

FRUGAL FORWARD-BACKWARD SPLITTING METHODS WITH DEVIATIONS

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Abstract. In this paper, we provide a general framework of frugal forward-backward splitting methods with deviations for finding zeros of the sum operator of a finite number of maximally monotone operators and cocoercive operators. Our framework encompasses the Douglas–Rachford splitting method with deviations. A unified weak convergence analysis is made under mild conditions. Numerical experiments on Markowitz portfolio optimization problem are given to demonstrate the effectiveness of deviations.

Keywords. Deviations; Frugal resolvent splitting; Forward-backward algorithm; Monotone inclusion.

1. INTRODUCTION

In this paper, we focus on the structured monotone inclusion problem in a real Hilbert space $\mathcal{H}$ which is to find $x \in \mathcal{H}$ such that

$$0 \in \left(\sum_{i=1}^n F_i + \sum_{i=1}^m B_i \right) (x), \quad (1.1)$$

where $F_1, \dots, F_n : \mathcal{H} \rightrightarrows \mathcal{H}$ are maximally monotone operators and $B_i : \mathcal{H} \rightarrow \mathcal{H}$ are $\frac{1}{L_i}$ -cocoercive operators, $i = 1, \dots, m$. The abstraction given by (1.1) covers fundamental problems arising in mathematical optimization involving objectives with finite-sum structure, such as composite minimization, structured saddle-point problems and structured variational inequalities [1, Examples 1, 2, 3].

Based on the splitting structural form of problem (1.1), researchers proposed some forward-backward-type splitting methods [12, 13, 16, 20, 23, 25, 26]. These methods use vector addition, scalar multiplication, the resolvents of the maximally monotone operators, and direct evaluations of the cocoercive operators. Frugal forward-backward splitting methods with the minimal lifting in the sense of Ryu [21] have recently received much attention [1, 2, 6, 15]. They calculate the resolvent of each maximally monotone operator and each cocoercive operator only once in each iteration. Moreover, these methods minimize storage requirements. In the recent seminal work, Morin et al. [15] showed that the minimal lifting number $n - 1$ of the frugal forward-backward splitting methods for (1.1) is achievable only if the first and last operator evaluations are resolvent evaluations.

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For solving the problem (1.1) with $n \geq 2$ and $m = n - 1$, some scholars designed several frugal forward-backward type splitting methods with the minimal lifting. Aragón-Artacho et al. [1] firstly extended the techniques from [14] to propose a distributed forward-backward splitting method. Based on the product space trick [17] and the degenerate preconditioned proximal point algorithms, Bredies et al. [6] developed parallel and sequential forward-backward splitting methods by generalizing Davis–Yin splitting method.

To systematically investigate the frugal forward-backward splitting methods with minimal lifting and efficiently analyze their convergence, there has been increasing interest recently in constructing algorithmic frameworks for solving (1.1). As far as we know, three frameworks were constructed. Aragón-Artacho et al. [2] firstly extended methodologies in [5] and introduced a graph-based framework by incorporating an additional spanning subgraph to model the cocoercive operators' role for problem (1.1) with $n \geq 2$ and $m = n - 1$. Then Åkerman et al. [3] proposed a more general framework for the problem (1.1) with $n \geq 1$ and $m \geq 1$ as follows

Algorithm 1.1.

- 1: **Input:** $\gamma \in (0, 1)$, $\mathbf{z}^0 = (z_1^0, \dots, z_{n-1}^0) \in \mathcal{H}^{n-1}$.
 2: **for** $k = 0, 1, 2, \dots$ **do**
 3: **for** $i = 1, \dots, n$ **do**

$$x_i^k = J_{d_i F_i} \left(-d_i \sum_{j=1}^{i-1} S_{ij} x_j^k + d_i \sum_{j=1}^{n-1} M_{ij} (z_j^k) - d_i \sum_{j=1}^m C_{ij} B_j \left(\sum_{h=1}^{i-1} Q_{jh} x_h^k \right) \right).$$

- 4: **end for**
 5: $z_i^{k+1} = z_i^k - \gamma \sum_{j=1}^n M_{ji} x_j^k, \quad \forall i \in [1, n-1]$.
 6: **end for**
-

In Algorithm 1.1, S, M, C , and Q are matrices satisfying some conditions, and $(d_1, \dots, d_n) = 2(\text{diag}(S))^{\odot(-1)}$. It was verified in [3] that Algorithm 1.1 encompasses Davis–Yin splitting method and the graph-based algorithmic framework in [2]. At almost the same time, Dao et al. [10] adopted approaches similar to those in [3] and devised a general framework for solving (1.1) with $n \geq 2$ and $m = n - 1$ by introducing matrices satisfying specific assumptions. It is worth mentioning that the stepsize in [2, 10] depends on $\min_i \{1/L_i\}$ while that in [3] involves each $1/L_i$, $i = 1, \dots, m$. Therefore, the latter is generally larger than the former.

Recently, the variants of the methods through deviations have attracted much attention since Sadeghi et al. [22] first proposed a forward-backward splitting algorithm with deviations by incorporating the deviation vectors into each iteration. To increase the flexibility in the process of learning to optimize [9] while guaranteeing convergence, a deviation-based proximal gradient method was developed in [7]. It learns the entire update function and then uses it in an optimization solver. Qin et al. [18] also investigated a Douglas–Rachford (DR) splitting algorithm with deviations whose scheme is given as

$$\begin{cases} x_1^k = J_{\gamma F_1} (z^k + v^k), \\ x_2^k = J_{\gamma F_2} (2x_1^k - (z^k + v^k)), \\ z^{k+1} = z^k - \lambda_k (x_1^k - x_2^k), \end{cases} \quad (1.2)$$

where $\lambda_k \in (0, 2)$, $\gamma > 0$ and $\{v^k\}$ is deviation vectors. In order to ensure the convergence of algorithm (1.2), the deviation vectors $\{v^k\}$ are required to satisfy the norm inequality

$$\frac{\lambda_{k+1}}{2 - \lambda_{k+1}} \|v^{k+1}\|^2 \leq \xi_k \lambda_k (2 - \lambda_k) \|x_2^k - x_1^k + \frac{1}{2 - \lambda_k} v^k\|^2, \quad (1.3)$$

where $\xi_k \in [0, 1)$. In addition, some other methods with deviations were also introduced, such as the Davis–Yin splitting method with deviations [11] and the forward-backward-half forward splitting algorithm with deviations as well [19]. Despite many similarities in the convergence analysis of these algorithms (e.g., the norm inequalities of deviations are introduced to obtain a nonincreasing sequence), their convergence analyses were performed on an inefficient, algorithm-by-algorithm basis in the literature.

In this work, we first propose a frugal forward-backward splitting algorithmic framework with deviations by incorporating deviation vectors into Algorithm 1.1. Our framework avoids the drawback of adding deviations to algorithms one by one. Then we derive a norm inequality to ensure the convergence. With the aid of the norm inequality, we establish the weak convergence results of the proposed algorithmic framework.

The structure of the paper is as follows. In Section 2, we introduce the notations and related concepts. In Section 3, we propose a frugal forward-backward splitting algorithmic framework with deviations and give the weak convergence analysis. In Section 4, the last section, we present numerical experiments.

