

NONLINEAR CHARACTERIZATIONS FOR SET OPTIMIZATION WITH A VARIABLE STRUCTURE AND APPLICATIONS

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Abstract. In this paper, we obtain some properties of a generalized Gerstewitz's function with respect to a variable structure. Then, by using the scalarization function and the image space analysis, necessary and sufficient optimality conditions for the solutions are established. In addition, we propose two nonlinear weak separation functions for the set optimization problems and introduce a generalized Lagrangian set-valued mapping to obtain the relationship between the saddle points and the existence of a nonlinear separation. Finally, as applications of the main results of this paper, we examine traffic network problems with interval-valued cost functions and medical image registration.

Keywords. Image space analysis; Medical image registration; Set optimization; Traffic network problems; Variable domination structure.

1. INTRODUCTION

Due to the extensive applications in many fields, such as image processing, optimal control and economics, the set optimization has received much attention during the last two decades, see, for example, [4, 15, 18, 21, 29, 36] and the references therein. There are (at least) three approaches for defining solution concepts for set-valued optimization problems: the vector approach [8, 31, 41], the set approach [7, 21], and the lattice approach [32]. In this work, we deal with set optimization with the set approach, where we compare the sets in linear spaces by some set ordered relations defined by means of convex cones. However, decisions may depend on more than one objective and what is preferred may vary with the actual state. In 1974, Yu [37] introduced the concept of variable ordering structure by using moving cones in vector optimization. In the past decades or so, variable ordering structure has gained interest, as seen in [10, 11, 12, 13, 14, 23, 26, 27, 29]. Eichfelder [11] examined the optimality concepts for vector optimization problem with a variable ordering structure which are motivated by an application in medical image registration. Köbis et al. [29] investigated a descent method to find approximations of a minimal solution structures with respect to variable domination structures and studied a set-valued optimization problem in medical image registration.

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There are some techniques for solving set-valued optimization problems based on the set optimization approach. Scalarization approach is one of the techniques to study optimality conditions and to develop algorithms for set-valued optimization problems; see [5, 21, 26, 33, 38]. Following this approach, solution concepts were introduced and characterized through diverse methods, often integrated with complementary frameworks such as duality theory, stability analysis and variational analysis; see [17, 19]. Crespi, Hamel, and Schrage [9] introduced a new set relation by using nonlinear functionals, established approximate solution concepts based on complete lattices, investigated the existence of solutions for set optimization problems within nonlinear functionals and studied the stability of these solutions. In the literature, two types of nonlinear scalarization functions have mainly been used to investigate set optimization problems: Gerstewitz's function and the oriented distance function; see, e.g., [2, 16, 22, 24]. Ansari, Hamel and Sharma [1] introduced a new generalized Gerstewitz's function, by which they presented minimal element theorems and Ekeland's variational principle. Han [20] introduced a Hausdorff-type distance based on Gerstewitz's function and applied it to characterize the optimality conditions for set optimization problems. A natural question arises that how can we introduce a generalized Gerstewitz's function with respect to a variable structure to establish optimality conditions for set optimization.

The image space analysis (ISA) plays a significant role in studying several optimization problems. It has been demonstrated as a very fruitful method to investigate duality, variational principles, penalty methods, gap functions, and error bounds, in optimization theory; see, e.g., [28, 30, 34, 39]. The infeasibility of such a system is characterized by the disjunction of two suitable subsets in the image space. This can be realized by allowing these subsets lie in two disjoint level sets of a linear or nonlinear separation function. Ansari, Köbis and Sharma [3] defined some separable sets in the scalarization image space by using oriented distance function and established optimality conditions for robust solutions for multiobjective optimization problems. Xu [35] introduced a nonlinear weak regular separation function and obtained the necessary and sufficient optimality conditions for the Ky Fan quasi-inequality with cone constraints. Thus, separation is vital in the process of implementation, and how to choose appropriate separation function is the key in the ISA.

Inspired by the work of Köbis et al. [26, 27] on a new variable order relation, in this paper, we adopt scalarization approach and image space analysis to investigate optimality conditions for set optimization problems with a binary variable set structure. In Section 2, we provide some notations which will be used in later sections. We recall a set order relation with respect to a variable set structure, introduced in [7]. In Section 3, we investigate a generalized Gerstewitz's function with respect to a variable structure. In Section 4, we establish necessary and sufficient optimality conditions for the set optimization problems with a variable structure. In Section 5, we establish optimality conditions and present the separation between two suitable sets in image space. Moreover, we construct a generalized Lagrangian set-valued mapping to discuss the properties of a saddle point. In Section 6, we illustrate the validity and effectiveness of the derived results by applying these results to traffic network problems with interval-valued cost functions and to a medical image registration problem. Section 7, the last section, ends this paper.

2. PRELIMINARIES

Throughout this paper, let X, Y and Z be real normed spaces, and the zero vector be denoted by $\mathbf{0}$. The family of nonempty subsets of Y is denoted by $\mathcal{P}_0(Y)$. Let $\mathbb{B}(\mathbf{0}, \varepsilon)$ denote the open ball with center at origin and radius $\varepsilon > 0$ in Y . For a set $Q \in \mathcal{P}_0(Y)$, $\text{int}Q$ denotes the topological interior of Q . Let $K \in \mathcal{P}_0(Y)$. A set $Q \in \mathcal{P}_0(Y)$ is said to be: K -proper if $Q + K \neq Y$; K -bounded if, for each neighborhood U of $\mathbf{0}$ in Y , there exists a positive number t such that $Q \subseteq tU + K$. The set Q is called a coradiant set if $\alpha Q \subseteq Q$ for all $\alpha \geq 1$. Let $\mathbb{R}^n$ be the n -dimensional Euclidean space. Denote

$$\mathbb{R}_+^n := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, 2, \dots, n\}$$

and

$$\mathbb{R}_{++}^n := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, 2, \dots, n\}.$$

Assume that $\mathcal{D} : Y \times Y \rightrightarrows Y$ is a set-valued mapping such that, for any $y_1, y_2 \in Y$, $\mathcal{D}(y_1, y_2) \in \mathcal{P}_0(Y)$ and $\text{int}\mathcal{D}(y_1, y_2) \neq \emptyset$. Let $k_0 \in Y \setminus \{\mathbf{0}\}$ be such that $\mathcal{D}(y_1, y_2) + [0, +\infty) \cdot k_0 \subseteq \mathcal{D}(y_1, y_2)$.

To study the set optimization problems with a variable set structure, we recall the following relation to compare sets in a linear topological space.

Definition 2.1. [27] Let $A, B \in \mathcal{P}_0(Y)$. The variable upper set less relation ($\preceq_{\mathcal{D}}^u$) is defined by

$$A \preceq_{\mathcal{D}}^u B \Leftrightarrow \text{for any } a \in A, \text{ there exists } b \in B \text{ such that } a \in b - \mathcal{D}(a, b).$$

Similarly, we introduce the following concept.

Definition 2.2. Let $A, B \in \mathcal{P}_0(Y)$. The variable strictly upper set less relation ($\prec_{\mathcal{D}}^u$) is defined by

$$A \prec_{\mathcal{D}}^u B \Leftrightarrow \text{for any } a \in A, \text{ there exists } b \in B \text{ such that } a \in b - \text{int}\mathcal{D}(a, b).$$

We say that $A \sim_{\mathcal{D}}^u B \Leftrightarrow A \preceq_{\mathcal{D}}^u B$ and $B \preceq_{\mathcal{D}}^u A$. We recall two binary relations with respect to a variable structure.

