

FROM THE VICOUS TWO-FLUID-MHD-POISSON SYSTEM TO ONE FLUID MHD

YOUNG-SAM KWON

Department of Mathematics, Dong-A University, Busan 604-714, Republic of Korea

Abstract. In this paper, we consider the compressible two fluid models of magnetohydrodynamic flows giving rise to a variety of mathematical problems in many areas. We study the convergence of a dissipative turbulent solution from the compressible two fluid models of magnetohydrodynamic flows to one fluid MHD model in the three dimensional periodic domain $\mathbb{T}^3$ with the well-prepared initial data.

Keywords. Dissipative turbulent solution; Poisson equation; Relative entropy; Two fluid MHD.

CONTENTS

1. Introduction	1033
2. The Main Result	1036
3. Proof of Theorem 2.1	1037
3.1. Uniform bounds	1037
3.2. Relative entropy	1038
3.3. Computation of the relative entropy	1042
4. Existence of Dissipative Turbulent Solutions	1046
References	1050

1. INTRODUCTION

The two-fluid plasma equations for a single ion species, with full transport terms, including magnetic field dependent ion and electron viscous stresses have been described in a variety of practical applications such as in plasma confinement, liquid-metal cooling of nuclear reactors, and electromagnetic casting. The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. The set of equations that describes MHD is a combination of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. These differential equations must be solved simultaneously, either analytically or numerically.

E-mail address: ykwon@dau.ac.kr

Received 13 June 2025; Accepted 13 August 2026; Published online 1 November 2026.

Here we consider the viscous compressible two fluid magnetohydrodynamic flows with “turbulent” component $\mathfrak{R}$ on the 3 dimensional torus $\Omega := \mathbb{T}^3$.

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \quad (1.1)$$

$$\partial_t n + \operatorname{div}(n \mathbf{u}) = 0, \quad (1.2)$$

$$\partial_t((\rho + n)\mathbf{u}) + \operatorname{div}((\rho + n)\mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho, n) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla(\operatorname{div} \mathbf{u}) \quad (1.3)$$

$$+ (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 - \operatorname{div} \mathfrak{R} - (\rho - n) \nabla \Phi, \quad (1.4)$$

$$\partial_t \mathbf{B} + (\operatorname{div} \mathbf{u}) \mathbf{B} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} = \nu \Delta \mathbf{B}, \quad (1.5)$$

$$- \varepsilon^2 \Delta \Phi = \rho - n,$$

where $\mathbf{u}$ is the vector field, ρ, n are the densities, $\mathbf{B}$ is the magnetic field, and $\gamma > 1$, $\beta > 1$, and the pressure $P(\rho, n)$ is defined as

$$P(\rho, n) = \rho^\gamma + n^\alpha.$$

The “turbulent” component $\mathfrak{R}$ is called *Reynolds stress* and it is a positively semi-definite tensor.

Formally, for a solution $(\rho_\varepsilon, n_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{B}_\varepsilon, \mathfrak{R}_\varepsilon, \Phi_\varepsilon)$ of system (1.1)–(1.5), if we further assume that the initial density $(\rho_{0,\varepsilon}, n_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \mathbf{B}_{0,\varepsilon})$ converge to $\rho_0, n_0, \mathbf{u}_0, \mathbf{B}_0$, and viscosities $\mu_\varepsilon, \lambda_\varepsilon, \nu_\varepsilon \rightarrow \mu, \lambda, \nu$, with $\mu > 0, \lambda > 0, \nu > 0$, then we can also expect that $(\rho_\varepsilon, n_\varepsilon, \mathbf{B}_\varepsilon, \mathfrak{R}_\varepsilon) \rightarrow (\rho, n, \mathbf{B}, 0)$ as $\varepsilon \rightarrow 0$. Moreover, system (1.1)–(1.5) converges to the classical compressible MHD on $(0, T^*]$:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \quad (1.6)$$

$$2 \left(\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) \right) + \nabla P(\rho) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla(\operatorname{div} \mathbf{u}) + (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2, \quad (1.7)$$

$$\partial_t \mathbf{B} + (\operatorname{div} \mathbf{u}) \mathbf{B} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} = \nu \Delta \mathbf{B}, \quad (1.8)$$

with $P(\rho) = \rho^\gamma + \rho^\alpha$.

We will explain the mathematical background of this model, namely the existence of a dissipative turbulent solution to system (1.1)–(1.5). The existence of dissipative turbulent solution to system (1.1)–(1.5) was first studied by Abbatiello, Feireisl, and Novotny [1] for the isentropic case. Jin, Kwon, and Novotny [12] derived a dissipative turbulent solution to the compressible two fluid models. The proof of the existence of a dissipative turbulent solution to system (1.1)–(1.5) is similar as the proof of [1].

The mathematical history of the quasi-neutral limit problem is as follows. For the results of smooth solutions, it was studied for the one-dimensional case in [3] and for the multidimensional case in [20]. In the same spirit of those results, Li, Peng, and Wang [18] studied the quasi-neutral limit problem from two-fluid Euler–Poisson equations to one-fluid Euler equations in the framework of locally smooth solutions. For the ionic Euler–Poisson system, the quasi-neutral limit was studied, for example, in [3, 10, 11, 19]. We also obtain the corresponding model for electrons and a few results on the quasi-neutral limit to the compressible Navier–Stokes equation are available [2, 6, 13, 14, 17]. Ju, Li, and Li [13] studied the quasi-neutral limit for local strong solutions to the Navier–Stokes–Fourier–Poisson system on $\mathbb{T}^3$. Chen, Donatelli, and Marcati [2] studied the quasi-neutral limit of a hydrodynamic model for charge-carrier transport in the framework of weak solutions. In [14], Ju and Li studied the combined quasi-neutral and zero-electron-mass limit of the Navier–Stokes–Fourier–Poisson system in the torus $\mathbb{T}^3$ and showed the limit is the incompressible Navier–Stokes equations. In [6], Donatelli

and Marcati gave some descriptions on the quasineutral limit for the full Navier-Stokes-Poisson system in $\mathbb{R}^3$. Very recently, Li, Ju, and Xu [17] improved the result in [14] to allow the temperature have a large variation. Also, there are many results on quasi-neutral limit to the electric Euler-Poisson and Navier-Stokes-Poisson system; see, e.g., [5, 15, 16].

In this paper, we study the convergence of dissipative turbulent solutions of a compressible two fluid model with the magnetic field to the strong solution of one fluid MHD, which is a first result for a weak solution of dissipative turbulent solutions of compressible two fluid model and it is related to the quasi-neutral limit of the above mentioned results.

The outline of this paper is as follows. In Section 2, we present the result of dissipative turbulent solutions for the compressible two-fluid model (1.1)–(1.5) and state our main results. Section 3 is devoted to proving the main Theorem 2.1. Finally, in Section 4, we give the proof existence of dissipative turbulent solution to (1.1)–(1.5).

Motivated by [1, Section 2], we introduce the dissipative turbulent solutions to problem (1.1)–(1.5) as follows.

Definition 1.1. We say that $(\rho, n, \mathbf{u}, \mathbf{B}, \mathfrak{R}, \Phi)$, belonging to the class

$$\begin{aligned} \rho &\in C_{\text{weak}}([0, T]; L^\gamma(\Omega)), n \in C_{\text{weak}}([0, T]; L^\alpha(\Omega)), \\ P(\rho, n) &\in L^1((0, T) \times \Omega), \gamma, \alpha > 1, \rho \geq 0, n \geq 0, \mathbf{u} \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)), \\ (\rho + n)\mathbf{u} &\in C_{\text{weak}}([0, T]; L^{\min\{\frac{2\gamma}{\gamma+1}, \frac{2\alpha}{\alpha+1}\}}(\Omega; \mathbb{R}^3)), \\ \mathbf{B} &\in C_{\text{weak}}([0, T]; L^2(\Omega; \mathbb{R}^3)) \cap L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)), \end{aligned} \quad (1.9)$$

is a dissipative turbulent solution to problem (1.1)–(1.5) iff

(1) The integral formulation of the continuity equations

$$\left[\int_{\Omega} r \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \left[r \partial_t \varphi + r \mathbf{u} \cdot \nabla \varphi \right] \, dx dt \quad (1.10)$$

holds for any $0 \leq \tau \leq T$, and any test function $\varphi \in C_c^1([0, T] \times \Omega)$,

$$r(0, \cdot) = r_0, \quad (1.11)$$

where r stands for ρ and n .

(2) There exists a tensor measure

$$\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\Omega; \mathbb{R}^{3 \times 3}_{\text{sym}})),$$

such that the integral identity

$$\begin{aligned} \left[\int_{\Omega} (\rho + n) \mathbf{u} \cdot \boldsymbol{\varphi} \, dx \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_{\Omega} \left[(\rho + n) \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + (\rho + n) \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \right. \\ &\quad \left. + P(\rho, n) \operatorname{div} \boldsymbol{\varphi} - (\mu \nabla \mathbf{u} + (\mu + \lambda) \operatorname{div} \mathbf{u} \mathbb{I}) : \nabla \boldsymbol{\varphi} \right. \\ &\quad \left. + [(\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2] \cdot \boldsymbol{\varphi} - (\rho - n) \nabla \Phi \cdot \boldsymbol{\varphi} \right] \, dx dt + \int_0^\tau \int_{\Omega} \nabla \boldsymbol{\varphi} : d\mathfrak{R}(t) \, dt \end{aligned} \quad (1.12)$$

holds for any $0 \leq \tau \leq T$ and any test function $\boldsymbol{\varphi} \in C^1([0, T] \times \Omega; \mathbb{R}^3)$,

$$(\rho + n) \mathbf{u}(0, \cdot) = \mathbf{m}_0 := (\rho_0 + n_0) \mathbf{u}_0. \quad (1.13)$$

Here we assume that all quantities appearing in (1.12) are at least integrable in $(0, T) \times \Omega$.