2. PRELIMINARIES

Let $\mathbb{N} = \{0, 1, \dots\}$ be the set of natural numbers and $\mathbb{N}_+ = \{1, 2, \dots\}$ be the set of non-zero natural numbers. Throughout this paper, we denote by $\mathcal{H}$ a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. Denote by Id the identity operator on $\mathcal{H}$. Given a linear, self-adjoint, and strongly positive operator $H : \mathcal{H} \rightarrow \mathcal{H}$, we define $\langle x, y \rangle_H = \langle x, Hy \rangle$ and $\|x\|_H = \sqrt{\langle x, Hx \rangle}$ for $\forall x, y \in \mathcal{H}$. We use $\omega_w(x^k) = \{x : \exists x^{k_j} \rightharpoonup x\}$ to denote the weak ω -limit set of the sequence $\{x^k\}$.

Fact 2.1. For all $a, b, c, d \in \mathcal{H}$, there holds

$$2\langle a - b, c - d \rangle = \|a - d\|^2 + \|b - c\|^2 - \|a - c\|^2 - \|b - d\|^2.$$

Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be an operator. Denote by $\text{Fix } T$ the set of fixed points of T , i.e., $\text{Fix } T = \{x \in \mathcal{H} : x = T(x)\}$.

Definition 2.1. An operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is said to be

- (i) β -Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|T(x) - T(y)\| \leq \beta \|x - y\|, \quad \forall x, y \in \mathcal{H},$$

and nonexpansive if $\beta = 1$.

- (ii) $\frac{1}{\beta}$ -cocoercive if there exists a constant $\beta > 0$, such that

$$\langle T(x) - T(y), x - y \rangle \geq \frac{1}{\beta} \|T(x) - T(y)\|^2, \quad \forall x, y \in \mathcal{H}.$$

By the Cauchy-Schwarz inequality, a $\frac{1}{\beta}$ -cocoercive operator is β -Lipschitz continuous.

Given a set-valued operator $F : \mathcal{H} \rightrightarrows \mathcal{H}$, the domain, the range, the graph, and the zeros of F are respectively denoted by $\text{dom} F = \{x \in \mathcal{H} : F(x) \neq \emptyset\}$, $\text{range} F = \{u \in \mathcal{H} : u \in F(x) \text{ for } \forall x \in \text{dom} F\}$, $\text{gra} F = \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in F(x)\}$, and $\text{zer} F = \{x \in \mathcal{H} : 0 \in F(x)\}$. The inverse operator of F , denoted by F^{-1} , is defined through $x \in F^{-1}(u) \Leftrightarrow u \in F(x)$.

Definition 2.2. ([4, Definition 20.1 and Definition 20.20]) A set-valued operator $F : \mathcal{H} \rightrightarrows \mathcal{H}$ is said to be

- (i) *monotone* if $\langle u - v, x - y \rangle \geq 0$ for all $(x, u), (y, v) \in \text{gra} F$.
- (ii) *maximally monotone* if there exists no monotone operator $B : \mathcal{H} \rightrightarrows \mathcal{H}$ such that $\text{gra} B$ properly contains $\text{gra} F$, i.e., for every $(x, u) \in \mathcal{H} \times \mathcal{H}$

$$(x, u) \in \text{gra} F \Leftrightarrow \langle u - v, x - y \rangle \geq 0, \quad \forall (y, v) \in \text{gra} F.$$

Given an operator $F : \mathcal{H} \rightrightarrows \mathcal{H}$, the resolvent of F with parameter $\lambda > 0$ is denoted by $J_{\lambda F} = (\text{Id} + \lambda F)^{-1}$. From [4, Corollary 23.11], if F is maximally monotone, then $J_{\lambda F}$ is single-valued and 1-cocoercive. For a matrix $P \in \mathbb{R}^{n \times m}$, we denote by P_{ij} its (i, j) component. The transpose of matrix P is denoted as $P^\top$. When $m = n$, we denote by $\text{slt}(P) \in \mathbb{R}^{n \times n}$ the strictly lower triangular matrix extracted from P , and by $\text{diag}(P) \in \mathbb{R}^n$ the vector extracted from the main diagonal of P . With a mild overload of notation, for $w \in \mathbb{R}^n$, we denote by $\text{diag}(w) \in \mathbb{R}^{n \times n}$ the diagonal matrix with the diagonal being w . Given a vector $g = (g_1, \dots, g_n)$ with all non-zero elements, the Hadamard inverse of g , denoted by $g^{\odot(-1)}$, is the vector defined element-wise by $(g^{\odot(-1)})_i = \frac{1}{g_i}$ for $i = 1, \dots, n$.

Given a matrix $P \in \mathbb{R}^{n \times m}$, we denote the Kronecker product of P and Id by

$$\mathbf{P} = P \otimes \text{Id} = \begin{bmatrix} P_{11} \text{Id} & P_{12} \text{Id} & \cdots & P_{1m} \text{Id} \\ P_{21} \text{Id} & P_{22} \text{Id} & \cdots & P_{2m} \text{Id} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} \text{Id} & P_{n2} \text{Id} & \cdots & P_{nm} \text{Id} \end{bmatrix}.$$

Note that $\mathbf{P}$ is a bounded linear operator from $\mathcal{H}^m$ to $\mathcal{H}^n$.

Definition 2.3. ((m, n) -nondecreasing vector). Let $n \in \mathbb{N}_+$ and $m \in \mathbb{N}$. We say that

$$A = (A_1, A_2, \dots, A_n) \in \{0, 1, \dots, m\}^n$$

is an (m, n) -nondecreasing vector if $A_1 = 0$, $A_n = m$ and $A_i \leq A_{i+1}$ for each $i \in [1, n-1]$.

Let $n \in \mathbb{N}_+$, $m \in \mathbb{N}$, and A be an (m, n) -nondecreasing vector. Denote by $\mathcal{S}(A)$ the set of matrices $R \in \mathbb{R}^{n \times m}$ that have a staircase structure w.r.t. A , i.e.,

$$R_{ij} = 0 \quad \text{for all } i = 1, \dots, n \text{ and } j > A_i.$$

Conversely, $\mathcal{S}^c(A)$ denotes the set of matrices $R \in \mathbb{R}^{n \times m}$ that have a complement staircase structure w.r.t. A , i.e., $R_{ij} = 0$ for all $i = 1, \dots, n$ and $j \leq A_i$.

Next, we use the concepts of staircase and complement staircase to define what is called a causal pair of matrices.

Definition 2.4. (Causal pair of matrices). A pair of matrices $C, Q^\top \in \mathbb{R}^{n \times m}$ is said to be causal if there exists an (m, n) -nondecreasing vector A such that $C \in \mathcal{S}(A)$ and $Q^\top \in \mathcal{S}^c(A)$.

Lemma 2.1. ([4, Lemma 2.47]) *Let Θ be a nonempty set of $\mathcal{H}$, and let $\{x^k\}$ be a sequence in $\mathcal{H}$. Assume that the following conditions hold:*

- (i) *for every $x \in \Theta$, $\lim_{k \rightarrow \infty} \|x^k - x\|$ exists;*
- (ii) *every weak sequential cluster point of $\{x^k\}$ belongs to Θ , i.e., $\omega_w(x^k) \subseteq \Theta$.*

Then $\{x^k\}$ converges weakly to a point in Θ .

