

NORMALIZED SOLUTIONS TO SCHRÖDINGER EQUATIONS WITH POTENTIALS AND GENERAL NONLINEARITIES

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Abstract. We investigate the existence of normalized solutions for the following Schrödinger equation with an L^2 -constraint

$$\begin{cases} -\Delta u + [V(x) + \lambda]u = f(u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c, \end{cases}$$

where $N \geq 3$, $c > 0$ is a prescribed mass, $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier, and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ exhibits the L^2 -supercritical growth. Without assuming $\mathcal{C}^1$ regularity on f , we establish the existence results in two scenarios concerning V : Non-potential case $V(x) \equiv 0$ and Potential case $V(x) \not\equiv 0$.

Keywords. General nonlinearities; L^2 -supercritical; Normalized solution; Potential; Schrödinger equation.

1. INTRODUCTION

This paper is concerned with the existence of normalized solutions for the following Schrödinger equation with an L^2 -constraint

$$\begin{cases} -\Delta u + [V(x) + \lambda]u = f(u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c, \end{cases} \quad (1.1)$$

where $N \geq 3$, $c > 0$ is a prescribed mass, $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier associated with the solution $u \in H^1(\mathbb{R}^N)$, and f satisfies the following basic assumption:

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and

$$\lim_{t \rightarrow 0} \frac{f(t)}{|t|^{1+\frac{4}{N}}} = 0, \quad \lim_{|t| \rightarrow +\infty} \frac{|f(t)|}{|t|^{2^*-1}} = 0.$$

We investigate two cases: the non-potential case $V(x) \equiv 0$ and the non-constant potential case $V(x) \not\equiv 0$ satisfying:

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Received 11 May 2025; Accepted 10 August 2026; Published online 1 November 2026.

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(V1) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$, $\min_{\mathbb{R}^N} V < \lim_{|x| \rightarrow \infty} V(x) = \sup_{\mathbb{R}^N} V$ and $\lim_{|x| \rightarrow \infty} \nabla V(x) \cdot x = 0$. Furthermore, there exists $\sigma_1 \in (0, 1)$ such that

$$\int_{\mathbb{R}^N} |V(x)| u^2 dx \leq \sigma_1 \|\nabla u\|_2^2, \quad \forall u \in H^1(\mathbb{R}^N).$$

Without loss of generality, we can assume that $\sup_{\mathbb{R}^N} V = 0$ in (V1). If not, we replace $(V(x), \lambda)$ with $(V(x) - \sup_{\mathbb{R}^N} V, \lambda + \sup_{\mathbb{R}^N} V)$.

Normalized solutions, which are solutions having a priori prescribed L^2 -norm (mass), correspond to critical points of the energy functional $\Phi : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined by:

$$\Phi(u) = \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} F(u) dx, \quad (1.2)$$

on the L^2 -constraint

$$\mathcal{S}_c := \{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = c\},$$

where $F(u) := \int_0^u f(t) dt$. It is standard that Φ is $\mathcal{C}^1$ on $H^1(\mathbb{R}^N)$ under assumptions (V1) (or $V(x) \equiv 0$) and (F1). These normalized solutions hold significant physical relevance as their L^2 -norm remains invariant under time evolution. Moreover, their variational characterization facilitates the analysis of orbital stability and instability properties (see [2, 18]). Following the pioneering work of Stuart [20] and Cazenave-Lions [5], the study of normalized solutions has emerged as an active research direction in nonlinear PDE analysis; see, e.g., [7, 13, 14, 15, 18, 19, 24].

The variational analysis of (1.1) fundamentally depends on the asymptotic behavior of the nonlinearity f relative to the L^2 -critical exponent $2 + \frac{4}{N}$, which arises from the Gagliardo-Nirenberg inequality [10, 17]. This behavior determines whether Φ is bounded below on $\mathcal{S}_c$, leading to two distinct research approaches.

The problem is termed L^2 -subcritical if $\inf_{\mathcal{S}_c} \Phi > -\infty$, allowing for the search for ground state solutions through constrained minimization combined with Lions' concentration-compactness principle. In contrast, the L^2 -supercritical regime is characterized by the condition:

$$(F2) \quad \lim_{|t| \rightarrow +\infty} \frac{F(t)}{|t|^{2+\frac{4}{N}}} = +\infty,$$

which results in $\inf_{\mathcal{S}_c} \Phi = -\infty$. This necessitates alternative minimax methods on $\mathcal{S}_c$ due to the failure of global minimization.

In comparison to the L^2 -subcritical case, the variational analysis in the L^2 -supercritical case is more complex, particularly in establishing the boundedness of sequences and verifying the convergence of the L^2 -norm to preserve the mass constraint $\|u\|_2^2 = c$. This complexity arises from the presence of the unknown Lagrange multiplier λ , as standard Nehari manifold techniques are inadequate, compounded by the lack of compactness for the embedding $H_{\text{rad}}^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$.

In the present paper, we focus on the L^2 -supercritical regime, addressing both the non-potential case $V(x) \equiv 0$ and the non-constant potential case $V(x) \not\equiv 0$ satisfying (V1). The subsequent two subsections will present previous works that motivate our current investigation and outline our main results for the non-potential and potential cases, respectively. Before proceeding, we introduce the following definitions.

We define the following Pohozaev functional:

$$\mathcal{P}(u) := \|\nabla u\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x) u^2 dx - \frac{N}{2} \int_{\mathbb{R}^N} [f(u)u - 2F(u)] dx, \quad \forall u \in H^1(\mathbb{R}^N). \quad (1.3)$$

It is well-known that any critical point of $\Phi|_{\mathcal{S}_c}$ satisfies the Pohozaev identity $\mathcal{P}(u) = 0$. We say that u is a ground state solution to (1.1) if it is a solution with minimal energy among all solutions belonging to $\mathcal{S}_c$.

Specifically, if $V(x) \equiv 0$, then (1.1) reduces to the autonomous equation:

$$\begin{cases} -\Delta u + \lambda u = f(u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c. \end{cases} \quad (1.4)$$

Similarly, we define the corresponding energy functional and Pohozaev functional as follows:

$$\Phi^\infty(u) = \frac{1}{2} \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} F(u) dx, \quad \forall u \in H^1(\mathbb{R}^N) \quad (1.5)$$

and

$$\mathcal{P}^\infty(u) := \|\nabla u\|_2^2 - \frac{N}{2} \int_{\mathbb{R}^N} [f(u)u - 2F(u)] dx, \quad \forall u \in H^1(\mathbb{R}^N). \quad (1.6)$$

1.1. Non-potential case $V(x) \equiv 0$. The seminal contribution to the L^2 -supercritical regime was established by Jeanjean [12] in 1997, where radial mountain-pass solutions to (1.4) were obtained under assumptions (H0) and (H1). Furthermore, the existence of ground state solutions was proven if (H2) is also satisfied. The assumptions (H0)-(H2) are defined as follows:

(H0) f is odd;

(H1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and there exist $\alpha, \beta \in \mathbb{R}$ satisfying $2 + \frac{4}{N} < \alpha \leq \beta < 2^* = \frac{2N}{N-2}$ such that

$$0 < \alpha F(t) \leq f(t)t \leq \beta F(t), \quad \forall t \in \mathbb{R} \setminus \{0\}; \quad (1.7)$$

(H2) the function $\tilde{F}(t) := f(t)t - 2F(t)$ is of class $\mathcal{C}^1$ and

$$\tilde{F}'(t)t > \left(2 + \frac{4}{N}\right) F(t), \quad \forall t \in \mathbb{R} \setminus \{0\}.$$

The pivotal innovation in [12] lies in the construction of a particular (PS) sequence $\{u_n\} \subset \mathcal{S}_c$ asymptotically satisfying the Pohozaev identity $\mathcal{P}^\infty(u_n) \rightarrow 0$, through the application of Ekeland principle to an augmented functional $(u, t) \in \mathcal{S}_c \times \mathbb{R} \mapsto \Phi^\infty(e^{Nt/2}u(e^t \cdot))$ which retains the same geometric structure as Φ^∞ on $\mathcal{S}_c$. The boundedness of $\{\|\nabla u_n\|_2\}$ follows from the key condition $\mathcal{P}^\infty(u_n) \rightarrow 0$ combined with the inequality $\alpha F(t) \leq f(t)t$ of (1.7) in (H1) through the relation

$$\begin{aligned} \Phi^\infty(u_n) + o(1) &= \Phi^\infty(u_n) - \frac{2}{(\alpha - 2)N} \mathcal{P}^\infty(u_n) \\ &= \frac{(\alpha - 2)N - 4}{2(\alpha - 2)N} \|\nabla u_n\|_2^2 + \frac{1}{\alpha - 2} \int_{\mathbb{R}^N} [f(u_n)u_n - \alpha F(u_n)] dx. \end{aligned}$$

By incorporating the fact that any solution must belong to the Pohozaev manifold

$$\mathcal{M}^\infty(c) := \{u \in \mathcal{S}_c : \mathcal{P}^\infty(u) = 0\}$$

into the variational procedure, Jeanjean further proved the mountain pass level coincides with the minimum

$$m^\infty(c) := \inf_{u \in \mathcal{M}^\infty(c)} \Phi^\infty(u),$$

provided f additionally satisfies (H2), thereby establishing the existence of ground state solutions. In the compactness analysis, Jeanjean introduced two distinctive strategies, fundamentally relying on conditions (H1) or (H2): the first ensures the positivity of the Lagrange multiplier λ through the compact embedding $H_{\text{rad}}^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ ($2 < p < 2^*$), combined with the inequality $f(t)t \leq \beta F(t)$ of (1.7) in (H1), while the second employs (H2) to establish that the ground state energy map $c \mapsto m^\infty(c)$ is nonincreasing with respect to $c > 0$ and strictly decreasing for some c . Since this groundbreaking work, both methods have become standard tools for addressing compactness issues in the context of normalized solutions.

It is noteworthy that the inequality $\alpha F(t) \leq f(t)t$ of (1.7) in (H1) is regarded as the natural L^2 -constrained counterpart to the classical Ambrosetti-Rabinowitz condition (referred to as the (AR) condition which is proposed by [1]) for unconstrained problems. Recently, the first and second authors in [8] weakened (H1) to

$$(H1') \quad 0 < \left(2 + \frac{4}{N}\right) F(t) \leq f(t)t < \frac{2N}{N-2} F(t) \text{ for all } t \in \mathbb{R} \setminus \{0\},$$

combined with appropriate L^2 -supercritical conditions at infinity. This work investigated the existence of radial mountain pass solutions for (1.4), but did not consider ground state solutions.

To obtain ground state solutions, Bieganowski-Mederski [4] introduced a novel minimization scheme on the intersection of closed L^2 -balls $\{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 \leq c\}$ and the Pohozaev manifold in $H^1(\mathbb{R}^N)$, replacing the standard constraint sphere $\mathcal{S}_c$. This innovation allowed for the relaxation of (H1) to the following condition:

$$(H1'') \quad 0 < \left(2 + \frac{4}{N}\right) F(t) \preceq f(t)t \preceq \frac{2N}{N-2} F(t) \text{ for all } t \in \mathbb{R} \setminus \{0\},$$

where $\preceq$ denotes the asymptotic non-strict inequality: for given functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$, $f_1(s) \preceq f_2(s)$ means that $f_1(s) \leq f_2(s)$ for all $s \in \mathbb{R}$ and for any $\gamma > 0$, there exists $|s| < \gamma$ such that $f_1(s) < f_2(s)$ (see Lemma 3.1 for essential properties). However, the analysis in [12] still required the stronger regularity assumption (H2).

In contrast, Jeanjean-Lu [15] removed the requirement $f \in \mathcal{C}^1$ and achieved another generalization by weakening the assumptions (H1) and (H2) to (F1), (F2), and the following conditions:

$$(H3) \quad 0 < f(t)t < \frac{2N}{N-2} F(t) \text{ for all } t \in \mathbb{R} \setminus \{0\};$$

$$(H4) \quad [f(t)t - 2F(t)]/|t|^{2+\frac{4}{N}} \text{ is strictly decreasing on } (-\infty, 0) \text{ and strictly increasing on } (0, +\infty).$$

By combining the Nehari-type techniques developed by Szulkin and Weth [21, 22] with Ghoussoub's minimax principle [11] on manifolds relying on complex topological arguments, they constructed a (PS) sequence that exactly satisfies the Pohozaev identity $\mathcal{P}^\infty(u_n) = 0$. Crucially, while eliminating the $\mathcal{C}^1$ -regularity requirements, their approach fundamentally depends on the strict monotonicity condition in (H4).