Definition 2.3. [13, Definition 2.3] Let $A, B \in 2^Y$, and $\mathcal{D}_1, \mathcal{P} : Y \rightrightarrows Y$ be set-valued mappings. The binary relations on 2^Y are defined as follows:

(a) The u-less order relation of type $\mathcal{D}_1(\preceq_u^{\mathcal{D}_1})$ is defined by

$$A \preceq_u^{\mathcal{D}_1} B \Leftrightarrow \text{for all } a \in A, B \cap (\{a\} + \mathcal{D}_1(a)) \neq \emptyset.$$

(b) The u-less order relation of type $\mathcal{P}(\preceq_u^{\mathcal{P}})$ is defined by

$$A \preceq_u^{\mathcal{P}} B \Leftrightarrow A \subseteq \bigcup_{b \in B} (\{b\} - \mathcal{P}(b)).$$

Remark 2.1. Let $A, B \in \mathcal{P}_0(Y)$. If $\mathcal{D}(\cdot, \cdot)$ is a given set-valued mapping that only depends on the first variable, that is, $\mathcal{D}(y_1, y_2) := \mathcal{D}_1(y_1)$ for all $y_1, y_2 \in Y$, then

$$A \preceq_{\mathcal{D}}^u B \Leftrightarrow A \preceq_u^{\mathcal{D}_1} B \Leftrightarrow \text{for any } a \in A, \text{ there exists } b \in B \text{ such that } a \in b - \mathcal{D}_1(a).$$

On the other hand, if $\mathcal{D}(\cdot, \cdot)$ is a given set-valued mapping that only depends on the second variable, that is, $\mathcal{D}(y_1, y_2) := \mathcal{P}(y_2)$ for all $y_1, y_2 \in Y$, then

$$A \preceq_{\mathcal{D}}^u B \Leftrightarrow A \preceq_u^{\mathcal{P}} B \Leftrightarrow \text{for any } a \in A, \text{ there exists } b \in B \text{ such that } a \in b - \mathcal{P}(b).$$

Definition 2.4. [25] A set-valued mapping $\Gamma : Y \times Y \rightrightarrows Y$ is said to be closed at $\hat{p} \in Y \times Y$ if for any sequence $((p_n, y_n))$ in the graph of Γ , defined by $\text{graph}\Gamma := \{(p, y) \in (Y \times Y) \times Y : y \in \Gamma(p)\}$, which converges to $(\hat{p}, \hat{y})$ implies that $\hat{y} \in \Gamma(\hat{p})$.

The mapping Γ is called closed on a nonempty set $E \subseteq Y \times Y$ if it is closed at every point in E .

Lemma 2.1. Let $Q \in \mathcal{P}_0(Y)$ with $\text{int}Q \neq \emptyset$, and $q_0 \in Y \setminus \{\mathbf{0}\}$. Then, $\text{int}Q \subseteq \text{int}Q + \bigcup_{\varepsilon > 0} \varepsilon q_0$.

Proof. Let $q \in \text{int}Q$. Then, there exists $\varepsilon_0 > 0$ such that $q + \mathbb{B}(\mathbf{0}, \varepsilon_0) \subseteq \text{int}Q$. Let $\varepsilon_1 = \frac{\varepsilon_0}{2\|q_0\|}$. Then, $-\varepsilon_1 q_0 \in \mathbb{B}(\mathbf{0}, \varepsilon_0)$, and so, $q - \varepsilon_1 q_0 \in \text{int}Q$, which implies that $q \in \text{int}Q + \varepsilon_1 q_0 \subseteq \text{int}Q + \bigcup_{\varepsilon > 0} \varepsilon q_0$. Hence, $\text{int}Q \subseteq \text{int}Q + \bigcup_{\varepsilon > 0} \varepsilon q_0$. $\square$

3. A NONLINEAR SCALARIZATION FUNCTION

Let $\mathcal{D} : Y \times Y \rightrightarrows Y$ be a set-valued mapping and $A, B \in \mathcal{P}_0(Y)$. In the later part of the paper, we denote $\mathcal{D}[A, B] := \bigcup_{a \in A, b \in B} \mathcal{D}(a, b)$ and $\bar{\mathcal{D}}[A, B] := \bigcap_{a \in A, b \in B} \mathcal{D}(a, b)$, as well as we assume that both of them are nonempty. Araya [7] investigated some scalarization functions with respect to a fixed solid closed convex cone. In a similar way, we define a scalarization function $h(\cdot, \cdot) : \mathcal{P}_0(Y) \times \mathcal{P}_0(Y) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ as

$$h(A, B) := \inf \{t \in \mathbb{R} : A \preceq_{\mathcal{D}}^u B + tk_0\}, \quad \text{for all } A, B \in \mathcal{P}_0(Y).$$

In order to derive some properties of the variable set relations, we consider the following assumptions related to the set-valued mapping $\mathcal{D} : Y \times Y \rightrightarrows Y$:

- (H₁) For all $y_1, y_2 \in Y$, $\mathcal{D}(y_1, y_2) + (0, +\infty)k_0 \subseteq \text{int} \mathcal{D}(y_1, y_2)$.
- (H₂) For all $y_1, y_2 \in Y$, $\mathcal{D}(y_1, y_2) + (-\infty, +\infty)k_0 = Y$.
- (H₃) For all $y_1, y_2 \in Y$, $\lambda \geq 0$, $\mathcal{D}(y_1, y_2) \subseteq \mathcal{D}(y_1, y_2 + \lambda k_0)$.
- (H₄) For all $y_1, y_2 \in Y$, $\lambda \geq 0$, $\mathcal{D}(y_1, y_2) \subseteq \mathcal{D}(y_1, y_2 - \lambda k_0)$.
- (H₅) For all $y_1, y_2 \in Y$, $\mathbf{0} \in \text{int}\mathcal{D}(y_1, y_2) + (-\infty, +\infty)k_0$.

Lemma 3.1. If (H₃) holds and $h(A, B) = t_0 \in \mathbb{R}$, then, for any $\varepsilon > 0$, $A \preceq_{\mathcal{D}}^u B + (t_0 + \varepsilon)k_0$.

Proof. Since $h(A, B) = t_0$, we deduce that, for any $\varepsilon > 0$, there exists ε_1 with $0 \leq \varepsilon_1 < \varepsilon$ such that $A \preceq_{\mathcal{D}}^u B + (\varepsilon_1 + t_0)k_0$. Then, for any $a \in A$, there exists $b_{\varepsilon_1} \in B$ such that

$$\begin{aligned} a &\in b_{\varepsilon_1} + (\varepsilon_1 + t_0)k_0 - \mathcal{D}(a, b_{\varepsilon_1} + (\varepsilon_1 + t_0)k_0) \\ &= b_{\varepsilon_1} + \varepsilon k_0 + (\varepsilon_1 - \varepsilon)k_0 + t_0 k_0 - \mathcal{D}(a, b_{\varepsilon_1} + (\varepsilon_1 + t_0)k_0). \end{aligned}$$

Since $\varepsilon_1 < \varepsilon$, by using (H₃), we obtain

$$a \in b_{\varepsilon_1} + \varepsilon k_0 + (\varepsilon_1 - \varepsilon)k_0 + t_0 k_0 - \mathcal{D}(a, b_{\varepsilon_1} + (\varepsilon + t_0)k_0). \quad (3.1)$$

Since $\varepsilon_1 - \varepsilon < 0$, it follows that

$$(\varepsilon_1 - \varepsilon)k_0 - \mathcal{D}(a, b_{\varepsilon_1} + (\varepsilon + t_0)k_0) \subseteq -\mathcal{D}(a, b_{\varepsilon_1} + (\varepsilon + t_0)k_0).$$

By combining this with (3.1), we have $a \in b_{\varepsilon_1} + \varepsilon k_0 + t_0 k_0 - \mathcal{D}(a, b_{\varepsilon_1} + (\varepsilon + t_0)k_0)$. Thus $A \preceq_{\mathcal{D}}^u B + (t_0 + \varepsilon)k_0$. $\square$

The following examples demonstrate that $h(A, B)$ may attain infinite values.