3. MAIN RESULTS

In this section, we first present a frugal forward-backward splitting algorithmic framework with deviations. Then we establish the weak convergence of the proposed framework. Throughout this section, the solution set of problem (1.1) is assumed to be nonempty, i.e., $\text{zer}(\sum_{i=1}^n F_i + \sum_{i=1}^m B_i) \neq \emptyset$. Define an operator $\mathbf{F} : \mathcal{H}^n \rightrightarrows \mathcal{H}^n$ by $\mathbf{F}(\mathbf{x}) = (F_1(x_1), \dots, F_n(x_n))$ for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{H}^n$. It follows from [4, Proposition 20.23] that $\mathbf{F}$ is maximally monotone and its resolvent $J_{\mathbf{F}} : \mathcal{H}^n \rightarrow \mathcal{H}^n$ is given by $J_{\mathbf{F}} = (J_{F_1}, \dots, J_{F_n})$. Similarly, one defines an operator $\mathbf{B} : \mathcal{H}^m \rightarrow \mathcal{H}^m$ by $\mathbf{B}(\mathbf{y}) = (B_1(y_1), \dots, B_m(y_m))$ for all $\mathbf{y} = (y_1, \dots, y_m) \in \mathcal{H}^m$. Let $L = \text{diag}(\frac{1}{L_1}, \dots, \frac{1}{L_m})$, where $\frac{1}{L_i}$ is the cocoercive constant of B_i for $i = 1, \dots, m$. From [15, Lemma 7.2], it follows that

$$\langle \mathbf{B}(\mathbf{x}) - \mathbf{B}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq \|\mathbf{B}(\mathbf{x}) - \mathbf{B}(\mathbf{y})\|_{\mathbf{L}}^2, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{H}^m. \quad (3.1)$$

3.1. Algorithm. Before giving our algorithm, we assume that the parameters θ , $\{\gamma_k\}$ and $\{\xi_k\}$ satisfy the following conditions.

Assumption 3.1. Let ε be a sufficiently small positive constant. For all $k \in \mathbb{N}$, assume that the follows hold:

- (i) $\theta \in (0, +\infty)$;
- (ii) $\varepsilon \leq \gamma_k \leq 1 - \varepsilon$;
- (iii) $0 \leq \xi_k \leq 1 - \varepsilon$.

We also need some matrices such that the following assumptions hold.

Assumption 3.2. The matrices $M \in \mathbb{R}^{n \times (n-1)}$, $S \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times m}$, and $Q \in \mathbb{R}^{m \times n}$ satisfy the following properties:

- (a) $\ker M^\top \subseteq \mathbb{R}e_n$, where $e_n = (1, \dots, 1)^\top \in \mathbb{R}^n$.
- (b) $C, Q^\top$ are causal and $C^\top e_n = Qe_n = e_m$, where $e_m = (1, \dots, 1)^\top \in \mathbb{R}^m$.
- (c) S is symmetric such that $e_n^\top S e_n = 0$ and $S - MM^\top - \frac{1}{2}(1 + \frac{1}{\theta})(C^\top - Q)^\top L^{-1}(C^\top - Q) \succeq 0$.

Now we present a frugal forward-backward splitting algorithmic framework with deviations for solving (1.1) with $n \geq 1$ and $m \geq 0$.

Algorithm 3.1. A Frugal Forward-Backward Algorithmic Framework with Deviations.

Pick: The parameters θ , $\{\gamma_k\}$ and $\{\xi_k\}$ satisfying Assumption 3.1, and the matrices $M \in \mathbb{R}^{n \times (n-1)}$, $S \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times m}$ and $Q \in \mathbb{R}^{m \times n}$ satisfying Assumption 3.2.

Let: $(d_1, \dots, d_n) = 2(\text{diag}(S))^{\odot(-1)}$.

1: **Input:** $\mathbf{z}^0 = (z_1^0, \dots, z_{n-1}^0) \in \mathcal{H}^{n-1}$, $\mathbf{u}^0 = (0, \dots, 0) \in \mathcal{H}^m$, and $\mathbf{v}^0 = (0, \dots, 0) \in \mathcal{H}^{n-1}$.

2: **for** $k = 0, 1, 2, \dots$ **do**

3: **for** $i = 1, \dots, n$ **do**

$$x_i^k = J_{d_i F_i} \left(-d_i \sum_{j=1}^{i-1} S_{ij} x_j^k + d_i \sum_{j=1}^{n-1} M_{ij} (z_j^k + v_j^k) - d_i \sum_{j=1}^m C_{ij} B_j \left(\sum_{h=1}^{i-1} Q_{jh} x_h^k + u_j^k \right) \right).$$

4: **end for**

5: $z_i^{k+1} = z_i^k - \gamma_k \sum_{j=1}^n M_{ji} x_j^k, \quad \forall i \in [1, n-1].$

6: Choose $\mathbf{u}^{k+1}$ and $\mathbf{v}^{k+1}$ such that the following inequality is satisfied:

$$\frac{\gamma_{k+1}}{1 - \gamma_{k+1}} \|\mathbf{v}^{k+1}\|^2 + \frac{\gamma_{k+1}(1 + \theta)}{2} \|\mathbf{u}^{k+1}\|_{\mathbf{L}^{-1}}^2 \leq \xi_k l_k^2, \quad (3.2)$$

where

$$l_k^2 = \frac{1 - \gamma_k}{\gamma_k} \|\mathbf{z}^{k+1} - \mathbf{z}^k + \frac{\gamma_k}{1 - \gamma_k} \mathbf{v}^k\|^2. \quad (3.3)$$

7: **end for**

Now we discuss the relation of Algorithm 3.1 and the Douglas–Rachford splitting method with deviations (1.2)–(1.3). Let $n = 2, m = 0$,

$$M = \sqrt{\frac{2}{\gamma}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad S = \begin{bmatrix} \frac{2}{\gamma} & -\frac{2}{\gamma} \\ -\frac{2}{\gamma} & \frac{2}{\gamma} \end{bmatrix},$$

and $d_1 = d_2 = \gamma$. Then Algorithm 3.1 becomes a Douglas–Rachford splitting method with deviations:

$$\begin{cases} x_1^k = J_{\gamma F_1}(\sqrt{2\gamma} z^k + \sqrt{2\gamma} v^k), \\ x_2^k = J_{\gamma F_2}(2x_1^k - (\sqrt{2\gamma} z^k + \sqrt{2\gamma} v^k)), \\ z^{k+1} = z^k - \gamma_k \sqrt{\frac{2}{\gamma}} (x_1^k - x_2^k). \end{cases} \quad (3.4)$$

To compare with that given by (1.2)–(1.3), we set $\bar{z}^k = \sqrt{2\gamma} z^k$, $\bar{v}^k = \sqrt{2\gamma} v^k$ and $\lambda_k = 2\gamma_k$. Then (3.4) can be reformulated as

$$\begin{cases} x_1^k = J_{\gamma F_1}(\bar{z}^k + \bar{v}^k), \\ x_2^k = J_{\gamma F_2}(2x_1^k - (\bar{z}^k + \bar{v}^k)), \\ \bar{z}^{k+1} = \bar{z}^k - \lambda_k (x_1^k - x_2^k), \end{cases}$$

which obviously equals to (1.2). From (3.2) and (3.3), it follows that the deviation vector $\bar{v}^k$ satisfies the norm inequality

$$\frac{\lambda_{k+1}}{2 - \lambda_{k+1}} \|\bar{v}^{k+1}\|^2 \leq \xi_k \lambda_k (2 - \lambda_k) \|x_2^k - x_1^k + \frac{1}{2 - \lambda_k} \bar{v}^k\|^2. \quad (3.5)$$

It is clear that the deviation inequalities (3.5) and (1.3) are equivalent. Therefore, our framework includes the Douglas–Rachford splitting method with deviations (1.2)–(1.3) as a special case.

