heat semigroup estimates, Xie and Zheng [23] proved that system (1.4) with $g(u) \leq a - bu^k$ admits a unique globally bounded classical solution under certain parameter conditions. Zhou, Li and Zhao [31] proved the existence of globally bounded classical solutions to system (1.4) with $g(u) \leq u(a - bu^k)$ under critical parameter conditions by means of weighted integral estimates, Moser iteration, and elliptic regularity theory. When $\tau_1 = \tau_2 = 1$, Ren and Liu [18] proved the global boundedness of solutions and analyzed the asymptotic behavior of system (1.4) with either $g(u) = \mu_0 - \mu_1 u^2$ or $g(u) = \mu_0 u - \mu_1 u^2$ in three-dimensional space under specific parameter conditions by constructing a coupled quantity and using a priori estimates. Liu and Liu [17] obtained an explicit threshold for the Logistic damping coefficient μ_1 and proved the global boundedness of classical solutions to system (1.4) with $g(u) = \mu_0 - \mu_1 u^2$ in four- and five-dimensional spaces. However, these research results concern non-singular chemotaxis systems with attractive-repulsive mechanisms and constant sensitivity. For further findings on such systems, we refer to [7, 19, 24] and the references therein. Next, we offer a series of findings for the attraction-repulsion chemotaxis system with singular sensitivities. In the scenario where $\chi_1(v) = \chi_1 > 0$ and $\chi_2(w) = \frac{\chi_2}{w} > 0$, when $\tau_1 = 1$, $\tau_2 = 0$, and $g(u) \leq a - bu^k$, by constructing a priori estimates, employing the Moser iteration technique, and utilizing elliptic regularity theory, Yan and Yang [25] proved the existence of globally bounded classical solutions to the system under specific conditions. In the scenario where $\chi_1(v) = \frac{\chi_1}{v} > 0$, $\chi_2(w) = \frac{\chi_2}{w} > 0$, and with $\tau_1 = \tau_2 = 1$ and $g(u) \leq r - \mu u^k$ for $k > 2$ or $k = 2$, Jiao, Jadlovská and Li [8] proved that system (1.4) possesses a global classical solution by constructing a novel triply multiplicative functional and applying heat semigroup estimates.

The following Table 1 presents the various systems reviewed in this section, the research findings closely associated with them, as well as the outstanding issues that remain unresolved.

TABLE 1. Research summary

w	N	τ_1	τ_2	$\chi_1(v)$	$\chi_2(w)$	Results	Reference
$w \equiv 0$	$N \geq 1$	1	0	χ_1	0	Global boundedness, Global classical solution	[22]
$w \equiv 0$	$N = 2$	0	0	$\chi_1 v^{-k}$	0	Global boundedness, Global classical solution	[13, 14, 29]
$w \equiv 0$	$N \geq 2$	1	0	$\chi_1 v^{-1}$	0	Global boundedness, Global classical solution	[2, 12, 28]
$w \neq 0$	$N \geq 1$	0	0	χ_1	χ_2	Global boundedness, Global classical solution	[19, 23, 17]
$w \neq 0$	$N \geq 1$	1	0	χ_1	χ_2	Global boundedness, Global classical solution	[19]
$w \neq 0$	$N \geq 2$	0	1	χ_1	χ_2	Global boundedness, Global classical solution	[16]
$w \neq 0$	$N \geq 2$	1	1	χ_1	χ_2	Global boundedness, Global classical solution	[18, 19, 7]
$w \neq 0$	$N \geq 2$	1	0	χ_1	$\chi_2 w^{-1}$	Global boundedness, Global classical solution	[25]
$w \neq 0$	$N \geq 1$	1	1	$\chi_1 v^{-1}$	$\chi_2 w^{-1}$	Global classical solution	[8, 10]
$w \neq 0$	$N \geq 1$	1	0	$\chi_1 v^{-1}$	$\chi_2 w^{-1}$	Global boundedness, Global classical solution?	Open problem
$w \neq 0$	$N \geq 1$	0	0	$\chi_1 v^{-1}$	$\chi_2 w^{-1}$	Global boundedness, Global classical solution?	Open problem

To the best of our knowledge, as mentioned above, in terms of the complexity arising from the singularity and the coupling of attraction-repulsion chemotactic mechanisms, the problem of the global existence and global boundedness of classical solutions to system (1.1) remains unsolved so far. Therefore, the purpose of this paper is to prove the global existence and global boundedness of classical solutions to system (1.1) when $\tau = 1$(i.e.,$\tau_1 = 1$ and $\tau_2 = 0$) or $\tau = 0$(i.e.,$\tau_1 = 0$ and $\tau_2 = 0$), namely the open problems mentioned in Table 1.

Our main findings are as follows.

Theorem 1.1. *Let $\tau = 1$, $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with a smooth boundary, $\lambda, \mu, \chi_1, \chi_2 > 0$ and (u_0, v_0) satisfy (1.2). Under the smallness assumptions $\chi_1 \chi_2 < \frac{1}{2}$ and $\chi_2 < \frac{2}{9}$, if λ exceeds the threshold $\lambda > \frac{\chi_1^2}{r} + \chi_2$ with $r \in (0,1)$, then system (1.1) possesses a global classical solution which is uniformly bounded.*

Remark 1.1. Theorem 1.1 indicates that, in high-dimensional spaces ($N \geq 3$), the interaction between the chemotactic sensitivity coefficients χ_1, χ_2 and the logistic source term coefficient λ is the key to ensuring the global boundedness of the solutions of system (1.1). On one hand, smaller values of χ_1 and χ_2 prevent excessive cell aggregation by regulating the attraction-repulsion balance; on the other hand, a sufficiently large value of λ effectively suppresses the chemotactic aggregation effect through the growth limitation mechanism. The combined action of the two suppresses the tendency of the solution to grow without bound, thereby facilitating the realization of global boundedness.

Remark 1.2. It is known from [28] that, when the repulsion mechanism in system (1.1) degenerates (i.e., $w \equiv 0$), the system admits globally bounded classical solutions if

$$\chi_1 \in \left(0, \min\left\{\frac{1}{2}, \frac{1}{2\sqrt{N-1}}\right\}\right)$$

and $N \geq 2$. Compared with the single singular chemotaxis system in [28], system(1.1) in this paper constructs a complete attraction-repulsion mechanism framework by introducing a singular repulsion term, which generalizes the results of [28] to a certain extent. Compared with the attraction-repulsion chemotaxis system in [16], system (1.1) is more complex due to the inclusion of dual singular terms. This directly renders a priori estimates employed in [16] inapplicable to system (1.1). Therefore, it is necessary to construct weighted integrals to obtain a uniform upper bound for v, and then conduct a priori estimates. Meanwhile, due to the presence of dual singular terms, the parameters χ_1 and χ_2 in system (1.1) need to constrain the influence of dual singular terms, which is significantly different from the case in [16] where only $\chi_1(v) \equiv \chi_1 > 0, \chi_2(v) \equiv \chi_2 > 0$ is required. Thus, this also generalizes the results in [16].

Remark 1.3. In Lemma 3.4 (see below), when estimating the integral term $\int_\Omega u^{\beta_1} |\nabla v|^2$, we apply Hölder's inequality with the conjugate exponents $\frac{N-2}{2}$ and $\frac{N}{2}$. This approach requires $N \geq 3$ and consequently fails for $N \leq 2$, where no alternative method is currently available. Thus, establishing the global boundedness of solutions to system (1.1) in the case of $N \leq 2$ remains an open problem.

Theorem 1.2. *Let $\tau = 0$, $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with a smooth boundary, $\lambda, \mu, \chi_1, \chi_2 > 0$ and (u_0, v_0) satisfy (1.2). If moreover $\chi_1 \in (0, \frac{1}{2})$, then system (1.1) possesses a global classical solution which is uniformly bounded.*

Remark 1.4. Compared with Theorem 1.1, Theorem 1.2 investigates system (1.1) in the case where $\tau = 0$. Though this simplifies the system, it weakens the parameter restrictions required by Theorem 1.1. Reference [23] demonstrated that when replacing the logistic source term $\lambda u - \mu u^2$ with $a - bu^2$ in system (1.1), the system possesses a unique global bounded classical solution under the conditions $\chi_1 > \chi_2, b \geq \frac{(N-2)_+}{N}$ and $N \geq 1$. In contrast to this result, Theorem

1.2 not only incorporates the influence of singular chemotaxis but also completely eliminates the restrictions on the logistic source parameters.