Condition (H4) has become a fundamental analytical tool in the study of L^2 -constrained elliptic problems, serving as the natural counterpart to the classical Nehari-type strict monotonicity condition in unconstrained settings, where the Pohozaev manifold $\mathcal{M}^\infty(c)$ plays an analogous role to the unconstrained Nehari manifold.

Despite this progress in this direction, a crucial gap still exists between the constrained and unconstrained framework: while the strictness requirement in the Nehari-type monotonicity condition has been substantially weakened for unconstrained problems, the analogous relaxation of strict monotonicity in (H4) for the constrained problem (1.4) remains unclear.

The primary objective of the present work is to address this problem. Specifically, we aim to establish the existence of ground state solutions to (1.4) by developing new analytical techniques that allow for the relaxation of assumptions (H3) and (H4) to the following conditions:

$$(F3) \quad 0 < f(t)t \preceq \frac{2N}{N-2}F(t) \text{ for all } t \in \mathbb{R} \setminus \{0\};$$

$$(F4) \quad [f(t)t - 2F(t)]/|t|^{2+\frac{4}{N}} \text{ is non-increasing on } (-\infty, 0) \text{ and non-decreasing on } (0, +\infty).$$

Our first result is as follows.

Theorem 1.1. *Let $N \geq 3$ and $c > 0$. Assume that f satisfies (F1)-(F4). Then (1.4) has a couple solution $(\bar{u}_c, \lambda_c) \in H^1(\mathbb{R}^N) \times (0, +\infty)$ such that*

$$\Phi^\infty(\bar{u}_c) = m^\infty(c) = \inf_{\mathcal{M}^\infty(c)} \Phi^\infty.$$

Remark 1.1. i) The proof of Theorem 1.1 is based on a direct minimization on $\mathcal{M}^\infty(c)$, facing two principal technical obstacles that require the implementation of new ideas. First, while any minimizing sequence $\{u_n\} \subset \mathcal{M}^\infty(c)$ for $m^\infty(c) = \inf_{\mathcal{M}^\infty(c)} \Phi^\infty$ admits a nontrivial weak limit $u \in H^1(\mathbb{R}^N)$ with $\|u\|_2^2 \in (0, c]$ up to a subsequence and translations, the absence of guaranteed mass conservation $u \in \mathcal{S}_c$ obstructs the verification of attainability even under the strict monotonicity assumption (H4). Second, the lack of derivative information $(\Phi^\infty)'(u_n)$ prevents the recovery of compactness through standard Lagrange multiplier analysis. These fundamental limitations, originally pointed out in [15, Remark 1.4 (iii)], explain why the topological minimax approach was adopted rather than direct minimization in [15].

ii) Inspired by the non-strict inequality property developed by Bieganowski-Mederski [4], we establish a robust inequality involving $\Phi^\infty(u)$, $\Phi^\infty(t^{N/2}u(t \cdot))$ and $\mathcal{P}^\infty(u)$ under the weakened monotonicity (see Lemmas 3.2 and 3.3). This allows us to demonstrate the strict monotonicity of $c \mapsto m^\infty(c)$ (see Lemma 3.8), ensuring the existence of minimizers. The final verification that minimizers solve (1.4) employs deformation techniques without requiring $f \in \mathcal{C}^1$ (see Lemma 3.10), circumventing the aforementioned obstacles under relaxed hypotheses.

1.2. Potential case $V(x) \not\equiv 0$. When the potential $V(x) \not\equiv 0$ satisfies (V1), the problem presents fundamentally new challenges compared to the non-potential case. The spatial dependence of $V(x)$ breaks translation invariance and introduces the nonhomogeneity, leading to an x -dependent linearization at zero:

$$\lim_{u \rightarrow 0} \frac{-V(x)u + f(u)}{u} = -V(x) + \lim_{u \rightarrow 0} \frac{f(u)}{u},$$

which depends explicitly on $x \in \mathbb{R}^N$. These features make the existing techniques for $V(x) \equiv 0$ inapplicable, resulting in limited mathematical progress for L^2 -supercritical problems with non-constant potentials despite their physical significance.

The first mathematical treatment of (1.1) with negative potentials is found in the work of Molle-Riey-Verzini [16], which dealt with the pure power nonlinearity $f(u) = |u|^{p-2}u$ with $2 + \frac{4}{N} < p < 2^*$. Under the smallness condition

$$\max \{ \|V\|_{N/2}, \|x \cdot \nabla V\|_N \} \leq L(N, p),$$

where $L(N, p)$ is an explicit constant, they established the existence of mountain pass solutions. A crucial element of their approach relies on the homogeneous scaling property characteristic of a power nonlinearity, which fundamentally fails for general nonlinearities. Subsequently,

Ding-Zhong [9] extended the result to broader classes of nonlinearities satisfying (H0) and (H1) proposed by [12] and the following stronger condition than (H2)

(H2') the function $\tilde{F}(t) := f(t)t - 2F(t)$ is of class $\mathcal{C}^1$ and $\tilde{F}'(t)t \geq \alpha F(t)$ for all $t \in \mathbb{R}$, which addressed the case $V(x) \not\equiv 0$, under modified assumptions: V is $\mathcal{C}^2$ (essentially), and satisfies a slightly stronger version of (V1), specifically where $\sigma_1 \in \left(0, \frac{N(\alpha-2)-4}{N(\alpha-2)}\right]$ instead of $\sigma_1 \in (0, 1)$, along with the following conditions:

(V2') there exists $0 \leq \sigma'_2 < \min \left\{ \frac{N(\alpha-2)(1-\sigma_1)}{4} - 1, \frac{N}{\beta} - \frac{N-2}{2} \right\}$ such that

$$\left| \int_{\mathbb{R}^N} [\nabla V(x) \cdot x] u^2 dx \right| \leq 2\sigma'_2 \|\nabla u\|_2^2, \quad \forall u \in H^1(\mathbb{R}^N);$$

(V3') there exists $\sigma'_3 \in \left[0, \frac{N\alpha}{2} - N - 2\right)$ such that

$$\int_{\mathbb{R}^N} Y^+(x) u^2 dx \leq \sigma'_3 \|\nabla u\|_2^2, \quad \forall u \in H^1(\mathbb{R}^N),$$

where $Y(x) := \left(\frac{N}{2}\alpha - N\right)W(x) + \nabla W(x) \cdot x$ with $W(x) := \frac{1}{2}\nabla V(x) \cdot x$.

Here, $g^\pm(x) = \max\{\pm g(x), 0\}$ represents the positive/negative part of any function $g(x)$ defined on $\mathbb{R}^N$. Their proof is based on a constrained minimization scheme on Pohozaev submanifolds, combined with Lagrange multiplier techniques. It is noteworthy that this method requires both the strong regularity assumption (H2') on f (which implies the strict monotonicity condition (H4)) and more restrictive assumptions on V , particularly the technical condition (V3').

Motivated by these developments, the second objective of this paper is to further investigate the existence of normalized solutions for the non-autonomous problem (1.1) involving potentials and general nonlinearities. In particular, we replace the strong regularity condition (H2') used in [9] with the weaker monotonicity condition (F4), while eliminating the essential requirement $V \in \mathcal{C}^2$.

To state our main result in this direction, we introduce the following assumptions:

(V2) there holds

$$\int_{\mathbb{R}^N} \left[V(x) + \frac{2}{N(\alpha-2)} \nabla V(x) \cdot x \right]^- u^2 dx \leq \frac{N(\alpha-2)-4}{N(\alpha-2)} \|\nabla u\|_2^2, \quad \forall u \in H^1(\mathbb{R}^N);$$

(V3) there exists $\sigma_2 \in \left(0, \frac{2N-\beta(N-2)}{N(\beta-2)}\right)$ such that

$$\int_{\mathbb{R}^N} \left[V(x) + \frac{\beta}{N(\beta-2)} \nabla V(x) \cdot x \right]^+ u^2 dx \leq \sigma_2 \|\nabla u\|_2^2, \quad \forall u \in H^1(\mathbb{R}^N).$$

Our second result is as follows.

Theorem 1.2. *Let $N \geq 3$ and $c > 0$. Assume that (V1)-(V3), (H1) and (F4) hold. Then (1.1) admits a mountain pass solution $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times (0, +\infty)$.*

Remark 1.2. There exist many functions satisfying (V1)-(V3). For example,

$$V(x) = -\frac{\sigma}{1+|x|^k}, \quad x \in \mathbb{R}^N, \quad k \geq 2, \quad \sigma > 0 \text{ is suitable small.}$$

Remark 1.3. i) In contrast to the Pohozaev submanifold minimization method used in [9], the proof of Theorem 1.2 is based on critical point theorems on manifolds developed in [8]. This allows us to construct a (PS) sequence $\{u_n\} \subset \mathcal{S}_c$ asymptotically satisfying the Pohozaev identity $\mathcal{P}(u_n) \rightarrow 0$ at the mountain pass level $M(c)$ (see Lemmas 4.1- 4.2). The boundedness of $\{\|\nabla u_n\|_2\}$ follows from $\mathcal{P}(u_n) \rightarrow 0$ combined with assumption (V1).

ii) The presence of the non-autonomous potential $V(x) \not\equiv 0$ introduces **new difficulties** for the compactness analysis. For example, the compact embedding $H_{\text{rad}}^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $2 < p < 2^*$ can no longer be applied in the non-radial context; the strict monotonicity of $c \mapsto m^\infty(c)$, valid for non-potential settings, fails here. To address these issues, we establish the sharp estimate: $M(c) < m^\infty(c)$ (see Lemma 4.4), where $m^\infty(c)$ is the ground state energy of the non-potential (limiting) problem, serving as the first energy threshold where the (PS) condition may fail. This strict inequality allows us to establish the positivity of the Lagrange multiplier $\lambda > 0$ under assumption (V3), enabling the convergence of the L^2 -norm.

iii) Notably, our method circumvents the technical complexities involved in Pohozaev manifold decomposition and topological minimax arguments, thereby eliminating the need for the oddness assumption (H0) on f and avoiding the imposition of higher-order regularity conditions on ∇V and f as required in [9]. This approach suggests potential applicability to other L^2 -constrained problems with external potentials and continuous L^2 -supercritical nonlinearities.

In Section 2, we present critical point theories on manifolds, which will be used for the proof of Theorem 1.2. In Section 3, we establish the existence of ground state solutions for autonomous problem (1.4), completing the proof of Theorem 1.1. In Section 4, we establish the existence of mountain pass solutions for non-autonomous problem (1.1), completing the proof of Theorem 1.2

Throughout the paper, we make use of the following notations:

- $H_{\text{rad}}^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) \mid u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^N\}$;
- $L^s(\mathbb{R}^N)$ ($1 \leq s < \infty$) denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^N} |u|^s dx)^{1/s}$;
- For any $N \geq 3$ there exists an optimal constant $\mathcal{S} > 0$ depending only on N , such that

$$\mathcal{S}\|u\|_{2^*}^2 \leq \|\nabla u\|_2^2, \quad \forall u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \quad (\text{Sobolev inequality}) \quad (1.8)$$

and $\mathcal{C}_{N,s} > 0$ such that

$$\|u\|_s^s \leq \mathcal{C}_{N,s} \|\nabla u\|_2^{\frac{N(s-2)}{2}} \|u\|_2^{\frac{2N-s(N-2)}{2}}, \quad \forall u \in H^1(\mathbb{R}^N) \quad (\text{Gagliardo-Nirenberg inequality}); \quad (1.9)$$

- For any $u \in H^1(\mathbb{R}^N)$ and $t > 0$, we set $u_t(x) := u(tx)$;
- For any $x \in \mathbb{R}^N$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}$ and $B_r = B_r(0)$;
- $C_1, C_2, \dots$ denote positive constants possibly different in different places.