Example 3.1. Let $Y = \ell^\infty$, $A = \{x = (a_1, a_2, \dots, a_m, \dots) \in \ell^\infty : a_m \in [-1, 0], m = 1, 2, \dots\}$, $B = \{x = (b_1, b_2, \dots, b_m, \dots) \in \ell^\infty : b_1 \geq 0, b_m \in [-1, 0], m = 2, 3, \dots\}$, $k_0 = (1, 0, 0, \dots, 0, \dots)$ and $\mathcal{D} : Y \times Y \rightrightarrows Y$ be defined by

$$\mathcal{D}(\alpha, \beta) := \{x = (x_1, x_2, \dots, x_m, \dots) \in \ell^\infty : x_1 \geq -|\max\{\alpha_1, \beta_1\}|, x_m \geq 0, m = 2, 3, \dots\}$$

for any $\alpha = (\alpha_1, \alpha_2, \dots)$, $\beta = (\beta_1, \beta_2, \dots) \in Y$. We can verify that, for any $t \in \mathbb{R}$ and for any $a = (a_1, a_2, \dots, a_m, \dots) \in A$, there exists $b_a = (\max\{0, -a_1 - t\}, 0, 0, \dots) \in B$ such that $a \in b_a + tk_0 - \mathcal{D}(a, b_a + tk_0)$. Then $A \preceq_{\mathcal{D}}^u B + tk_0$ and thus $h(A, B) = \inf \mathbb{R} = -\infty$.

Example 3.2. Let $Y = \mathbb{R}^2$, $A = -\mathbb{R}_+^2$, $B = \mathbb{R}_+^2$, $k_0 = (0, 1)$ and $\mathcal{D} : Y \times Y \rightrightarrows Y$ be defined by

$$\mathcal{D}(y_1, y_2) := \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha \leq 0, \beta \geq \min\{\|y_1\|, \|y_2\|\}\}, \text{ for all } y_1, y_2 \in Y.$$

Then, $h(A, B) = \inf \emptyset = +\infty$.

The following propositions answer the question “Under which conditions, $h(\cdot, \cdot)$ does not attain $-\infty$ or $+\infty$?”.

Proposition 3.1. Let $A, B \in \mathcal{P}_0(Y)$ and (H_2) hold. Assume that $\mathcal{D}(y_1, y_2) + \mathcal{D}(y_2, y_3) \subseteq \mathcal{D}(y_1, y_3)$ for all $y_1, y_2 \in Y$ and $y_3 \in B + (-\infty, +\infty)k_0$. If there exists $y_0 \in Y$ such that B is $(-\mathcal{D}[\{y_0\}, Y])$ -proper, then $h(A, B) > -\infty$.

Proof. Assume to the contrary that $h(A, B) = -\infty$. Then, for any $t \in \mathbb{R}$, there exists $t_1 < t$ such that $A \preceq_{\mathcal{D}}^u B + t_1 k_0$. Consequently, for any $a \in A$, there exists $b_{at_1} \in B$ such that $a \in b_{at_1} + t_1 k_0 - \mathcal{D}(a, b_{at_1} + t_1 k_0)$. Since $t_1 - t < 0$ and $[0, +\infty)k_0 + \mathcal{D}(a, b_{at_1} + t_1 k_0) \subseteq \mathcal{D}(a, b_{at_1} + t_1 k_0)$, we infer $(t_1 - t)k_0 - \mathcal{D}(a, b_{at_1} + t_1 k_0) \subseteq -\mathcal{D}(a, b_{at_1} + t_1 k_0)$. Then,

$$\begin{aligned} a &\in b_{at_1} + tk_0 + (t_1 - t)k_0 - \mathcal{D}(a, b_{at_1} + t_1 k_0) \\ &\subseteq b_{at_1} + tk_0 - \mathcal{D}(a, b_{at_1} + t_1 k_0), \end{aligned}$$

which implies that

$$a - tk_0 - \mathcal{D}(y_0, a) \subseteq b_{at_1} - \mathcal{D}(y_0, a) - \mathcal{D}(a, b_{at_1} + t_1 k_0). \quad (3.2)$$

Since $b_{at_1} + t_1 k_0 \in B + (-\infty, +\infty)k_0$, we have

$$-\mathcal{D}(y_0, a) - \mathcal{D}(a, b_{at_1} + t_1 k_0) \subseteq -\mathcal{D}(y_0, b_{at_1} + t_1 k_0).$$

By combining this with (3.2), we obtain $a - tk_0 - \mathcal{D}(y_0, a) \subseteq b_{at_1} - \mathcal{D}(y_0, b_{at_1} + t_1 k_0)$. Thus $a - tk_0 - \mathcal{D}(y_0, a) \subseteq B - \mathcal{D}[\{y_0\}, Y] \subseteq Y$ for all $t \in \mathbb{R}$. Then, from (H_2) ,

$$\bigcup_{t \in \mathbb{R}} (a - tk_0 - \mathcal{D}(y_0, a)) = a - (-\infty, +\infty)k_0 - \mathcal{D}(y_0, a) = Y,$$

therefore, $B - \mathcal{D}[\{y_0\}, Y] = Y$, which is a contradiction. $\square$

Proposition 3.2. Let $A, B \in \mathcal{P}_0(Y)$ and (H_5) hold. Assume that $\mathcal{D}(y_1, y_3)$ is a coradiant set and $\mathcal{D}(y_1, y_2) + \mathcal{D}(y_2, y_3) \subseteq \mathcal{D}(y_1, y_3)$ for any $y_1, y_2 \in Y$, $y_3 \in B + (-\infty, +\infty)k_0$. If A is $(-\tilde{\mathcal{D}}[A, A])$ -bounded and B is $\tilde{\mathcal{D}}[B, Y]$ -bounded, then $h(A, B) < +\infty$.

Proof. Let $a_0 \in A$ and $b_0 \in B$. From (H_5) , we see that $\mathbf{0} \in \text{int}\mathcal{D}(a_0, b_0) + (-\infty, +\infty)k_0$. Thus, there exists $t_0 \in \mathbb{R}$ such that $t_0k_0 - \text{int}\mathcal{D}(a_0, b_0)$ is a neighborhood of zero. Since A is $-\bar{\mathcal{D}}[A, A]$ -bounded, there exists $\lambda_0 > 1$ such that $A \subseteq \lambda_0(t_0k_0 - \text{int}\mathcal{D}(a_0, b_0)) - \bar{\mathcal{D}}[A, A]$. Since $\mathcal{D}(a_0, b_0)$ is a coradiant set, we obtain

$$A \subseteq \lambda_0 t_0 k_0 - \mathcal{D}(a_0, b_0) - \bar{\mathcal{D}}[A, A]. \quad (3.3)$$

Similarly, $\mathbf{0} \in \text{int}\mathcal{D}(b_0, b_0) + (-\infty, +\infty)k_0$. Then, there exists $t_1 \in \mathbb{R}$ such that $t_1k_0 + \text{int}\mathcal{D}(b_0, b_0)$ is a neighborhood of zero. As B is $\bar{\mathcal{D}}[B, Y]$ -bounded, there exists $\lambda_1 > 1$ such that

$$B \subseteq \lambda_1(t_1k_0 + \text{int}\mathcal{D}(b_0, b_0)) + \bar{\mathcal{D}}[B, Y].$$

In view of $\mathcal{D}(b_0, b_0)$ being a coradiant set, we have $\mathbf{0} \in B - B \subseteq B - \lambda_1 t_1 k_0 - \mathcal{D}(b_0, b_0) - \bar{\mathcal{D}}[B, Y]$, which implies that

$$-\mathcal{D}(a_0, b_0) \subseteq B - \lambda_1 t_1 k_0 - \mathcal{D}(b_0, b_0) - \bar{\mathcal{D}}[B, Y].$$