Remark 1.5. According to [8], when the third equation of system (1.1) is replaced by $w_t = \Delta w - w + u$ with $\tau = 1$, it has been proven that the system possesses a global classical solution under appropriate parameter conditions. Theorem 1.1 and Theorem 1.2 establish the existence of uniformly bounded global classical solutions to system (1.1) for the cases $\tau = 1$ and $\tau = 0$, respectively. Although system (1.1) represents a simplified version of the model in [8], Theorem 1.1 extends previous work in characterizing solution properties by supplementing the analysis of solution boundedness, which was not addressed in the original study.

This paper investigates the uniform global boundedness of solutions for system (1.1). The proof of the theorem is accomplished by constructing a priori estimates and then employing Moser's iteration technique and the parabolic-elliptic regularity theory, with relevant methods drawing from references [16, 19, 22, 23]. In view of the existence of the two singular chemotactic terms $-\chi_1\nabla\cdot(\frac{u}{v}\nabla v)$ and $+\chi_2\cdot\nabla(\frac{u}{w}\nabla w)$ in system (1.1), we initially attempted to follow the variable transformation approach used in [9] and [26], hoping to transform the singular terms into nonsingular forms such as $-\chi_1\nabla\cdot(U\nabla V)$ and $+\chi_2\nabla\cdot(U\nabla W)$, so as to analyze the boundedness through traditional approaches. However, it seems impossible to find a substitution that can simultaneously eliminate the singularity of these two chemotactic terms. This essential difficulty forces us to break through the limitations of previous methods. Different from previous studies focusing on finding appropriate variable substitutions, we break the routine and find a new idea. We adopt different treatment strategies for the cases of $\tau = 1$ and $\tau = 0$.

For the case of $\tau = 1$, firstly, we construct a weighted integral $\int_\Omega u^{-\tau}v^{-\theta}$ and conduct its estimation to obtain a uniform positive lower bound for v. In the specific calculation, by flexibly adjusting coefficients, skillfully applying Young's inequality, utilizing the elliptic property of the third equation in system (1.1), and combining with heat semigroup estimates, we derive the uniform positive lower bound of v with respect to time. This key estimation essentially eliminates the singularity caused by the term $\frac{1}{v}$, creating favorable conditions for establishing a priori estimates. Secondly, based on the uniform positive lower bound of v, we establish a priori estimates and prove the boundedness of $\|u(\cdot,t)\|L^\beta(\Omega)+\|\nabla v(\cdot,t)\|L^{2\beta}$. Finally, by applying Moser's iteration and the parabolic-elliptic regularity theory, we complete the proof of Theorem 1.1.

For the case of $\tau = 0$, firstly, it is necessary to prove the boundedness of $\|u\|_{L^\xi(\Omega)}$. A crucial step therein is to derive a differential inequality $\int_\Omega u^\xi v^{-2}|\nabla v|^2 \le \xi^2\int_\Omega u^{\xi-2}|\nabla u|^2+2\int_\Omega u^\xi$ from the second equation of (1.1). This effectively addresses the cross-term $\int_\Omega u^\xi v^{-2}|\nabla v|^2$ that arises in the boundedness proof of $\|u\|_{L^\xi(\Omega)}$. Through this mechanism, we eliminate the analytical obstacles brought about by the $\frac{1}{v}$ singularity. Then, we complete the proof of Theorem 1.2 by utilizing Moser's iteration and elliptic regularity theory.

The paper is organized as follows. In Section 2, we give the local existence of solutions to system (1.1) and some inequalities used in subsequent analytical proofs. In Section 3, we present the proof of Theorem 1.1. In Section 4, we present the proof of Theorem 1.2. In Section 5, the last section, we summarize the main results of this paper.

2. Preliminaries

In this section, we state the local existence of solutions to system (1.1) and some inequalities that are utilized in the proof later.

Lemma 2.1. *(Local Existence) Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with a smooth boundary, (u_0, v_0, w_0) satisfying (1.2) and positive parameters $\lambda, \mu, \chi_1, \chi_2$. Then there exists $T_{max} \in (0, \infty]$ such that system (1.1) has a unique nonnegative classical solution*

$$(u, v, w) \in \left(C^0\left(\bar{\Omega} \times [0, T_{\max})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{\max})\right)\right)^3.$$

Furthermore, if $T_{\max} = \infty$ or $T_{\max} < \infty$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \to \infty \text{ as } t \nearrow \infty.$$

Proof. Define $R := 1 + \|u_0\|_{L^\infty(\Omega)}$ and let $T \in (0,1)$ be a small value to be specified later. Within the Banach space $A := C^0(\bar{\Omega} \times [0,T])$, we consider the closed bounded convex subset

$$R_T = \left\{ u \in A \mid \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq R,\ \inf_{\Omega} v(\cdot,t) \geq \zeta_1,\ \inf_{\Omega} w(\cdot,t) \geq \zeta_2, \text{ for all } t \in (0,T) \right\}.$$

We also define $\phi : R_T \to R_T$ such that, for any given $\bar{u} \in R_T$, $\phi(\bar{u}) = u$, where u denotes the first component of the solution to

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (\frac{u}{v} \nabla v) + \chi_2 \nabla \cdot (\frac{u}{w} \nabla w) + \lambda u - \mu u \bar{u}, & x \in \Omega, t \in (0,T), \\ \tau v_t = \Delta v - v + \bar{u}, & x \in \Omega, t \in (0,T), \\ 0 = \Delta w - w + \bar{u}, & x \in \Omega, t \in (0,T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \\ (u, \tau v)(x,0) = (u_0(x), \tau v_0(x)), & x \in \Omega. \end{cases} \tag{2.1}$$

Based on the standard L^p theory of elliptic equations and the third equation in the above system, we have $\|w(\cdot,t)\|_{W^{2,p}(\Omega \times (0,T))} \leq c_1$, $1 < p < \infty$, where c_1 (along with all the constants $c_2, c_3, \ldots$ appear later) is positive. By the Sobolev embedding theorem, $W^{2,p}(\Omega) \hookrightarrow C^{1+s_1}$ for some $s_1 \in (0,1)$, which derives $\|w(\cdot,t)\|_{L^\infty(\Omega \times (0,T))} \leq c_2$ and $\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \leq c_2$. Similarly, for $\tau = 0$, we have $\|v(\cdot,t)\|_{L^\infty(\Omega \times (0,T))} \leq c_3$ and $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leq c_3$. For $\tau = 1$, applying the standard Schauder and L^p theory of parabolic equation to the second equation of (2.1), for some $s_2 \in (0,1)$, we have $\|v(\cdot,t)\|_{C^{s_2,\frac{s_2}{2}}(\bar{\Omega} \times [0,T])} \leq c_4$ and $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leq c_5$. With the fact $-\chi_1 v^{-1} \nabla v + \chi_2 w^{-1} \nabla w \in L^\infty(\Omega \times (0,T))$ and $\lambda - \mu \bar{u} \in L^\infty(\Omega \times (0,T))$, we can use parabolic regularity to obtain

$$\|u(\cdot,t)\|_{C^{s_3,\frac{s_3}{2}}(\bar{\Omega} \times [0,T])} \leq c_6 \text{ for some } s_3 \in (0,1), \tag{2.2}$$

which derives $\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + c_7 t^{\frac{s_3}{2}}$, $t \in (0,T)$. We fix $T < c_7^{-\frac{2}{s_3}}$, which implies that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + 1, \quad t \in (0,T).$$

Therefore, $u \in R_T$ and ϕ maps R_T into itself. Moreover, for any $\bar{u} \in R_T$ and for some $s_4 \in (0,1)$, we have $\|\phi(\bar{u})\|_{C^{s_4,\frac{s_4}{2}}(\bar{\Omega} \times [0,T])} \leq c_8$, due to (2.2). Thus, by the compact embedding theorem, ϕ is relatively compact in A.