2. CRITICAL POINT THEORIES ON MANIFOLDS

Before presenting critical point theories on manifolds, we formalize necessary structures. Let H be a real Hilbert space with norm $\|\cdot\|_H$ and inner product $(\cdot, \cdot)_H$, and let E be a real Banach space, with norm $\|\cdot\|_E$, embedded continuously into H : $E \hookrightarrow H \hookrightarrow E^*$, where E^* is the dual space of E . Thus H is identified with its dual space. We assume in the sequel that E and H are infinite dimensional spaces. Define the manifold $M := \{u \in E : \|u\|_H = 1\}$ equipped with the subspace topology from E , which is the trace of the unit sphere of H in E and is unbounded

in general. Moreover M is a submanifold of E of codimension 1 and its tangent space at a given point $u \in M$ can be considered as a closed subspace of E of codimension 1, namely

$$T_u M := \{v \in E : (u, v)_H = 0\}.$$

For $\varphi \in \mathcal{C}^1(E, \mathbb{R})$, we denote by $\varphi|_M$ the trace of φ on M . Then $\varphi|_M$ is a $\mathcal{C}^1$ functional on M , and for any $u \in M$, $\langle \varphi|'_M(u), v \rangle = \langle \varphi'(u), v \rangle$ for all $v \in T_u M$. For any $u \in M$, we define the norm $\|\varphi|'_M(u)\|$ by

$$\|\varphi|'_M(u)\| = \sup_{v \in T_u M, \|v\|_E=1} |\langle \varphi'(u), v \rangle|.$$

Let $E \times \mathbb{R}$ be equipped with the scalar product

$$((u, \tau), (v, \sigma))_{E \times \mathbb{R}} := (u, v)_H + \tau\sigma, \quad \forall (u, \tau), (v, \sigma) \in E \times \mathbb{R},$$

and corresponding norm

$$\|(u, \tau)\|_{E \times \mathbb{R}} := \sqrt{\|u\|_H^2 + \tau^2}, \quad \forall (u, \tau) \in E \times \mathbb{R}.$$

For $\tilde{\varphi} \in \mathcal{C}^1(E \times \mathbb{R}, \mathbb{R})$, analogous definitions hold, moreover, for any $(u, \tau) \in M \times \mathbb{R}$,

$$\langle \tilde{\varphi}|'_{M \times \mathbb{R}}(u, \tau), (v, \sigma) \rangle := \langle \tilde{\varphi}'(u, \tau), (v, \sigma) \rangle, \quad \forall (v, \sigma) \in \tilde{T}_{(u, \tau)}(M \times \mathbb{R}),$$

where $\tilde{T}_{(u, \tau)}(M \times \mathbb{R}) := \{(v, \sigma) \in E \times \mathbb{R} : (u, v)_H = 0\}$. For any $(u, \tau) \in M \times \mathbb{R}$, we define the norm $\|\tilde{\varphi}|'_{M \times \mathbb{R}}(u, \tau)\|$ by

$$\|\tilde{\varphi}|'_{M \times \mathbb{R}}(u, \tau)\| = \sup_{(v, \sigma) \in \tilde{T}_{(u, \tau)}(M \times \mathbb{R}), \|(v, \sigma)\|_{E \times \mathbb{R}}=1} |\langle \tilde{\varphi}'(u, \tau), (v, \sigma) \rangle|.$$

Next, we present the critical point theories on manifolds, which will be used in the proof of Theorem 1.2.

Lemma 2.1. [8] *Let $\varphi \in \mathcal{C}^1(E, \mathbb{R})$, $S \subset M$, $\tilde{a} \in \mathbb{R}$, $\varepsilon, \delta > 0$ such that*

$$u \in M \cap \varphi^{-1}([\tilde{a} - 2\varepsilon, \tilde{a} + 2\varepsilon]) \cap S_{2\delta} \Rightarrow \|\varphi|'_M(u)\| \geq \frac{8\varepsilon}{\delta}.$$

Then, there exists $\eta \in \mathcal{C}([0, 1] \times M, M)$ such that

- (i) $\eta(t, u) = u$, if $t = 0$, or if $u \notin M \cap \varphi^{-1}([\tilde{a} - 2\varepsilon, \tilde{a} + 2\varepsilon]) \cap S_{2\delta}$;
- (ii) $\eta(1, \varphi^{\tilde{a}+\varepsilon} \cap S) \subset \varphi^{\tilde{a}-\varepsilon}$;
- (iii) for every $t \in [0, 1]$, $\eta(t, \cdot) : M \rightarrow M$ is a homeomorphism;
- (iv) $\|\eta(t, u) - u\| \leq \delta$, $\forall u \in M$, $t \in [0, 1]$;
- (v) for every $u \in M$, $\varphi(\eta(t, u))$ is non-increasing on $t \in [0, 1]$;
- (vi) $\varphi(\eta(t, u)) < \tilde{a}$, $\forall u \in M \cap \varphi^{\tilde{a}} \cap S_\delta$, $t \in [0, 1]$.

Lemma 2.2. [3] *Let $\{u_n\} \subset M$ be a bounded sequence in E . Then the following are equivalent:*

- (i) $\|\varphi|'_M(u_n)\| \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) $\varphi'(u_n) - \langle \varphi'(u_n), u_n \rangle u_n \rightarrow 0$ in E' as $n \rightarrow \infty$.

Lemma 2.3. [8] *Assume that $\theta_1, \theta_2 \in \mathbb{R}$ and $\tilde{\varphi} \in \mathcal{C}^1(E \times \mathbb{R}, \mathbb{R})$ satisfies*

$$\tilde{a} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \max_{t \in [0, 1]} \tilde{\varphi}(\tilde{\gamma}(t)) > \tilde{b} := \sup_{\tilde{\gamma} \in \tilde{\Gamma}} \max \{\tilde{\varphi}(\tilde{\gamma}(0)), \tilde{\varphi}(\tilde{\gamma}(1))\}, \quad (2.1)$$

where

$$\tilde{\Gamma} := \{\tilde{\gamma} \in \mathcal{C}([0, 1], M \times \mathbb{R}) : \tilde{\varphi}(\tilde{\gamma}(0)) \leq \theta_1, \tilde{\varphi}(\tilde{\gamma}(1)) < \theta_2\}.$$

Let $\{\tilde{\gamma}_n\} \subset \tilde{\Gamma}$ be such that $\sup_{t \in [0,1]} \tilde{\Phi}(\tilde{\gamma}_n(t)) \leq \tilde{a} + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then there exists a sequence $\{(v_n, \tau_n)\} \subset M \times \mathbb{R}$ satisfying

- (i) $\tilde{a} - \frac{2}{n} \leq \tilde{\Phi}(v_n, \tau_n) \leq \tilde{a} + \frac{2}{n}$;
- (ii) $\min_{t \in [0,1]} \|(v_n, \tau_n) - \tilde{\gamma}_n(t)\|_{E \times \mathbb{R}} \leq \frac{2}{\sqrt{n}}$;
- (iii) $\|\tilde{\Phi}'_{M \times \mathbb{R}}(v_n, \tau_n)\| \leq \frac{8}{\sqrt{n}}$.

3. GROUND STATE SOLUTIONS FOR AUTONOMOUS PROBLEM (1.4)

In this section, we establish the existence of ground state solutions for (1.4), thereby completing the proof of Theorem 1.1.

Lemma 3.1. [4, Lemma 2.1] *Let $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ such that $g(s) \leq 0$ and $|g(s)| \leq C(|s|^2 + |s|^{2^*})$ for any $s \in \mathbb{R}$ and some $C > 0$. Then $g(s) \leq 0$ for $s \in \mathbb{R}$ if and only if*

$$\int_{\mathbb{R}^N} g(u) dx < 0, \quad \forall u \in H^1(\mathbb{R}^N) \setminus \{0\}.$$

Lemma 3.2. *Assume that (F1), (F2) and (F4) hold. Then*

- (i) *for any $\tau \in \mathbb{R}$ and $t > 0$, there holds*

$$g(t, \tau) := N(1 - t^2)f(\tau)\tau - (2N + 4 - 2Nt^2)F(\tau) + 4t^{-N}F\left(t^{\frac{N}{2}}\tau\right) \geq 0; \quad (3.1)$$

- (ii) *for any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, there holds*

$$\int_{\mathbb{R}^N} g(t, u) dx > 0, \quad \forall t \in (0, 1) \cup (1, +\infty). \quad (3.2)$$

Proof. (i). Set $\zeta(\tau) := [f(\tau)\tau - 2F(\tau)]|\tau|^{-2-\frac{4}{N}}$. It is evident that (3.1) holds for $\tau = 0$. For $\tau \neq 0$, from (F4), one has

$$\begin{aligned} \frac{dg(t, \tau)}{dt} &= 2Nt \left\{ \frac{f\left(t^{\frac{N}{2}}\tau\right)\left(t^{\frac{N}{2}}\tau\right) - 2F\left(t^{\frac{N}{2}}\tau\right)}{t^{N+2}} - [f(\tau)\tau - 2F(\tau)] \right\} \\ &= 2Nt|\tau|^{2+\frac{4}{N}} \left[\zeta\left(t^{\frac{N}{2}}\tau\right) - \zeta(\tau) \right] \\ &\begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1, \end{cases} \end{aligned} \quad (3.3)$$

which yields that $g(t, \tau) \geq g(1, \tau) = 0$ for $t > 0$. It follows that (3.1) holds.

- (ii). For any given $t \in (0, 1) \cup (1, +\infty)$, set $h_t(\tau) := g(t, \tau)$. It follows from (3.3) that

$$h_t(\tau) = 2N|\tau|^{2+\frac{4}{N}} \int_t^1 s \left[\zeta(\tau) - \zeta\left(s^{\frac{N}{2}}\tau\right) \right] ds. \quad (3.4)$$

By (3.1), (F1), (F2) and (F4), $\zeta(\tau)$ is non-increasing on $(-\infty, 0)$ and non-decreasing on $(0, +\infty)$, and

$$\zeta(\tau) > 0, \quad \forall \tau \neq 0, \quad \lim_{|\tau| \rightarrow 0} \zeta(\tau) = 0, \quad \lim_{|\tau| \rightarrow +\infty} \zeta(\tau) = +\infty. \quad (3.5)$$

Therefore, for any $\gamma > 0$, (3.5) shows that there exist $0 < \tau_1 < \tau_2 < \gamma$ such that

$$\zeta(\tau_1) < \zeta(\tau) < \zeta(\tau_2), \quad \forall \tau_1 < \tau < \tau_2. \quad (3.6)$$

Then it follows from (3.4) and (3.6) that $h_t(\tau_2) > 0$ if $t \in (0, 1)$; and $h_t(\tau_1) > 0$ if $t \in (1, +\infty)$. Thus, we have

$$-h_t(\tau) \preceq 0, \quad \forall \tau \in \mathbb{R}, \quad t \in (0, 1) \cup (1, +\infty). \quad (3.7)$$

Hence, by Lemma 3.1, (3.7) implies that (3.2) holds. $\square$

Lemma 3.3. *Assume that (F1) holds. Then there holds*

$$\Phi^\infty(u) = \Phi^\infty\left(t^{\frac{N}{2}}u_t\right) + \frac{1-t^2}{2}\mathcal{P}^\infty(u) + \frac{1}{4}\int_{\mathbb{R}^N} g(t, u)dx, \quad \forall u \in \mathcal{S}_c, \quad t > 0. \quad (3.8)$$

Proof. By (1.2), we have

$$\Phi^\infty\left(t^{\frac{N}{2}}u_t\right) = \frac{t^2}{2}\|\nabla u\|_2^2 - t^{-N}\int_{\mathbb{R}^N} F\left(t^{\frac{N}{2}}u\right)dx. \quad (3.9)$$

From (1.2), (1.3), (3.1) and (3.9), one has

$$\begin{aligned} & \Phi^\infty(u) - \Phi^\infty\left(t^{\frac{N}{2}}u_t\right) \\ &= \frac{1-t^2}{2}\|\nabla u\|_2^2 - \int_{\mathbb{R}^N} \left[F(u) - t^{-N}F\left(t^{\frac{N}{2}}u\right)\right]dx \\ &= \frac{1-t^2}{2}\left[\|\nabla u\|_2^2 - \frac{N}{2}\int_{\mathbb{R}^N} [f(u)u - 2F(u)]dx\right] \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^N} \left[N(1-t^2)f(u)u - (2N+4-2Nt^2)F(u) + 4t^{-N}F\left(t^{\frac{N}{2}}u\right)\right]dx \\ &= \frac{1-t^2}{2}\mathcal{P}^\infty(u) + \frac{1}{4}\int_{\mathbb{R}^N} g(t, u)dx. \end{aligned}$$

$\square$

From Lemmas 3.2 and 3.3, we have the following corollary.