By combining this with (3.3), we get

$$\begin{aligned} A &\subseteq \lambda_0 t_0 k_0 + B - \lambda_1 t_1 k_0 - \mathcal{D}(b_0, b_0) - \mathcal{D}(a_0, b_0) - \bar{\mathcal{D}}[B, Y] - \bar{\mathcal{D}}[A, A] \\ &\subseteq B + (\lambda_0 t_0 - \lambda_1 t_1)k_0 - \mathcal{D}(a_0, b_0) - \bar{\mathcal{D}}[B, Y] - \bar{\mathcal{D}}[A, A]. \end{aligned}$$

Then, for any $a \in A$, there exists $b_a \in B$ such that

$$a \in b_a + (\lambda_0 t_0 - \lambda_1 t_1)k_0 - \mathcal{D}(a_0, b_0) - \mathcal{D}(b_0, b_a + (\lambda_0 t_0 - \lambda_1 t_1)k_0) - \mathcal{D}(a, a_0).$$

Since $b_0, b_a \in B$, we have

$$-\mathcal{D}(a, a_0) - \mathcal{D}(a_0, b_0) - \mathcal{D}(b_0, b_a + (\lambda_0 t_0 - \lambda_1 t_1)k_0) \subseteq -\mathcal{D}(a, b_a + (\lambda_0 t_0 - \lambda_1 t_1)k_0).$$

Therefore, $a \in b_a + (\lambda_0 t_0 - \lambda_1 t_1)k_0 - \mathcal{D}(a, b_a + (\lambda_0 t_0 - \lambda_1 t_1)k_0)$. Hence, $A \preceq_{\mathcal{D}}^u B + (\lambda_0 t_0 - \lambda_1 t_1)k_0$. According to the definition of $h(A, B)$, we have $h(A, B) \leq \lambda_0 t_0 - \lambda_1 t_1 < +\infty$. $\square$

Proposition 3.3. Let $A, B \in \mathcal{P}_0(Y)$ and $t \in \mathbb{R}$.

- (a) If $A \preceq_{\mathcal{D}}^u B + tk_0$, then $h(A, B) \leq t$.
- (b) Assume that (H_3) holds, B is compact and $\mathcal{D}(\cdot, \cdot)$ is closed on $A \times Y$. If $h(A, B) \leq t$, then $A \preceq_{\mathcal{D}}^u B + tk_0$.

Proof. We only prove (b), since (a) is true from the definition of $h(A, B)$. Suppose that $h(A, B) \leq t$. Then, there are three cases to be considered:

CASE 1. $h(A, B) = t$. It follows from Lemma 3.1 that $A \preceq_{\mathcal{D}}^u B + (t + \varepsilon)k_0$ for any $\varepsilon > 0$, which implies that $A \preceq_{\mathcal{D}}^u B + (t + \varepsilon_n)k_0$ for any $\varepsilon_n \rightarrow 0^+$. Thus, for any $a \in A$, there exists $b_{an} \in B$ such that $a \in b_{an} + (t + \varepsilon_n)k_0 - \mathcal{D}(a, b_{an} + (t + \varepsilon_n)k_0)$. Since B is compact, there exists a subsequence $\{b_{an_k}\}$ of $\{b_{an}\}$ converging to $b_a \in B$. Then, $b_{an_k} + (t + \varepsilon_{n_k})k_0 \rightarrow b_a + tk_0$. As $\mathcal{D}(\cdot, \cdot)$ is closed at $(a, b_a + tk_0)$, one has $b_{an_k} + (t + \varepsilon_{n_k})k_0 - a \rightarrow b_a + tk_0 - a \in \mathcal{D}(a, b_a + tk_0)$. Therefore, $a \in b_a + tk_0 - \mathcal{D}(a, b_a + tk_0)$, which implies that $A \preceq_{\mathcal{D}}^u B + tk_0$.

CASE 2. $h(A, B) = t_0 < t$. By using similar argument as in CASE 1, we can show that $A \preceq_{\mathcal{D}}^u B + t_0k_0$. Since $t_0 < t$, it follows by Lemma 3.1 that $A \preceq_{\mathcal{D}}^u B + tk_0$.

CASE 3. $h(A, B) = -\infty$. Then there exists $t_1 < 0$ with $t_1 < t$ such that $A \preceq_{\mathcal{D}}^u B + t_1k_0$. For any $a \in A$, there exists $b_1 \in B$ satisfying $a \in b_1 + t_1k_0 - \mathcal{D}(a, b_1 + t_1k_0)$. From (H_3) , we have $-\mathcal{D}(a, b_1 + t_1k_0) \subseteq -\mathcal{D}(a, b_1 + tk_0)$. Therefore,

$$a \in b_1 + tk_0 + (t_1 - t)k_0 - \mathcal{D}(a, b_1 + tk_0) \subseteq b_1 + tk_0 - \mathcal{D}(a, b_1 + tk_0),$$

which implies that $A \preceq_{\mathcal{D}}^u B + tk_0$. $\square$

Remark 3.1. Let the set-valued mapping $\mathcal{D}_1 : Y \rightrightarrows Y$ be such that $\mathcal{D}(y_1, y_2) = \mathcal{D}_1(y_1)$ for any $y_1, y_2 \in Y$. If B is compact and $h(A, B) \leq t$, then by adapting the proof of Proposition 3.3 (b), we have $A \preceq_{\mathcal{D}}^u B + tk_0$.

As a consequence of Proposition 3.3 applied to $t = 0$, we obtain the following result.

Corollary 3.1. Let $A, B \in \mathcal{P}_0(Y)$.

- (a) If $A \preceq_{\mathcal{D}}^u B$, then $h(A, B) \leq 0$.
- (b) Assume that (H_3) holds, B is compact and $\mathcal{D}(\cdot, \cdot)$ is closed on $A \times Y$. If $h(A, B) \leq 0$, then $A \preceq_{\mathcal{D}}^u B$.

Proposition 3.4. Let $A, B \in \mathcal{P}_0(Y)$, $t \in \mathbb{R}$ and (H_1) hold.

- (a) If (H_4) holds and $A \prec_{\mathcal{D}}^u B + tk_0$, then $h(A, B) < t$.
- (b) If (H_3) holds and $h(A, B) < t$, then $A \prec_{\mathcal{D}}^u B + tk_0$.

Proof. (a) Let $A \prec_{\mathcal{D}}^u B + tk_0$. Then, for any $a \in A$, there exists $b_0 \in B$ such that $a \in b_0 + tk_0 - \text{int}\mathcal{D}(a, b_0 + tk_0)$. By Lemma 2.1, we have

$$a \in b_0 + tk_0 - \bigcup_{\varepsilon > 0} (\varepsilon k_0 + \text{int}\mathcal{D}(a, b_0 + tk_0)).$$

Consequently, there exists $\varepsilon_0 > 0$ such that $a \in b_0 + tk_0 - \varepsilon_0 k_0 - \text{int}\mathcal{D}(a, b_0 + tk_0)$. According to (H_4) , we have $a \in b_0 + (t - \varepsilon_0)k_0 - \mathcal{D}(a, b_0 + (t - \varepsilon_0)k_0)$. Hence, $A \preceq_{\mathcal{D}}^u B + (t - \varepsilon_0)k_0$, which implies that $h(A, B) \leq t - \varepsilon_0 < t$.

(b) Let $h(A, B) < t$. If $h(A, B) = t_0 < t$, then by Lemma 3.1, there exists $t_1 \in \mathbb{R}$ such that $t_0 < t_1 < t$ and $A \preceq_{\mathcal{D}}^u B + t_1 k_0$. Therefore, for any $a \in A$, there exists $b_1 \in B$ such that

$$a \in b_1 + t_1 k_0 - \mathcal{D}(a, b_1 + t_1 k_0). \quad (3.4)$$

Since $t_1 - t < 0$, from (H_1) , we have

$$(t_1 - t)k_0 - \mathcal{D}(a, b_1 + t_1 k_0) \subseteq -\text{int}\mathcal{D}(a, b_1 + t_1 k_0), \quad (3.5)$$

and from (H_3) , one sees that $\mathcal{D}(a, b_1 + t_1 k_0) \subseteq \mathcal{D}(a, b_1 + tk_0)$, which together with (3.5) leads to

$$\begin{aligned} b_1 + t_1 k_0 - \mathcal{D}(a, b_1 + t_1 k_0) &= b_1 + tk_0 + (t_1 - t)k_0 - \mathcal{D}(a, b_1 + t_1 k_0) \\ &\subseteq b_1 + tk_0 - \text{int}\mathcal{D}(a, b_1 + tk_0). \end{aligned}$$

Combining this with (3.4), we obtain $a \in b_1 + tk_0 - \text{int}\mathcal{D}(a, b_1 + tk_0)$. Hence, $A \prec_{\mathcal{D}}^u B + tk_0$.