(3) There exists an energy defect measure

$$\mathfrak{E} \in L^\infty(0, T; \mathcal{M}^+(\Omega))$$

such that

$$\begin{aligned} & \left[\int_{\Omega} \left(\frac{1}{2}(\rho + n)|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{B}|^2 + \frac{1}{\gamma-1}\rho^\gamma + \frac{1}{\alpha-1}n^\alpha + \frac{\varepsilon^2}{2}|\nabla\Phi|^2 \right) dx \right]_{t=0}^{t=\tau} \\ & + \int_0^\tau \int_{\Omega} (\mu \nabla \mathbf{u} + (\mu + \lambda) \operatorname{div} \mathbf{u} \mathbb{I}) : \nabla \mathbf{u} + \nu |\nabla \mathbf{B}|^2 dx dt + \int_{\Omega} d\mathfrak{E}(\tau) \leq 0. \end{aligned} \quad (1.14)$$

for a.e. $0 \leq \tau \leq T$.

(4) Compatibility condition between the energy defect $\mathfrak{E}$ and the Reynolds defect $\mathfrak{R}$ are verified,

$$\underline{d}\mathfrak{E} \leq \operatorname{Tr}[\mathfrak{R}] \leq \bar{d}\mathfrak{E}, \text{ for certain constants } 0 < \underline{d} \leq \bar{d}. \quad (1.15)$$

(5) The Maxwell equation verifies

$$\begin{aligned} & \left[\int_{\Omega} \mathbf{B} \cdot \boldsymbol{\varphi} dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega} \mathbf{B} \cdot \partial_t \boldsymbol{\varphi} dx dt \\ & - \int_0^\tau \int_{\Omega} \left((\operatorname{div} \mathbf{u})\mathbf{B} + (\mathbf{u} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{u} \right) \cdot \boldsymbol{\varphi} dx dt + \nu \int_0^\tau \int_{\Omega} \mathbf{B} \cdot \Delta \boldsymbol{\varphi} dx dt, \\ & \int_0^\tau \int_{\Omega} \mathbf{B} \cdot \nabla \phi dx dt = 0, \end{aligned} \quad (1.16)$$

for a.e. $0 \leq \tau \leq T$, all $\boldsymbol{\varphi} \in [\mathcal{D}([0, T] \times \Omega)]^3$, and $\phi \in \mathcal{D}([0, T] \times \Omega)$.

(6) The Poisson equation (1.5) is satisfied in $\mathcal{D}'(\Omega)$ with the initial data

$$\Phi_0 = -\frac{1}{\varepsilon^2} \Delta^{-1}(\rho_0 - n_0).$$

2. THE MAIN RESULT

In this section, we introduce our main result of the convergence of viscous two fluid models (1.1)–(1.5) to one fluid model in the periodic domain $\mathbb{T}^3$. We first state the existence and regularity of a local smooth for MHD.

Proposition 2.1. (See [7, 9].) Assume that the initial data $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ satisfies

$$0 < \underline{\rho} < \rho_0 \in H^2(\Omega), \mathbf{u}_0 \in H^2(\Omega; \mathbb{R}^3), \mathbf{B}_0 \in H^2(\Omega; \mathbb{R}^3), \operatorname{div} \mathbf{B}_0 = 0 \quad (2.1)$$

and the compatibility condition verifies

$$\mu \Delta \mathbf{u}_0 + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u}_0 + \nabla P(\rho_0) + \frac{1}{2} \nabla |\nabla \mathbf{B}_0|^2 - \mathbf{B}_0 \cdot \nabla \mathbf{B}_0 = \sqrt{\rho_0} \mathbf{g} \quad (2.2)$$

with $\|\mathbf{g}\|_{L^3} \leq C$. Then there exist a $T^* \in (0, \infty)$ and a unique solution $(\rho, \mathbf{u}, \mathbf{B})$ to the compressible MHD (1.6)–(1.8) on $(0, T]$ satisfying the following estimates

$$\begin{aligned} & \rho \in H^2(\Omega), \partial_t \rho \in H^1(\Omega), \mathbf{u} \in H^2(\Omega; \mathbb{R}^3), \partial_t \mathbf{u} \in H^1(\Omega; \mathbb{R}^3), \\ & \mathbf{B} \in H^2(\Omega; \mathbb{R}^3), \partial_t \mathbf{B} \in H^1(\Omega; \mathbb{R}^3), \end{aligned} \quad (2.3)$$

with $C(T) > 0$ a constant for any $0 < T < T^*$.

The following main result is as follows.

Theorem 2.1. *Let $\Omega := \mathbb{T}^3$ be the three dimensional torus and $(\rho_\varepsilon, n_\varepsilon, \mathbf{u}_\varepsilon, \mathfrak{R}_\varepsilon, \mathbf{B}_\varepsilon, \Phi_\varepsilon)$ be a dissipative turbulent solution of system (1.1)–(1.5) in the sense of Definition 1.1 with the initial data $(\rho_{0,\varepsilon}, n_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \mathbf{B}_{0,\varepsilon}, \Phi_{0,\varepsilon})$. Assume that $(\rho_{0,\varepsilon}, n_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \mathbf{B}_{0,\varepsilon}, \Phi_{0,\varepsilon})$ satisfies*

$$\int_{\Omega} |\rho_{0,\varepsilon} - \rho_0|^2 dx + \int_{\Omega} |n_{0,\varepsilon} - n_0|^2 dx \leq C\varepsilon, \quad (2.4)$$

$$\|\sqrt{\rho_{0,\varepsilon}} \mathbf{u}_{0,\varepsilon} - \sqrt{\rho_0} \mathbf{u}_0\|_{L^2(\Omega)}^2 + \|\sqrt{n_{0,\varepsilon}} \mathbf{u}_{0,\varepsilon} - \sqrt{n_0} \mathbf{u}_0\|_{L^2(\Omega)}^2 + \|\mathbf{B}_{0,\varepsilon} - \mathbf{B}_0\|_{L^2(\Omega)}^2 \leq C\varepsilon, \quad (2.5)$$

$$\varepsilon \|\nabla \Phi_{0,\varepsilon}\|_{L^2(\Omega)}^2 \leq C, \quad (2.6)$$

where $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ satisfies the conditions (2.1)–(2.2). Then, for $0 < T < T^*$,

$$\|\rho_\varepsilon - \rho\|_{L^\infty(0,T;(L^2+L^\gamma)(\Omega))} + \|n_\varepsilon - n\|_{L^\infty(0,T;(L^2+L^\alpha)(\Omega))} \leq C\sqrt{\varepsilon}, \quad (2.7)$$

$$\|\sqrt{\rho_\varepsilon} \mathbf{u}_\varepsilon - \sqrt{\rho} \mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\sqrt{n_\varepsilon} \mathbf{u}_\varepsilon - \sqrt{n} \mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))} \leq C\sqrt{\varepsilon}, \quad (2.8)$$

$$\|\mathbf{B}_\varepsilon - \mathbf{B}\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathfrak{R}_\varepsilon\|_{L^\infty(0,T;\mathcal{M}(\Omega;\mathbb{R}_{\text{sym}}^{3 \times 3}))} + \varepsilon \|\nabla \Phi_\varepsilon\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}, \quad (2.9)$$

where $(\rho, \mathbf{u}, \mathbf{B})$ is the strong solution to the system (1.6)–(1.8) with the initial condition $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$.

Remark 2.1. The above result shows that when ε tends to 0 and a dissipative turbulent solution $[\rho_\varepsilon, n_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{B}_\varepsilon, \mathfrak{R}_\varepsilon, \Phi_\varepsilon]$ strongly converges to $[\rho, \mathbf{u}, \mathbf{B}]$ on $[0, T^*)$, then the Reynolds tensor $\mathfrak{R}$ to the target system (1.6)–(1.8) on $[0, T^*)$ vanishes and it means that the Reynolds tensor $\mathfrak{R}$ disappears whenever the densities and the velocity converge strongly.

Remark 2.2. A compressible two-fluid MHD model is used when a single-fluid approximation is not enough. That happens when the species have different velocities, temperatures, or response times, so effects like charge separation, Hall currents, ion-neutral drift, and finite relaxation between components become important. You can see the model in [8].

3. PROOF OF THEOREM 2.1

In this section, we are going to give a rigorous proof of Theorem 2.1. We may assume the viscosities $\mu_\varepsilon = \mu, \lambda_\varepsilon = \lambda, \nu_\varepsilon = \nu$. To begin with, we derive some uniform estimates from the energy inequality.