Next, we prove that ϕ is continuous. For $\tau = 1$, give $\bar{u}_i \in A$ and $u_i = \phi(\bar{u}_i)$ $(i = 1,2)$. Let v_i and w_i $(i = 1,2)$ satisfy

$$\begin{cases} v_{it} = \Delta v_i - v_i + \bar{u}_i, & x \in \Omega, t \in (0,T), \\ \frac{\partial v_i}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \\ v_i(x,0) = v_0, & x \in \Omega \end{cases} \tag{2.3}$$

and

$$\begin{cases} \Delta w_i = w_i - \bar{u}_i, & x \in \Omega, t \in (0,T), \\ \frac{\partial w_i}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T). \end{cases} \tag{2.4}$$

Due to the parabolic and elliptic regularity, for all $t_0 \in (0,T)$ and for some $s_5, s_6 \in (0,1)$, we get

$$u_i, v_i \in C^{2+s_5, 1+\frac{s_5}{2}}(\bar{\Omega} \times [0,T]) \quad \text{and} \quad w_i \in C^{2+s_6}(\bar{\Omega}),$$

where $i = 1,2$. Setting $z = \phi(\bar{u}_1) - \phi(\bar{u}_2) = u_1 - u_2$, we see from system (2.1) that

$$\begin{cases} z_t = \Delta z - \chi_1 v_1^{-1} \nabla v_1 \cdot \nabla z + \chi_2 w_1^{-1} \nabla w_1 \cdot \nabla z + h_1(x,t) z + h_2(x,t), & x \in \Omega, t \in (0,T), \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \\ z(x,0) = v_0, & x \in \Omega, \end{cases} \tag{2.5}$$

where

$$h_1(x,t) = \chi_1 v_1^{-2} |\nabla v_1|^2 - \chi_2 w_1^{-2} |\nabla w_1|^2 - \chi_1 v_1^{-1} \Delta v_1 + \chi_2 w_1^{-1} \Delta w_1 + \lambda - \mu \bar{u}_1$$

and

$$\begin{aligned} h_2(x,t) = &- \chi_1 \nabla((v_1^{-1} - v_2^{-1}) u_2 \nabla v_1) - \chi_1 \nabla(v_2^{-1} u_2 \nabla(v_1 - v_2)) \\ &+ \chi_2 \nabla((w_1^{-1} - w_2^{-1}) u_2 \nabla w_1) + \chi_1 \nabla(w_2^{-1} u_2 \nabla(w_1 - w_2)) + \mu u_2 (\bar{u}_2 - \bar{u}_1). \end{aligned}$$

Given $h_1(x,t), \chi_1 v_1^{-1} \nabla v_1$ and $\chi_2 w_1^{-1} \nabla w_1 \in L^\infty(\Omega \times (0,T))$, by virtue of the parabolic and elliptic regularity, applying the L^p theory to (2.3), (2.4), and (2.5) yields

$$\begin{aligned} \|z\|_{W_p^{2,1}(\Omega\times(0,T))} &\le c_9 \|h_2\|_{L^p(\Omega\times(0,T))} \\ &\le c_{10}(\|v_1 - v_2\|_{W^{2,p}(\Omega\times(0,T))} + \|w_1 - w_2\|_{W^{2,p}(\Omega\times(0,T))} + \|\bar{u}_1 - \bar{u}_2\|_{L^p(\Omega\times(0,T))}) \end{aligned}$$

and

$$\begin{aligned} &\|v_1 - v_2\|_{W^{2,p}(\Omega\times(0,T))} + \|w_1 - w_2\|_{W^{2,p}(\Omega\times(0,T))} \\ \le &c_{11} \|\bar{u}_1 - \bar{u}_2\|_{L^p(\Omega\times(0,T))} \le c_{12} \|\bar{u}_1 - \bar{u}_2\|_{C^0(\bar{\Omega}\times[0,T])}. \end{aligned}$$

For some $s_7 \in (0,1)$ and large enough p, since $W_p^{2,1}(\Omega) \hookrightarrow C^{s_7, \frac{s_7}{2}}(\bar{\Omega})$, we can get

$$\|z\|_{C^0(\bar{\Omega}\times[0,T])} \le c_{13} \|z\|_{C^{s_7,\frac{s_7}{2}}(\bar{\Omega}\times[0,T])} \le c_{14} \|z\|_{W_p^{2,1}(\bar{\Omega}\times[0,T])} \le c_{15} \|\bar{u}_1 - \bar{u}_2\|_{C^0(\bar{\Omega}\times[0,T])}.$$

Thus, ψ is continuous. For $\tau = 0$, the same method can be applied to the result. Then, based on Schauder fixed point theorem, there exists a fixed point $\bar{u} \in R_T$ such that $\phi(\bar{u}) = \bar{u}$. Substituting $\bar{u}$ into (2.1), we can establish the existence of solutions v and w. Furthermore, by incorporating the regularity theories of parabolic and elliptic equations, it can be proved that (u,v,w) is a local classical solution to system (1.1).

Concerning the uniqueness of the solution, we suppose that (u_1, v_1, w_1) and (u_2, v_2, w_2) are two nonnegative classical solutions to system (1.1) with identical initial values. Let $z_1 = u_1 - u_2$,

$z_2 = v_1 - v_2$, and $z_3 = w_1 - w_2$. Through simple calculations, we can derive $z_1 = z_2 = z_3 \equiv 0$, thereby proving the uniqueness of the classical solution. The proof is completed. □

To derive a uniform lower bound for v, we present a pointwise lower bound estimate concerning the Neumann heat semigroup $(e^{\sigma\Delta})_{\sigma\geq 0}$ on Ω.

Lemma 2.2. *Let $\sigma \geq 0$. Then, for all nonnegative $z \in C^0(\overline{\Omega})$,*

$$(e^{\sigma\Delta}z)(x) \geq \frac{1}{(4\pi\rho)^{\frac{N}{2}}} e^{-\frac{(diam\Omega)^2}{4\sigma}} \int_\Omega z dx$$

for all $x \in \Omega$.

Proof. See [6, lemma 3.1]. □

The following form of Gagliardo-Nirenberg inequality will be repeatedly utilized in subsequent discussions.

Lemma 2.3. *(Gagliardo-Nirenberg Inequality [5]) Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ and $0 < s \leq q \leq \frac{2N}{N-2}$, Then, for all $\eta \in W^{1,2} \cap L^s(\Omega)$, there exists a constant $C > 0$ such that*

$$\|\eta\|_{L^q(\Omega)} \leq C\|\nabla\eta\|^a_{L^2(\Omega)}\|\eta\|^{1-a}_{L^s(\Omega)} + C\|\eta\|_{L^s(\Omega)},$$

with $a := \frac{\frac{N}{s}-\frac{N}{q}}{1-\frac{N}{2}+\frac{N}{s}} \in (0,1)$ holds.

The following is a specific form of the Gronwall's inequality, which will be frequently used in the subsequent proofs.

Lemma 2.4. *(Gronwall's Inequality [30, Lemma 2.2]) Let h be a positive absolutely continuous function defined on $(0,\infty)$ which satisfies*

$$\begin{cases} h'(t) + \varepsilon h^p \leq \varsigma \\ h(0) = h_0 \end{cases}$$

with some constants $\varepsilon > 0$, $\varsigma > 0$,and $p > 0$. Then, for $t > 0$, $h(t) \leq \max\{y_0, (\frac{\varsigma}{\varepsilon})^{\frac{1}{p}}\}$.

In subsequent analysis, we make use of a conclusion regarding the boundedness in ordinary differential inequalities.

Lemma 2.5. *([23, lemma 2.4]) Let $T > 0$, and consider a nonnegative absolutely continuous function g defined on $[0,T)$. Suppose that, for almost every $t \in (0,T)$, $g'(t) + \delta g(t) \leq h(t)$ with $\delta > 0$ and a nonnegative function $h \in L^1_{loc}([0,T))$. In addition, assume that there exists $C > 0$ such that*

$$\int_t^{t+1} h(s)\,ds \leq C \quad \text{for all } t \in [0, T-1).$$

Then

$$g(t) \leq \max\left\{g(0) + C, \frac{\sigma}{\delta} + 2C\right\} \quad \text{for all } t \in (0,T).$$

3. THE CASE $\tau = 1$

In this section, we furnish a proof regarding the uniform global boundedness of solutions to system (1.1) in the case where $\tau = 1$, that is, the proof of Theorem 1.1. Proving global boundedness starts with establishing basic L^1 boundedness of the solution.