If $h(A, B) = -\infty$, then there exists $t_2 < 0$ with $t_2 < t$ such that $A \preceq_{\mathcal{D}}^u B + t_2 k_0$. Similar to the above proof, we can obtain $A \prec_{\mathcal{D}}^u B + tk_0$. $\square$

Remark 3.2. Suppose that (H_1) holds. If $\mathcal{D}(\cdot, \cdot)$ depends only on the first variable, then $h(A, B) < t$ if and only if $A \prec_{\mathcal{D}}^u B + tk_0$.

Indeed, let the set-valued mapping $\mathcal{D}_1 : Y \rightrightarrows Y$ be such that $\mathcal{D}(y_1, y_2) = \mathcal{D}_1(y_1)$ for any $y_1, y_2 \in Y$. If $h(A, B) < t$, then there exists $t_1 < t$ such that $A \preceq_{\mathcal{D}}^u B + t_1 k_0$. Then, for any $a \in A$, there exists $b_1 \in B$ such that $a \in b_1 + t_1 k_0 - \mathcal{D}_1(a)$. Combining this with (H_1) , we have $a \in b_1 + tk_0 - \text{int}\mathcal{D}_1(a)$. Thus $A \prec_{\mathcal{D}}^u B + tk_0$.

Conversely, since $\mathcal{D}(y_1, y_2) = \mathcal{D}_1(y_1)$ for all $y_1, y_2 \in Y$, we conclude that (H_4) holds. If $A \prec_{\mathcal{D}}^u B + tk_0$, then by using similar argument as in the proof of Proposition 3.4 (a), we obtain $h(A, B) < t$.

Proposition 3.4 reduces to the following corollary by considering $t = 0$.

Corollary 3.2. *Let $A, B \in \mathcal{P}_0(Y)$ and (H_1) hold.*

- (a) *If (H_3) holds and $h(A, B) < 0$, then $A \prec_{\mathcal{D}}^u B$.*
- (b) *If (H_4) holds and $A \prec_{\mathcal{D}}^u B$, then $h(A, B) < 0$.*

Definition 3.1. Let $\mathcal{S} \subseteq \mathcal{P}_0(Y)$. A function $T : \mathcal{P}_0(Y) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is said to be

- (a) increasing on $\mathcal{S}$ if $A, B \in \mathcal{S}$ and $A \preceq_{\mathcal{D}}^u B$ implies $T(A) \leq T(B)$;
- (b) decreasing on $\mathcal{S}$ if $A, B \in \mathcal{S}$ and $A \preceq_{\mathcal{D}}^u B$ implies $T(B) \leq T(A)$;
- (c) strictly increasing on $\mathcal{S}$ if $A, B \in \mathcal{S}$ and $A \prec_{\mathcal{D}}^u B$ implies $T(A) < T(B)$;
- (d) strictly decreasing on $\mathcal{S}$ if $A, B \in \mathcal{S}$ and $A \prec_{\mathcal{D}}^u B$ implies $T(B) < T(A)$.

We now study monotonicity properties of $h(\cdot, \cdot)$.

Proposition 3.5. Let $A \in \mathcal{P}_0(Y)$. If, for any $y_1, y_2 \in Y, y_3 \in A, \lambda \in \mathbb{R}$,

$$\mathcal{D}(y_1, y_2) = \mathcal{D}(y_1 + \lambda k_0, y_2 + \lambda k_0) \text{ and } \mathcal{D}(y_3, y_1) + \mathcal{D}(y_1, y_2) \subseteq \mathcal{D}(y_3, y_2),$$

then $h(A, \cdot)$ is decreasing on $\mathcal{P}_0(Y)$.

Proof. Let $B_1, B_2 \in \mathcal{P}_0(Y)$ be such that $B_1 \preceq_{\mathcal{D}}^u B_2$. In what follows, we prove that $h(A, B_2) \leq h(A, B_1)$. There are three cases to be considered:

CASE 1. $h(A, B_1) = +\infty$. It is clear to see that $h(A, B_2) \leq h(A, B_1)$.

CASE 2. $h(A, B_1) = -\infty$. Then, there exists $t_n \rightarrow -\infty$ such that $A \preceq_{\mathcal{D}}^u t_n k_0 + B_1$. Therefore, for any $a \in A$, there exists $b_{an} \in B_1$ such that

$$a \in b_{an} + t_n k_0 - \mathcal{D}(a, b_{an} + t_n k_0). \quad (3.6)$$

Since $B_1 \preceq_{\mathcal{D}}^u B_2$, there exists $b_{2n} \in B_2$ such that

$$b_{an} \in b_{2n} - \mathcal{D}(b_{an}, b_{2n}) \subseteq b_{2n} - \mathcal{D}(b_{an} + t_n k_0, b_{2n} + t_n k_0). \quad (3.7)$$

In view of the hypothesis, we have

$$\mathcal{D}(a, b_{an} + t_n k_0) + \mathcal{D}(b_{an} + t_n k_0, b_{2n} + t_n k_0) \subseteq \mathcal{D}(a, b_{2n} + t_n k_0). \quad (3.8)$$

Then, by combining (3.6), (3.7) and (3.8), one has $a \in b_{2n} + t_n k_0 - \mathcal{D}(a, b_{2n} + t_n k_0)$. Thus $A \preceq_{\mathcal{D}}^u B_2 + t_n k_0$, which implies $h(A, B_2) \leq t_n \rightarrow -\infty$.

CASE 3. $h(A, B_1) = t_1 \in \mathbb{R}$. Then, from the definition of infimum, we know that there exists $\varepsilon_n \geq 0$ with $\varepsilon_n \rightarrow 0$ such that $A \preceq_{\mathcal{D}}^u B_1 + (t_1 + \varepsilon_n)k_0$. In other words, for any $a \in A$, there exists $b_{an} \in B_1$ such that

$$a \in b_{an} + (t_1 + \varepsilon_n)k_0 - \mathcal{D}(a, b_{an} + (t_1 + \varepsilon_n)k_0). \quad (3.9)$$

Since $B_1 \preceq_{\mathcal{D}}^u B_2$, there exists $b_{2n} \in B_2$ such that

$$b_{an} \in b_{2n} - \mathcal{D}(b_{an}, b_{2n}) \subseteq b_{2n} - \mathcal{D}(b_{an} + (t_1 + \varepsilon_n)k_0, b_{2n} + (t_1 + \varepsilon_n)k_0). \quad (3.10)$$

By combining (3.9) and (3.10), we obtain $a \in b_{2n} + (t_1 + \varepsilon_n)k_0 - \mathcal{D}(a, b_{2n} + (t_1 + \varepsilon_n)k_0)$. Thus $A \preceq_{\mathcal{D}}^u B_2 + (t_1 + \varepsilon_n)k_0$, which implies that $h(A, B_2) \leq t_1 + \varepsilon_n = h(A, B_1) + \varepsilon_n$. Since $\varepsilon_n \rightarrow 0$, we have $h(A, B_2) \leq h(A, B_1)$. $\square$