3.1. Uniform bounds. In this subsection, we are going to derive some estimates with the energy inequality (1.14). Energy inequality (1.14) provides that

$$\begin{aligned} & \int_{\Omega} \left(\frac{1}{2}(\rho_\varepsilon + n_\varepsilon)|\mathbf{u}_\varepsilon|^2 + \frac{1}{\gamma-1}\rho_\varepsilon^\gamma + \frac{1}{\alpha-1}n_\varepsilon^\alpha + \frac{1}{2}|\mathbf{B}_\varepsilon|^2 + \frac{\varepsilon^2}{2}|\nabla \Phi_\varepsilon|^2 \right) dx \\ & + \int_0^\tau \int_{\Omega} \mu |\nabla \mathbf{u}_\varepsilon|^2 + (\mu + \lambda)(\operatorname{div} \mathbf{u}_\varepsilon)^2 dx dt \leq E_{0,\varepsilon} \leq C. \end{aligned} \quad (3.1)$$

Thus we have the uniform estimates with the Korn's inequality and list them here for later use

$$\operatorname{ess\,sup}_{t \in (0, T)} \|\sqrt{\rho_\varepsilon} \mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)} \leq C, \quad \operatorname{ess\,sup}_{t \in (0, T)} \|\sqrt{n_\varepsilon} \mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)} \leq C, \quad (3.2)$$

$$\operatorname{ess\,sup}_{t \in (0, T)} \|\varepsilon \nabla \Phi_\varepsilon\|_{L^2(\Omega)} \leq C, \quad (3.3)$$

$$\operatorname{ess\,sup}_{t \in (0, T)} \|\rho_\varepsilon^\gamma\|_{L^1(\Omega)} \leq C, \quad \operatorname{ess\,sup}_{t \in (0, T)} \|n_\varepsilon^\alpha\|_{L^1(\Omega)} \leq C, \quad (3.4)$$

$$\operatorname{ess\,sup}_{t \in (0, T)} \|\mathbf{B}_\varepsilon\|_{L^2(\Omega)} \leq C. \quad (3.5)$$

$$\|\mathbf{u}_\varepsilon\|_{L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3))} \leq C, \quad (3.6)$$

$$\|\mathbf{B}_\varepsilon\|_{L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3))} \leq C. \quad (3.7)$$

Let us denote by

$$H(\rho_\varepsilon, n_\varepsilon) = \frac{1}{\gamma-1} (\rho_\varepsilon^\gamma - \rho^\gamma - \gamma \rho^{\gamma-1} (\rho_\varepsilon - \rho)) + \frac{1}{\alpha-1} (n_\varepsilon^\alpha - \rho^\alpha - \alpha \rho^{\alpha-1} (n_\varepsilon - \rho)).$$

From the typical result of the convex function H , it follows that

$$H(\rho_\varepsilon, n_\varepsilon) \geq \begin{cases} C(K)(|\rho_\varepsilon - \rho|^2 + |n_\varepsilon - \rho|^2), & \text{if } \rho_\varepsilon, n_\varepsilon \in K \\ C(K)(1 + \rho_\varepsilon^\gamma + n_\varepsilon^\alpha), & \text{if } \rho_\varepsilon, n_\varepsilon \in (0, \infty) \times (0, \infty) \setminus K \end{cases} \quad (3.8)$$

for any compact subset $K \subset (0, \infty) \times (0, \infty)$ and some $C(K) > 0$. The following notations are used later:

$$[H]_{\text{ess}} = H1_{\{\rho/2 < \rho_\varepsilon < 2\rho, n/2 < n_\varepsilon < 2n\}}, \quad H = [H]_{\text{ess}} + [H]_{\text{res}}.$$

3.2. Relative entropy. We now derive the relative entropy inequality, which is the core of the proof. For some convenient notations, Let us define by

$$G_1(\rho_\varepsilon) = \frac{1}{\gamma-1} (\rho_\varepsilon^\gamma - \rho^\gamma - \gamma \rho^{\gamma-1} (\rho_\varepsilon - \rho)),$$

$$G_2(n_\varepsilon) = \frac{1}{\alpha-1} (n_\varepsilon^\alpha - \rho^\alpha - \alpha \rho^{\alpha-1} (n_\varepsilon - \rho)),$$

and the relative entropy:

$$\mathcal{E}_\varepsilon(\tau) = \int_\Omega \left(\frac{1}{2} (\rho_\varepsilon + n_\varepsilon) |\mathbf{u}_\varepsilon - \mathbf{u}|^2 + G_1(\rho_\varepsilon) + G_2(n_\varepsilon) + \frac{1}{2} |\mathbf{B}_\varepsilon - \mathbf{B}|^2 + \frac{1}{2} \varepsilon^2 |\nabla \Phi_\varepsilon|^2 \right) dx \quad (3.9)$$

for any $\tau \in [0, T]$ where $(\rho_\varepsilon, n_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{B}_\varepsilon, \mathfrak{R}_\varepsilon, \Phi_\varepsilon)$ is a dissipative turbulent solution of the system (1.1)–(1.5) and $(\rho, \mathbf{u}, \mathbf{B})$ is the solution of the system (1.6)–(1.8). Take $\frac{1}{2} |\mathbf{u}|^2$ and $p'_j(r)$, $j = 1, 2$, as a test function in (1.1) to obtain

$$\int_\Omega \frac{1}{2} r_\varepsilon |\mathbf{u}|^2 dx = \int_\Omega \frac{1}{2} r_{0,\varepsilon} |\mathbf{u}_0|^2 dx + \int_0^\tau \int_\Omega \left(r_\varepsilon \mathbf{u} \cdot \partial_t \mathbf{u} + r_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u} \cdot \mathbf{u} \right) dx dt \quad (3.10)$$

and

$$\int_\Omega r_\varepsilon p'_j(r) dx = \int_\Omega r_{0,\varepsilon} p'_j(r_0) dx + \int_0^\tau \int_\Omega \left(r_\varepsilon \partial_t p'_j(r) + r_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla p'_j(r) \right) dx dt, \quad (3.11)$$

where r represents for ρ or n and

$$p_1(\rho) = \frac{1}{\gamma-1} \rho^\gamma, \quad p_2(n) = \frac{1}{\alpha-1} n^\alpha.$$

Next, we choose $\mathbf{u}$ as a test function to the moment equation (1.3) and it provides

$$\begin{aligned}
& - \int_{\Omega} (\rho_{\varepsilon} + n_{\varepsilon}) \mathbf{u}_{\varepsilon} \cdot \mathbf{u}(\tau) dx = - \int_{\Omega} (\rho_{0,\varepsilon} + n_{0,\varepsilon}) \mathbf{u}_{0,\varepsilon} \cdot \mathbf{u}_0 dx \\
& - \int_0^{\tau} \int_{\Omega} \left((\rho_{\varepsilon} + n_{\varepsilon}) \mathbf{u}_{\varepsilon} \cdot \partial_t \mathbf{u} + (\rho_{\varepsilon} + n_{\varepsilon}) \mathbf{u}_{\varepsilon} \otimes \mathbf{u}_{\varepsilon} : \nabla \mathbf{u} \right) dx dt \\
& + \int_0^{\tau} \int_{\Omega} \mu \nabla \mathbf{u}_{\varepsilon} : \nabla \mathbf{u} + (\mu + \lambda) \operatorname{div} \mathbf{u}_{\varepsilon} \operatorname{div} \mathbf{u} - (\rho_{\varepsilon}^{\gamma} + n^{\alpha}) \operatorname{div} \mathbf{u} \Big) dx dt \\
& + \int_0^{\tau} \int_{\Omega} \left(\varepsilon^2 \mathbf{D} \mathbf{u} : (\nabla \Phi_{\varepsilon} \otimes \nabla \Phi_{\varepsilon}) + \frac{\varepsilon^2}{2} |\nabla \Phi_{\varepsilon}|^2 \operatorname{div} \mathbf{u} \right) dx dt, \\
& - \int_0^{\tau} \int_{\Omega} \nabla \mathbf{u} : d \mathfrak{R}_{\varepsilon}(t) dt - \int_{\Omega} \left[((\mathbf{B}_{\varepsilon} \cdot \nabla) \mathbf{B}_{\varepsilon} - \frac{1}{2} \nabla |\mathbf{B}_{\varepsilon}|^2) \cdot \mathbf{u} \right] dx dt, \tag{3.12}
\end{aligned}$$

where $\mathbf{D} \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ while the equation (1.5) together with using the integration by part yields that

$$\begin{aligned}
& - \int_0^{\tau} \int_{\Omega} (\rho_{\varepsilon} - n_{\varepsilon}) \nabla \Phi_{\varepsilon} \cdot \mathbf{u} dx dt = \int_0^{\tau} \int_{\Omega} \varepsilon^2 \Delta \Phi_{\varepsilon} \nabla \Phi_{\varepsilon} \cdot \mathbf{u} dx dt \\
& = - \varepsilon^2 \int_0^{\tau} \int_{\Omega} \left(\mathbf{D} \mathbf{u} : (\nabla \Phi_{\varepsilon} \otimes \nabla \Phi_{\varepsilon}) + \frac{1}{2} |\nabla \Phi_{\varepsilon}|^2 \operatorname{div} \mathbf{u} \right) dx dt.
\end{aligned}$$

Thus, we deduce, after adding (1.14), (3.10), (3.11), and (3.12), the following inequality:

$$\begin{aligned}
& \int_{\Omega} \left(\frac{1}{2}(\rho_{\varepsilon} + n_{\varepsilon})|\mathbf{u}_{\varepsilon} - \mathbf{u}|^2 + p_1(\rho_{\varepsilon}) - \rho_{\varepsilon} p'_1(\rho) + p_2(n_{\varepsilon}) - n_{\varepsilon} p'_2(\rho) \right. \\
& \quad \left. + \frac{1}{2}|\mathbf{B}_{\varepsilon} - \mathbf{B}|^2 + \frac{\varepsilon^2}{2}|\nabla \Phi_{\varepsilon}|^2 \right) dx \\
& \quad + \int_0^{\tau} \int_{\Omega} \left(\mu |\nabla \mathbf{u}_{\varepsilon} - \nabla \mathbf{u}|^2 + (\mu + \lambda) |\operatorname{div} \mathbf{u}_{\varepsilon} - \operatorname{div} \mathbf{u}|^2 + \nu |\nabla \mathbf{B}_{\varepsilon} - \nabla \mathbf{B}|^2 \right) dx dt \\
& \leq \int_{\Omega} \left(\frac{1}{2}(\rho_{0,\varepsilon} + n_{0,\varepsilon})|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0|^2 + p_1(\rho_{0,\varepsilon}) - \rho_{0,\varepsilon} p'_1(\rho_0) + p_2(n_{0,\varepsilon}) - n_{0,\varepsilon} p'_2(\rho_0) \right. \\
& \quad \left. + \frac{1}{2}|\mathbf{B}_{0,\varepsilon} - \mathbf{B}_0|^2 + \frac{\varepsilon^2}{2}|\nabla \Phi_{0,\varepsilon}|^2 \right) dx \\
& \quad + \int_0^{\tau} \int_{\Omega} \left((\rho_{\varepsilon} + n_{\varepsilon})(\partial_t \mathbf{u} + \mathbf{u}_{\varepsilon} \cdot \nabla \mathbf{u}) \cdot (\mathbf{u} - \mathbf{u}_{\varepsilon}) \right) dx dt \\
& \quad + \mu \int_0^{\tau} \int_{\Omega} \nabla \mathbf{u} : \nabla (\mathbf{u} - \mathbf{u}_{\varepsilon}) dx dt + (\mu + \lambda) \int_0^{\tau} \int_{\Omega} \operatorname{div} \mathbf{u} (\operatorname{div} \mathbf{u} - \operatorname{div} \mathbf{u}_{\varepsilon}) dx dt \\
& \quad - \int_0^{\tau} \int_{\Omega} (\rho_{\varepsilon} \partial_t p_1(\rho) + \rho_{\varepsilon} \mathbf{u}_{\varepsilon} \cdot \nabla p'_1(\rho)) dx dt - \int_0^{\tau} \int_{\Omega} \rho_{\varepsilon}^{\gamma} \operatorname{div} \mathbf{u} dx dt \\
& \quad - \int_0^{\tau} \int_{\Omega} (n_{\varepsilon} \partial_t p_2(\rho) + n_{\varepsilon} \mathbf{u}_{\varepsilon} \cdot \nabla p'_2(\rho)) dx dt - \int_0^{\tau} \int_{\Omega} n_{\varepsilon}^{\alpha} \operatorname{div} \mathbf{u} dx dt \\
& \quad + \int_0^{\tau} \int_{\Omega} \left(\varepsilon^2 \mathbf{D} \mathbf{u} : (\nabla \Phi_{\varepsilon} \otimes \nabla \Phi_{\varepsilon}) + \frac{\varepsilon^2}{2} |\nabla \Phi_{\varepsilon}|^2 \operatorname{div} \mathbf{u} \right) dx dt, \\
& \quad - \nu \int_0^{\tau} \int_{\Omega} (2 \nabla \mathbf{B}_{\varepsilon} \cdot \nabla \mathbf{B} + |\nabla \mathbf{B}|^2) dx dt, \\
& \quad - \int_{\Omega} \left[((\mathbf{B}_{\varepsilon} \cdot \nabla) \mathbf{B}_{\varepsilon} - \frac{1}{2} \nabla |\mathbf{B}_{\varepsilon}|^2) \cdot \mathbf{u} + \mathbf{B}_{\varepsilon} \cdot \mathbf{B} - \mathbf{B}_{0,\varepsilon} \cdot \mathbf{B}_0 - \frac{1}{2} |\mathbf{B}|^2 + \frac{1}{2} |\mathbf{B}_0|^2 \right] dx. \tag{3.13}
\end{aligned}$$

Note that

$$\begin{aligned}
& \int_{\Omega} \rho^{\gamma} dx - \int_{\Omega} \rho_0^{\gamma} dx = \int_0^{\tau} \int_{\Omega} \partial_t \rho^{\gamma} dx dt \\
& = \int_0^{\tau} \int_{\Omega} \left(\rho \partial_t p'_1(\rho) + \rho \nabla p'_1(\rho) \cdot \mathbf{u} + \rho^{\gamma} \operatorname{div} \mathbf{u} \right) dx dt, \tag{3.14}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega} \rho^{\alpha} dx - \int_{\Omega} \rho_0^{\alpha} dx = \int_0^{\tau} \int_{\Omega} \partial_t \rho^{\alpha} dx dt \\
& = \int_0^{\tau} \int_{\Omega} \left(\rho \partial_t p'_2(\rho) + \rho \nabla p'_2(\rho) \cdot \mathbf{u} + \rho^{\alpha} \operatorname{div} \mathbf{u} \right) dx dt. \tag{3.15}
\end{aligned}$$

Using the identities (3.14) and (3.15), the relative entropy in (3.13) can be written as follows:

$$\begin{aligned} & \left[\mathcal{E}_\varepsilon(t) \right]_0^\tau + \int_0^\tau \int_\Omega \left(\mu |\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}|^2 + (\mu + \lambda) |\operatorname{div} \mathbf{u}_\varepsilon - \operatorname{div} \mathbf{u}|^2 + \nu |\nabla \mathbf{B}_\varepsilon - \nabla \mathbf{B}|^2 \right) dx dt \\ & \leq \sum_{j=1}^8 A_\varepsilon^j, \end{aligned} \quad (3.16)$$

where

$$\begin{aligned} A_\varepsilon^1 &= \int_0^\tau \int_\Omega \left(\rho_\varepsilon + n_\varepsilon \right) (\partial_t \mathbf{u} + \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}) \cdot (\mathbf{u} - \mathbf{u}_\varepsilon) dx dt \\ A_\varepsilon^2 &= \mu \int_0^\tau \int_\Omega \nabla \mathbf{u} : \nabla (\mathbf{u} - \mathbf{u}_\varepsilon) dx dt + (\mu + \lambda) \int_0^\tau \int_\Omega \operatorname{div} \mathbf{u} (\operatorname{div} \mathbf{u} - \operatorname{div} \mathbf{u}_\varepsilon) dx dt \\ A_\varepsilon^3 &= \int_0^\tau \int_\Omega \left((\rho - \rho_\varepsilon) \partial_t p'_1(\rho) + \nabla p'_1(\rho) \cdot (\rho \mathbf{u} - \rho_\varepsilon \mathbf{u}_\varepsilon) - \operatorname{div} \mathbf{u} (\rho_\varepsilon^\gamma - \rho^\gamma) \right) dx dt \\ A_\varepsilon^4 &= \int_0^\tau \int_\Omega \left((\rho - n_\varepsilon) \partial_t p'_2(\rho) + \nabla p'_2(\rho) \cdot (\rho \mathbf{u} - n_\varepsilon \mathbf{u}_\varepsilon) - \operatorname{div} \mathbf{u} (n_\varepsilon^\alpha - \rho^\alpha) \right) dx dt \\ A_\varepsilon^5 &= \int_0^\tau \int_\Omega \left(\varepsilon^2 \mathbf{D} \mathbf{u} : (\nabla \Phi_\varepsilon \otimes \nabla \Phi_\varepsilon) + \frac{\varepsilon^2}{2} |\nabla \Phi_\varepsilon|^2 \operatorname{div} \mathbf{u} \right) dx dt \\ A_\varepsilon^6 &= -\nu \int_0^\tau \int_\Omega \left(2 \nabla \mathbf{B}_\varepsilon \cdot \nabla \mathbf{B} + |\nabla \mathbf{B}|^2 \right) dx dt, \\ A_\varepsilon^7 &= - \int_\Omega \left[((\mathbf{B}_\varepsilon \cdot \nabla) \mathbf{B}_\varepsilon - \frac{1}{2} \nabla |\mathbf{B}_\varepsilon|^2) \cdot \mathbf{u} + \mathbf{B}_\varepsilon \cdot \mathbf{B} - \mathbf{B}_{0,\varepsilon} \cdot \mathbf{B}_0 - \frac{1}{2} |\mathbf{B}|^2 + \frac{1}{2} |\mathbf{B}_0|^2 \right] dx, \\ A_8 &= - \int_0^\tau \int_\Omega \nabla \mathbf{u} : d\mathfrak{R}(t) dt. \end{aligned}$$

To compute the part of magnetic field $A_\varepsilon^6, A_\varepsilon^7$, we first adapt $\mathbf{B}$ to the Maxwell equation (1.16)

$$\begin{aligned} \int_\Omega \mathbf{B}_\varepsilon \cdot \mathbf{B}(\tau, \cdot) dx &= \int_\Omega \mathbf{B}_\varepsilon \cdot \mathbf{B}(\tau, \cdot) dx + \int_0^\tau \int_\Omega \left(\mathbf{B}_\varepsilon \cdot \partial_t \mathbf{B} - \nu \nabla \mathbf{B}_\varepsilon \cdot \nabla \mathbf{B} \right) dx dt \\ &+ \int_0^\tau \int_\Omega \left(\mathbf{B} \cdot \nabla \mathbf{u}_\varepsilon \cdot \mathbf{B}_\varepsilon - \mathbf{B} \cdot (\nabla \mathbf{B}_\varepsilon) \mathbf{u}_\varepsilon - (\mathbf{B}_\varepsilon \cdot \mathbf{B}) \operatorname{div} \mathbf{u}_\varepsilon \right) dx dt. \end{aligned} \quad (3.17)$$

We can use $\frac{1}{2} |\mathbf{B}|^2$ to the Maxwell equation (1.8) and then we have

$$\begin{aligned} \frac{1}{2} \int_\Omega |\mathbf{B}|^2(\tau, \cdot) dx &= \frac{1}{2} \int_\Omega |\mathbf{B}|^2(0, \cdot) dx - \nu \int_0^\tau \int_\Omega |\nabla \mathbf{B}|^2 dx dt \\ &+ \int_0^\tau \int_\Omega \left(\mathbf{B} \cdot (\nabla \mathbf{B}) \mathbf{u} - \mathbf{u} \cdot (\nabla \mathbf{B}) \mathbf{B} \right) dx dt. \end{aligned} \quad (3.18)$$

Using (3.17), (3.18) and equation (1.8), we can do a simple calculation and obtain

$$\begin{aligned}
A_\varepsilon^6 + A_\varepsilon^7 &= \int_0^\tau \int_\Omega \left((\mathbf{B}_\varepsilon - \mathbf{B})^T \nabla \mathbf{u} (\mathbf{B}_\varepsilon - \mathbf{B}) - \frac{1}{2} |\mathbf{B}_\varepsilon - \mathbf{B}|^2 \operatorname{div} \mathbf{u} \right) dx dt \\
&\quad + \int_0^\tau \int_\Omega \left((\mathbf{B}_\varepsilon - \mathbf{B})^T \nabla (\mathbf{u}_\varepsilon - \mathbf{u}) \mathbf{B} + \frac{1}{2} \nabla (\mathbf{B}(\mathbf{B} - \mathbf{B}_\varepsilon)) (\mathbf{u}_\varepsilon - \mathbf{u}) \right) dx dt \\
&\quad - \int_0^\tau \int_\Omega (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \left((\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 \right) dx dt.
\end{aligned} \tag{3.19}$$