Lemma 3.1. *Let $\tau = 1$ and $N \geq 1$. Then, for all $t \in (0, T_{max})$, the solution of (1.1) has*

$$\int_\Omega u(\cdot,t) \leq m_u := \max\left\{\int_\Omega u_0, \frac{\lambda}{\mu}|\Omega|\right\}, \tag{3.1}$$

$$\int_\Omega v(\cdot,t) \leq m_v := \max\{\int_\Omega v_0, m_u\}, \tag{3.2}$$

and

$$\int_\Omega |\nabla v|^2 \leq K_1 := \max\left\{\int_\Omega |\nabla v_0|^2 + \frac{\lambda+1}{2\mu}m_u, \frac{5(\lambda+1)}{4\mu}m_u\right\}. \tag{3.3}$$

Proof. Integrating the first equation in (1.1), Hölder inequality yields

$$\frac{d}{dt}\int_\Omega u = \lambda\int_\Omega u - \mu\int_\Omega u^2 \leq \lambda\int_\Omega u - \mu|\Omega|^{-1}(\int_\Omega u)^2, \tag{3.4}$$

which implies (3.1) holds by Gronwall's inequality. Integrating the second equation of (1.1) and utilizing (3.1) yield (3.2). By integrating the first formula in (3.4) from t to $t+1$ and utilizing (3.1), we sees that

$$\int_t^{t+1}\int_\Omega u^2(\cdot,t) \leq \frac{\lambda+1}{\mu}m_u. \tag{3.5}$$

Testing the second equation of (1.1) against $-2\Delta v$, performing integration by parts, and applying Young's inequality, we derive

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla v|^2 &= -2\int_\Omega |\Delta v|^2 - 2\int_\Omega |\nabla v|^2 - 2\int_\Omega u\Delta v \\ &\leq -2\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega u^2.\end{aligned}$$

Combining this with(3.5) and applying Lemma 2.5 yield (3.3). □

To establish an upper bound estimate for v, the key step lies in deriving control of the weighted integral $\int_\Omega u^{-r}v^{-\theta}$ with some $r, \theta > 0$.

Lemma 3.2. *Let $N \geq 1$, $\lambda > \frac{\chi_1^2}{r} + \chi_2$, $\chi_1 > 0, 0 < \chi_2 < \frac{2}{9}$ and $\chi_1\chi_2 < \frac{1}{2}$. If $\theta \in (\chi_1^2, \frac{\chi_1}{2\chi_2})$, then, for each $r \in (0,1)$ and $t \in (0, T_{max})$, there exists a constant $K_2 > 0$ such that*

$$\int_\Omega u^{-r}v^{-\theta} \leq K_2. \tag{3.6}$$

Proof. A straightforward calculation utilizing (1.1) derives

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u^{-r}v^{-\theta}\\
\le&-r(r+1)\int_\Omega u^{-r-2}v^{-\theta}|\nabla u|^2-r\chi_2\int_\Omega u^{-r-1}v^{-\theta}\nabla\cdot(\frac{u}{w}\nabla w)\\
&+[r\chi_1\theta-\theta(\theta+1)]\int_\Omega u^{-r}v^{-\theta-2}|\nabla v|^2+[r\chi_1(r+1)-2r\theta]\int_\Omega u^{-r-1}v^{-\theta-1}\nabla u\cdot\nabla v\\
&+r\mu\int_\Omega u^{-r+1}v^{-\theta}+(\theta-r\lambda)\int_\Omega u^{-r}v^{-\theta}-\theta\int_\Omega u^{-r+1}v^{-\theta-1}.
\end{aligned}
\tag{3.7}
$$

Exploiting nonnegation of u together with third equation of system (1.1) yields

$$
\frac{|\nabla w|^2}{w^2}\le 1-\Delta\ln w.
$$

An application of integration by parts gives

$$
\begin{aligned}
&-r\chi_2\int_\Omega u^{-r-1}v^{-\theta}\nabla\cdot(\frac{u}{w}\nabla w)\\
=&-r\chi_2\int_\Omega u^{-r}v^{-\theta}w^{-1}\Delta w-r\chi_2\int_\Omega u^{-r-1}v^{-\theta}w^{-1}\nabla u\cdot\nabla w+r\chi_2\int_\Omega u^{-r}v^{-\theta}w^{-2}|\nabla w|^2\\
\le&-r^2\chi_2\int_\Omega u^{-r-1}v^{-\theta}w^{-1}\nabla u\cdot\nabla w-r\chi_2\theta\int_\Omega u^{-r}v^{-\theta-1}w^{-1}\nabla v\cdot\nabla w\\
&-r\chi_2\int_\Omega u^{-r}v^{-\theta}w^{-2}|\nabla w|^2-r\chi_2\int_\Omega u^{-r-1}v^{-\theta}w^{-1}\nabla u\cdot\nabla w\\
&+r\chi_2\int_\Omega u^{-r}v^{-\theta}-r\chi_2\int_\Omega u^{-r}v^{-\theta}\Delta\ln w\\
=&-(2r^2\chi_2+r\chi_2)\int_\Omega u^{-r-1}v^{-\theta}w^{-1}\nabla u\cdot\nabla w-2r\chi_2\theta\int_\Omega u^{-r}v^{-\theta-1}w^{-1}\nabla v\cdot\nabla w\\
&-r\chi_2\int_\Omega u^{-r}v^{-\theta}w^{-2}|\nabla w|^2+r\chi_2\int_\Omega u^{-r}v^{-\theta}.
\end{aligned}
\tag{3.8}
$$

Inserting (3.8) into (3.7) yields

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u^{-r}v^{-\theta}+r\lambda\int_\Omega u^{-r}v^{-\theta}\\
\le&-r(r+1)\int_\Omega u^{-r-2}v^{-\theta}|\nabla u|^2+[r\chi_1(r+1)-2r\theta]\int_\Omega u^{-r-1}v^{-\theta-1}\nabla u\cdot\nabla v\\
&+[r\chi_1\theta-\theta(\theta+1)]\int_\Omega u^{-r}v^{-\theta-2}|\nabla v|^2-r\chi_2(2r+1)\int_\Omega u^{-r-1}v^{-\theta}w^{-1}\nabla u\cdot\nabla w\\
&-2r\chi_2\theta\int_\Omega u^{-r}v^{-\theta-1}w^{-1}\nabla v\cdot\nabla w-r\chi_2\int_\Omega u^{-r}v^{-\theta}w^{-2}|\nabla w|^2\\
&+r\mu\int_\Omega u^{-r+1}v^{-\theta}+(\theta+r\chi_2)\int_\Omega u^{-r}v^{-\theta}-\theta\int_\Omega u^{-r+1}v^{-\theta-1}.
\end{aligned}
\tag{3.9}
$$

Using Young's inequality, we have

$$
\begin{aligned}
&[r\chi_1(r+1)-2r\theta]\int_\Omega u^{-r-1}v^{-\theta-1}\nabla u\cdot\nabla v\\
\le&\frac{r(r+1)}{2}\int_\Omega u^{-r-2}v^{-\theta}|\nabla u|^2+\frac{r[\chi_1(r+1)-2\theta]^2}{2(r+1)}\int_\Omega u^{-r}v^{-\theta-2}|\nabla v|^2
\end{aligned}
$$

$$\begin{aligned}&-r\chi_2(2r+1)\int_\Omega u^{-r-1}v^{-\theta}w^{-1}\nabla u\cdot\nabla w\\&\le\frac{r\chi_1(2r+1)^2}{2}\int_\Omega u^{-r-2}v^{-\theta}|\nabla u|^2+\frac{r\chi_2}{2}\int_\Omega u^{-r}v^{-\theta}w^{-2}|\nabla w|^2\end{aligned}\tag{3.10}$$

and

$$-2r\chi_2\theta\int_\Omega u^{-r}v^{-\theta-1}w^{-1}\nabla v\cdot\nabla w\le r\chi_1\theta\int_\Omega u^{-r}v^{-\theta-2}|\nabla v|^2+\frac{r\chi_2^2\theta}{\chi_1}\int_\Omega u^{-r}v^{-\theta}w^{-2}|\nabla w|^2.\tag{3.11}$$

Let $f(\theta):=\frac{r[\chi_1(r+1)-2\theta]^2}{2(r+1)}+[2r\chi_1\theta-\theta(\theta+1)],\theta>0$. It can be reformulated as

$$2(r+1)f(\theta)=2(r-1)\theta^2-2(r+1)\theta+r\chi_1^2(r+1)^2.$$

Since $\triangle_\theta=4(r+1)^2[1+2r\chi_1^2(1-r)]>0$ with $r\in(0,1)$, it follows that $f(\theta)<0$ holds if $\theta>\theta_-:=\frac{(r+1)(\sqrt{1+2r\chi_1^2(1-r)}-1)}{2(1-r)}$. Because $\sqrt{1+2r\chi_1^2(1-r)}<r\chi_1^2(1-r)+1$, we have $\theta_-<\chi_1^2$. Thus, choosing $\theta>\chi_1^2$ guarantees $f(\theta)<0$.