Proposition 3.6. *Let $B \in \mathcal{P}_0(Y)$. If, for any $y_1, y_2 \in Y$, $y_3 \in B + (-\infty, +\infty)k_0$, $\mathcal{D}(y_1, y_2) + \mathcal{D}(y_2, y_3) \subseteq \mathcal{D}(y_1, y_3)$, then $h(\cdot, B)$ is increasing on $\mathcal{P}_0(Y)$.*

Proof. Let $A_1, A_2 \in \mathcal{P}_0(Y)$ be such that $A_1 \preceq_{\mathcal{D}}^u A_2$. Then, for any $a \in A_1$, there exists $a_2 \in A_2$ such that

$$a \in a_2 - \mathcal{D}(a, a_2). \quad (3.11)$$

There are three cases to be considered:

CASE 1. $h(A_2, B) = t_2 \in \mathbb{R}$. Then there exists $\varepsilon_n \geq 0$ with $\varepsilon_n \rightarrow 0$ such that $A_2 \preceq_{\mathcal{D}}^u B + (t_2 + \varepsilon_n)k_0$. Thus there exists $b_{2\varepsilon} \in B$ such that

$$a_2 \in b_{2\varepsilon} + (t_2 + \varepsilon_n)k_0 - \mathcal{D}(a_2, b_{2\varepsilon} + (t_2 + \varepsilon_n)k_0). \quad (3.12)$$

Since $b_{2\varepsilon} + (t_2 + \varepsilon_n)k_0 \in B + (-\infty, +\infty)k_0$, we have

$$\mathcal{D}(a, a_2) + \mathcal{D}(a_2, b_{2\varepsilon} + (t_2 + \varepsilon_n)k_0) \subseteq \mathcal{D}(a, b_{2\varepsilon} + (t_2 + \varepsilon_n)k_0).$$

Therefore, it follows from (3.11) and (3.12) that $a \in b_{2\varepsilon} + (t_2 + \varepsilon_n)k_0 - \mathcal{D}(a, b_{2\varepsilon} + (t_2 + \varepsilon_n)k_0)$, and hence, $A_1 \preceq_{\mathcal{D}}^u B + (t_2 + \varepsilon_n)k_0$. Thus $h(A_1, B) \leq t_2 + \varepsilon_n$. Since $\varepsilon_n \rightarrow 0$, we see $h(A_1, B) \leq h(A_2, B)$.

CASE 2. $h(A_2, B) = +\infty$. It is clear to see that $h(A_1, B) \leq h(A_2, B)$.

CASE 3. $h(A_2, B) = -\infty$. Since $A_1 \preceq_{\mathcal{D}}^u A_2$, for any $a \in A_1$, there exists $a_2 \in A_2$ such that $a \in a_2 - \mathcal{D}(a, a_2)$. We can find a sequence $\{t_n\} \subseteq \mathbb{R}$ with $t_n \rightarrow -\infty$ such that $A_2 \preceq_{\mathcal{D}}^u B + t_n k_0$. Then there exists $b_{an} \in B$, such that $a_2 \in b_{an} + t_n k_0 - \mathcal{D}(a_2, b_{an} + t_n k_0)$. Thus, $a \in b_{an} + t_n k_0 - \mathcal{D}(a, b_{an} + t_n k_0)$, which implies $A_1 \preceq_{\mathcal{D}}^u t_n k_0 + B$. Hence, $h(A_1, B) \leq t_n \rightarrow -\infty$. $\square$

Proposition 3.7. Let A_1, A_2 and $B \in \mathcal{P}_0(Y)$, $k_0 \in \text{int}\tilde{\mathcal{D}}[A_1, B]$, and (H_2) – (H_4) hold. Assume that $\mathcal{D}(\cdot, \cdot)$ is closed on $A_2 \times Y$, $\mathcal{D}(y_1, y_2)$ is a coradiant set and $\mathcal{D}(y_1, y_2) + \mathcal{D}(y_2, y_3) \subseteq \mathcal{D}(y_1, y_3)$ for any $y_1, y_2 \in Y, y_3 \in B + (-\infty, +\infty)k_0$. If A_1 is B -bounded, B is compact and there exists $y_0 \in Y$ such that B is $(-\mathcal{D}[\{y_0\}, Y])$ -proper, then $A_1 \prec_{\mathcal{D}}^u A_2$ implies $h(A_1, B) < h(A_2, B)$.

Proof. Since $A_1 \prec_{\mathcal{D}}^u A_2$, for any $a \in A_1$, there exists $a_2 \in A_2$ such that $a \in a_2 - \text{int}\mathcal{D}(a, a_2)$. According to Proposition 3.1, we deduce that $h(A_2, B) \neq -\infty$. We consider the following two cases: CASE 1. $h(A_2, B) = t_2 \in \mathbb{R}$. Similar to the proof of part (b) of Proposition 3.3, we can demonstrate that $A_2 \preceq_{\mathcal{D}}^u B + t_2 k_0$. Consequently, there exists $b_{2t} \in B$ such that $a_2 \in b_{2t} + t_2 k_0 - \mathcal{D}(a_2, b_{2t} + t_2 k_0)$. Therefore,

$$a \in b_{2t} + t_2 k_0 - \text{int}\mathcal{D}(a, a_2) - \mathcal{D}(a_2, b_{2t} + t_2 k_0). \quad (3.13)$$

Given that $b_{2t} + t_2 k_0 \in B + (-\infty, +\infty)k_0$, we obtain $\mathcal{D}(a, a_2) + \mathcal{D}(a_2, b_{2t} + t_2 k_0) \subseteq \mathcal{D}(a, b_{2t} + t_2 k_0)$. Since $-\text{int}\mathcal{D}(a, a_2) - \mathcal{D}(a_2, b_{2t} + t_2 k_0)$ is an open set, we have

$$-\text{int}\mathcal{D}(a, a_2) - \mathcal{D}(a_2, b_{2t} + t_2 k_0) \subseteq -\text{int}\mathcal{D}(a, b_{2t} + t_2 k_0).$$

Then, based on (3.13), it can be inferred that $a \in b_{2t} + t_2 k_0 - \text{int}\mathcal{D}(a, b_{2t} + t_2 k_0)$. Consequently, by Lemma 2.1, we have $a \in b_{2t} + t_2 k_0 - \bigcup_{\varepsilon > 0} \varepsilon k_0 - \text{int}\mathcal{D}(a, b_{2t} + t_2 k_0)$, and therefore, there exists a positive real number $\varepsilon_0 > 0$ such that

$$a \in b_{2t} + t_2 k_0 - \varepsilon_0 k_0 - \text{int}\mathcal{D}(a, b_{2t} + t_2 k_0).$$

Since $t_2 - \varepsilon_0 < t_2$, we have $a \in b_{2t} + t_2 k_0 - \varepsilon_0 k_0 - \mathcal{D}(a, b_{2t} + (t_2 - \varepsilon_0)k_0)$. This shows that $A_1 \preceq_{\mathcal{D}}^u B + (t_2 - \varepsilon_0)k_0$, from which it follows that $h(A_1, B) \leq t_2 - \varepsilon_0 < t_2$.