3.3. Computation of the relative entropy. In this subsection, we prove Theorem 2.1 by calculating the relative entropy calculated above. We first compute the residue and essential part together with (3.8). Using the Holler inequality, it follows that

$$\begin{aligned}
\left| \int_\Omega \left[\frac{\rho_\varepsilon - \rho}{\rho} \right]_{\text{ess}} \Psi(\mathbf{u}, \mathbf{B}) \cdot (\mathbf{u}_\varepsilon - \mathbf{u}) dx \right| \\
\leq \| [\rho_\varepsilon - \rho]_{\text{ess}} \|_{L^2(\Omega)} \left\| \frac{\Psi(\mathbf{u}, \mathbf{B})}{\rho} \right\|_{L^3(\Omega)} \| \mathbf{u}_\varepsilon - \mathbf{u} \|_{L^6(\Omega)} \\
\leq C(\theta) \mathcal{E}_\varepsilon(\tau) + \theta \| \mathbf{u}_\varepsilon - \mathbf{u} \|_{L^6(\Omega)}^2
\end{aligned} \tag{3.20}$$

for a small number $\theta > 0$, where $\Psi(\mathbf{u}, \mathbf{B}) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2$. The residue part can be estimated as follows:

$$\begin{aligned}
\left| \int_\Omega \left[\frac{\rho_\varepsilon - \rho}{\rho} \right]_{\text{res}} \Psi(\mathbf{u}, \mathbf{B}) \cdot (\mathbf{u}_\varepsilon - \mathbf{u}) dx \right| \\
\leq \| [\rho_\varepsilon^{\gamma/2}]_{\text{res}} \|_{L^2(\Omega)} \left\| \frac{\Psi(\mathbf{u}, \mathbf{B})}{\rho} \right\|_{L^p(\Omega)} \| \sqrt{\rho_\varepsilon} (\mathbf{u}_\varepsilon - \mathbf{u}) \|_{L^2(\Omega)} \\
\quad + \| [1]_{\text{res}} \|_{L^2(\Omega)} \| \Psi(\mathbf{u}, \mathbf{B}) \|_{L^3(\Omega)} \| \mathbf{u}_\varepsilon - \mathbf{u} \|_{L^6(\Omega)} \\
\leq C(\theta) \mathcal{E}_\varepsilon(\tau) + \theta \| \mathbf{u}_\varepsilon - \mathbf{u} \|_{L^6(\Omega)}^2,
\end{aligned} \tag{3.21}$$

for a small number $\theta > 0$, where p verifies $\frac{1}{\gamma} + \frac{1}{p} = \frac{1}{2}$. For the n_ε , we have the same results as follows:

$$\left| \int_\Omega \left[\frac{n_\varepsilon - \rho}{\rho} \right]_{\text{ess}} \Psi(\mathbf{u}, \mathbf{B}) \cdot (\mathbf{u}_\varepsilon - \mathbf{u}) dx \right| \leq C(\theta) \mathcal{E}_\varepsilon(\tau) + \theta \| \mathbf{u}_\varepsilon - \mathbf{u} \|_{L^6(\Omega)}^2 \tag{3.22}$$

and

$$\left| \int_\Omega \left[\frac{n_\varepsilon - \rho}{\rho} \right]_{\text{res}} \Psi(\mathbf{u}, \mathbf{B}) \cdot (\mathbf{u}_\varepsilon - \mathbf{u}) dx \right| \leq C(\theta) \mathcal{E}_\varepsilon(\tau) + \theta \| \mathbf{u}_\varepsilon - \mathbf{u} \|_{L^6(\Omega)}^2, \tag{3.23}$$

for a small number $\theta > 0$. Let us next control the velocity term in A_ε^1 . Following the facts (3.20), (3.21), (3.22), and (3.23), we see that

$$\begin{aligned}
& \int_0^\tau \int_\Omega (\rho_\varepsilon + n_\varepsilon) (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) dx dt \\
&= \int_0^\tau \int_\Omega \left(\frac{\rho_\varepsilon + n_\varepsilon}{2\rho} \right) (\mathbf{u}_\varepsilon - \mathbf{u}) \cdot \nabla (\rho^\gamma + n^\alpha) dx dt \\
&\quad + \int_0^\tau \int_\Omega \frac{\rho_\varepsilon + n_\varepsilon - \rho - \rho}{2\rho} (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \Psi(\mathbf{u}, \mathbf{B}) dx dt + \int_0^\tau \int_\Omega (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \Psi(\mathbf{u}, \mathbf{B}) dx dt \\
&= \int_0^\tau \int_\Omega \left(\frac{\rho_\varepsilon + n_\varepsilon}{2\rho} \right) (\mathbf{u}_\varepsilon - \mathbf{u}) \cdot \nabla (\rho^\gamma + n^\alpha) dx dt \\
&\quad + \int_0^\tau \int_\Omega \left[\frac{\rho_\varepsilon + n_\varepsilon - \rho - \rho}{2\rho} (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \Psi(\mathbf{u}, \mathbf{B}) \right]_{\text{ess}} dx dt \\
&\quad + \int_0^\tau \int_\Omega \left[\frac{\rho_\varepsilon + n_\varepsilon - \rho - \rho}{2\rho} (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \Psi(\mathbf{u}, \mathbf{B}) \right]_{\text{res}} dx dt \\
&\quad + \int_0^\tau \int_\Omega (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \left(\mu \Delta \mathbf{u} + (\mu + \nu) \nabla \operatorname{div} \mathbf{u} \right) dx dt \\
&\quad + \int_0^\tau \int_\Omega (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \left((\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 \right) dx dt \\
&\leq C(\theta) \int_0^\tau \mathcal{E}_\varepsilon(t) dt + \theta \int_0^\tau \|\mathbf{u}_\varepsilon - \mathbf{u}\|_{L^6(\Omega)}^2 dt \\
&\quad + \int_0^\tau \int_\Omega \left(\frac{\rho_\varepsilon + n_\varepsilon}{2\rho} \right) (\mathbf{u}_\varepsilon - \mathbf{u}) \cdot \nabla (\rho^\gamma + n^\alpha) dx dt \\
&\quad + \int_0^\tau \int_\Omega (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \left(\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} \right) dx dt \\
&\quad + \int_0^\tau \int_\Omega (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \left((\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 \right) dx dt \tag{3.24}
\end{aligned}$$

where we used Hollder's inequality, integration by parts, and the property in (3.8). Thus, we obtain

$$\begin{aligned}
A_\varepsilon^1 + A_\varepsilon^2 &\leq \theta \|\nabla(\mathbf{u}_\varepsilon - \mathbf{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2 + C(\theta) \int_0^\tau \mathcal{E}_\varepsilon(t) dt \\
&\quad + \int_0^\tau \int_\Omega \left(\frac{\rho_\varepsilon + n_\varepsilon}{2\rho} \right) (\mathbf{u}_\varepsilon - \mathbf{u}) \cdot \nabla (\rho^\gamma + n^\alpha) dx dt \\
&\quad + \int_0^\tau \int_\Omega (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \left((\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 \right) dx dt, \tag{3.25}
\end{aligned}$$

for a small $\theta > 0$. To control the second part of (3.25), we compute

$$\begin{aligned}
& \int_0^\tau \int_\Omega \left(\frac{\rho_\varepsilon + n_\varepsilon}{2\rho} \right) (\mathbf{u}_\varepsilon - \mathbf{u}) \cdot \nabla (\rho^\gamma + n^\alpha) dx dt \\
&= \int_0^\tau \int_\Omega \left(\frac{\rho_\varepsilon + n_\varepsilon - \rho - \rho}{2\rho} \right) (\mathbf{u}_\varepsilon - \mathbf{u}) \cdot \nabla (\rho^\gamma + n^\alpha) dx dt \\
&+ \int_0^\tau \int_\Omega \mathbf{u}_\varepsilon \cdot \nabla (\rho^\gamma + n^\alpha) dx dt + \int_0^\tau \int_\Omega \operatorname{div} \mathbf{u} (\rho^\gamma + n^\alpha) dx dt \\
&\leq \theta \|\nabla(\mathbf{u}_\varepsilon - \mathbf{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2 + C(\theta) \int_0^\tau \mathcal{E}_\varepsilon(t) dt + \int_0^\tau \int_\Omega \operatorname{div} \mathbf{u} (\rho^\gamma + n^\alpha) dx dt \\
&+ \int_0^\tau \int_\Omega \rho \mathbf{u}_\varepsilon \cdot \nabla p'_1(\rho) dx dt + \int_0^\tau \int_\Omega n \mathbf{u}_\varepsilon \cdot \nabla p'_2(\rho) dx dt.
\end{aligned} \tag{3.26}$$

In order to handle the pressure parts $A_\varepsilon^3, A_\varepsilon^4$. Observe that

$$\int_0^\tau \int_\Omega (\rho - \rho_\varepsilon) (\partial_t p'_1(\rho) + \nabla p'_1(\rho) \cdot \mathbf{u}) dx dt = - \int_0^\tau \int_\Omega \operatorname{div} \mathbf{u} (\rho_\varepsilon - \rho) \gamma \rho^{\gamma-1} dx dt, \tag{3.27}$$

and

$$\int_0^\tau \int_\Omega (\rho - n_\varepsilon) (\partial_t p'_2(\rho) + \nabla p'_2(\rho) \cdot \mathbf{u}) dx dt = - \int_0^\tau \int_\Omega \operatorname{div} \mathbf{u} (n_\varepsilon - \rho) \alpha \rho^{\alpha-1} dx dt. \tag{3.28}$$