For the convenience of the convergence analysis, we introduce an operator $T : \mathcal{H}^{n-1} \rightarrow \mathcal{H}^{n-1}$ of the form

$$T(\mathbf{z}) = \mathbf{z} - \gamma \mathbf{M}^* \mathbf{x}, \quad \mathbf{x} = J_{\mathbf{DF}}(\mathbf{DNx} + \mathbf{DMz} - \mathbf{DCB}(\mathbf{Qx})),$$

where $N = -\text{slt}(S)$ and $D = \text{diag}(d_1, \dots, d_n)$. Note that Algorithm 3.1 can be recast as

$$\mathbf{z}^{k+1} = \mathbf{z}^k - \gamma_k \mathbf{M}^* \mathbf{x}^k, \quad \mathbf{x}^k = J_{\mathbf{DF}}(\mathbf{D}\mathbf{N}\mathbf{x}^k + \mathbf{D}\mathbf{M}(\mathbf{z}^k + \mathbf{v}^k) - \mathbf{DCB}(\mathbf{Q}\mathbf{x}^k + \mathbf{u}^k)). \quad (3.6)$$

Remark 3.1. (1) A simple choice of S satisfying Assumption 3.2(c) is $S = MM^\top + PP^\top + \frac{1}{2}(1 + \frac{1}{\theta})(C^\top - Q)^\top L^{-1}(C^\top - Q)$, where $P \in \mathbb{R}^{n \times (n-1)}$ is such that $\mathbb{R}e_n \subseteq \ker P^\top$. In Algorithm 3.1, $d_i, i = 1, \dots, n$ is the stepsize. Obviously, to obtain large stepsizes, $P = 0$ is the optimal option for S .

(2) By $N = -\text{slt}(S)$ and $D = \text{diag}(d_1, \dots, d_n)$, we have $S = 2D^{-1} - N - N^\top$. According to Assumption 3.2(c), the matrices $M \in \mathbb{R}^{n \times (n-1)}$, $N \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times m}$ and $Q \in \mathbb{R}^{m \times n}$ satisfy the following properties:

(a) $e_n^\top (D^{-1} - N)e_n = 0$;

(b) $2D^{-1} - N - N^\top \succeq MM^\top + \frac{1}{2}(1 + \frac{1}{\theta})(C^\top - Q)^\top L^{-1}(C^\top - Q)$.

(3) The concept of causal pairs of matrices captures the full class of matrices $C, Q^\top \in \mathbb{R}^{n \times m}$ such that the term $\mathbf{CB}(\mathbf{Q}\mathbf{x}^k + \mathbf{u}^k)$ in (3.6) can be computed only using a single evaluation of each operator B_i and simple algebraic operations. Furthermore, $\mathbf{CB}(\mathbf{Q})$ is strictly lower triangular for all choices of B_i according to Definition 2.4.

The next lemma gives the relation of the fixed point set of T and the zeros of $\sum_{i=1}^n F_i + \sum_{i=1}^m B_i$. One can easily show it by employing arguments which are similar to those used in the proof of [10, Lemma 3.5] and [24, Lemma 3.1].

Lemma 3.1. (Fixed points and zeros) Suppose that Assumption 3.2(a) and (b) hold, and let

$$\Omega = \{(\mathbf{z}, x) \in \mathcal{H}^{n-1} \times \mathcal{H} : \mathbf{x} = J_{\mathbf{DF}}(\mathbf{D}\mathbf{N}\mathbf{x} + \mathbf{D}\mathbf{M}\mathbf{z} - \mathbf{DCB}(\mathbf{Q}\mathbf{x})) \text{ where } \mathbf{x} = (x, \dots, x) \in \mathcal{H}^n\}.$$

Then the following assertions hold.

- (a) If $\mathbf{z} \in \text{Fix } T$, then there exists $x \in \mathcal{H}$ such that $(\mathbf{z}, x) \in \Omega$.
- (b) If $x \in \text{zer}(\sum_{i=1}^n F_i + \sum_{i=1}^m B_i)$, then there exists $\mathbf{z} \in \mathcal{H}^{n-1}$ such that $(\mathbf{z}, x) \in \Omega$.
- (c) If $(\mathbf{z}, x) \in \Omega$, then $\mathbf{z} \in \text{Fix } T$ and $x \in \text{zer}(\sum_{i=1}^n F_i + \sum_{i=1}^m B_i)$. Consequently,

$$\text{Fix } T \neq \emptyset \Leftrightarrow \Omega \neq \emptyset \Leftrightarrow \text{zer}(\sum_{i=1}^n F_i + \sum_{i=1}^m B_i) \neq \emptyset.$$

3.2. Convergence analysis. Now we present a lemma which is key for the convergence analysis of Algorithm 3.1.

Lemma 3.2. Let $F_1, \dots, F_n : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone, and let $B_i : \mathcal{H} \rightarrow \mathcal{H}$ be $\frac{1}{L_i}$ -cocoercive, $i = 1, \dots, m$ with $\text{zer}(\sum_{i=1}^n F_i + \sum_{i=1}^m B_i) \neq \emptyset$. Suppose that Assumption 3.1 and Assumption 3.2(c) hold. Let $\{\mathbf{z}^k\} \subseteq \mathcal{H}^{n-1}$ and $\{\mathbf{x}^k\} \subseteq \mathcal{H}^n$ be sequences generated by Algorithm 3.1, and $\{\mathbf{u}^k\} \subseteq \mathcal{H}^m$ and $\{\mathbf{v}^k\} \subseteq \mathcal{H}^{n-1}$ satisfy (3.2). Let $\mathbf{z} = (z_1, \dots, z_{n-1}) \in \mathcal{H}^{n-1}$ be an arbitrary point in $\text{Fix } T$. Then, for all $k \in \mathbb{N}$,

$$\|\mathbf{z}^{k+1} - \mathbf{z}\|^2 + l_k^2 \leq \|\mathbf{z}^k - \mathbf{z}\|^2 + \frac{\gamma_k}{1 - \gamma_k} \|\mathbf{v}^k\|^2 + \frac{\gamma_k(1 + \theta)}{2} \|\mathbf{u}^k\|_{\mathbf{L}^{-1}}^2. \quad (3.7)$$

Furthermore,

$$\|\mathbf{z}^{k+1} - \mathbf{z}\|^2 + l_k^2 \leq \|\mathbf{z}^k - \mathbf{z}\|^2 + \xi_{k-1} l_{k-1}^2 \quad (3.8)$$

holds for all $k \in \mathbb{N}_+$.