On the other hand, we need to get $\frac{r\chi_2^2\theta}{\chi_1}<\frac{r\chi_2}{2}$, which is equivalent to $\theta<\frac{\chi_1}{2\chi_2}$. The condition $\chi_1\chi_2<\frac{1}{2}$ implies $\chi_1^2<\frac{\chi_1}{2\chi_2}$. Hence, the interval $(\chi_1^2,\frac{\chi_1}{2\chi_2})$ is nonempty, and any θ chosen in this interval satisfies both $f(\theta)<0$ and $\theta<\frac{\chi_1}{2\chi_2}$.

Next, we need to ensure $\frac{r\chi_2(2r+1)^2}{2}<\frac{r(r+1)}{2}$, which is equivalent to $\varphi(r):=4\chi_2r^2+(4\chi_2-1)r+\chi_2-1<0$, Because of $\triangle_r=8\chi_2+1>0$, $r_\pm=\frac{1-4\chi_2\pm\sqrt{8\chi_2+1}}{8\chi_2}$. Thus $\varphi(r)<0$ if $r\in(r_-,r_+)$. Through a simple calculation, we can get $(r_-,r_+)\cap(0,1)=(0,1)$ if $\chi_2\in(0,\frac{2}{9})$. Thus, the combination of (3.9), (3.10), and (3.11) allows us to obtain that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^{-r}v^{-\theta}+(r\lambda-\theta-r\chi_2)\int_\Omega u^{-r}v^{-\theta}\\&\le r\mu\int_\Omega u^{-r+1}v^{-\theta}-\theta\int_\Omega u^{-r+1}v^{-\theta-1}.\end{aligned}\tag{3.12}$$

Exploiting Young's inequality together with (3.1) estimates

$$\begin{aligned}r\mu\int_\Omega u^{-r+1}v^{-\theta}&=r\mu\int_\Omega(u^{-r+1}v^{-\theta-1})^{\frac{\theta}{\theta+1}}u^{\frac{-r+1}{\theta+1}}\\&\le\theta\int_\Omega u^{-r+1}v^{-\theta-1}+\left[\frac{r\mu}{\theta}\right]^\theta r\mu\int_\Omega u^{-r+1}\\&\le\theta\int_\Omega u^{-r+1}v^{-\theta-1}+c_1,\end{aligned}\tag{3.13}$$

where $c_1:=\left[\frac{r\mu}{\theta}\right]^\theta r\mu[(1-r)m_u+r|\Omega|]>0$. By integrating (3.12) and (3.13), we infer that

$$\frac{d}{dt}\int_\Omega u^{-r}v^{-\theta}+c_2\int_\Omega u^{-r}v^{-\theta}\le c_1,$$

where $c_2:=r\lambda-\theta-r\chi_2>0$. Utilizing Gronwall's inequality leads to

$$\int_\Omega u^{-r}v^{-\theta}\le\max\left\{\int_\Omega u_0^{-r}v_0^{-\theta},\frac{c_1}{c_2}\right\},$$

for all $t\in(0,T_{\max})$. This yields (3.6) by taking $K_2:=\max\{\int_\Omega u_0^{-r}v_0^{-\theta},\frac{c_1}{c_2}\}$. □

We derive an upper bound estimate for v, which will be employed to establish L^β-norm for u.

Lemma 3.3. *Under the assumption $N \geq 1$ and Lemma 3.2, for all $t \in (0, T_{max})$, there exists a constant $\delta > 0$ such that $v(\cdot,t) \geq \delta$.*

Proof. Let $l \in (0, \frac{r}{\theta+1})$. Applying Hölder inequality leads to

$$\int_\Omega u^{-l} \leq (\int_\Omega u^{-r}v^{-\theta})^{\frac{l}{r}} (\int_\Omega v^{\frac{\theta l}{r-l}})^{\frac{r-l}{r}}. \tag{3.14}$$

By applying the heat semigroup estimates [21, Lemma 1.3], we derive by (3.1) that

$$\begin{aligned}\|v\|_{L^{\frac{\theta l}{r-l}}(\Omega)} \leq & \|v_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}(u-\bar{u})\|_{L^{\frac{\theta l}{r-l}}(\Omega)} \\ & + \int_0^t \|e^{(t-s)(\Delta-1)}\bar{u}\|_{L^{\frac{\theta l}{r-l}}(\Omega)} \\ \leq & \|v_0\|_{L^\infty(\Omega)} + c_1 \int_0^t (1+(t-s)^{-\frac{N}{2}(1-\frac{r-l}{\theta l})})e^{-(\lambda_1+1)(t-s)}\|u-\bar{u}\|_{L^1(\Omega)} \\ & + \frac{m_u}{|\Omega|}\int_0^t e^{-(t-s)} \\ \leq & c_2,\end{aligned}$$

where constants $c_1, c_2 > 0$, $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u dx$, and $-\frac{N}{2}(1-\frac{r-l}{\theta l}) > -1$. By using (3.6) and (3.14), we have $\int_\Omega u^{-l} \leq K_2^{\frac{l}{r}} c_2^{\frac{r-l}{r}}$. By using Hölder inequality and the above formula, we conclude

$$\int_\Omega u \geq |\Omega|^{\frac{l+1}{l}} \left(\int_\Omega u^{-l}\right)^{-\frac{1}{l}} \geq |\Omega|^{\frac{l+1}{l}} K_2^{-\frac{1}{r}} c_2^{-\frac{r-l}{rl}} := m_1. \tag{3.15}$$

Combining Lemma 2.2 with (3.15), we estimate

$$\begin{aligned}v(\cdot,t) = & e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s) \\ \geq & \int_0^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{(diam\Omega)^2}{4(t-s)}-(t-s)} (\int_\Omega u(\cdot,s)) \\ \geq & m_1 \int_0^{t_0} \frac{1}{(4\pi s)^{\frac{N}{2}}} e^{-\frac{(diam\Omega)^2}{4s}-s} \\ = & : \delta_1,\end{aligned}$$

for all $t \in (t_0, T_{\max})$. Relying on the positivity of v and utilizing the comparison principle, we conclude $v(\cdot,t) \geq \inf_{x\in\bar{\Omega}} v_0 e^{-t_0} =: \delta_2$, for all $t \in (0, t_0]$, which directly yields (3.13) by taking $\delta := \min\{\delta_1, \delta_2\}$. □

To handle the integral term $\int_\Omega u^\beta |\nabla v|^2$ and $\int_\Omega u^2 |\nabla v|^{2\beta-2}$, we use the second equation from system (1.1) to obtain a key result. However, due to the close resemblance between the proof and the one in [27, Lemma 4.3], we omit it here to preclude redundancy.

Lemma 3.4. *There exists a constant $K_3 > 0$, for any $\alpha > 1$ and $t \in (0, T_{\max})$, such that*

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2\alpha} + 2\alpha \int_\Omega |\nabla v|^{2\alpha} + \frac{1}{\alpha}\int_\Omega |\nabla|\nabla v|^\alpha|^2 \leq (\frac{4\alpha}{\alpha-1} + \alpha n)\int_\Omega u^2 |\nabla v|^{2\alpha-2} + K_3. \tag{3.16}$$

By applying the aforementioned lemma, we derive the estimate for $\|u\|_{L^\beta(\Omega)} + \|\nabla v\|_{L^{2\beta}(\Omega)}$.