CASE 2. $h(A_2, B) = +\infty$. Since $k_0 \in \text{int}\tilde{\mathcal{D}}[A_1, B]$, we deduce that $k_0 - \text{int}\tilde{\mathcal{D}}[A_1, B]$ is a neighborhood of zero. Since A_1 is B -bounded, there exists $t_1 \geq 1$ such that $A_1 \subseteq t_1(k_0 - \text{int}\tilde{\mathcal{D}}[A_1, B]) + B$. Since $\tilde{\mathcal{D}}[A_1, B]$ is a coradiant set, one has $A_1 \subseteq t_1 k_0 - \tilde{\mathcal{D}}[A_1, B] + B$. Then, for any $a_1 \in A_1$, there exists $b_0 \in B$ such that $a_1 \in b_0 + t_1 k_0 - \mathcal{D}(a_1, b_0) \subseteq b_0 + t_1 k_0 - \mathcal{D}(a_1, b_0 + t_1 k_0)$. Thus, $A_1 \preceq_{\mathcal{D}}^u B + t_1 k_0$, which implies that $h(A_1, B) \leq t_1 < h(A_2, B)$. $\square$

Proposition 3.8. *Let (H_1) and (H_3) hold, and $\mathbf{0} \in \mathcal{D}(a, a)$ for any $a \in A$. If there exists $\bar{a} \in A$ such that, for any $a \in A$, $a - \bar{a} \notin \text{int}\mathcal{D}(\bar{a}, a)$, then $h(A, A) = 0$.*

Proof. Since $a - a \in \mathcal{D}(a, a)$ for any $a \in A$, we have $A \preceq_{\mathcal{D}}^u A$, which implies that $h(A, A) \leq 0$. Given that $a - \bar{a} \notin \text{int}\mathcal{D}(\bar{a}, a)$, we have $\bar{a} \notin a - \text{int}\mathcal{D}(\bar{a}, a)$. Then, $A \not\preceq_{\mathcal{D}}^u A$. From Corollary 3.2(a), we have $h(A, A) \geq 0$. Thus $h(A, A) = 0$. $\square$

Proposition 3.9. *Let $A, B \in \mathcal{P}_0(Y)$. Then $h(A, B) = \sup_{a \in A} \inf_{b \in B} \inf \{t \in \mathbb{R} : a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\}$.*

Proof. We first prove that

$$h(A, B) = \sup_{a \in A} \inf \{t \in \mathbb{R} : \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\}. \quad (3.14)$$

It can be seen from the definition of $h(A, B)$ that

$$h(A, B) = \inf \{t \in \mathbb{R} : \forall a \in A, \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\}. \quad (3.15)$$

For any given $a' \in A$, we have

$$\begin{aligned} & \{t \in \mathbb{R} : \forall a \in A, \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} \\ & \subseteq \{t \in \mathbb{R} : \exists b' \in B \text{ such that } a' \in b' + tk_0 - \mathcal{D}(a', b' + tk_0)\}. \end{aligned}$$

Therefore, for all $a' \in A$,

$$\begin{aligned} & \inf \{t \in \mathbb{R} : \forall a \in A, \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} \\ & \geq \inf \{t \in \mathbb{R} : \exists b' \in B \text{ such that } a' \in b' + tk_0 - \mathcal{D}(a', b' + tk_0)\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} & \inf \{t \in \mathbb{R} : \forall a \in A, \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} \\ & \geq \sup_{a \in A} \inf \{t \in \mathbb{R} : \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\}. \end{aligned} \quad (3.16)$$

Next, we prove that the strict inequality in (3.16) does not hold. Otherwise, there exists $\bar{s} \in \mathbb{R}$ such that

$$\inf \{t \in \mathbb{R} : \forall a \in A, \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} > \bar{s} \quad (3.17)$$

and

$$\sup_{a \in A} \inf \{t \in \mathbb{R} : \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} < \bar{s}. \quad (3.18)$$

From (3.18), for any $a \in A$, there exist $s_a < \bar{s}$ and $b_a \in B$ such that $a \in b_a + s_a k_0 - \mathcal{D}(a, b_a + s_a k_0)$. Noticing that $[0, +\infty)k_0 + \mathcal{D}(a, b_a + s_a k_0) \subseteq \mathcal{D}(a, b_a + s_a k_0)$, we can infer that

$$s_a k_0 - \mathcal{D}(a, b_a + s_a k_0) = \bar{s} k_0 + (s_a - \bar{s})k_0 - \mathcal{D}(a, b_a + s_a k_0) \subseteq \bar{s} k_0 - \mathcal{D}(a, b_a + s_a k_0).$$

Hence, $a \in b_a + \bar{s} k_0 - \mathcal{D}(a, b_a + s_a k_0)$. It follows that

$$\inf \{t \in \mathbb{R} : \forall a \in A, \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} \leq \bar{s},$$

which contradicts (3.17). Therefore,

$$\begin{aligned} & \inf \{t \in \mathbb{R} : \forall a \in A, \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} \\ &= \sup_{a \in A} \inf \{t \in \mathbb{R} : \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\}. \end{aligned}$$

Then, from (3.15), it can be seen that (3.14) holds.

Next, we prove that, for any $a \in A$,

$$\begin{aligned} & \inf \{t \in \mathbb{R} : \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} \\ &= \inf_{b \in B} \inf \{t \in \mathbb{R} : a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\}. \end{aligned} \quad (3.19)$$

Let $b' \in B$. Then,

$$\{t \in \mathbb{R} : a \in b' + tk_0 - \mathcal{D}(a, b' + tk_0)\} \subseteq \{t \in \mathbb{R} : \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\},$$

and therefore,

$$\begin{aligned} & \inf \{t \in \mathbb{R} : a \in b' + tk_0 - \mathcal{D}(a, b' + tk_0)\} \\ & \geq \inf \{t \in \mathbb{R} : \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\}, \end{aligned}$$

which implies

$$\begin{aligned} & \inf_{b \in B} \inf \{t \in \mathbb{R} : a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} \\ & \geq \inf \{t \in \mathbb{R} : \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\}. \end{aligned} \quad (3.20)$$

Similar to the above proof, next we prove that the strict inequality in (3.20) does not hold. If there exists $\bar{\alpha} \in \mathbb{R}$ such that

$$\inf \{t \in \mathbb{R} : \exists b \in B \text{ such that } a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} < \bar{\alpha}, \quad (3.21)$$

and

$$\inf_{b \in B} \inf \{t \in \mathbb{R} : a \in b + tk_0 - \mathcal{D}(a, b + tk_0)\} > \bar{\alpha}, \quad (3.22)$$

then, from (3.21), there exist $t_0 < \bar{\alpha}$ and $b_0 \in B$ such that

$$a \in b_0 + t_0 k_0 - \mathcal{D}(a, b_0 + t_0 k_0). \quad (3.23)$$

Since $t_0 k_0 - \mathcal{D}(a, b_0 + t_0 k_0) = \bar{\alpha} k_0 + (t_0 - \bar{\alpha}) k_0 - \mathcal{D}(a, b_0 + t_0 k_0) \subseteq \bar{\alpha} k_0 - \mathcal{D}(a, b_0 + t_0 k_0)$, from (3.23), we obtain $a \in b_0 + \bar{\alpha} k_0 - \mathcal{D}(a, b_0 + t_0 k_0)$. Then,

$$\inf \{t \in \mathbb{R} : a \in b_0 + tk_0 - \mathcal{D}(a, b_0 + t_0 k_0)\} \leq \bar{\alpha},$$

which contradicts (3.22). Hence, (3.19) holds. $\square$

As a consequence of Proposition 3.9, we obtain the following result.

Corollary 3.3. *If $\mathcal{D}(\cdot, \cdot)$ is a given set-valued mapping that only depends on the first variable, that is, $\mathcal{D}(y_1, y_2) = \mathcal{D}_1(y_1)$ for all $y_1, y_2 \in Y$, then $h(A, B) = \sup_{a \in A} \inf_{b \in B} \{t \in \mathbb{R} : a \in b + tk_0 - \mathcal{D}_1(a)\}$.*

4. OPTIMALITY CONDITIONS FOR SET-VALUED OPTIMIZATION PROBLEMS WITH A VARIABLE STRUCTURE

In this section, we investigate some optimality conditions for solutions of set optimization problems by using scalarization function $h(\cdot, \cdot)$ with respect to a variable structure.