Proof. Let $\mathbf{x} = J_{\mathbf{DF}}(\mathbf{y})$ with $\mathbf{y} = \mathbf{DMz} + \mathbf{DNx} - \mathbf{DCB}(\mathbf{Qx})$ and $\mathbf{x}^k = J_{\mathbf{DF}}(\mathbf{y}^k)$ with $\mathbf{y}^k = \mathbf{DM}(\mathbf{z}^k + \mathbf{v}^k) + \mathbf{DNx}^k - \mathbf{DCB}(\mathbf{Qx}^k + \mathbf{u}^k)$. Since $\mathbf{D}^{-1}(\mathbf{y} - \mathbf{x}) \in \mathbf{F}(\mathbf{x})$ and $\mathbf{D}^{-1}(\mathbf{y}^k - \mathbf{x}^k) \in \mathbf{F}(\mathbf{x}^k)$, the monotonicity of $\mathbf{F}$ gives

$$\begin{aligned}
0 &\leq \langle \mathbf{x} - \mathbf{x}^k, \mathbf{D}^{-1}(\mathbf{y} - \mathbf{x}) - \mathbf{D}^{-1}(\mathbf{y}^k - \mathbf{x}^k) \rangle \\
&= \langle \mathbf{x} - \mathbf{x}^k, \mathbf{D}^{-1}(\mathbf{DMz} + \mathbf{DNx} - \mathbf{DCB}(\mathbf{Qx}) - \mathbf{x}) \rangle \\
&\quad - \langle \mathbf{x} - \mathbf{x}^k, \mathbf{D}^{-1}(\mathbf{DM}(\mathbf{z}^k + \mathbf{v}^k) + \mathbf{DNx}^k - \mathbf{DCB}(\mathbf{Qx}^k + \mathbf{u}^k) - \mathbf{x}^k) \rangle \\
&= \langle \mathbf{M}^* \mathbf{x} - \mathbf{M}^* \mathbf{x}^k, \mathbf{z} - (\mathbf{z}^k + \mathbf{v}^k) \rangle + \langle \mathbf{x} - \mathbf{x}^k, (\mathbf{N} - \mathbf{D}^{-1})\mathbf{x} - (\mathbf{N} - \mathbf{D}^{-1})\mathbf{x}^k \rangle \\
&\quad + \langle \mathbf{x} - \mathbf{x}^k, \mathbf{CB}(\mathbf{Qx}^k + \mathbf{u}^k) - \mathbf{CB}(\mathbf{Qx}) \rangle.
\end{aligned} \tag{3.9}$$

The first term on the RHS of (3.9) can be expressed as

$$\begin{aligned}
&\langle \mathbf{M}^* \mathbf{x} - \mathbf{M}^* \mathbf{x}^k, \mathbf{z} - (\mathbf{z}^k + \mathbf{v}^k) \rangle \\
&= \langle \frac{1}{\gamma}(\text{Id} - T)(\mathbf{z}) - \frac{1}{\gamma_k}(\mathbf{z}^k - \mathbf{z}^{k+1}), \mathbf{z} - (\mathbf{z}^k + \mathbf{v}^k) \rangle \\
&= \frac{1}{\gamma_k} \langle \mathbf{z}^{k+1} - \mathbf{z}^k, \mathbf{z} - (\mathbf{z}^k + \mathbf{v}^k) \rangle \\
&= \frac{1}{2\gamma_k} \left(\|\mathbf{z}^{k+1} - \mathbf{z}^k - \mathbf{v}^k\|^2 + \|\mathbf{z}^k - \mathbf{z}\|^2 - \|\mathbf{z}^{k+1} - \mathbf{z}\|^2 - \|\mathbf{v}^k\|^2 \right),
\end{aligned} \tag{3.10}$$

where the third equality comes from Fact 2.1. The second term on the RHS of (3.9) can be rewritten as

$$\begin{aligned}
&\langle \mathbf{x} - \mathbf{x}^k, (\mathbf{N} - \mathbf{D}^{-1})\mathbf{x} - (\mathbf{N} - \mathbf{D}^{-1})\mathbf{x}^k \rangle \\
&= \frac{1}{2} \langle \mathbf{x} - \mathbf{x}^k, (\mathbf{MM}^* + 2\mathbf{N} - 2\mathbf{D}^{-1})(\mathbf{x} - \mathbf{x}^k) \rangle - \frac{1}{2} \|\mathbf{M}^* \mathbf{x} - \mathbf{M}^* \mathbf{x}^k\|^2 \\
&= \frac{1}{2} \langle \mathbf{x} - \mathbf{x}^k, (\mathbf{MM}^* + \mathbf{N} + \mathbf{N}^* - 2\mathbf{D}^{-1})(\mathbf{x} - \mathbf{x}^k) \rangle - \frac{1}{2\gamma_k^2} \|\mathbf{z}^{k+1} - \mathbf{z}^k\|^2.
\end{aligned} \tag{3.11}$$

To estimate the last term on the RHS of (3.9), Young's inequality and (3.1) give

$$\begin{aligned}
&\langle \mathbf{x} - \mathbf{x}^k, \mathbf{CB}(\mathbf{Qx}^k + \mathbf{u}^k) - \mathbf{CB}(\mathbf{Qx}) \rangle \\
&= - \langle \mathbf{C}^*(\mathbf{x} - \mathbf{x}^k) - [\mathbf{Qx} - (\mathbf{Qx}^k + \mathbf{u}^k)], (\mathbf{L}^{-1})^{\frac{1}{2}} \mathbf{L}^{\frac{1}{2}} [\mathbf{B}(\mathbf{Qx}) - \mathbf{B}(\mathbf{Qx}^k + \mathbf{u}^k)] \rangle \\
&\quad - \langle \mathbf{Qx} - (\mathbf{Qx}^k + \mathbf{u}^k), \mathbf{B}(\mathbf{Qx}) - \mathbf{B}(\mathbf{Qx}^k + \mathbf{u}^k) \rangle \\
&\leq \frac{1}{4} \|(\mathbf{C}^* - \mathbf{Q})\mathbf{x} - (\mathbf{C}^* - \mathbf{Q})\mathbf{x}^k + \mathbf{u}^k\|_{\mathbf{L}^{-1}}^2 + \|\mathbf{B}(\mathbf{Qx}) - \mathbf{B}(\mathbf{Qx}^k + \mathbf{u}^k)\|_{\mathbf{L}}^2 \\
&\quad - \|\mathbf{B}(\mathbf{Qx}) - \mathbf{B}(\mathbf{Qx}^k + \mathbf{u}^k)\|_{\mathbf{L}}^2 \\
&= \frac{1}{4} \|(\mathbf{C}^* - \mathbf{Q})\mathbf{x} - (\mathbf{C}^* - \mathbf{Q})\mathbf{x}^k + \mathbf{u}^k\|_{\mathbf{L}^{-1}}^2 \\
&\leq \frac{1}{4} \left(1 + \frac{1}{\theta}\right) \left\langle \mathbf{x} - \mathbf{x}^k, (\mathbf{C}^* - \mathbf{Q})^* \mathbf{L}^{-1} (\mathbf{C}^* - \mathbf{Q})(\mathbf{x} - \mathbf{x}^k) \right\rangle + \frac{1}{4} (1 + \theta) \|\mathbf{u}^k\|_{\mathbf{L}^{-1}}^2.
\end{aligned} \tag{3.12}$$