Thus, the sum of $\sum_{j=1}^4 A_\varepsilon^j$ provides that

$$\begin{aligned}
\sum_{j=1}^4 A_\varepsilon^j &\leq \theta \|\nabla(\mathbf{u}_\varepsilon - \mathbf{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2 + C(\theta) \int_0^\tau \mathcal{E}_\varepsilon(t) dt \\
&- \int_0^\tau \int_\Omega \operatorname{div} \mathbf{u} \left(\rho_\varepsilon^\gamma - \rho^\gamma - \gamma \rho^{\gamma-1} (\rho_\varepsilon - \rho) \right) dx dt \\
&- \int_0^\tau \int_\Omega \operatorname{div} \mathbf{u} \left(n_\varepsilon^\alpha - \rho^\alpha - \alpha \rho^{\alpha-1} (n_\varepsilon - \rho) \right) dx dt \\
&\leq \theta \|\nabla(\mathbf{u}_\varepsilon - \mathbf{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2 + C(\theta) \int_0^\tau \mathcal{E}_\varepsilon(t) dt
\end{aligned} \tag{3.29}$$

while

$$\begin{aligned}
& \int_0^\tau \int_\Omega (\rho - \rho_\varepsilon) \mathbf{u}_\varepsilon \cdot \nabla p'_1(\rho) dx dt \\
&= \int_0^\tau \int_\Omega ([\rho - \rho_\varepsilon]_{\text{ess}} + [\rho - \rho_\varepsilon]_{\text{res}}) (\mathbf{u}_\varepsilon - \mathbf{u}) \cdot \nabla p'_1(\rho) dx dt \\
&+ \int_0^\tau \int_\Omega ([\rho - \rho_\varepsilon]_{\text{ess}} + [\rho - \rho_\varepsilon]_{\text{res}}) \mathbf{u} \cdot \nabla p'_1(\rho) dx dt \\
&\leq \theta \|\nabla(\mathbf{u}_\varepsilon - \mathbf{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2 + C(\theta) \int_0^\tau \mathcal{E}_\varepsilon(t) dt
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned} & \int_0^\tau \int_\Omega (\rho - n_\varepsilon) \mathbf{u}_\varepsilon \cdot \nabla p'_2(\rho) dx dt \\ & \leq \theta \|\nabla(\mathbf{u}_\varepsilon - \mathbf{u})\|_{L^2(\Omega; \mathbb{R}^3)}^2 + C(\theta) \int_0^\tau \mathcal{E}_\varepsilon(t) dt \end{aligned} \quad (3.31)$$

where we used the same method of (3.20), (3.21), (3.22), and (3.23). It is easily seen to show that

$$A_\varepsilon^5 \leq C \int_0^\tau \mathcal{E}_\varepsilon(t) dx.$$

We apply Poincare' inequality and Hölder's inequality to estimate A_ε^6 and A_ε^7 .

$$\begin{aligned} A_\varepsilon^6 + A_\varepsilon^7 & \leq C(\theta) \int_0^\tau \mathcal{E}_\varepsilon(t) dx + \theta \int_0^\tau \int_\Omega |\nabla \mathbf{B}_\varepsilon - \nabla \mathbf{B}|^2 dx dt \\ & \quad + \theta \int_0^\tau \int_\Omega |\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}|^2 dx dt - \int_0^\tau \int_\Omega (\mathbf{u} - \mathbf{u}_\varepsilon) \cdot \left((\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 \right) dx dt, \end{aligned} \quad (3.32)$$

for a small number $\theta > 0$. Summing up all estimates in previous computation and taking a small suitable $\theta > 0$, the relative entropy arrives at

$$\begin{aligned} & \mathcal{E}_\varepsilon(\tau) + C \int_0^\tau \int_\Omega \left(\mu |\nabla \mathbf{u}_\varepsilon - \nabla \mathbf{u}|^2 + (\mu + \nu) |\operatorname{div} \mathbf{u}_\varepsilon - \operatorname{div} \mathbf{u}|^2 + \nu |\nabla \mathbf{B}_\varepsilon - \nabla \mathbf{B}|^2 \right) dx dt \\ & \leq C \mathcal{E}_\varepsilon(0) + C \int_0^\tau \mathcal{E}_\varepsilon(t) dx, \end{aligned} \quad (3.33)$$

which implies that

$$\mathcal{E}_\varepsilon(\tau) \leq C \mathcal{E}_\varepsilon(0) \exp(\tau C)$$

for any $\tau \in [0, T]$.

For the estimates of the initial data, we use the assumptions (2.4)-(2.6) to obtain

$$\begin{aligned} & \int_\Omega \rho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0|^2 dx \\ & \leq C \int_\Omega |\sqrt{\rho_{0,\varepsilon}} \mathbf{u}_{0,\varepsilon} - \sqrt{\rho_0} \mathbf{u}_0|^2 dx + C \int_\Omega |\sqrt{\rho_{0,\varepsilon}} - \sqrt{\rho_0}|^2 dx \\ & \leq C \int_\Omega |\sqrt{\rho_{0,\varepsilon}} \mathbf{u}_{0,\varepsilon} - \sqrt{\rho_0} \mathbf{u}_0|^2 dx + C \int_\Omega |\rho_{0,\varepsilon} - \rho_0|^2 dx \leq C\varepsilon, \\ & \int_\Omega n_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0|^2 dx \leq C\varepsilon, \quad \int_\Omega |\mathbf{B}_{0,\varepsilon} - \mathbf{B}_0|^2 dx \leq C\varepsilon, \quad \int_\Omega |\varepsilon \nabla \Phi_{0,\varepsilon}|^2 \leq C\varepsilon, \end{aligned}$$

and

$$\int_\Omega \left(G_1(\rho_{0,\varepsilon}) + G_2(n_{0,\varepsilon}) \right) dx \leq C\varepsilon.$$

Thus

$$\mathcal{E}_\varepsilon(0) \leq C\varepsilon$$

and the Gronwall inequality gives the results of Theorem 2.1. Indeed,

$$\begin{aligned} \int_{\Omega} |\sqrt{\rho_{\varepsilon}} \mathbf{u}_{\varepsilon} - \sqrt{\rho} \mathbf{u}|^2 dx &\leq C \int_{\Omega} \rho_{\varepsilon} |\mathbf{u}_{\varepsilon} - \mathbf{u}|^2 dx + C \int_{\Omega} |\sqrt{\rho_{\varepsilon}} - \sqrt{\rho}|^2 dx \leq C\varepsilon, \\ \int_{\Omega} |\sqrt{n_{\varepsilon}} \mathbf{u}_{\varepsilon} - \sqrt{\rho} \mathbf{u}|^2 dx &\leq C \int_{\Omega} n_{\varepsilon} |\mathbf{u}_{\varepsilon} - \mathbf{u}|^2 dx + C \int_{\Omega} |\sqrt{n_{\varepsilon}} - \sqrt{\rho}|^2 dx \leq C\varepsilon, \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} \left([\rho_{\varepsilon} - \rho]_{\text{ess}}^2 + [\rho_{\varepsilon} - \rho]_{\text{res}}^{\gamma} \right) dx &\leq \int_{\Omega} G_1(\rho_{\varepsilon}) dx \leq C\varepsilon, \\ \int_{\Omega} \left([n_{\varepsilon} - \rho]_{\text{ess}}^2 + [n_{\varepsilon} - \rho]_{\text{res}}^{\alpha} \right) dx &\leq \int_{\Omega} G_2(n_{\varepsilon}) dx \leq C\varepsilon, \\ \int_{\Omega} |\mathbf{B}_{\varepsilon} - \mathbf{B}|^2 dx &\leq C\varepsilon, \quad \int_{\Omega} |\varepsilon \nabla \Phi_{\varepsilon}|^2 \leq C\varepsilon \end{aligned}$$

and we get

$$\int_{\Omega} d\mathfrak{R}(\tau) \leq C \int_{\Omega} d\mathfrak{E}(\tau) \leq C\varepsilon$$

where we used (1.15). Hence we complete our proof of Theorem 2.1. $\square$

4. EXISTENCE OF DISSIPATIVE TURBULENT SOLUTIONS

In this section, we construct a dissipative turbulent solution of the system (1.1)–(1.5) in the three dimensional torus $\Omega := \mathbb{T}^3$. We introduce a sequence of finite-dimensional spaces $X_n \subset L^2(\Omega; \mathbb{R}^3)$,

$$X_n = \text{span} \left\{ \mathbf{w}_i \mid \mathbf{w}_i \in C_c^{\infty}(\Omega; \mathbb{R}^d), i = 1, \dots, n \right\}$$

Without loss of generality, we may assume that $\mathbf{w}_i$ are orthonormal with respect to the standard scalar product in $L^2(\Omega)$. We suppose that the initial data is smooth. Specifically,

$$\rho_0, n_0 \in C^1(\Omega).$$

Let us consider a parabolic approximation of the equation of continuity by following Crippa, Donadello, and Spinolo [4],

$$\partial_t r + \text{div}(r\mathbf{u}) = \beta \Delta r \text{ in } (0, T) \times \Omega, \quad \beta > 0, \quad (4.1)$$

and the initial condition

$$r(0, \cdot) = r_0, \quad (4.2)$$

where r stands for ρ, n .

- The weak formulation of the equation of continuity reads:

$$\left[\int_{\Omega} r \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^{\tau} \int_{\Omega} [r \partial_t \varphi + r \mathbf{u} \cdot \nabla \varphi - \beta \nabla r \cdot \nabla \varphi] \, dx dt, \quad r(0, \cdot) = r_0, \quad (4.3)$$

for any test function

$$\varphi \in L^2(0, T; W^{1,2}(\Omega)), \quad \partial_t \varphi \in L^1(0, T; L^2(\Omega)),$$

where r stands for ρ, n .