Lemma 3.5. *Let $N \geq 3$ and Lemma 3.2 hold. Then, there exists a constant $K_4 > 0$, for any $\beta > \frac{N+2}{2}$ and $t \in (0, T_{\max})$, such that*

$$\int_\Omega u^\beta + \int_\Omega |\nabla v|^{2\beta} \leq K_4. \tag{3.17}$$

Proof. Testing the first equation in (1.1) against $u^{\beta-1}$ and integrating by parts yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^\beta = &-\beta(\beta-1)\int_\Omega u^{\beta-2}|\nabla u|^2 + \chi_1\beta(\beta-1)\int_\Omega u^{\beta-1}v^{-1}\nabla u\cdot\nabla v \\ &+\beta\chi_2\int_\Omega u^{\beta-1}w^{-1}\nabla u\cdot\nabla w - \beta\chi_2\int_\Omega u^\beta w^{-2}|\nabla w|^2 \\ &+\beta(\chi_2+\lambda)\int_\Omega u^\beta - \beta(\chi_2+\mu)\int_\Omega u^{\beta+1}.\end{aligned} \tag{3.18}$$

Utilizing Young's inequality and Lemma 3.3, we estimate

$$\int_\Omega u^{\beta-1}v^{-1}\nabla u\cdot\nabla v \leq \frac{1}{4\chi_1}\int_\Omega u^{\beta-2}|\nabla u|^2 + \frac{\chi_1}{\delta^2}\int_\Omega u^\beta|\nabla v|^2 \tag{3.19}$$

and

$$\int_\Omega u^{\beta-1}w^{-1}\nabla u\cdot\nabla w \leq \frac{1}{4}\int_\Omega u^{\beta-2}|\nabla u|^2 + \int_\Omega u^\beta w^{-2}|\nabla w|^2, \tag{3.20}$$

where $\frac{\chi_2}{4} < \frac{\beta-1}{4}$ since $\chi_2 < \frac{4}{9}$. Substituting (3.19) and (3.20) into (3.18) and incorporating (3.16) yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^\beta + \frac{d}{dt}\int_\Omega |\nabla v|^{2\beta} + \frac{\beta(\beta-1)}{2}\int_\Omega u^{\beta-2}|\nabla u|^2 + 2\beta\int_\Omega |\nabla v|^{2\beta} + \frac{1}{\beta}\int_\Omega |\nabla|\nabla v|^\beta|^2 \\ \leq &\frac{\chi_1^2\beta(\beta-1)}{\delta^2}\int_\Omega u^\beta|\nabla v|^2 + (\frac{4\beta}{\beta-1}+\beta n)\int_\Omega u^2|\nabla v|^{2\beta-2} \\ &+\beta(\chi_2+\lambda)\int_\Omega u^\beta - \beta(\chi_2+\mu)\int_\Omega u^{\beta+1} + K_3.\end{aligned} \tag{3.21}$$

Subsequently, we focus on the first two terms located on the right-hand side of the aforementioned formula. For $N \geq 3$, by selecting β_1 sufficiently large such that $\beta_1 \geq \frac{N-2}{2}$, Hölder inequality yields

$$\int_\Omega u^{\beta_1}|\nabla v|^2 \leq (\int_\Omega u^{\frac{\beta_1 N}{N-2}})^{\frac{N-2}{N}}(\int_\Omega |\nabla v|^N)^{\frac{2}{N}}. \tag{3.22}$$

By making use of Lemma 2.3, along with (3.1) and (3.3), we derive

$$\begin{aligned}(\int_\Omega u^{\frac{\beta_1 N}{N-2}})^{\frac{N-2}{N}} &= \|u^{\frac{\beta_1}{2}}\|^{\frac{1}{2}}_{L^{\frac{2N}{N-2}}(\Omega)} \\ &\leq c_1\|\nabla u^{\frac{\beta_1}{2}}\|^{\frac{a_1}{2}}_{L^2(\Omega)}\|u^{\frac{\beta_1}{2}}\|^{\frac{1}{2}(1-a_1)}_{L^{\frac{2}{\beta_1}}(\Omega)} + c_1\|u^{\frac{\beta_1}{2}}\|^{\frac{1}{2}}_{L^{\frac{2}{\beta_1}}(\Omega)} \\ &\leq c_1 m_u^{\frac{\beta_1}{4}(1-a_1)}\|\nabla u^{\frac{\beta_1}{2}}\|^{\frac{a_1}{2}}_{L^2(\Omega)} + c_1 m_u^{\frac{\beta_1}{4}}\end{aligned} \tag{3.23}$$

and

$$(\int_\Omega |\nabla v|^N)^{\frac{2}{N}} = \||\nabla v|^{\beta_1}\|^{\frac{2}{\beta_1}}_{L^{\frac{N}{\beta_1}}(\Omega)} \le c_2 \|\nabla|\nabla v|^{\beta_1}\|^{\frac{2a_2}{\beta_1}}_{L^2(\Omega)} \||\nabla v|^{\beta_1}\|^{\frac{2}{\beta_1}(1-a_2)}_{L^{\frac{2}{\beta_1}}(\Omega)} + c_2\||\nabla v|^{\beta_1}\|^{\frac{2}{\beta_1}}_{L^{\frac{2}{\beta_1}}(\Omega)}$$
$$\le c_2 K_1^{1-a_2} \|\nabla|\nabla v|^{\beta_1}\|^{\frac{a_1}{2}}_{L^2(\Omega)} + c_2 K_1, \tag{3.24}$$

where $a_1 = \frac{\frac{\beta_1 N}{2}-1}{1-\frac{N}{2}-\frac{\beta_1 N}{2}} \in (0,1)$, $a_2 = \frac{\frac{\beta_1 N}{2}-\beta_1}{1-\frac{N}{2}-\frac{\beta_1 N}{2}} \in (0,1)$ and c_1, c_2 are positive constants. By choosing $\beta_1 \ge \max\{\frac{N-2}{2}, \frac{4}{3}\}$ and combining equations (3.22), (3.23), and (3.24), along with repeatedly invoking Young's inequality, we obtain

$$\frac{\chi_1^2 \beta_1(\beta_1-1)}{\delta^2}\int_\Omega u^{\beta_1}|\nabla v|^2 \le \frac{2(\beta_1-1)}{\beta_1}\int_\Omega |\nabla u^{\frac{\beta_1}{2}}|^2 + \frac{1}{2\beta_1}\int_\Omega |\nabla|\nabla v|^{\beta_1}|^2 + c_3, \tag{3.25}$$

where $c_3 = (4\beta_1)^{\frac{4a_2}{\beta_1(4-a_1)-4a_2}} (\frac{\beta_1}{\beta_1-1})^{\frac{\beta_1 a_1}{\beta_1(4-a_1)-4a_2}} \left[\frac{\chi_1^2\beta_1(\beta_1-1)}{\delta^2} c_1 m_u^{\frac{\beta_1(1-a_1)}{4}}\right]^{\frac{4\beta_1}{\beta_1(4-a_1)-4a_2}} + (\frac{\beta_1}{\beta_1-1})^{\frac{a_1}{4-a_1}}$ $\left[\frac{\chi_1^2\beta_1(\beta_1-1)}{\delta^2} c_1 c_2 m_u^{\frac{\beta_1(1-a_1)}{4}} K_1\right]^{\frac{4}{4-a_1}} + (4\beta_1)^{\frac{a_2}{\beta_1-a_2}} \left[\frac{\chi_1^2\beta_1(\beta_1-1)}{\delta^2} c_1 c_2 m_u^{\frac{\beta_1}{4}} K_1^{1-a_2}\right]^{\frac{\beta_1}{\beta_1-a_2}} + c_1 c_2 m_u^{\frac{\beta_1}{4}} K_1$. By choosing $\beta_2 \ge \frac{N+2}{2} (N \ge 3)$, we utilize Young's inequality to obtain

$$\int_\Omega u^2 |\nabla v|^{2\beta_2-2} \le \int_\Omega u^2 + \int_\Omega |\nabla v|^{\frac{2(\beta_2-1)\beta_2}{\beta_2-2}}. \tag{3.26}$$