By substituting (3.10), (3.11), and (3.12) into (3.9), and multiplying by $2\gamma_k$, one finds

$$\begin{aligned}
& \|\mathbf{z}^{k+1} - \mathbf{z}\|^2 \\
& \leq \|\mathbf{z}^k - \mathbf{z}\|^2 - \frac{1}{\gamma_k} \|\mathbf{z}^{k+1} - \mathbf{z}^k\|^2 + \|\mathbf{z}^{k+1} - \mathbf{z}^k - \mathbf{v}^k\|^2 - \|\mathbf{v}^k\|^2 + \frac{\gamma_k(1+\theta)}{2} \|\mathbf{u}^k\|_{\mathbf{L}^{-1}}^2 \\
& \quad + \gamma_k \langle \mathbf{x} - \mathbf{x}^k, [\mathbf{M}\mathbf{M}^* + \mathbf{N} + \mathbf{N}^* - 2\mathbf{D}^{-1} + \frac{1+\theta}{2\theta} (\mathbf{C}^* - \mathbf{Q})^* \mathbf{L}^{-1} (\mathbf{C}^* - \mathbf{Q})] (\mathbf{x} - \mathbf{x}^k) \rangle \\
& = \|\mathbf{z}^k - \mathbf{z}\|^2 + \frac{\gamma_k - 1}{\gamma_k} \|\mathbf{z}^{k+1} - \mathbf{z}^k\|^2 + \frac{\gamma_k}{1 - \gamma_k} \|\mathbf{v}^k\|^2 + \frac{\gamma_k}{1 - \gamma_k} \|\mathbf{v}^k\|^2 + \frac{\gamma_k(1+\theta)}{2} \|\mathbf{u}^k\|_{\mathbf{L}^{-1}}^2 \\
& \quad + \gamma_k \langle \mathbf{x} - \mathbf{x}^k, [\mathbf{M}\mathbf{M}^* + \mathbf{N} + \mathbf{N}^* - 2\mathbf{D}^{-1} + \frac{1+\theta}{2\theta} (\mathbf{C}^* - \mathbf{Q})^* \mathbf{L}^{-1} (\mathbf{C}^* - \mathbf{Q})] (\mathbf{x} - \mathbf{x}^k) \rangle.
\end{aligned}$$

According to (b) of Remark 3.1(2) and Assumption 3.1(ii), we obtain

$$\begin{aligned}
& \|\mathbf{z}^{k+1} - \mathbf{z}\|^2 + \frac{1 - \gamma_k}{\gamma_k} \|\mathbf{z}^{k+1} - \mathbf{z}^k + \frac{\gamma_k}{1 - \gamma_k} \mathbf{v}^k\|^2 \\
& \leq \|\mathbf{z}^k - \mathbf{z}\|^2 + \frac{\gamma_k}{1 - \gamma_k} \|\mathbf{v}^k\|^2 + \frac{\gamma_k(1+\theta)}{2} \|\mathbf{u}^k\|_{\mathbf{L}^{-1}}^2.
\end{aligned}$$

From (3.3), we see inequality (3.7) holds. Finally, (3.8) follows from (3.2). $\square$

The following theorem is our main result regarding the convergence of Algorithm 3.1.

Theorem 3.1. *Let $F_1, \dots, F_n : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone, and let $B_i : \mathcal{H} \rightarrow \mathcal{H}$ be $\frac{1}{L_i}$ -cocoercive for $i = 1, \dots, m$ with $\text{zer}(\sum_{i=1}^n F_i + \sum_{i=1}^m B_i) \neq \emptyset$. Suppose Assumptions 3.1 and 3.2 hold. Let $\{\mathbf{x}^k\} \subseteq \mathcal{H}^n$ and $\{\mathbf{z}^k\} \subseteq \mathcal{H}^{n-1}$ be generated by Algorithm 3.1, and $\{\mathbf{u}^k\} \subseteq \mathcal{H}^m$ and $\{\mathbf{v}^k\} \subseteq \mathcal{H}^{n-1}$ satisfy (3.2). Then the following assertions hold:*

- (i) *The sequence $\{l_k^2\}$ is summable and the sequences $\{\mathbf{u}^k\}$ and $\{\mathbf{v}^k\}$ converge to zero.*
- (ii) *For all $\mathbf{z} \in \text{Fix } T$, the sequence $\{\|\mathbf{z}^k - \mathbf{z}\|\}$ converges.*
- (iii) *As $k \rightarrow \infty$, we have $\mathbf{z}^{k+1} - \mathbf{z}^k \rightarrow 0$ and $\sum_{i=1}^n s_i x_i^k \rightarrow 0$ for all $s \in \mathbb{R}^n$ with $\sum_{i=1}^n s_i = 0$.*
- (iv) *The sequence $\{\mathbf{z}^k\}$ converges weakly to a point in $\text{Fix } T$.*
- (v) *The sequence $\{\mathbf{x}^k\}$ converges weakly to a point $(\bar{x}, \dots, \bar{x}) \in \mathcal{H}^n$ such that*

$$\bar{x} \in \text{zer}(\sum_{i=1}^n F_i + \sum_{i=1}^m B_i).$$

Proof. (i) Now, we start by proving (i) via (3.8). Let $\mathbf{z} \in \text{Fix } T$ and $N \in \mathbb{N}_+$. We sum (3.8) for $k = 1, 2, \dots, N$ to obtain

$$\|\mathbf{z}^{N+1} - \mathbf{z}\|^2 + l_N^2 + \sum_{k=1}^{N-1} (1 - \xi_k) l_k^2 \leq \|\mathbf{z}^1 - \mathbf{z}\|^2 + \xi_0 l_0^2.$$

Then, rearranging the terms gives

$$\begin{aligned}
\sum_{k=1}^N (1 - \xi_k) l_k^2 & \leq \|\mathbf{z}^1 - \mathbf{z}\|^2 - \|\mathbf{z}^{N+1} - \mathbf{z}\|^2 - \xi_N l_N^2 + \xi_0 l_0^2 \\
& \leq \|\mathbf{z}^1 - \mathbf{z}\|^2 + \xi_0 l_0^2.
\end{aligned} \tag{3.13}$$

Since the RHS of (3.13) is independent of N , we conclude that $\sum_{k=1}^\infty (1 - \xi_k) l_k^2 < \infty$. Thanks to $\{\xi_k\} \subseteq [0, 1 - \varepsilon]$ from Assumption 3.1(iii), we have $\sum_{k=1}^\infty l_k^2 < \infty$, which implies

$$\lim_{k \rightarrow \infty} l_k^2 = 0. \tag{3.14}$$

Then, from (3.2), Assumption 3.1(i) and (ii), we deduce that $\lim_{k \rightarrow \infty} \mathbf{u}^k = 0$ and $\lim_{k \rightarrow \infty} \mathbf{v}^k = 0$.

(ii) From (3.8) and Assumption 3.1(iii), we deduce that $\{\|\mathbf{z}^k - \mathbf{z}\|^2 + l_{k-1}^2\}$ is nonincreasing and hence convergent. Furthermore, from (3.14), one sees that $\{\|\mathbf{z}^k - \mathbf{z}\|\}$ converges, which implies that $\{\mathbf{z}^k\}$ is bounded.