- For $\mathbf{u} \in C([0, T]; X_n)$, the approximate momentum balance reads

$$\begin{aligned} \left[\int_{\Omega} (\rho + n) \mathbf{u} \cdot \boldsymbol{\varphi} dx \right]_{t=0}^{t=\tau} &= \int_0^{\tau} \int_{\Omega} \left[(\rho + n) \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + (\rho + n) \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \right. \\ &\quad \left. + P(\rho, n) \operatorname{div} \boldsymbol{\varphi} - (\mu \nabla \mathbf{u} + (\mu + \lambda) \operatorname{div} \mathbf{u} \mathbb{I}) : \nabla \boldsymbol{\varphi} \right. \\ &\quad \left. + [(\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2] \cdot \boldsymbol{\varphi} - (\rho - n) \nabla \Phi \cdot \boldsymbol{\varphi} \right] dx dt + \int_0^{\tau} \int_{\Omega} \nabla \boldsymbol{\varphi} : d\mathfrak{R}(t) dt \end{aligned} \quad (4.4)$$

holds for any $0 \leq \tau \leq T$ and any $\boldsymbol{\varphi} \in C^1([0, T]; X_n)$, with the initial condition

$$(\rho + n) \mathbf{u}(0, \cdot) = (\rho_0 + n_0) \mathbf{u}_0, \quad \mathbf{u}_0 \in X_n. \quad (4.5)$$

- The Maxwell equation

$$\begin{cases} \partial_t \mathbf{B} + (\operatorname{div} \mathbf{u}) \mathbf{B} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} = \nu \Delta \mathbf{B}, \\ \operatorname{div} \mathbf{B} = 0, \quad \mathbf{B}(0, x) = \mathbf{B}_0 \end{cases} \quad (4.6)$$

is a linear parabolic type problem in $\mathbf{B}$ and thus we can solve (4.6) by the standard Faedo-type problems for $\mathbf{u} \in X_n$.

- The Poisson equation (1.5) is solvable because it is the elliptic equation.

We report useful results.

Lemma 4.1 (Crippa et al [4, Lemma 3.2]). *Let $\Omega := \mathbb{T}^3 \subset \mathbb{R}^3$ be the three dimensional torus. Given $\mathbf{u} \in C([0, T]; X_n)$, the initial-boundary value problem (4.1)–(4.2) admits a weak solution r specified in (4.3), unique in the class*

$$r \in L^2(0, T; W^{1,2}(\Omega)) \cap C([0, T]; L^2(\Omega)).$$

The norm in the aforementioned spaces is bounded only in terms of the data ρ_0 and

$$\sup_{t \in (0, T)} \|\mathbf{u}(t, \cdot)\|_{X_n},$$

where r stands for ρ, n .

Next, let us introduce a result of Crippa et al. [4, Lemma 3.4].

Lemma 4.2 (Maximum principle). *Under the hypotheses of Lemma (4.1), the solution ρ satisfies*

$$\|r\|_{L^\infty((0, \tau) \times \Omega)} \leq \max \{ \|r_0\|_{L^\infty(\Omega)} \} \exp \left(\tau \|\operatorname{div} \mathbf{u}\|_{L^\infty((0, \tau) \times \Omega)} \right),$$

where r stands for ρ, n .

Lemma 4.3 (Renormalization). *Under the conditions of Lemma 4.2, letting $B \in C^2(\mathbb{R})$, we obtain the (integrated) renormalized equation*

$$\int_{\Omega} B(r) dx \Big|_0^{\tau} + \varepsilon \int_0^{\tau} \int_{\Omega} |\nabla r|^2 B''(r) dx dt + \int_0^{\tau} \int_{\Omega} \left(r B'(r) - B(r) \right) \operatorname{div} \mathbf{u} dx dt \quad (4.7)$$

holds for any $0 \leq \tau \leq T$, where r stands for ρ, n .

Using Lemma 4.3, we obtain strict positivity of ρ on condition that ρ_0 enjoys the same property.

Corollary 4.1. *Under the hypotheses of Lemma 4.1, it holds that*

$$\operatorname{ess\,inf}_{t,x} r(t,x) \geq \min \left\{ \min_{\Omega} r_0 \right\} \exp \left(-T \|\operatorname{div} \mathbf{u}\|_{L^\infty((0,T) \times \Omega)} \right),$$

where r stands for ρ, n .

For fixed parameter $n, \varepsilon > 0$, the first level approximation are solutions $[\rho, n, \mathbf{u}]$ of the problem of the parabolic problem (4.1)–(4.2), and the Galerkin approximation (4.4), (4.5). For $\mathbf{u} \in C([0, T]; X_n)$, we identify the unique solution $r = r[\mathbf{u}, \mathbf{B}]$ of (4.1)–(4.2), where r stands for r and plug it as r in (4.4) by the fixed point Theorem. Thus we have the following approximate solutions.

Proposition 4.1 (Approximate solutions). *Let $\Omega := \mathbb{T}^3$ be the three dimensional torus. Assume that the boundary data belong to the following class*

$$\rho_0, n_0 \in C^1(\Omega), \quad \rho_0 \geq \underline{\rho} > 0, \quad n_0 \geq \underline{n} > 0, \quad \mathbf{u}_0 \in X_n.$$

Then, for each fixed $\varepsilon > 0, n > 0$, there exists a solution $(\rho, n, \mathbf{u}, \mathbf{B}, \Phi)$ of approximate problem (4.1–4.2) and (4.4), (4.5) such that

$$\begin{aligned} & \left[\int_{\Omega} \left[\frac{1}{2}(\rho + n)|\mathbf{u}|^2 + G_1(\rho) + G_2(n) + \frac{1}{2}|\mathbf{B}|^2 + \frac{\varepsilon^2}{2}|\nabla \Phi|^2 \right] dx \right]_{t=0}^{t=\tau} \\ & + \int_0^\tau \int_{\Omega} (\mu \nabla \mathbf{u} + (\mu + \lambda) \operatorname{div} \mathbf{u} \mathbb{I}) : \nabla \mathbf{u} + \nu |\nabla \mathbf{B}|^2 dx dt \\ & + \beta \int_0^\tau \int_{\Omega} \rho^{\gamma-2} |\nabla \rho|^2 + n^{\alpha-2} |\nabla n|^2 dx dt \leq 0 \end{aligned} \quad (4.8)$$

holds for any $0 \leq \tau \leq T$.

From the energy inequality (4.8), we can derive the uniform estimates

$$\|\mathbf{u}_\beta\|_{L^2(0,T;W_0^{1,2}(\Omega))} \leq C, \quad (4.9)$$

$$0 < \underline{r} \leq r(t,x) \leq \bar{r}, \quad (4.10)$$

$$\beta \int_0^T \int_{\Omega} |\nabla r_\beta|^2 dx dt \leq C, \quad (4.11)$$

$$\sup_{\tau \in [0,T]} \|\mathbf{u}_\beta(\tau, \cdot)\|_{W^{1,\infty}(\Omega; \mathbb{R}^3)} \leq C, \quad (4.12)$$

$$\sup_{\tau \in [0,T]} \|\mathbf{B}_\beta(\tau, \cdot)\|_{L^2(\Omega; \mathbb{R}^3)} \leq C, \quad (4.13)$$

$$\|\mathbf{B}_\beta(\tau, \cdot)\|_{L^2([0,T]; W_0^{1,2}(\Omega; \mathbb{R}^3))} \leq C, \quad (4.14)$$

$$\operatorname{ess\,sup}_{t \in (0,T)} \|\varepsilon \nabla \Phi_\beta\|_{L^2(\Omega)} \leq C, \quad (4.15)$$

where r_ε stands for $\rho_\varepsilon, n_\varepsilon$.

Let us first prove the convergence of Φ_β together with estimate (4.14) and the standard L^p elliptic regularity of (1.5). It is obviously seen to show that

$$\nabla \Phi_\beta \rightarrow \nabla \Phi \text{ weakly in } C([0, T]; W^{1, \min\{\gamma, \alpha\}}(\Omega)). \quad (4.16)$$

Using the compactness $W^{1,\min\{\gamma,\alpha\}}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 < p < \frac{3\min\{\gamma,\alpha\}}{3-\min\{\gamma,\alpha\}}$, we obtain

$$\nabla\Phi_\beta \rightarrow \nabla\Phi \text{ in } C([0, T]; L^p(\Omega)) \quad (4.17)$$

for $1 < \min\{\gamma, \alpha\} < 3$. Moreover, if $\min\{\gamma, \alpha\} \geq 3$, then we can show that

$$\nabla\Phi_\beta \rightarrow \nabla\Phi \text{ in } C([0, T]; L^{\min\{\gamma,\alpha\}}(\Omega)). \quad (4.18)$$

In view of the uniform bounds (4.9-4.14), we can prove the convergence up to subsequence

$$\begin{aligned} r_\beta &\rightarrow r \text{ weakly-}^*(*) \text{ in } L^\infty((0, T) \times \Omega) \\ &\text{and weakly in } C_{\text{weak}}([0, T]; L^r(\Omega)) \text{ for any } 1 < r < \infty, \end{aligned} \quad (4.19)$$

$$\mathbf{u}_\beta \rightarrow \mathbf{u} \text{ weakly-}^*(*) \text{ in } L^\infty(0, T; W^{1,\infty}(\Omega; \mathbb{R}^3)), \quad (4.20)$$

$$(\rho_\beta + n_\beta)\mathbf{u}_\beta \rightarrow \mathbf{m} \text{ weakly-}^*(*) \text{ in } L^\infty((0, T) \times \Omega; \mathbb{R}^3), \quad (4.21)$$

$$\begin{aligned} \mathbf{B}_\beta &\rightarrow \mathbf{B} \text{ weakly-}^*(*) \text{ in } L^2([0, T]; W^1(\Omega)) \cap L^\infty([0, T]; L^2(\Omega)), \\ &\text{strongly } L^2([0, T] \times \Omega), \end{aligned} \quad (4.22)$$

where r_β stands for ρ_β, n_β and we have used Aubin-Lions Lemma to (1.4) to show (3.5) while the Arzela-Ascoli theorem yields

$$\mathbf{m} = (n + \rho)\mathbf{u} \text{ a.e. in } (0, T) \times \Omega. \quad (4.23)$$