Corollary 3.1. *Assume that (F1), (F2) and (F4) hold. Then, for $u \in \mathcal{M}^\infty(c)$, there holds*

$$\Phi^\infty(u) = \max_{t>0} \Phi^\infty\left(t^{\frac{N}{2}}u_t\right). \quad (3.10)$$

Lemma 3.4. *Assume that (F1), (F2) and (F4) hold. Then, for any $u \in \mathcal{S}_c$, there exists a unique $t_u > 0$ such that $t_u^{\frac{N}{2}}u_{t_u} \in \mathcal{M}^\infty(c)$.*

Proof. Let $u \in \mathcal{S}_c$ be fixed and define a function $\zeta(t) := \Phi^\infty\left(t^{\frac{N}{2}}u_t\right)$ on $[0, \infty)$. Clearly, by (3.9), we have

$$\begin{aligned} \zeta'(t) = 0 &\Leftrightarrow t^2\|\nabla u\|_2^2 - \frac{N}{2}t^{-N}\int_{\mathbb{R}^N} \left[f\left(t^{\frac{N}{2}}u\right)t^{\frac{N}{2}}u - 2F\left(t^{\frac{N}{2}}u\right)\right]dx = 0 \\ &\Leftrightarrow \mathcal{P}^\infty\left(t^{\frac{N}{2}}u_t\right) = 0 \Leftrightarrow t^{\frac{N}{2}}u_t \in \mathcal{M}^\infty(c). \end{aligned}$$

It is easy to verify that $\zeta(0) = 0$, $\zeta(t) > 0$ for $t > 0$ small and $\zeta(t) < 0$ for t large. Therefore $\max_{t \in [0, \infty)} \zeta(t)$ is achieved at a $t_0 = t_u > 0$ so that $\zeta'(t_0) = 0$ and $t_0^{\frac{N}{2}}u_{t_0} \in \mathcal{M}^\infty(c)$.

Next, we claim that t_u is unique for any $u \in \mathcal{S}_c$. In fact, for any given $u \in \mathcal{S}_c$, let $t_1, t_2 > 0$ such that $\mathcal{P}^\infty\left(t_1^{\frac{N}{2}}u_{t_1}\right) = \mathcal{P}^\infty\left(t_2^{\frac{N}{2}}u_{t_2}\right) = 0$. Jointly with (3.8), we have

$$\Phi^\infty\left(t_1^{\frac{N}{2}}u_{t_1}\right) = \Phi^\infty\left(t_2^{\frac{N}{2}}u_{t_2}\right) + \frac{t_1^2 - t_2^2}{2t_1^2} \mathcal{P}^\infty\left(t_1^{\frac{N}{2}}u_{t_1}\right) + \frac{1}{4} \int_{\mathbb{R}^N} g\left(\frac{t_2}{t_1}, t_1^{\frac{N}{2}}u\right) dx$$

and

$$\Phi^\infty\left(t_2^{\frac{N}{2}}u_{t_2}\right) = \Phi^\infty\left(t_1^{\frac{N}{2}}u_{t_1}\right) + \frac{t_2^2 - t_1^2}{2t_2^2} \mathcal{P}^\infty\left(t_2^{\frac{N}{2}}u_{t_2}\right) + \frac{1}{4} \int_{\mathbb{R}^N} g\left(\frac{t_1}{t_2}, t_2^{\frac{N}{2}}u\right) dx.$$

The combination of the above two inequalities implies that $t_1 = t_2$. Therefore, $t_u > 0$ is unique for any $u \in \mathcal{S}_c$. $\square$

From Corollary 3.1 and Lemma 3.4, we have the following lemma.

Lemma 3.5. *Assume that (F1), (F2) and (F4) hold. Then*

$$\inf_{u \in \mathcal{M}^\infty(c)} \Phi^\infty(u) := m^\infty(c) = \inf_{u \in \mathcal{S}_c} \max_{t > 0} \Phi^\infty\left(t^{\frac{N}{2}}u_t\right).$$

Analogous to [23, Lemma 2.7, Lemma 2.8], we can prove the following lemma.

Lemma 3.6. *Assume that (F1) holds. If $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^N)$, then*

$$\Phi^\infty(u_n) = \Phi^\infty(\bar{u}) + \Phi^\infty(u_n - \bar{u}) + o(1)$$

and

$$\mathcal{P}^\infty(u_n) = \mathcal{P}^\infty(\bar{u}) + \mathcal{P}^\infty(u_n - \bar{u}) + o(1).$$

Lemma 3.7. *Assume that (F1), (F2) and (F4) hold. Then*

- (i) *there exists $\vartheta_c > 0$ such that $\|\nabla u\|_2 \geq \vartheta_c$, $\forall u \in \mathcal{M}^\infty(c)$;*
- (ii) *$m^\infty(c) > 0$.*

Proof. i). Since $\mathcal{P}^\infty(u) = 0$, $\forall u \in \mathcal{M}^\infty(c)$, then it follows from (F1), (1.6), (1.8) and (1.9) that

$$\begin{aligned} \|\nabla u\|_2^2 &= \frac{N}{2} \int_{\mathbb{R}^N} [f(u)u - 2F(u)] dx \\ &\leq \frac{c^{-\frac{2}{N}}}{\mathcal{C}_{N, 2+\frac{4}{N}}} \|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} + C_1 \|u\|_2^{2^*} \\ &\leq \frac{1}{2} \|\nabla u\|_2^2 + C_1 \mathcal{S}^{-\frac{N}{N-2}} \|\nabla u\|_2^{2^*}, \quad \forall u \in \mathcal{M}^\infty(c), \end{aligned}$$

which implies that there exists $\vartheta_c > 0$ such that $\|\nabla u\|_2 \geq \vartheta_c$ for all $u \in \mathcal{M}^\infty(c)$.

ii). For $u \in \mathcal{M}^\infty(c)$, let $t_u = \frac{1}{\|\nabla u\|_2} \left[\frac{(N-2)\mathcal{S}^{\frac{N}{N-2}}}{4NC_2} \right]^{\frac{N-2}{4}}$. From (F1), (1.5), (1.8), (1.9) and (3.10), we have

$$\begin{aligned} \Phi^\infty(u) &\geq \Phi^\infty\left(t_u^{\frac{N}{2}} u_{t_u}\right) = \frac{t_u^2}{2} \|\nabla u\|_2^2 - \frac{1}{t_u^N} \int_{\mathbb{R}^N} F\left(t_u^{\frac{N}{2}} u\right) dx \\ &\geq \frac{t_u^2}{2} \|\nabla u\|_2^2 - \frac{t_u^2 c^{-\frac{2}{N}}}{4\mathcal{C}_{N,2+\frac{4}{N}}} \|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} - C_2 t_u^{2^*} \|u\|_{2^*}^{2^*} \\ &\geq \frac{t_u^2}{4} \|\nabla u\|_2^2 - C_2 t_u^{2^*} \mathcal{S}^{-\frac{N}{N-2}} \|\nabla u\|_2^{2^*} \\ &= \frac{1}{2N} \left[\frac{(N-2)\mathcal{S}^{\frac{N}{N-2}}}{4NC_2} \right]^{\frac{N-2}{2}}, \quad \forall u \in \mathcal{M}^\infty(c). \end{aligned}$$

This shows that $m^\infty(c) > 0$. □

Lemma 3.8. Assume that (F1)-(F4) hold. Then the function $c \mapsto m^\infty(c)$ is nonincreasing on $(0, +\infty)$. In particular, if $m^\infty(c_0)$ is achieved, then $m^\infty(c_0) > m^\infty(c)$ for any $c > c_0$.

Proof. For any $c > c_0 > 0$, it follows from the definition of $m^\infty(c_0)$ that there exists $\{u_n\} \subset \mathcal{M}^\infty(c_0)$ such that

$$\Phi^\infty(u_n) < m^\infty(c_0) + \frac{1}{n}, \quad \forall n \in \mathbb{N}. \quad (3.11)$$

Let $\theta := \sqrt{c/c_0}$ and $v_n(x) := \theta^{\frac{2-N}{2}} u_n(x/\theta)$. Then $\|\nabla v_n\|_2^2 = \|\nabla u_n\|_2^2$ and $\|v_n\|_2^2 = c$. By Lemma 3.4, there exists $t_n > 0$ such that $t_n^{\frac{N}{2}} (v_n)_{t_n} \in \mathcal{M}^\infty(c)$. By (F3), we have $f(\tau)\tau \preceq \frac{2N}{N-2} F(\tau)$ for all $\tau \in \mathbb{R} \setminus \{0\}$. It follows that

$$\frac{d}{dt} \left[\frac{F(t\tau)}{t^{\frac{2N}{N-2}}} \right] \preceq 0, \quad \forall t > 0, \tau \neq 0, \quad (3.12)$$

which implies that

$$F\left(t_n^{\frac{N}{2}} \theta^{\frac{2-N}{2}} u_n\right) \geq \theta^{-N} F\left(t_n^{\frac{N}{2}} u_n\right), \quad \forall u_n(x) \neq 0. \quad (3.13)$$

Hence, it follows from (1.5), (3.10), (3.11) and (3.13) that

$$\begin{aligned} m^\infty(c) &\leq \Phi^\infty\left(t_n^{\frac{N}{2}} (v_n)_{t_n}\right) \\ &= \frac{t_n^2}{2} \|\nabla v_n\|_2^2 - t_n^{-N} \int_{\mathbb{R}^N} F\left(t_n^{\frac{N}{2}} v_n\right) dx \\ &= \frac{t_n^2}{2} \|\nabla u_n\|_2^2 - \theta^N t_n^{-N} \int_{\mathbb{R}^N} F\left(t_n^{\frac{N}{2}} \theta^{\frac{2-N}{2}} u_n\right) dx \\ &\leq \frac{t_n^2}{2} \|\nabla u_n\|_2^2 - t_n^{-N} \int_{\mathbb{R}^N} F\left(t_n^{\frac{N}{2}} u_n\right) dx \\ &= \Phi^\infty\left(t_n^{\frac{N}{2}} (u_n)_{t_n}\right) \leq \Phi^\infty(u_n) < m^\infty(c_0) + \frac{1}{n}, \end{aligned}$$

which shows that $m^\infty(c) \leq m^\infty(c_0)$ by letting $n \rightarrow \infty$.

If $m^\infty(c_0)$ is achieved, i.e., there exists $\tilde{u} \in \mathcal{M}^\infty(c_0)$ such that $\Phi^\infty(\tilde{u}) = m^\infty(c_0)$. Let $\tilde{v}(x) := \theta^{\frac{2-N}{2}} \tilde{u}(x/\theta)$. Then $\|\nabla \tilde{v}\|_2^2 = \|\nabla \tilde{u}\|_2^2$ and $\|\tilde{v}\|_2^2 = c$. By Lemma 3.4, we have there exists $\tilde{t} > 0$ such that $\tilde{t}^{\frac{N}{2}} \tilde{v}_{\tilde{t}} \in \mathcal{M}^\infty(c)$. By (3.12), we have

$$F\left(\tilde{t}^{\frac{N}{2}} \theta^{\frac{2-N}{2}} \tau\right) \geq \theta^{-N} F\left(\tilde{t}^{\frac{N}{2}} \tau\right), \quad \forall \tau \in \mathbb{R} \setminus \{0\}. \quad (3.14)$$

Hence, it follows from (1.5), (3.10), (3.14) and Lemma 3.1 that

$$\begin{aligned} m^\infty(c) &\leq \Phi^\infty\left(\tilde{t}^{\frac{N}{2}} \tilde{v}_{\tilde{t}}\right) \\ &= \frac{\tilde{t}^2}{2} \|\nabla \tilde{v}\|_2^2 - \tilde{t}^{-N} \int_{\mathbb{R}^N} F\left(\tilde{t}^{\frac{N}{2}} \tilde{v}\right) dx \\ &= \frac{\tilde{t}^2}{2} \|\nabla \tilde{u}\|_2^2 - \theta^N \tilde{t}^{-N} \int_{\mathbb{R}^N} F\left(\tilde{t}^{\frac{N}{2}} \theta^{\frac{2-N}{2}} \tilde{u}\right) dx \\ &< \frac{\tilde{t}^2}{2} \|\nabla \tilde{u}\|_2^2 - \tilde{t}^{-N} \int_{\mathbb{R}^N} F\left(\tilde{t}^{\frac{N}{2}} \tilde{u}\right) dx \\ &= \Phi^\infty\left(\tilde{t}^{\frac{N}{2}} \tilde{u}_{\tilde{t}}\right) \leq \Phi^\infty(\tilde{u}) = m^\infty(c_0), \end{aligned}$$

which shows that $m^\infty(c) < m^\infty(c_0)$. □

Lemma 3.9. Assume that (F1)-(F4) hold. Then $m^\infty(c)$ is achieved.