(iii) From (3.3), (3.14), Assumption 3.1(ii), and $\lim_{k \rightarrow \infty} \mathbf{v}^k = 0$, we deduce that $\lim_{k \rightarrow \infty} (\mathbf{z}^{k+1} - \mathbf{z}^k) = 0$. According to Assumption 3.2(a), let $s \in \{a \in \mathbb{R}^n : \sum_{i=1}^n a_i = 0\} = (\ker M^\top)^\perp = \text{range } M$. Then, there exists $v \in \mathbb{R}^{n-1}$ such that $s = -Mv$ and hence

$$\begin{aligned} \sum_{i=1}^n s_i x_i^k &= (s^\top \otimes \text{Id}) \mathbf{x}^k = -\left((v^\top M^\top) \otimes \text{Id}\right) \mathbf{x}^k \\ &= -(v^\top \otimes \text{Id}) \mathbf{M}^* \mathbf{x}^k = \frac{1}{\gamma_k} (v^\top \otimes \text{Id}) (\mathbf{z}^{k+1} - \mathbf{z}^k) \rightarrow 0. \end{aligned}$$

(iv) and (v) Let $\mathbf{x}^k = J_{\mathbf{DF}}(\mathbf{y}^k)$, where $\mathbf{y}^k = \mathbf{DM}(\mathbf{z}^k + \mathbf{v}^k) + \mathbf{DNx}^k - \mathbf{DCB}(\mathbf{Qx}^k + \mathbf{u}^k)$. We claim that $\{\mathbf{x}^k\}$ is bounded. To see this, let $d_1 M_1$ denote the first row of the matrix DM . By Assumption 3.2(b) and Remark 3.1(2), we have

$$x_1^k = J_{d_1 F_1}(y_1^k) = J_{d_1 F_1}(d_1 \mathbf{M}_1(\mathbf{z}^k + \mathbf{v}^k)).$$

From the nonexpansivity of the resolvents and boundedness of $\{\mathbf{z}^k\}$ and $\{\mathbf{v}^k\}$, one sees that $\{x_1^k\}$ is also bounded. By (iii), $\{\mathbf{x}^k\}$ is bounded, as claimed. Let $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_{n-1}) \in \mathcal{H}^{n-1}$ be an arbitrary weak cluster point of $\{\mathbf{z}^k\}$. Then there exists a point $\bar{\mathbf{x}} \in \mathcal{H}^n$ such that $(\bar{\mathbf{z}}, \bar{\mathbf{x}})$ is a weak cluster point of $\{(\mathbf{z}^k, \mathbf{x}^k)\}$, where $\bar{\mathbf{x}} = (\bar{x}, \dots, \bar{x})$ according to (iii). Denote $\mathbf{w}^k = \mathbf{DM}(\mathbf{z}^k + \mathbf{v}^k) + \mathbf{DNx}^k$. Then $\mathbf{y}^k = \mathbf{w}^k - \mathbf{DCB}(\mathbf{Qx}^k + \mathbf{u}^k)$ and $\mathbf{w} = \mathbf{DM}\bar{\mathbf{z}} + \mathbf{DN}\bar{\mathbf{x}}$ is a weak cluster point of $\{\mathbf{w}^k\}$. Using $\text{range } M = \{a \in \mathbb{R}^n : \sum_{i=1}^n a_i = 0\}$, we deduce $\sum_{i=1}^n \frac{w_i^k}{d_i} = \sum_{i,j=1}^n N_{ij} x_j^k$ from $\mathbf{D}^{-1} \mathbf{w}^k = \mathbf{M}(\mathbf{z}^k + \mathbf{v}^k) + \mathbf{Nx}^k$. Then,

$$\begin{aligned} \Phi \begin{pmatrix} \frac{1}{d_1}(w_1^k - x_1^k) \\ \frac{1}{d_2}(w_2^k - x_2^k) + b_2^k \\ \vdots \\ \frac{1}{d_{n-1}}(w_{n-1}^k - x_{n-1}^k) + b_{n-1}^k \\ x_n^k \end{pmatrix} &\ni \begin{pmatrix} x_1^k - x_n^k \\ x_2^k - x_n^k \\ \vdots \\ x_{n-1}^k - x_n^k \\ \sum_{i=1}^n \frac{1}{d_i}(w_i^k - x_i^k) + \sum_{i=2}^n b_i^k \end{pmatrix} \\ &= \begin{pmatrix} x_1^k - x_n^k \\ x_2^k - x_n^k \\ \vdots \\ x_{n-1}^k - x_n^k \\ \sum_{i,j=1}^n N_{ij} x_j^k - \sum_{i=1}^n \frac{1}{d_i} x_i^k + \sum_{i=2}^n b_i^k \end{pmatrix}, \end{aligned} \quad (3.15)$$

where $b_i^k = \sum_{j=1}^m C_{ij} B_j(x_i^k) - \sum_{j=1}^m C_{ij} B_j(\sum_{h=1}^{i-1} Q_{jh} x_h^k + u_j^k)$, $i = 2, \dots, n$ and the operator $\Phi : \mathcal{H}^n \rightrightarrows \mathcal{H}^n$ is defined by

$$\Phi = \begin{pmatrix} F_1^{-1} \\ (F_2 + \sum_{j=1}^m C_{2j} B_j)^{-1} \\ \vdots \\ (F_{n-1} + \sum_{j=1}^m C_{n-1,j} B_j)^{-1} \\ F_n + \sum_{j=1}^m C_{nj} B_j \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\text{Id} \\ 0 & 0 & 0 & \cdots & 0 & -\text{Id} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & -\text{Id} \\ \text{Id} & \text{Id} & \text{Id} & \cdots & \text{Id} & 0 \end{pmatrix}.$$

According to (i), (iii), Assumption 3.2(b), and the Lipschitz continuity of B_i , we have $\lim_{k \rightarrow \infty} b_i^k = 0$ for $i = 2, \dots, n$. As the sum of two maximally monotone operators is again maximally monotone provided that one of the operators has full domain [4, Corollary 24.4 (i)], it follows that Φ is maximally monotone. Consequently, its graph is sequentially closed in the weak-strong topology [4, Proposition 20.32]. Note also that the RHS of (3.15) converges strongly to zero as a consequence of (i), (iii), and (a) of Remark 3.1(2). Taking the limit along a subsequence of $\{\mathbf{z}^k, \mathbf{x}^k\}$ which converges weakly to $(\bar{\mathbf{z}}, \bar{\mathbf{x}})$ in (3.15), and using the weak-strong topology of Φ , we obtain

$$\Phi \begin{pmatrix} \frac{1}{d_1}(w_1 - \bar{x}) \\ \vdots \\ \frac{1}{d_{n-1}}(w_{n-1} - \bar{x}) \\ \bar{x} \end{pmatrix} \ni \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Then, it follows

$$\begin{cases} F_1(\bar{x}) \ni \frac{1}{d_1}(w_1 - \bar{x}), \\ (F_i + \sum_{j=1}^m C_{ij}B_j)(\bar{x}) \ni \frac{1}{d_i}(w_i - \bar{x}), \quad \forall i \in [2, n-1], \\ (F_n + \sum_{j=1}^m C_{nj}B_j)(\bar{x}) \ni -\sum_{i=1}^{n-1} \frac{1}{d_i}(w_i - \bar{x}), \end{cases} \quad (3.16)$$

which together with Lemma 3.1(c) yields $\bar{\mathbf{z}} \in \text{Fix } T$. Hence $\omega_w(\mathbf{z}^k) \subseteq \text{Fix}(T)$. Using (ii) and Lemma 2.1, it follows that $\{\mathbf{z}^k\}$ converges weakly to a point in $\text{Fix } T$. From (3.16), we also have $\bar{x} = J_{d_1 F_1}(w_1) = J_{d_1 F_1}(d_1 \mathbf{M}_1 \bar{\mathbf{z}})$ and $\bar{x} \in \text{zer}(\sum_{i=1}^n F_i + \sum_{i=1}^m B_i)$. Thus $\bar{\mathbf{x}} = (\bar{x}, \dots, \bar{x})$ is the unique weak sequential cluster point of $\{\mathbf{x}^k\}$ and $\{\mathbf{x}^k\}$ converges weakly to $\bar{\mathbf{x}}$. $\square$