In a similar manner, due to $\frac{2}{\beta_2} \le \frac{2(\beta_2-1)}{\beta_2-2} \le \frac{2N}{N-2}$, an application of Lemma 2.3 leads to

$$\begin{aligned}(\int_\Omega |\nabla v|^{\frac{2(\beta_2-1)\beta_2}{\beta_2-2}}) &= \||\nabla v|^{\beta_2}\|^{\frac{\beta_2-2}{2(\beta_2-1)}}_{L^{\frac{2(\beta_2-1)}{\beta_2-2}}(\Omega)} \\ &\le c_4 \|\nabla|\nabla v|^{\beta_2}\|^{\frac{a_3(\beta_2-2)}{2(\beta_2-1)}}_{L^2(\Omega)} \||\nabla v|^{\beta_2}\|^{\frac{(1-a_3)(\beta_2-2)}{2(\beta_2-1)}}_{L^{\frac{2}{\beta_2}}(\Omega)} + c_4 \||\nabla v|^{\beta_2}\|^{\frac{\beta_2-2}{2(\beta_2-1)}}_{L^{\frac{2}{\beta_2}}(\Omega)} \\ &\le c_4 K_1^{\frac{\beta_2(1-a_3)(\beta_2-2)}{4(\beta_2-1)}} \|\nabla|\nabla v|^{\beta_2}\|^{\frac{a_3(\beta_2-2)}{2(\beta_2-1)}}_{L^2(\Omega)} + c_4 K_1^{\frac{\beta_2(\beta_2-2)}{4(\beta_2-1)}},\end{aligned} \tag{3.27}$$

where $a_3 = \frac{\frac{\beta_2 N}{2} - \frac{N(\beta_2-2)}{2(\beta_2-1)}}{1-\frac{N}{2}+\frac{\beta_2 N}{2}} \in (0,1)$, $c_4 > 0$ is a constant and K_1 is given in (3.3). By virtue of Young's inequality, (3.26) and (3.27), it holds that

$$(\frac{4\beta_2}{\beta_2-1} + \beta_2 N)\int_\Omega u^2|\nabla v|^{2\beta_2-2} \le (\frac{4\beta_2}{\beta_2-1} + \beta_2 N)\int_\Omega u^{\beta_2} + \frac{1}{2\beta_2}\int_\Omega |\nabla|\nabla v|^{\beta_2}|^2 + c_5, \tag{3.28}$$

where

$$c_5 = (2\beta_2)^{\frac{a_3(\beta_2-2)}{4(\beta_2-1)-a_3(\beta_2-2)}} [(\frac{4\beta_2}{\beta_2-1} + \beta_2 N) c_4 K_1^{\frac{\beta_2(1-a_3)(\beta_2-2)}{4(\beta_2-1)}}]^{\frac{4(\beta_2-1)}{4(\beta_2-1)-a_3(\beta_2-2)}} + (\frac{4\beta_2}{\beta_2-1} + \beta_2 N) c_4 K_1^{\frac{\beta_2(\beta_2-2)}{4(\beta_2-1)}}.$$

Then, by selecting $\beta \geq \max\{\beta_1, \beta_2\}$ and combining equations (3.21), (3.25), and(3.28), we arrive at

$$\begin{aligned} &\frac{d}{dt}\left\{\int_\Omega u^\beta + \int_\Omega |\nabla v|^{2\beta}\right\} + 2\beta\left\{\int_\Omega u^\beta + \int_\Omega |\nabla v|^{2\beta}\right\} \\ \leq &\beta(\chi_2 + \lambda + \frac{4}{\beta - 1} + N + 2)\int_\Omega u^\beta - \beta(\chi_2 + \mu)\int_\Omega u^{\beta+1} + K_3 + c_3 + c_5. \end{aligned} \tag{3.29}$$

Utilizing Young's inequality shows

$$\beta(\chi_2 + \lambda + \frac{4}{\beta - 1} + N + 2)\int_\Omega u^\beta \leq \beta(\chi_2 + \mu)\int_\Omega u^{\beta+1} + c_6, \tag{3.30}$$

where $c_6 = \beta(\chi_2 + \lambda + \frac{4}{\beta-1} + N + 2)^\beta(\chi_2 + \mu)^{-\beta+1}$. By inserting (3.30) into (3.29), we derive

$$\frac{d}{dt}\left\{\int_\Omega u^\beta + \int_\Omega |\nabla v|^{2\beta}\right\} + 2\beta\left\{\int_\Omega u^\beta + \int_\Omega |\nabla v|^{2\beta}\right\} \leq c_7,$$

where $c_7 := K_3 + c_3 + c_5 + c_6 > 0$ is a constant. Finally, using Gronwall's inequality yields (3.17) with $K_4 = \max\left\{\int_\Omega u_0^\beta + \int_\Omega |\nabla v_0|^{2\beta}, \frac{c_7}{2\beta}\right\}$. □

Proof of Theorem 1.1. Combining Lemma 3.5 with standard parabolic regularity theory [21] and elliptic regularity, it follows that there exist constants $K_5, K_6 > 0$, for all $t \in (0, T_{\max})$, such that

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq K_5 \tag{3.31}$$

and $\|w(\cdot,t)\|_{W^{2,\beta}(\Omega)} \leq K_6$. By the Sobolev embedding theorem, $W^{2,\beta}(\Omega)$ is embedded into $C^1(\overline{\Omega})$ with $\beta > N$, which implies that there exists a constant $K_7 > 0$, for all $t \in (0, T_{\max})$, such that $\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq K_7$. Substituting (3.19) and (3.20) into (3.18) and incorporating (3.31), we see that a constant $c_1 > 0$ can be found such that

$$\frac{d}{dt}\int_\Omega u^\beta + \frac{2(\beta-1)}{\beta}\int_\Omega |\nabla u^{\frac{\beta}{2}}|^2 + \int_\Omega u^\beta \leq c_1\beta(\beta-1)\int_\Omega u^\beta. \tag{3.32}$$

We use Lemma 2.3, Young's inequality, and (3.1) to obtain

$$\begin{aligned} c_1\beta(\beta-1)\int_\Omega u^\beta &\leq c_2\beta(\beta-1)\left(\|\nabla u^{\frac{\beta}{2}}\|_{L^2(\Omega)}^{\frac{N}{2+N}}\|u^{\frac{\beta}{2}}\|_{L^1(\Omega)}^{\frac{2}{2+N}} + \|u^{\frac{\beta}{2}}\|_{L^1(\Omega)}\right)^2 \\ &\leq c_2\beta(\beta-1)\left(\frac{1}{\beta^2 c_2}\|\nabla u^{\frac{\beta}{2}}\|_{L^2(\Omega)} + c_3\|u^{\frac{\beta}{2}}\|_{L^1(\Omega)}\right)^2 \\ &\leq \frac{2(\beta-1)}{\beta}\|\nabla u^{\frac{\beta}{2}}\|_{L^2(\Omega)}^2 + c_4\beta(\beta-1)\|u^{\frac{\beta}{2}}\|_{L^1(\Omega)}^2, \end{aligned} \tag{3.33}$$

where $c_2 > 0$ is a constant, $c_3 := (\frac{\beta^2 c_2 N}{N+2})^{\frac{N}{2}}\frac{2}{2+N} + 1$, and $c_4 := 2c_2c_3$. Substituting (3.33) into (3.32) yields

$$\frac{d}{dt}\int_\Omega u^\beta + \int_\Omega u^\beta \leq c_4\beta(\beta-1)(\int_\Omega u^{\frac{\beta}{2}})^2.$$

Applying Moser iteration technique [20] in conjunction with the above inequality yields that there exists a constant $K_8 > 0$, for all $t \in (0, T_{\max})$, such that $\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq K_8$. Finally, an application of Lemma 2.1 finishes the proof. □

4. THE CASE $\tau = 0$

In this section, we establish the global boundedness of solutions to (1.1) when $\tau = 0$, that is, the proof of Theorem 1.2. Primarily, we present the subsequent L^ξ estimates.