Proof. In view of Lemmas 3.4 and 3.7, we have $\mathcal{M}^\infty(c) \neq \emptyset$ and $m^\infty(c) > 0$. Let $\{u_n\} \subset \mathcal{M}^\infty(c)$ be such that $\Phi^\infty(u_n) \rightarrow m^\infty(c)$. It follows from (1.5) and (1.6) that

$$m^\infty(c) + o(1) = \frac{1}{2} \|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} F(u_n) dx \quad (3.15)$$

and

$$0 = \|\nabla u_n\|_2^2 - \frac{N}{2} \int_{\mathbb{R}^N} [f(u_n) u_n - 2F(u_n)] dx. \quad (3.16)$$

We first prove that $\sup_{n \in \mathbb{N}} \|\nabla u_n\|_2 < +\infty$. Otherwise, we may assume that $\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 = +\infty$. By (F1), there exists $C_1 > 0$ such that

$$F(t) \leq C_1 |t|^{2+\frac{4}{N}} + \mathcal{S}^{\frac{N}{N-2}} |t|^{2^*}, \quad \forall t \in \mathbb{R}. \quad (3.17)$$

Let $t_n := \|\nabla u_n\|_2$ and $v_n(x) := t_n^{-\frac{N}{2}} u_n(x/t_n)$. Then it is easy to check that $\|\nabla v_n\|_2^2 = 1$ and $\|v_n\|_2^2 = c$. If $\delta := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{B_1(y)} |v_n|^2 dx = 0$, then in light of Lions' concentration compactness principle [25, Lemma 1.21], $\|v_n\|_s \rightarrow 0$ for $s \in (2, 2^*)$. Choose $R = \sqrt{4m^\infty(c) + 2}$.

It follows from (F1), (1.5), (1.8), (3.10) and (3.17) that

$$\begin{aligned}
 m^\infty(c) + o(1) &= \Phi^\infty(u_n) \geq \Phi^\infty\left((R/t_n)^{\frac{N}{2}}(u_n)_{R/t_n}\right) \\
 &= \Phi^\infty\left(R^{\frac{N}{2}}(v_n)_R\right) \\
 &= \frac{R^2}{2}\|\nabla v_n\|_2^2 - \frac{1}{R^N} \int_{\mathbb{R}} F\left(R^{\frac{N}{2}}v_n\right) dx \\
 &\geq \frac{R^2}{2}\|\nabla v_n\|_2^2 - C_1 R^2 \|v_n\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} - \mathcal{S}^{\frac{N}{N-2}} \|v_n\|_{2^*}^{2^*} \\
 &\geq \frac{R^2}{2} - 1 - o(1) = 2m^\infty(c) - o(1).
 \end{aligned}$$

This contradiction shows that $\delta > 0$. Without loss of generality, we may assume the existence of $y_n \in \mathbb{R}$ such that $\int_{B_1(y_n)} |v_n|^2 dx > \frac{\delta}{2}$. Let $\hat{v}_n(x) = v_n(x + y_n)$. Passing to a subsequence, we have

$$\hat{v}_n \rightharpoonup \bar{v} \in H^1(\mathbb{R}^N) \setminus \{0\}, \quad \hat{v}_n(x) \rightarrow \bar{v}(x), \quad \text{a.e. } x \in \mathbb{R}^N. \quad (3.18)$$

By (F2), (1.5), (3.15) and (3.18), we obtain

$$\begin{aligned}
 \frac{m^\infty(c) + o(1)}{t_n^2} &= t_n^{-2} \Phi^\infty(u_n) = t_n^{-2} \Phi^\infty\left(t_n^{\frac{N}{2}}(v_n)_{t_n}\right) \\
 &= \frac{1}{2} \|\nabla v_n\|_2^2 - \frac{1}{t_n^{N+2}} \int_{\mathbb{R}^N} F\left(t_n^{\frac{N}{2}}v_n\right) dx \\
 &= \frac{1}{2} - \int_{\mathbb{R}^N} \frac{F\left(t_n^{\frac{N}{2}}v_n\right)}{t_n^{N+2}} dx \\
 &= \frac{1}{2} - \int_{\mathbb{R}^N} \frac{F\left(t_n^{\frac{N}{2}}\hat{v}_n\right)}{t_n^{N+2}} dx \rightarrow -\infty, \quad n \rightarrow \infty.
 \end{aligned}$$

This contradiction shows that $\{\|\nabla u_n\|_2\}$ is bounded, so $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Let $\delta' := \limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}} \int_{B_1(y)} |u_n|^2 dx$. We show that $\delta' > 0$. Otherwise, in light of Lions' concentration compactness principle [25, Lemma 1.21], $\|u_n\|_s \rightarrow 0$ for $s \in (2, 2^*)$. It follows from (F1), (3.15) and (3.16) that

$$m^\infty(c) + o(1) = \frac{N}{2} \int_{\mathbb{R}} \left[f(u_n)u_n - \left(2 + \frac{4}{N}\right) F(u_n) \right] dx = o(1),$$

which contradicts the fact that $m^\infty(c) > 0$. Thus $\delta' > 0$. Without loss of generality, we may assume the existence of $\hat{y}_n \in \mathbb{R}$ such that $\int_{B_1(\hat{y}_n)} |u_n|^2 dx > \frac{\delta'}{2}$. Let $\hat{u}_n(x) = u_n(x + \hat{y}_n)$. Then we have

$$\|\hat{u}_n\|_2^2 = c, \quad \mathcal{P}^\infty(\hat{u}_n) = 0, \quad \Phi^\infty(\hat{u}_n) \rightarrow m^\infty(c), \quad \int_{B_1(0)} |\hat{u}_n|^2 dx > \frac{\delta'}{2}. \quad (3.19)$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u}, & \text{in } H^1(\mathbb{R}^N); \\ \hat{u}_n \rightarrow \hat{u}, & \text{in } L_{\text{loc}}^s(\mathbb{R}^N), \quad \forall s \in [1, 2^*); \\ \hat{u}_n \rightarrow \hat{u}, & \text{a.e. on } \mathbb{R}^N. \end{cases} \quad (3.20)$$

Let $w_n = \hat{u}_n - \hat{u}$. Then (3.20) and Lemma 3.6 yield

$$\Phi^\infty(\hat{u}_n) = \Phi^\infty(\hat{u}) + \Phi^\infty(w_n) + o(1) \quad (3.21)$$

and

$$\mathcal{P}^\infty(\hat{u}_n) = \mathcal{P}^\infty(\hat{u}) + \mathcal{P}^\infty(w_n) + o(1). \quad (3.22)$$

Set

$$Q(u) := \frac{N}{4} \int_{\mathbb{R}^N} \left[f(u)u - \left(2 + \frac{4}{N} \right) F(u) \right] dx. \quad (3.23)$$

By (F1), we have

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}^N} \frac{F\left(t^{\frac{N}{2}}u\right)}{t^{N+2}} dx = 0, \quad \forall u \in H^1(\mathbb{R}^N).$$

For any $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, the above equality implies that there exists $\hat{t}(u) \in (0, 1)$ such that

$$Nt^2 \int_{\mathbb{R}^N} [f(u)u - 2F(u)] dx > 4t^{-N} \int_{\mathbb{R}^N} F\left(t^{\frac{N}{2}}u\right) dx, \quad \forall 0 < t < \hat{t}(u),$$

which, together with (3.1), (3.2) and (3.23), implies that

$$\begin{aligned} Q(u) &= \frac{N}{4} \int_{\mathbb{R}^N} \left[f(u)u - \left(2 + \frac{4}{N} F(u) \right) \right] dx \\ &= \frac{1}{4} \left[\int_{\mathbb{R}^N} g(t, u) dx + Nt^2 \int_{\mathbb{R}^N} [f(u)u - 2F(u)] dx - 4t^{-N} \int_{\mathbb{R}^N} F\left(t^{\frac{N}{2}}u\right) dx \right] \\ &> 0, \quad \forall u \in H^1(\mathbb{R}^N) \setminus \{0\}, \quad 0 < t < \hat{t}(u). \end{aligned} \quad (3.24)$$

From (1.5), (1.6), (3.21), (3.22) and (3.23) that

$$m^\infty(c) - Q(\hat{u}) + o(1) = Q(w_n) \quad (3.25)$$

and

$$\mathcal{P}^\infty(w_n) = -\mathcal{P}^\infty(\hat{u}) + o(1). \quad (3.26)$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then going to this subsequence, we have

$$\|\hat{u}\|_2^2 = c, \quad \Phi^\infty(\hat{u}) = m(c), \quad \mathcal{P}^\infty(\hat{u}) = 0, \quad (3.27)$$

which implies the conclusion of Lemma 3.9 holds. Next, we assume that $w_n \neq 0$. By (3.19) and (3.20), one has $c = \|\hat{u}_n\|_2^2 = \|\hat{u}\|_2^2 + \|w_n\|_2^2 + o(1)$. This implies that $\|\hat{u}\|_2^2 := \hat{c} \leq c$ and $\|w_n\|_2^2 := \tilde{c}_n \leq c$ for large $n \in \mathbb{N}$. We claim that $\mathcal{P}^\infty(\hat{u}) \leq 0$. Otherwise, if $\mathcal{P}^\infty(\hat{u}) > 0$, then (3.26) implies $\mathcal{P}^\infty(w_n) < 0$ for large n . In view of Lemma 3.4, there exists $t_n > 0$ such that $t_n^{\frac{N}{2}}(w_n)_{t_n} \in \mathcal{M}^\infty(\tilde{c}_n)$. From (1.5), (1.6), (3.1), (3.8), (3.25), (3.26) and Lemma 3.8, we obtain

$$\begin{aligned} m^\infty(c) - Q(\hat{u}) + o(1) &= Q(w_n) \\ &= \Phi^\infty(w_n) - \frac{1}{2} \mathcal{P}^\infty(w_n) \\ &\geq \Phi^\infty\left(t_n^{\frac{N}{2}}(w_n)_{t_n}\right) - \frac{t_n^2}{2} \mathcal{P}^\infty(w_n) \\ &\geq m^\infty(\tilde{c}) \geq m^\infty(c), \end{aligned}$$

which implies $\mathcal{P}^\infty(\hat{u}) \leq 0$ due to (3.24). Since $\hat{u} \neq 0$ and $\mathcal{P}^\infty(\hat{u}) \leq 0$, in view of Lemma 3.4, there exists $\hat{t} > 0$ such that $\hat{t}^{N/2}\hat{u}_{\hat{t}} \in \mathcal{M}^\infty(\hat{c})$. (1.5), (1.6), (3.1), (3.8), (3.15), (3.16), (3.24), Lemma 3.8, Fatou's lemma and the weak semicontinuity of norm, one has

$$\begin{aligned} m^\infty(c) &= \lim_{n \rightarrow \infty} \left[\Phi^\infty(\hat{u}_n) - \frac{1}{2} \mathcal{P}^\infty(\hat{u}_n) \right] \\ &= \frac{N}{4} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left[f(\hat{u}_n) \hat{u}_n - \left(2 + \frac{4}{N} F(\hat{u}_n) \right) \right] dx \\ &\geq \frac{N}{4} \int_{\mathbb{R}^N} \left[f(\hat{u}) \hat{u} - \left(2 + \frac{4}{N} F(\hat{u}) \right) \right] dx \\ &= \Phi^\infty(\hat{u}) - \frac{1}{2} \mathcal{P}^\infty(\hat{u}) \\ &\geq \Phi^\infty\left(\hat{t}^{\frac{N}{2}} \hat{u}_{\hat{t}}\right) - \frac{\hat{t}^2}{2} \mathcal{P}^\infty(\hat{u}) \\ &\geq m^\infty(\hat{c}) \geq m^\infty(c), \end{aligned}$$

which implies

$$\|\hat{u}\|_2^2 = \hat{c}, \quad \Phi^\infty(\hat{u}) = m^\infty(\hat{c}) = m^\infty(c), \quad \mathcal{P}^\infty(\hat{u}) = 0.$$

This shows $m^\infty(\hat{c})$ is achieved. In view of Lemma 3.8, $\hat{c} = c$. Thus, (3.27) holds also, i.e. the conclusion of Lemma 3.9 holds. $\square$

Lemma 3.10. Assume that (F1)-(F4) hold. If $\bar{u} \in \mathcal{M}^\infty(c)$ and $\Phi^\infty(\bar{u}) = m^\infty(c)$, then $\bar{u}$ is a critical point of Φ^∞ on $\mathcal{S}_c$, i.e. $\Phi^\infty|'_{\mathcal{S}_c}(\bar{u}) = 0$.