4. NUMERICAL EXPERIMENTS

Numerical experiments in [3] illustrate that among some schemes in Algorithm 1.1, the split-forward-backward+ (SFB+) and the adapted graph forward-backward splitting (aGFB) achieve favorable numerical results. Therefore, in this section, we consider split-forward-backward+ with deviations (dev_SFB+) and the adapted graph forward-backward splitting with deviations (dev_aGFB).

4.1. Problem description. A classic problem for testing optimization algorithms is the Markowitz portfolio optimization in [8]. Here we focus on this problem of the form:

$$\min_{x \in \Delta} \frac{1}{2} x^\top \Lambda x - r^\top x + \frac{\delta}{2} \|x\|^2 + \sum_{i=1}^n |x_i - (x_0)_i| + \sum_{i=1}^n |x_i - (x_0)_i|^{3/2}, \quad (4.1)$$

where $\delta > 0$, $r \in \mathbb{R}^n$ is a vector of estimated assets returns, $\Lambda \in \mathbb{R}^{n \times n}$ is the estimated covariance matrix of returns (which is symmetric positive semidefinite), $\Delta = \{x \in \mathbb{R}^n : x_i \geq 0, \forall i = 1, \dots, n, x_1 + \dots + x_n = 1\}$ is the standard simplex, and $x_0 \in \mathbb{R}^n$ is the initial position. We introduce the indicator function $\mathbb{I}_\Delta$ of the constraint set Δ to reformulate (4.1) as an unconstrained optimization problem and then split the objective function into the following form:

$$\min_{x \in \mathbb{R}^n} \underbrace{\frac{1}{2} x^\top \Lambda x - r^\top x}_{f_1(x)} + \underbrace{\frac{\delta}{2} \|x\|^2}_{f_2(x)} + \underbrace{\sum_{i=1}^n |x_i - (x_0)_i|}_{g_1(x)} + \underbrace{\sum_{i=1}^n |x_i - (x_0)_i|^{3/2}}_{g_2(x)} + \underbrace{\mathbb{I}_\Delta(x)}_{g_3(x)}. \quad (4.2)$$

Let $F_i = \partial g_i$ for $i = 1, 2, 3$ and $B_i = \nabla f_i$ for $i = 1, 2$. Then, optimization problem (4.2) is equivalent to the inclusion problem: find $x \in \mathbb{R}^n$ such that $0 \in (F_1 + F_2 + F_3 + B_1 + B_2)(x)$, where F_i , $i = 1, 2, 3$ are the maximally monotone operators, as shown in [4, Theorem 20.48]. By [4, Corollary 18.17], B_i , $i = 1, 2$ are $\frac{1}{L_i}$ -cocoercive, where L_1 is the largest eigenvalue of Λ and $L_2 = \delta$.

4.2. Experiment settings and results. In the experiment, the estimated returns r and the matrix Λ are derived from real data. Details are available in data¹ for 200 market days, and $n = 53$ assets. Set $\delta = 6$. Consider the following two cases of the problem (4.1):

Case 1: the initial state x_0 is chosen randomly;

Case 2: x_0 is the output of Case 1 for the same optimization problem but 20 days later.

We test aGFB and SFB+ and their formulas with deviations to show the effectiveness of the deviations. For the four algorithms, we take the initial point $z_i^0 = (0, \dots, 0) \in \mathbb{R}^n$, $i = 1, 2$, and choose the matrices M , C and Q as in [3]. We take $\gamma_k = 0.9$ for all algorithms. For dev_SFB+ and dev_aGFB, θ is tuned manually to a nearly optimal value. Initialize $u_i^0 = (0, \dots, 0) \in \mathbb{R}^n$ and $v_i^0 = (0, \dots, 0) \in \mathbb{R}^n$ for $i = 1, 2$. The algorithms terminate when $\|x_3^k - x^*\| < 10^{-8}$, where x^* is the exact solution.

There are some ways for selecting deviation vectors that satisfy the deviation condition. In order to select relatively optimal deviation vectors, we refer to [7] and adopt a three-layer convolutional neural network model. Figure 1 presents the decay of the errors of four algorithms with the numbers of the iterations. It demonstrates the improvement in the iterative methods achieved by the appropriate selection of the deviations. We randomly choose x_0 and report arithmetical averages of the required numbers of iterations taken to run the algorithms 50 times in Table 1. It can be observed from Figure 1 and Table 1 that for Case 1, the dev_aGFB and dev_SFB+ algorithms perform significantly better than their counterparts (aGFB and SFB+) while for Case 2, the improvements of dev_aGFB and dev_SFB+ on aGFB and SFB+ are slightly. Furthermore, dev_aGFB behaves best among four algorithms.

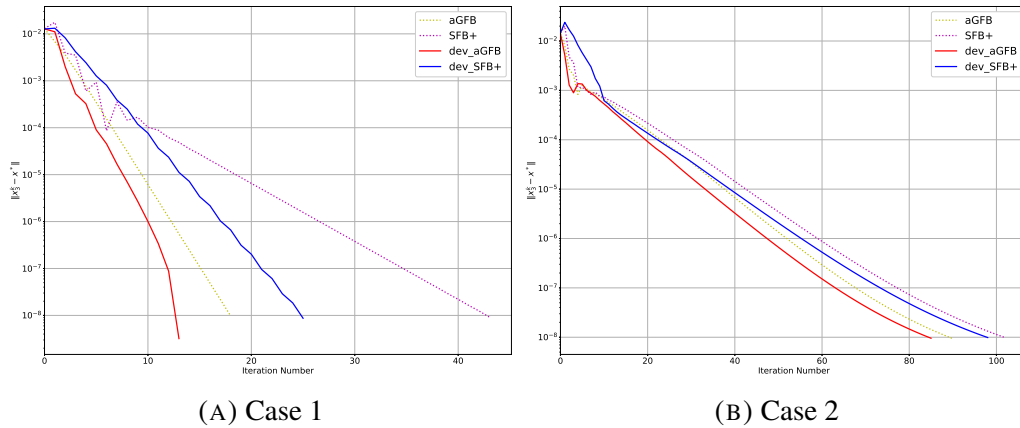


FIGURE 1. The decay of the errors of four algorithms with the numbers of the iterations under Case 1 and Case 2.

¹Code and data can be downloaded from <https://github.com/TraDE-OPT/portopt-siopt21>.

TABLE 1. Comparisons of the numbers of iterations of the four algorithms.

	Algorithms			
	aGFB	SFB+	dev_aGFB	dev_SFB+
Case 1	28	44	23.64	26.34
Case 2	89.06	91.16	85.12	87.56

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