We will pass to the limits $\beta \rightarrow 0$, $n \rightarrow \infty$ and obtain the existence of the following dissipative turbulent solutions based on the convergence of density, magnetic field, and velocity. Indeed, the limit process of $\beta \rightarrow 0$, $n \rightarrow \infty$ is able to be carried out as the process of Jin, Kwon, Nacasova, and Novotny [12] with the convergence of (4.19)–(4.22). Using the convergence of (4.19) and (4.20), we obtain the continuity equation

$$\left[\int_\Omega r \varphi \, dx \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[r \partial_t \varphi + r \mathbf{u} \cdot \nabla \varphi \right] \, dx dt = 0, \quad r(0, \cdot) = r_0 \quad (4.24)$$

for any $\varphi \in C^1([0, T] \times \Omega)$, where r stands for ρ, n . We also can derive the moment equation and Maxwell equation such that there exists a tensor measure

$$\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\Omega; \mathbb{R}_{\text{sym}}^{3 \times 3})),$$

such that the integral identity

$$\begin{aligned} \left[\int_\Omega (\rho + n) \mathbf{u} \cdot \boldsymbol{\varphi} \, dx \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega \left[(\rho + n) \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + (\rho + n) \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \right. \\ &\quad \left. + P(\rho, n) \operatorname{div} \boldsymbol{\varphi} - (\mu \nabla \mathbf{u} + (\mu + \lambda) \operatorname{div} \mathbf{u} \mathbb{I}) : \nabla \boldsymbol{\varphi} \right. \\ &\quad \left. + [(\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2] \cdot \boldsymbol{\varphi} - (\rho - n) \nabla \Phi \cdot \boldsymbol{\varphi} \right] \, dx dt + \int_0^\tau \int_\Omega \nabla \boldsymbol{\varphi} : d\mathfrak{R}(t) \, dx dt \end{aligned} \quad (4.25)$$

holds for any $0 \leq \tau \leq T$ and any test function $\boldsymbol{\varphi} \in C^1([0, T] \times \Omega; \mathbb{R}^3)$, $\boldsymbol{\varphi}|_{\partial\Omega} = 0$,

$$(\rho + n) \mathbf{u}(0, \cdot) = \mathbf{m}_0 := (\rho_0 + n_0) \mathbf{u}_0, \quad (4.26)$$

and the Maxwell equation verifies

$$\begin{aligned} \left[\int_{\Omega} \mathbf{B} \cdot \boldsymbol{\varphi} \, dx \right]_{t=0}^{t=\tau} &= \int_0^{\tau} \int_{\Omega} \mathbf{B} \cdot \partial_t \boldsymbol{\varphi} \, dx \, dt \\ &- \int_0^{\tau} \int_{\Omega} \left((\operatorname{div} \mathbf{u}) \mathbf{B} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} \right) \cdot \boldsymbol{\varphi} \, dx \, dt + \nu \int_0^{\tau} \int_{\Omega} \mathbf{B} \cdot \Delta \boldsymbol{\varphi} \, dx \, dt, \\ &\int_0^{\tau} \int_{\Omega} \mathbf{B} \cdot \nabla \phi \, dx \, dt = 0, \end{aligned} \quad (4.27)$$

for a.e. $0 \leq \tau \leq T$, all $\boldsymbol{\varphi} \in [\mathcal{D}([0, T] \times \Omega)]^3$, and $\phi \in \mathcal{D}([0, T] \times \Omega)$.

Finally, we also obtain the energy inequality. We can construct an energy defect measure

$$\mathfrak{E} \in L^{\infty}(0, T; \mathcal{M}(\Omega))$$

such that

$$\begin{aligned} &\left[\int_{\Omega} \left(\frac{1}{2}(\rho + n)|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{B}|^2 + G_1(\rho) + G_2(n) + \frac{\varepsilon^2}{2}|\nabla \Phi|^2 \right) dx \right]_{t=0}^{t=\tau} \\ &+ \int_0^{\tau} \int_{\Omega} (\mu \nabla \mathbf{u} + (\mu + \lambda) \operatorname{div} \mathbf{u} \mathbb{I}) : \nabla \mathbf{u} + \nu |\nabla \mathbf{B}|^2 \, dx \, dt + \int_{\Omega} \mathcal{D} \, \mathfrak{E}(\tau) \leq 0. \end{aligned} \quad (4.28)$$

for a.e. $0 \leq \tau \leq T$. Then we obtain global existence of dissipative solutions to (1.1)–(1.4). $\square$

Theorem 4.1 (Global existence of dissipative solutions). *Let $\Omega := \mathbb{T}^3$ be the three dimensional torus. Assume that the boundary data belong to the following class*

$$\begin{aligned} \rho_0 &\in L^{\gamma}(\Omega), \quad n_0 \in L^{\alpha}(\Omega), \\ \rho_0 &\geq 0, \quad n_0 \geq 0, \quad \mathbf{m}_0 \in L^{\frac{2\min\{\alpha, \gamma\}}{\min\{\alpha, \gamma\}+1}}(\Omega; \mathbb{R}^3), \\ &\int_{\Omega} \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\rho_0 + n_0} + G_1(\rho_0) + G_2(n_0) + \frac{1}{2}|\mathbf{B}_0|^2 + \frac{\varepsilon^2}{2}|\nabla \Phi_0|^2 \right] dx < \infty. \end{aligned}$$

Then problem (1.1)–(1.5) admits at least one dissipative solution $[\rho, n, \mathbf{u}, \mathbf{B}, \mathfrak{R}, \Phi]$ in $(0, T) \times \Omega$ in the sense specified in Definition 1.1.

Acknowledgements

The author was supported by Dong-A university.

REFERENCES

- [1] A. Abbatiello, E. Feireisl, A. Novotný, Generalized solutions to models of compressible viscous fluids, *Discrete Cont. Dyn.* 41 (2019), 1-28.
- [2] L. Chen, D. Donatelli, P. Marcati, Incompressible type limit analysis of a hydrodynamic model for charge-carrier transport, *SIAM J. Math. Anal.* 45 (2013), 915-933.
- [3] S. Cordier, E. Grenier, Quasineutral limit of an Euler-Poisson system arising from plasma physics, *Comm. Part. Diff. Equations* 25 (2000), 1099-1113.
- [4] G. Crippa, C. Donadello, L. V. Spinolo, A note on the initial-boundary value problem for continuity equations with rough coefficients, In: *HYP 2012 Conference Proceedings*, AIMS Series in Appl. Math. 8 (2014), 957-966.
- [5] D. Donatelli, E. Feireisl, A. Novotný, Scale analysis of a hydrodynamic model of plasma, *Math. Models Methods Appl. Sci.* 25 (2015), 371-394.

- [6] D. Donatelli, P. Marcati, The Quasineutral Limit for the Navier-Stokes-Fourier-Poisson System, In: G.Q. Chen, H. Holden, K. Karlsen, (eds) *Hyperbolic Conservation Laws and Related Analysis with Applications*, Springer Proceedings in Mathematics & Statistics, vol 49. Springer, Berlin, Heidelberg, 2014.
- [7] J. I. Diaz, M.B. Lerena, On the inviscid and non-resistive limit for the equations of incompressible magneto-hydrodynamics, *Math. Models Methods Appl. Sci.* 12 (2002), 1401-1419.
- [8] N.M. Ferraro, S.C. Jardin, Calculations of two-fluid magnetohydrodynamic axisymmetric steady-states, *J. Comput. Phys.* 228 (2009), 7742-7770.
- [9] J. S. Fan, W. H. Yu, Strong solution to the compressible MHD equations with vacuum, *Nonlinear Anal. Real World Appl.* 10 (2009), 392-409.
- [10] D. Gérard-Varet, D. Han-Kwan, F. Rousset, Quasineutral limit of the Euler-Poisson system for ions in a domain with boundaries II, *J. Éc. polytech. Math.* 1 (2014), 343-386.
- [11] D. Gérard-Varet, D. Han-Kwan, F. Rousset, Quasineutral limit of the Euler-Poisson system for ions in a domain with boundaries. *Indiana Univ. Math. J.* 62 (2013), no. 2, 359-402.
- [12] B. Jin, Y.-S. Kwon, S. Nečasová, A. Novotný, Existence and stability of dissipative turbulent solutions to a simple bi-fluid model of compressible fluids, *J. Elliptic Parabol.* 7 (2021), 537-570.
- [13] Q. Ju, F. Li, H. Li, The quasineutral limit of compressible Navier-Stokes-Poisson system with heat conductivity and general initial data, *J. Differential Equations* 247 (2009), 203-224.
- [14] Q. Ju, Y. Li, Asymptotic limits of the full Navier-Stokes-Fourier-Poisson system, *J. Differential Equations* 254 (2013), 2587-2602.
- [15] Q. Ju, F. Li, S. Wang, Convergence of the Navier-Stokes-Poisson system to the incompressible Navier-Stokes equations, *J. Math. Phys.* 49 (2008), 073515.
- [16] Q. Ju, Y. Li, S. Wang, Rate of convergence from the Navier-Stokes-Poisson system to the incompressible Euler equations, *J. Math. Phys.* 50 (2009), 013533.
- [17] Y. Li, Q. Ju, W.-Q Xu, Quasi-neutral limit of the full Navier-Stokes-Fourier-Poisson system, *J. Differential Equations* 258 (2015), 3661-3687.
- [18] Y. Li, Y. Peng, Y. Wang, From two-fluid Euler-Poisson equations to one-fluid Euler equations, *Asymptotic Anal.* 85 (2013), 12-148.
- [19] X. Pu, B. Guo, Quasineutral limit of the pressureless Euler-Poisson equation for ions, *Quart. Appl. Math.* 74 (2016), 245-273.
- [20] S. Wang, Quasineutral limit of Euler-Poisson system with and without viscosity, *Comm. Part. Diff. Equations* 29 (2004), 419-456.