Proof. We prove this lemma by using the method introduced in [6]. Assume that $\Phi^\infty|'_{\mathcal{S}_c}(\bar{u}) \neq 0$. Then there exist $\delta > 0$ and $\rho > 0$ such that $\|u - \bar{u}\| \leq 3\delta \Rightarrow \|\Phi^\infty|'_{\mathcal{S}_c}(u)\| \geq \rho$. It is easy to see that

$$\left\| \nabla \left(t^{\frac{N}{2}} \bar{u}_t \right) - \nabla \bar{u} \right\|_2^2 = (t^2 + 1) \|\nabla \bar{u}\|_2^2 - 2t^{\frac{N}{2}} \int_{\mathbb{R}^N} \nabla \bar{u}_t \cdot \nabla \bar{u} dx \rightarrow 0, \quad t \rightarrow 1.$$

Thus, there exists $\delta_1 \in (0, 1/4)$ such that

$$|t - 1| < \delta_1 \Rightarrow \left\| t^{\frac{N}{2}} \bar{u}_t - \bar{u} \right\| < \delta. \quad (3.28)$$

In view of Lemma 3.2 ii) and Lemma 3.3, one has

$$\int_{\mathbb{R}^N} g(t, \bar{u}) dx > 0, \quad \forall t \in (0, 1) \cup (1, +\infty) \quad (3.29)$$

and

$$\Phi^\infty\left(t^{\frac{N}{2}} \bar{u}_t\right) = \Phi^\infty(\bar{u}) - \int_{\mathbb{R}^N} g(t, \bar{u}) dx = m^\infty(c) - \frac{1}{4} \int_{\mathbb{R}^N} g(t, \bar{u}) dx, \quad \forall t > 0. \quad (3.30)$$

It follows from (F1), (F2) and (1.6) that there exist $T_1 \in (0, 1/2)$ and $T_2 \in (3/2, +\infty)$ such that

$$\mathcal{P}^\infty\left(T_1^{\frac{N}{2}} \bar{u}_{T_1}\right) > 0, \quad \mathcal{P}^\infty\left(T_2^{\frac{N}{2}} \bar{u}_{T_2}\right) < 0 \quad (3.31)$$

Let

$$\varepsilon := \min \left\{ \frac{1}{12} \int_{\mathbb{R}^N} g(T_1, \bar{u}) dx, \frac{1}{12} \int_{\mathbb{R}^N} g(T_2, \bar{u}) dx, 1, \frac{\rho \delta}{8} \right\}, \quad S := \{v \in \mathcal{S}_c : \|v - \bar{u}\| < \delta\}.$$

Then $\varepsilon \in (0, 1)$, and Lemma 2.1 yields a deformation $\eta \in \mathcal{C}([0, 1] \times \mathcal{S}_c, \mathcal{S}_c)$ such that

- i) $\eta(1, u) = u$ if $\Phi^\infty(u) < m^\infty(c) - 2\varepsilon$ or $\Phi^\infty(u) > m^\infty(c) + 2\varepsilon$;
- ii) $\eta\left(1, (\Phi^\infty)^{m^\infty(c)+\varepsilon} \cap S\right) \subset (\Phi^\infty)^{m^\infty(c)-\varepsilon}$;
- iii) $\Phi^\infty(\eta(1, u)) \leq \Phi^\infty(u)$, $\forall u \in \mathcal{S}_c$.

By Corollary 3.1, $\Phi^\infty\left(t^{\frac{N}{2}} \bar{u}_t\right) \leq \Phi^\infty(\bar{u}) = m^\infty(c)$ for $t > 0$. Then it follows from (3.28) and ii) that

$$\Phi^\infty\left(\eta\left(1, t^{\frac{N}{2}} \bar{u}_t\right)\right) \leq m^\infty(c) - \varepsilon, \quad \forall t > 0, \quad |t - 1| < \delta_1. \quad (3.32)$$

On the other hand, by iii), (3.8), (3.29) and (3.30), one has

$$\begin{aligned} \Phi\left(\eta\left(1, t^{\frac{N}{2}} \bar{u}_t\right)\right) &\leq \Phi^\infty\left(t^{\frac{N}{2}} \bar{u}_t\right) = m^\infty(c) - \frac{1}{4} \int_{\mathbb{R}^N} g(t, \bar{u}) dx \\ &\leq m^\infty(c) - \delta_2, \quad \forall t > 0, \quad |t - 1| \geq \delta_1, \end{aligned} \quad (3.33)$$

where

$$\delta_2 := \frac{1}{4} \min \left\{ \int_{\mathbb{R}^N} g(1 - \delta_1, \bar{u}) dx, \int_{\mathbb{R}^N} g(1 + \delta_1, \bar{u}) dx \right\} > 0.$$

Combining (3.32) with (3.33), we have

$$\max_{t \in [T_1, T_2]} \Phi^\infty\left(\eta\left(1, t^{\frac{N}{2}} \bar{u}_t\right)\right) < m^\infty(c).$$

Define $\Psi_0(t) := \mathcal{P}^\infty\left(\eta\left(1, t^{\frac{N}{2}} \bar{u}_t\right)\right)$ for $t > 0$. It follows from (3.30), i) and the definition of ε that $\eta\left(1, t^{\frac{N}{2}} \bar{u}_t\right) = t^{\frac{N}{2}} \bar{u}_t$ for $t = T_1$ and $t = T_2$, which, together with (3.31), implies

$$\Psi_0(T_1) = \mathcal{P}^\infty(\sqrt{T_1} \bar{u}_{T_1}) > 0, \quad \Psi_0(T_2) = \mathcal{P}^\infty(\sqrt{T_2} \bar{u}_{T_2}) < 0.$$

Since $\Psi_0(t)$ is continuous on $(0, \infty)$, then $\eta\left(1, t^{\frac{N}{2}} \bar{u}_t\right) \cap \mathcal{M}^\infty(c) \neq \emptyset$ for some $t_0 \in (T_1, T_2)$, contradicting to the definition of $m^\infty(c)$. $\square$

Based on Lemmas 3.9 and 3.10, the proof of Theorem 1.1 follows directly.

4. MOUNTAIN PASS SOLUTIONS FOR NON-AUTONOMOUS PROBLEM (1.1)

In this section, based on the critical point theories on manifolds established in Section 2, particularly Lemma 2.3, we prove the existence of mountain pass solutions for non-autonomous problem (1.1), thus completing the proof of Theorem 1.2.

To apply Lemma 2.3, we let $E = H^1(\mathbb{R}^N)$ and $H = L^2(\mathbb{R}^N)$. Define the norms of E and H by

$$\|u\|_E := \left[\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx \right]^{1/2}, \quad \|u\|_H := \frac{1}{\sqrt{c}} \left(\int_{\mathbb{R}^N} u^2 dx \right)^{1/2}, \quad \forall u \in E.$$

Identifying H with its dual space yields continuous injections $E \hookrightarrow H \hookrightarrow E^*$. We set the constraint manifold

$$M := \left\{ u \in E : \|u\|_2^2 = \int_{\mathbb{R}^N} u^2 dx = c \right\}.$$

Obviously, under (F1), $\Phi \in \mathcal{C}^1(E, \mathbb{R})$, and

$$\langle \Phi'(u), u \rangle = \|\nabla u\|_2^2 + \int_{\mathbb{R}^N} V(x)u^2 dx - \int_{\mathbb{R}^N} f(u)u dx.$$

Let us define a continuous map $\beta : H^1(\mathbb{R}^N) \times \mathbb{R} \rightarrow H^1(\mathbb{R}^N)$ by

$$\beta(v, t)(x) := e^{Nt/2}v(e^t x) \text{ for } v \in H^1(\mathbb{R}^N), \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^N,$$

and consider the following auxiliary functional:

$$\tilde{\Phi}(v, t) := \Phi(\beta(v, t)) = \frac{e^{2t}}{2} \|\nabla v\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(e^{-t}v)v^2 dx - \frac{1}{e^{Nt}} \int_{\mathbb{R}^N} F(e^{Nt/2}v) dx. \quad (4.1)$$

We see that $\tilde{\Phi}'$ is of class $\mathcal{C}^1$, and for any $(w, s) \in H^1(\mathbb{R}^N) \times \mathbb{R}$,

$$\begin{aligned} \langle \tilde{\Phi}'(v, t), (w, s) \rangle &= \langle \tilde{\Phi}'(v, t), (w, 0) \rangle + \langle \tilde{\Phi}'(v, t), (0, s) \rangle \\ &= e^{2t} \int_{\mathbb{R}^N} \nabla v \cdot \nabla w dx + e^{2t} s \|\nabla v\|_2^2 + \int_{\mathbb{R}^N} V(e^{-t}v)vw dx \\ &\quad - \frac{s}{2} \int_{\mathbb{R}^N} [\nabla V(e^{-t}v) \cdot (e^{-t}v)]v^2 dx - \frac{1}{e^{Nt}} \int_{\mathbb{R}^N} f(e^{Nt/2}v)e^{Nt/2}w dx \\ &\quad + \frac{Ns}{2e^{Nt}} \int_{\mathbb{R}^N} [2F(e^{Nt/2}v) - f(e^{Nt/2}v)e^{Nt/2}v] dx \\ &= \langle \Phi'(\beta(v, t)), \beta(w, t) \rangle + s \mathcal{P}(\beta(v, t)). \end{aligned} \quad (4.2)$$

Let

$$u(x) := \beta(v, t)(x) = e^{Nt/2}v(e^t x), \quad \phi(x) := \beta(w, t)(x) = e^{Nt/2}w(e^t x). \quad (4.3)$$

Then

$$(u, \phi)_H = \frac{1}{c} \int_{\mathbb{R}^N} u(x)\phi(x) dx = \frac{1}{c} \int_{\mathbb{R}^N} v(x)w(x) dx = (v, w)_H.$$

This shows that

$$\phi \in T_u(\mathcal{S}_c) \Leftrightarrow (w, s) \in \tilde{T}_{(v, t)}(\mathcal{S}_c \times \mathbb{R}), \quad \forall t, s \in \mathbb{R}. \quad (4.4)$$

It follows from (4.2), (4.3) and (4.4) that $|\mathcal{P}(u)| = |\langle \tilde{\Phi}'(v, t), (0, 1) \rangle| \leq \|\tilde{\Phi}'|_{\mathcal{S}_c \times \mathbb{R}}(v, t)\|$ and

$$\begin{aligned} \|\Phi'|_{\mathcal{S}_c}(u) \| &= \sup_{\phi \in T_u(\mathcal{S}_c)} \frac{1}{\|\phi\|_E} |\langle \Phi'(u), \phi \rangle| \\ &= \sup_{\phi \in T_u(\mathcal{S}_c)} \frac{1}{\sqrt{\|\nabla \phi\|_2^2 + \|\phi\|_2^2}} |\langle \Phi'(\beta(v, t)), \beta(w, t) \rangle| \\ &= \sup_{\phi \in T_u(\mathcal{S}_c)} \frac{1}{\sqrt{\|\nabla \phi\|_2^2 + \|\phi\|_2^2}} |\langle \tilde{\Phi}'(v, t), (w, 0) \rangle| \\ &\leq \sup_{(w, 0) \in \tilde{T}_{(v, t)}(\mathcal{S}_c \times \mathbb{R})} \frac{e^{|t|}}{\|(w, 0)\|_{E \times \mathbb{R}}} |\langle \tilde{\Phi}'(v, t), (w, 0) \rangle| \\ &\leq e^{|t|} \|\tilde{\Phi}'|_{\mathcal{S}_c \times \mathbb{R}}(v, t)\|. \end{aligned}$$