Lemma 4.1. *Let $N \geq 1$, the parameter $\lambda, \mu, \chi_2 > 0$ and $\chi_1 \in (0, \frac{1}{2})$. Then, for any $\xi \in (1, \frac{1}{2\chi_1})$ and $t \in (0, T_{\max})$, there exists a constant $C_1 > 0$ such that*

$$\|u(\cdot,t)\|_{L^\xi(\Omega)} \leq C_1. \tag{4.1}$$

Proof. Testing first equation of system (1.1) by $u^{\xi-1}$and performing integration by parts leads us to

$$\begin{aligned} \frac{1}{\xi}\frac{d}{dt}\int_\Omega u^\xi = &-(\xi-1)\int_\Omega u^{\xi-2}|\nabla u|^2 + \chi_1(\xi-1)\int_\Omega u^{\xi-1}v^{-1}\nabla u\cdot\nabla v \\ &-\chi_2(\xi-1)\int_\Omega u^{\xi-1}w^{-1}\nabla u\cdot\nabla w + \lambda\int_\Omega u^\xi - \mu\int_\Omega u^{\xi+1}. \end{aligned} \tag{4.2}$$

Exploiting Young's inequality, we have

$$\chi_1(\xi-1)\int_\Omega u^{\xi-1}v^{-1}\nabla u\cdot\nabla v \leq \frac{\xi-1}{4}\int_\Omega u^{\xi-2}|\nabla u|^2 + \chi_1^2(\xi-1)\int_\Omega u^\xi v^{-2}|\nabla v|^2. \tag{4.3}$$

Observe that

$$\int_\Omega u^{\xi+1}v^{-1} - \int_\Omega u^\xi = \int_\Omega \nabla(u^\xi v^{-1})\nabla v = \xi\int_\Omega u^{\xi-1}v^{-1}\nabla u\cdot\nabla v - \int_\Omega u^\xi v^{-2}|\nabla v|^2.$$

Applying Young's inequality yields

$$\begin{aligned} \int_\Omega u^\xi v^{-2}|\nabla v|^2 &\leq \xi\int_\Omega u^{\xi-1}v^{-1}\nabla u\cdot\nabla v + \int_\Omega u^\xi \\ &\leq \frac{\xi^2}{2}\int_\Omega u^{\xi-2}|\nabla u|^2 + \frac{1}{2}\int_\Omega u^\xi v^{-2}|\nabla v|^2 + \int_\Omega u^\xi, \end{aligned}$$

which means

$$\int_\Omega u^\xi v^{-2}|\nabla v|^2 \leq \xi^2\int_\Omega u^{\xi-2}|\nabla u|^2 + 2\int_\Omega u^\xi. \tag{4.4}$$

For $b \in (0,1)$, utilizing Young's inequality allows us to obtain

$$\begin{aligned} &-(1-b)\chi_2(\xi-1)\int_\Omega u^{\xi-1}w^{-1}\nabla u\cdot\nabla w \\ &\leq \frac{\xi-1}{4}\int_\Omega u^{\xi-2}|\nabla u|^2 + \chi_2^2(1-b)^2(\xi-1)\int_\Omega u^\xi w^{-2}|\nabla w|^2, \end{aligned} \tag{4.5}$$

and simple calculations yield

$$\begin{aligned}
& -b\chi_2(\xi-1)\int_\Omega u^{\xi-1}w^{-1}\nabla u\cdot\nabla w \\
=& \frac{b\chi_2(\xi-1)}{\xi}\int_\Omega u^\xi\nabla(w^{-1}\nabla w) \\
=& -\frac{b\chi_2(\xi-1)}{\xi}\int_\Omega u^\xi w^{-2}|\nabla w|^2+\frac{b\chi_2(\xi-1)}{\xi}\int_\Omega u^\xi w^{-1}\Delta w \\
=& -\frac{b\chi_2(\xi-1)}{\xi}\int_\Omega u^\xi w^{-2}|\nabla w|^2+\frac{b\chi_2(\xi-1)}{\xi}\int_\Omega u^\xi-\frac{b\chi_2(\xi-1)}{\xi}\int_\Omega u^{\xi+1} \\
\le& -\frac{b\chi_2(\xi-1)}{\xi}\int_\Omega u^\xi w^{-2}|\nabla w|^2+\frac{b\chi_2(\xi-1)}{\xi}\int_\Omega u^\xi.
\end{aligned} \tag{4.6}$$

Let $b:=\frac{2\xi\chi_2+1-\sqrt{4\xi\chi_2+1}}{2\xi\chi_2}$. It follow that $\frac{b\chi_2(\xi-1)}{\xi}=\chi_2^2(1-b)^2(\xi-1)$. Adding (4.5) and (4.6) yields

$$-\chi_2(\xi-1)\int_\Omega u^{\xi-1}w^{-1}\nabla u\cdot\nabla w\le\frac{\xi-1}{4}\int_\Omega u^{\xi-2}|\nabla u|^2+\frac{b\chi_2(\xi-1)}{\xi}\int_\Omega u^\xi. \tag{4.7}$$

Since $\xi\in(1,\frac{1}{2\chi_1})$ with $\chi_1\in(0,\frac{1}{2})$, combining this with (4.2), (4.3), (4.4), and (4.7), we have

$$\frac{d}{dt}\int_\Omega u^\xi+\int_\Omega u^\xi\le c_1, \tag{4.8}$$

where $c_1:=\{\mu^{-\xi}[2\chi_1^2\xi(\xi-1)+b\chi_2(\xi-1)+\lambda\xi+1]^{\xi+1}(\xi+1)^{-\xi-1}\}|\Omega|>0$. Employing Gronwall's inequality shows (4.1) with $C_1:=\max\{c_1,\int_\Omega u_0^\xi\}$. This completes proving. □

Proof of Theorem 1.2. Since $\frac{4}{\xi^2}\int_\Omega|\nabla u^{\frac{\xi}{2}}|^2=\int_\Omega u^{\xi-2}|\nabla u|^2$, for all $\xi\in(1,\frac{1}{2\chi_1})$ and $t\in(0,T_{\max})$, the combination of (4.2), (4.3), (4.4), and (4.7) allows us to obtain

$$\frac{d}{dt}\int_\Omega u^\xi+\int_\Omega u^\xi+\frac{\xi-1}{\xi}\int_\Omega|\nabla u^{\frac{\xi}{2}}|^2\le c_1\xi\int_\Omega u^\xi, \tag{4.9}$$

where $c_1:=(\frac{b\chi_2(\xi-1)}{\xi}+2\chi_1^2(\xi-1)+\lambda)\xi+1$. Similar to formula (3.33), Gagliardo-Nirenberg inequality and Young's inequality can be employed to derive

$$c_1\xi\int_\Omega u^\xi\le\frac{\xi-1}{\xi}\|\nabla u^{\frac{\xi}{2}}\|^2_{L^2(\Omega)}+c_3\xi(\xi-1)\|u^{\frac{\xi}{2}}\|^2_{L^1(\Omega)}, \tag{4.10}$$

where $c_3:=[\frac{2c_2N\xi^2}{(\xi-1)(2+N)}]^{\frac{N}{2}}\frac{4c_2}{2+N}+2c_2>0$ with $c_2>0$ being a constant. Plugging (4.10) into (4.9) gives

$$\frac{d}{dt}\int_\Omega u^\xi+\int_\Omega u^\xi\le c_2\xi(\int_\Omega u^{\frac{\xi}{2}})^2.$$

Using the Moser iteration technique [20] and Lemmas 4.1 and 2.1, we complete the proof immediately. □

5. Conclusion

In conclusion, we proved the existence of globally bounded classical solutions to attraction-repulsion chemotaxis system (1.1) with singular sensitivities under homogeneous Neumann boundary conditions. By constructing appropriate weighted integrals and a priori estimates, and combining analytical methods including heat semigroup estimates, Moser iteration, and parabolic-elliptic regularity theory, we proved the uniform global boundedness of classical solutions for both cases $\tau = 1$ and $\tau = 0$ under explicit parameter conditions. Compared with the existing literature, our results not only extend the existing boundedness theory to the setting with both singular attraction and repulsion terms, but also relax the constraints on the logistic source coefficients in certain parameter regimes.

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