Lemma 4.1. Assume that (V1), (F1) and (F2) hold. Then

i) there exists $\vartheta_c > 0$ small enough such that $\Phi(u) > 0$ if $u \in A_{\frac{4\vartheta_c}{1-\sigma_1}}$ and

$$0 < \sup_{u \in A_{\vartheta_c}} \Phi(u) < \kappa_c := \inf \left\{ \Phi(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = \frac{4\vartheta_c}{1-\sigma_1} \right\}, \quad (4.5)$$

where

$$A_{\vartheta_c} = \left\{ u \in \mathcal{S}_c : \|\nabla u\|_2^2 \leq \vartheta_c \right\} \quad \text{and} \quad A_{\frac{4\vartheta_c}{1-\sigma_1}} = \left\{ u \in \mathcal{S}_c : \|\nabla u\|_2^2 \leq \frac{4\vartheta_c}{1-\sigma_1} \right\};$$

ii) there holds:

$$M(c) := \inf_{\gamma \in \Gamma_c} \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \kappa_c > \sup_{\gamma \in \Gamma_c} \max \{ \Phi(\gamma(0)), \Phi(\gamma(1)) \}, \quad (4.6)$$

where

$$\Gamma_c := \left\{ \gamma \in \mathcal{C}([0,1], \mathcal{S}_c) : \|\nabla \gamma(0)\|_2^2 \leq \vartheta_c, \Phi(\gamma(1)) < 0 \right\}. \quad (4.7)$$

Proof. i). By (V1), (F1), (F3), (1.2), (1.8) and (1.9), one has

$$\Phi(u) = \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} F(u) dx \leq \frac{1}{2} \|\nabla u\|_2^2, \quad \forall u \in \mathcal{S}_c$$

and

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} F(u) dx \\ &\geq \frac{1-\sigma_1}{2} \|\nabla u\|_2^2 - \frac{1-\sigma_1}{4\mathcal{C}_{N,2+4/N}} c^{-\frac{2}{N}} \|u\|_{2+\frac{4}{N}}^{2+\frac{4}{N}} - C_1 \|u\|_2^{2^*} \\ &\geq \|\nabla u\|_2^2 \left[\frac{1-\sigma_1}{4} - C_1 \mathcal{S}^{-\frac{2^*}{2}} \|\nabla u\|_2^{2^*-2} \right], \quad \forall u \in \mathcal{S}_c. \end{aligned}$$

The above inequalities show that there exists $\vartheta_c > 0$ such that i) holds.

ii). For any given $w \in \mathcal{S}_c$, we have $\|t^{\frac{N}{2}} w_t\|_2 = \|w\|_2$, so $t^{\frac{N}{2}} w_t \in \mathcal{S}_c$ for every $t > 0$. Then (V1), (F2) and (1.2) yield

$$\begin{aligned} \Phi\left(t^{\frac{N}{2}} w_t\right) &= \frac{t^2}{2} \|\nabla w\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(t^{-1}x) w^2 dx - \frac{1}{t^N} \int_{\mathbb{R}^N} F\left(t^{\frac{N}{2}} w\right) dx \\ &\leq \frac{t^2}{2} \|\nabla w\|_2^2 - \frac{1}{t^N} \int_{\mathbb{R}^N} F\left(t^{\frac{N}{2}} w\right) dx \\ &= \frac{t^2}{2} \left[\|\nabla w\|_2^2 - \int_{\mathbb{R}^N} \frac{F\left(t^{\frac{N}{2}} w\right)}{t^{N+2}} dx \right] \rightarrow -\infty \quad \text{as } t \rightarrow +\infty. \end{aligned}$$

Thus we can deduce that there exist $t_1 > 0$ small enough and $t_2 > 0$ large enough such that

$$\left\| \nabla \left(t_1^{\frac{N}{2}} w_{t_1} \right) \right\|_2^2 = t_1^2 \|\nabla w\|_2^2 \leq \vartheta_c, \quad \text{and} \quad \Phi\left(t_2^{\frac{N}{2}} w_{t_2}\right) < 0. \quad (4.8)$$

Let $\gamma_0(t) := [t_1 + (t_2 - t_1)t]^{\frac{N}{2}} w_{t_1 + (t_2 - t_1)t}$. Then $\gamma_0 \in \Gamma_c$, so $\Gamma_c \neq \emptyset$. Now using the intermediate value theorem, for any $\gamma \in \Gamma_c$, there exists $t_0 \in (0, 1)$, depending on γ , such that $\|\nabla \gamma(t_0)\|_2^2 =$

$\frac{4\vartheta_c}{1-\sigma_1}$ and

$$\max_{t \in [0,1]} \Phi(\gamma(t)) \geq \Phi(\gamma(t_0)) \geq \inf \left\{ \Phi(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = \frac{4\vartheta_c}{1-\sigma_1} \right\},$$

which, together with the arbitrariness of $\gamma \in \Gamma_c$, implies

$$M(c) = \inf_{\gamma \in \Gamma_c} \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \inf \left\{ \Phi(u) : u \in \mathcal{S}_c, \|\nabla u\|_2^2 = \frac{4\vartheta_c}{1-\sigma_1} \right\}. \quad (4.9)$$

Hence, (4.6) follows directly from (4.5) and (4.9), and the proof is completed. $\square$

Lemma 4.2. Assume that (V1), (F1) and (F2) hold. Then there exists a sequence $\{u_n\} \subset \mathcal{S}_c$ such that

$$\Phi(u_n) \rightarrow M(c) > 0, \quad \Phi|'_{\mathcal{S}_c}(u_n) \rightarrow 0 \quad \text{and} \quad \mathcal{P}(u_n) \rightarrow 0. \quad (4.10)$$

Proof. Let

$$\tilde{\Gamma}_c := \{ \tilde{\gamma} \in \mathcal{C}([0,1], \mathcal{S}_c \times \mathbb{R}) : \tilde{\gamma}(0) = (\tilde{\gamma}_1(0), 0), \|\nabla \tilde{\gamma}_1(0)\|_2^2 \leq \vartheta_c, \tilde{\Phi}(\tilde{\gamma}(1)) < 0 \}$$

and

$$m(c) := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{t \in [0,1]} \tilde{\Phi}(\tilde{\gamma}(t)).$$

For any $\tilde{\gamma} \in \tilde{\Gamma}_c$, it is easy to see that $\gamma = \beta \circ \tilde{\gamma} \in \Gamma_c$ defined by (4.7). Then it follows from (4.1) and (4.6) that

$$\begin{aligned} \inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{t \in [0,1]} \tilde{\Phi}(\tilde{\gamma}(t)) &= \inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{t \in [0,1]} \Phi(\beta(\tilde{\gamma}(t))) \\ &\geq \inf_{\gamma \in \Gamma_c} \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \kappa_c \\ &> \sup_{\gamma \in \Gamma_c} \max \{ \Phi(\gamma(0)), \Phi(\gamma(1)) \} \\ &\geq \sup_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max \{ \tilde{\Phi}(\tilde{\gamma}(0)), \tilde{\Phi}(\tilde{\gamma}(1)) \}. \end{aligned}$$

This shows that $m(c) \geq M(c)$ and (2.1) holds.

On the other hand, for any $\gamma \in \Gamma_c$, let $\tilde{\gamma}(t) := (\gamma(t), 0)$. It is easy to verify that $\tilde{\gamma} \in \tilde{\Gamma}_c$ and $\Phi(\gamma(t)) = \tilde{\Phi}(\tilde{\gamma}(t))$, and so, we trivially have $m(c) \leq M(c)$. Thus $m(c) = M(c)$. For any $n \in \mathbb{N}$, (4.6) implies that there exists $\gamma_n \in \Gamma_c$ such that

$$\max_{t \in [0,1]} \Phi(\gamma_n(t)) \leq M(c) + \frac{1}{n}.$$

Set $\tilde{\gamma}_n(t) := (\gamma_n(t), 0)$. Then applying Lemma 2.3 to $\tilde{\Phi}$, there exists a sequence $\{(v_n, t_n)\} \subset \mathcal{S}_c \times \mathbb{R}$ satisfying

- (i) $M(c) - \frac{2}{n} \leq \tilde{\Phi}(v_n, t_n) \leq M(c) + \frac{2}{n}$;
- (ii) $\min_{t \in [0,1]} \|(v_n, t_n) - (\gamma_n(t), 0)\|_{E \times \mathbb{R}} \leq \frac{2}{\sqrt{n}}$;
- (iii) $\left\| \tilde{\Phi}|'_{\mathcal{S}_c \times \mathbb{R}}(v_n, t_n) \right\| \leq \frac{8}{\sqrt{n}}$.

Let $u_n = \beta(v_n, t_n)$. It follows from (3.5), (3.6) and (i)-(iii) that (4.10) holds. $\square$

Lemma 4.3. [9, Lemma 3.3] Assume that (V1), (F1) and (F2) hold. If there exist $u \in H^1(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ such that

$$-\Delta u + [V(x) + \lambda]u = f(u), \quad x \in \mathbb{R}^N,$$

then $\mathcal{P}(u) = 0$, where $\mathcal{P}$ is defined by (1.3).

Lemma 4.4. Assume that (V1) and (F1)-F4) hold. Then, for any $c > 0$,

$$M(c) < m^\infty(c).$$

Proof. In view of Theorem 1.1, there exists $u_c \in \mathcal{S}_c$ such that $\Phi^\infty(u_c) = m^\infty(c)$. Since (1.4) is autonomous, $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}^-)$ and $\min_{\mathbb{R}^N} V < 0$, then there exist $\bar{x} \in \mathbb{R}^N$ and $\bar{r} > 0$ such that

$$V(x) < 0, \quad |u_c(x)| > 0, \quad a.e. \quad |x - \bar{x}| \leq \bar{r}. \quad (4.11)$$

Similar to (4.8), there exist $t_1 > 0$ small enough and $t_2 > 1$ large enough such that

$$\left\| \nabla \left(t_1^{\frac{N}{2}}(u_c)_{t_1} \right) \right\|_2^2 = t_1^2 \|\nabla u_c\|_2^2 \leq \vartheta_c, \quad \text{and} \quad \Phi \left(t_2^{\frac{N}{2}}(u_c)_{t_2} \right) < 0.$$

Let

$$\hat{\gamma}(t) := [(1-t)t_1 + tt_2]^{\frac{N}{2}}(u_c)_{(1-t)t_1 + tt_2}, \quad \forall t \in [0, 1].$$

Then $\hat{\gamma} \in \Gamma_c$ defined by (4.7). It is easy to see that $\Phi(\hat{\gamma}(t))$ is continuous on $t \in [0, 1]$. Hence, we can choose $t_0 \in [0, 1]$ such that $\Phi(\hat{\gamma}(t_0)) = \max_{t \in [0, 1]} \Phi(\hat{\gamma}(t))$. Set $t_* := (1-t_0)t_1 + t_0t_2$. Then $t_* \in [t_1, t_2]$. By (3.10) and the definition of $M(c)$, we have

$$\Phi \left(t_*^{\frac{N}{2}}(u_c)_{t_*} \right) = \Phi(\hat{\gamma}(t_0)) = \max_{t \in [0, 1]} \Phi(\hat{\gamma}(t)) \geq M(c). \quad (4.12)$$

We have two cases to distinguish:

Case i). $t^* = 1$. From (1.2), (1.5), (3.10), (4.11) and (4.12), we have

$$\begin{aligned} m^\infty(c) &= \Phi^\infty(u_c) \\ &\geq \Phi^\infty \left(t_*^{\frac{N}{2}}(u_c)_{t_*} \right) \\ &= \Phi \left(t_*^{\frac{N}{2}}(u_c)_{t_*} \right) - \frac{1}{2} \int_{\mathbb{R}^N} V(t_*^{-1}x) u_c^2 dx \\ &\geq M(c) - \frac{1}{2} \int_{\mathbb{R}^N} V(x) u_c^2 dx \\ &> M(c). \end{aligned}$$

Case ii). $t^* \in (0, 1) \cup (1, +\infty)$. From (1.2), (1.5), (3.8), (4.12) and Lemma 3.2 ii), we have

$$\begin{aligned} m^\infty(c) &= \Phi^\infty(u_c) \\ &= \Phi^\infty \left(t_*^{\frac{N}{2}}(u_c)_{t_*} \right) + \frac{1}{4} \int_{\mathbb{R}^N} g(t^*, u_c) dx \\ &= \Phi \left(t_*^{\frac{N}{2}}(u_c)_{t_*} \right) - \frac{1}{2} \int_{\mathbb{R}^N} V(t_*^{-1}x) u_c^2 dx + \frac{1}{4} \int_{\mathbb{R}^N} g(t^*, u_c) dx \\ &\geq M(c) + \frac{1}{4} \int_{\mathbb{R}^N} g(t^*, u_c) dx \\ &> M(c). \end{aligned}$$

In both cases, we obtain that $M(c) < m^\infty(c)$ for $c > 0$. $\square$

Proof of Theorem 1.2. It is easy to see that (H1) implies (F1)-(F3). In view of Lemmas 4.2 and 4.4, there exists $\{u_n\} \subset \mathcal{S}_c$ such that

$$\|u_n\|_2^2 = c, \quad \Phi(u_n) \rightarrow M(c) \in (0, m^\infty(c)), \quad \Phi(u_n)|'_{\mathcal{S}_c} \rightarrow 0, \quad \mathcal{P}(u_n) \rightarrow 0, \quad (4.13)$$

which, together with (1.2) and (1.3), yields that

$$M(c) + o(1) = \frac{1}{2} \|\nabla u_n\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u_n^2 dx - \int_{\mathbb{R}^N} F(u_n) dx \quad (4.14)$$

and

$$o(1) = \|\nabla u_n\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x) u_n^2 dx - \frac{N}{2} \int_{\mathbb{R}^N} [f(u_n) u_n - 2F(u_n)] dx. \quad (4.15)$$

It follows from (H1), (V1), (4.14) and (4.15) that

$$\begin{aligned} M(c) + o(1) &= \frac{N(\alpha - 2) - 4}{2N(\alpha - 2)} \|\nabla u_n\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} \left[V(x) + \frac{2(\nabla V(x) \cdot x)}{N(\alpha - 2)} \right] u_n^2 dx \\ &\quad + \frac{N[N(\alpha - 2) - 4]}{2(\alpha - 2)} \int_{\mathbb{R}^N} [f(u_n) u_n - \alpha F(u_n)] dx \\ &\geq \frac{N(\alpha - 2) - 4}{2N(\alpha - 2)} \|\nabla u_n\|_2^2 - C_1. \end{aligned}$$

This shows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^N)$ such that, passing to a subsequence,

$$\begin{cases} u_n \rightharpoonup \hat{u}, & \text{in } H^1(\mathbb{R}^N); \\ u_n \rightarrow \hat{u}, & \text{in } L^s_{\text{loc}}(\mathbb{R}^N), \forall s \in [1, 2^*); \\ u_n \rightarrow \hat{u}, & \text{a.e. on } \mathbb{R}^N. \end{cases} \quad (4.16)$$

We claim that $\hat{u} \neq 0$. If not, then (V1), (1.5), (1.6) and (4.13) imply that

$$\Phi^\infty(u_n) = M(c) + o(1), \quad \mathcal{P}^\infty(u_n) = o(1). \quad (4.17)$$

By (V1), (H1) and (4.14), $\|\nabla u_n\|_2^2 \geq 2M(c) + o(1)$, which, together with (4.17) and Lemma 3.4, implies that there exists $t_n = 1 + o(1)$ such that $t_n^{\frac{N}{2}}(u_n)_{t_n} \in \mathcal{M}^\infty(c)$. From (4.13), (4.17), Lemmas 3.2, 3.3 and 3.5, we have

$$\begin{aligned} M(c) &= \lim_{n \rightarrow \infty} \Phi^\infty(u_n) \\ &\geq \lim_{n \rightarrow \infty} \left[\Phi^\infty \left(t_n^{\frac{N}{2}}(u_n)_{t_n} \right) + \frac{1 - t_n^2}{2} \mathcal{P}^\infty(u_n) \right] \\ &= \lim_{n \rightarrow \infty} \Phi^\infty \left(t_n^{\frac{N}{2}}(u_n)_{t_n} \right) \\ &\geq \lim_{n \rightarrow \infty} m^\infty(c) > M(c). \end{aligned}$$

This contradiction shows that $\hat{u} \neq 0$. By Lemma 2.2, one has $\Phi'(u_n) + \lambda_n u_n \rightarrow 0$, where

$$\lambda_n = -\frac{1}{\|u_n\|_2^2} \langle \Phi'(u_n), u_n \rangle = \frac{1}{c} \left[-\|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} V(x) u_n^2 dx + \int_{\mathbb{R}^N} f(u_n) u_n dx \right]. \quad (4.18)$$

By (H1), one has

$$f(u_n)u_n \geq \frac{\beta}{\beta-2}[f(u_n)u_n - 2F(u_n)]. \quad (4.19)$$

From (V1), (V3), (4.14), (4.15), (4.18) and (4.19), we have

$$\begin{aligned} \lambda_n c &= -\|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} V(x)u_n^2 dx + \int_{\mathbb{R}^N} f(u_n)u_n dx \\ &\geq \frac{2N-\beta(N-2)}{N(\beta-2)}\|\nabla u_n\|_2^2 - \int_{\mathbb{R}^N} \left[V(x) + \frac{\beta}{N(\beta-2)}\nabla V(x) \cdot x \right] u_n^2 dx + o(1) \\ &\geq \left[\frac{2N-\beta(N-2)}{N(\beta-2)} - \sigma_2 \right] \|\nabla u_n\|_2^2 + o(1) \\ &\geq 2 \left[\frac{2N-\beta(N-2)}{N(\beta-2)} - \sigma_2 \right] M(c) + o(1). \end{aligned} \quad (4.20)$$

Since $\{\|\nabla u_n\|_2\}$ is bounded, it follows from (4.18) that $\{\lambda_n\}$ is also bounded. Thus, we may thus assume, passing to a subsequence if necessary, that $\lambda_n \rightarrow \lambda_c$. Jointly with (V3) and (4.20), we have $\lambda_c > 0$. By a standard argument, we can deduce $\Phi'(\hat{u}) + \lambda_c \hat{u} = 0$. Hence, by Lemma 4.3, one has

$$\lambda_c \|\hat{u}\|_2^2 = -\|\nabla \hat{u}\|_2^2 - \int_{\mathbb{R}^N} V(x)\hat{u}^2 dx + \int_{\mathbb{R}^N} f(\hat{u})\hat{u} dx \quad (4.21)$$

and

$$\mathcal{P}(\hat{u}) = \|\nabla \hat{u}\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x) \cdot x) \hat{u}^2 dx - \frac{N}{2} \int_{\mathbb{R}^N} [f(\hat{u})\hat{u} - 2F(\hat{u})] dx = 0. \quad (4.22)$$

It follows from (V2), (H1), (1.2) and (4.22) that

$$\begin{aligned} \Phi(\hat{u}) &= \Phi(\hat{u}) - \frac{2}{N(\alpha-2)} \mathcal{P}(\hat{u}) \\ &= \frac{N(\alpha-2)-4}{2N(\alpha-2)} \|\nabla \hat{u}\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} \left[V(x) + \frac{2(\nabla V(x) \cdot x)}{N(\alpha-2)} \right] \hat{u}^2 dx \\ &\quad + \frac{N[N(\alpha-2)-4]}{2(\alpha-2)} \int_{\mathbb{R}^N} [f(\hat{u})\hat{u} - \alpha F(\hat{u})] dx \geq 0. \end{aligned} \quad (4.23)$$

Set $v_n := u_n - \hat{u}$. By (V1), (4.13) and (4.16), we have

$$\int_{\mathbb{R}^N} V(x)v_n^2 dx = o(1), \quad (4.24)$$

$$\|\nabla u_n\|_2^2 = \|\nabla \hat{u}\|_2^2 + \|\nabla v_n\|_2^2 + o(1) \quad (4.25)$$

and

$$\|v_n\|_2^2 = \|u_n\|_2^2 - \|\hat{u}\|_2^2 + o(1) = c - \|\hat{u}\|_2^2 + o(1). \quad (4.26)$$

Hence, by (1.2), (1.3), (4.18), (4.21), (4.25), (4.26) and Lemma 3.6, we have

$$\lambda_c \|v_n\|_2^2 = -\|\nabla v_n\|_2^2 - \int_{\mathbb{R}^N} V(x)v_n^2 dx + \int_{\mathbb{R}^N} f(v_n)v_n dx + o(1), \quad (4.27)$$

$$\Phi(u_n) = \Phi(\hat{u}) + \Phi(v_n) + o(1) \quad (4.28)$$

and

$$\mathcal{P}(u_n) = \mathcal{P}(\hat{u}) + \mathcal{P}(v_n) + o(1). \quad (4.29)$$

By (4.26), for $n \in \mathbb{N}$ large enough, we have $c_n := \|v_n\|_2^2 \leq c$. Hence, from (4.15), (4.22) and (4.29), we can deduce

$$o(1) = \mathcal{P}(u_n) = \mathcal{P}(\hat{u}) + \mathcal{P}(v_n) + o(1) = \mathcal{P}(v_n) + o(1). \quad (4.30)$$

Since $\{\|\nabla v_n\|_2^2\}$ is bounded, we may thus assume, passing to a subsequence if necessary, $\lim_{n \rightarrow \infty} \|\nabla v_n\|_2^2 = l$. If $l > 0$, then (4.30) and Lemma 3.4 implies that there exists $\tilde{t}_n = 1 + o(1)$ such that $\tilde{t}_n^{\frac{N}{2}}(v_n)_{\tilde{t}_n} \in \mathcal{M}^\infty(c)$. From (1.2), (1.5), (4.13), (4.16), (4.24), (4.28), Lemma 3.2, 3.3 and Lemma 3.8, we have

$$\begin{aligned} M(c) - \Phi(\hat{u}) &= \lim_{n \rightarrow \infty} \Phi(u_n) - \Phi(\hat{u}) \\ &= \lim_{n \rightarrow \infty} \Phi(v_n) = \lim_{n \rightarrow \infty} \Phi^\infty(v_n) \\ &\geq \lim_{n \rightarrow \infty} \left[\Phi^\infty\left(\tilde{t}_n^{\frac{N}{2}}(v_n)_{\tilde{t}_n}\right) + \frac{1 - \tilde{t}_n^2}{2} \mathcal{P}^\infty(v_n) \right] \\ &= \lim_{n \rightarrow \infty} \Phi^\infty\left(\tilde{t}_n^{\frac{N}{2}}(v_n)_{\tilde{t}_n}\right) \\ &\geq \lim_{n \rightarrow \infty} m^\infty(c_n) \geq m^\infty(c) > M(c), \end{aligned}$$

which contradicts with (4.23). So $l = 0$, i.e. $\lim_{n \rightarrow \infty} \|\nabla v_n\|_2^2 = 0$. By (H1) and (1.9), it is easy to see that

$$\int_{\mathbb{R}^N} f(v_n) v_n dx = o(1), \quad \int_{\mathbb{R}^N} F(v_n) dx = o(1). \quad (4.31)$$

From (4.24), (4.27) and (4.31), we have

$$\lambda_c \|v_n\|_2^2 = -\|\nabla v_n\|_2^2 - \int_{\mathbb{R}^N} V(x) v_n^2 dx + \int_{\mathbb{R}^N} f(v_n) v_n dx + o(1) = o(1).$$

Since $\lambda_c > 0$, then $\|v_n\|_2^2 = o(1)$. Hence, it follows from (1.2), (4.13), (4.24), (4.28) and (4.31) that

$$\begin{aligned} M(c) - \Phi(\hat{u}) &= \lim_{n \rightarrow \infty} \Phi(u_n) - \Phi(\hat{u}) \\ &= \lim_{n \rightarrow \infty} \Phi(v_n) \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \|\nabla v_n\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x) v_n^2 dx - \int_{\mathbb{R}^N} F(v_n) dx \right] \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} \|\nabla v_n\|_2^2 = 0, \end{aligned}$$

which implies

$$\|\hat{u}\|_2^2 = c, \quad \Phi(\hat{u}) = M(c), \quad \Phi'(\hat{u}) + \lambda_c \hat{u} = 0.$$

□

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China (Grant No.12371181 and No.12471175), the Scientific Research Project of the Hunan Provincial Department of Education (No. 24B0021), the Natural Science Foundation of Hunan Province (Grant No. 2021JJ40703).

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