Let $S \subseteq X$ and $F : X \rightrightarrows Y$ be a set-valued mapping. Denote $\mathcal{F}(S) := \{F(x) : x \in S\}$ and the image of S under F by $F(S) := \bigcup_{x \in S} F(x)$. We consider the following set optimization problem:

$$\text{Minimize } F(x) \quad \text{subject to } x \in S. \quad (\text{SOP})$$

Definition 4.1. [26, 27] Let $H \in \mathcal{P}_0(Y)$. An element $\bar{x} \in S$ is said to be a

- (a) minimal solution of (SOP) with respect to the variable upper set less relation if, for any $x \in S$, $F(x) \preceq_{\mathcal{D}}^u F(\bar{x})$ implies $F(\bar{x}) \preceq_{\mathcal{D}}^u F(x)$;
- (b) weakly optimal solution of (SOP) with respect to the variable upper set less relation if there does not exist $x \in S \setminus \{\bar{x}\}$ such that $F(x) \prec_{\mathcal{D}}^u F(\bar{x})$;
- (c) H -minimal solution of (SOP) with respect to the variable upper set less relation if, for any $x \in S$, $F(x) + H \preceq_{\mathcal{D}}^u F(\bar{x})$ implies $F(\bar{x}) \preceq_{\mathcal{D}}^u F(x) + H$.

Let $\text{Min}(F, S, \mathcal{D})$, $\text{WMin}(F, S, \mathcal{D})$ and $H\text{-Min}(F, S, \mathcal{D})$ denote the sets of minimal solutions, weakly optimal solutions and H -minimal solutions of (SOP) with respect to the variable upper set less relation, respectively.

We establish optimality conditions for (SOP).

Theorem 4.1. Assume that $\bar{x} \in S$, $F(\bar{x})$ is compact, (H_1) and (H_3) hold, as well as $\mathcal{D}$ is closed on $F(S) \times Y$. Let $\mathbf{0} \in \mathcal{D}(y, y)$ for any $y \in F(\bar{x})$. Let $\mathcal{D}(y_1, y_2) + \mathcal{D}(y_2, y_3) \subseteq \mathcal{D}(y_1, y_3)$ for any $y_1, y_2 \in Y$, $y_3 \in F(\bar{x}) + (-\infty, +\infty)k_0$. If there exists $\bar{y} \in F(\bar{x})$ such that for any $y \in F(\bar{x})$, $y - \bar{y} \notin \text{int}\mathcal{D}(\bar{y}, y)$, then $\bar{x} \in \text{Min}(F, S, \mathcal{D})$ implies that there exists a function $T : \mathcal{P}_0(Y) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ which is increasing on $\mathcal{P}_0(Y)$ and satisfies the following conditions:

- (a) $T(F(\bar{x})) = 0$,
- (b) $T(F(x)) > 0$ for all $x \in S \setminus \{\bar{x}\}$ with $F(x) \approx_{\mathcal{D}}^u F(\bar{x})$.

Proof. Define $T : \mathcal{P}_0(Y) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ as $T(\cdot) = h(\cdot, F(\bar{x}))$. Then, from Proposition 3.6, $T(\cdot)$ is increasing on $\mathcal{P}_0(Y)$. It follows from Proposition 3.8 that $T(F(\bar{x})) = 0$. Since $\mathbf{0} \in \mathcal{D}(y, y)$ for all $y \in F(\bar{x})$, we obtain $F(\bar{x}) \preceq_{\mathcal{D}}^u F(\bar{x})$. Then, $F(\bar{x}) \approx_{\mathcal{D}}^u F(\bar{x})$. Due to $\bar{x} \in \text{Min}(F, S, \mathcal{D})$, we obtain $F(x) \not\preceq_{\mathcal{D}}^u F(\bar{x})$ for all $x \in S \setminus \{\bar{x}\}$ with $F(x) \approx_{\mathcal{D}}^u F(\bar{x})$. Then, it follows from Corollary 3.1(b) that $h(F(x), F(\bar{x})) > 0$ for all $x \in S \setminus \{\bar{x}\}$ with $F(x) \approx_{\mathcal{D}}^u F(\bar{x})$. $\square$

The following example illustrates Theorem 4.1.

Example 4.1. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$, $S = [0, 2] \times [0, 2]$, $k_0 = (0, 1)$, and the set-valued mappings $F : X \rightrightarrows Y$ and $\mathcal{D} : Y \times Y \rightrightarrows Y$ be defined by

$$F(x) := \begin{cases} [0, 1] \times [0, 1], & x \in [-1, 1] \times [-1, 1], \\ [0, \|x\|_{\infty}] \times [0, \|x\|_{\infty}], & \text{otherwise,} \end{cases}$$

and

$$\mathcal{D}(y_1, y_2) = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha + \beta \geq s_1 - s_2, \beta \geq 0\},$$

respectively, where $y_1 = (s_1, t_1)$, $y_2 = (s_2, t_2) \in Y$. Then, $\mathcal{D}$ is closed on $F(S) \times Y$. We can easily verify that (H_1) and (H_3) hold. Let $\bar{x} = (1, 1)$. Then $F(\bar{x})$ is compact and there exists $\bar{y} = (1, 1) \in F(\bar{x})$ such that $y - \bar{y} \notin \text{int}\mathcal{D}(\bar{y}, y)$ for any $y \in F(\bar{x})$. For any $x \in [0, 2] \times [0, 2]$, if

$x \in [0, 1] \times [0, 1]$, then $F(x) \sim_{\mathcal{D}}^u F(1)$; if $x \notin [0, 1] \times [0, 1]$, taking $y_0 = (\|x\|_\infty, \|x\|_\infty)$, one gets for any $y \in F(\bar{x})$, $y_0 \notin y - \mathcal{D}(y_0, y)$, thus $F(x) \not\sim_{\mathcal{D}}^u F(\bar{x})$. Therefore $\bar{x} = (1, 1) \in \text{Min}(F, S, \mathcal{D})$. Let $T(\cdot) = h(\cdot, F(1))$. Then $T(\cdot)$ is increasing on $\mathcal{P}_0(Y)$ and $T(F(\bar{x})) = h(F(\bar{x}), F(\bar{x})) = 0$.

Next, we verify that $h(F(x), F(\bar{x})) > 0$ for any $x \notin [0, 1] \times [0, 1]$. Let $y_1 = (\|x\|_\infty, \|x\|_\infty)$. Then, we can verify that

$$\{t \in \mathbb{R} : \exists y_0 \in F((1, 1)) \text{ such that } y_1 - y_0 \in tk_0 - \mathcal{D}(y_1, y_0)\} \subseteq \{t \in \mathbb{R} : t \geq \|x\|_\infty - 1\}.$$

According to Proposition 3.9 and (3.19), we have

$$h(F(x), F((1, 1))) \geq \inf \{t \in \mathbb{R} : \exists y_0 \in F((1, 1)) \text{ such that } y_1 - y_0 \in tk_0 - \mathcal{D}(y_1, y_0 + tk_0)\}.$$

As $k_0 = (0, 1)$, it can be seen that $\mathcal{D}(y_1, y_0 + tk_0) = \mathcal{D}(y_1, y_0)$. Therefore,

$$\begin{aligned} h(F(x), F((1, 1))) &\geq \inf \{t \in \mathbb{R} : y - y_0 \in tk_0 - \mathcal{D}(y_1, y_0)\} \\ &\geq \inf \{t \in \mathbb{R} : t \geq \|x\|_\infty - 1\} \\ &= \|x\|_\infty - 1 > 0. \end{aligned}$$

Theorem 4.2. Assume that $\bar{x} \in S$ and $\mathbf{0} \in \mathcal{D}(y, y)$ for any $y \in F(\bar{x})$. If there exists a function $T : \mathcal{P}_0(Y) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ which is increasing on $\mathcal{F}(S)$ and satisfies the following conditions

- (i) $T(F(\bar{x})) = 0$,
- (ii) $T(F(x)) > 0$ for all $x \in S \setminus \{\bar{x}\}$ with $F(x) \approx_{\mathcal{D}}^u F(\bar{x})$,

then $\bar{x} \in \text{Min}(F, S, \mathcal{D})$.

Proof. Since $\mathbf{0} \in \mathcal{D}(y, y)$ for any $y \in F(\bar{x})$, we obtain $F(\bar{x}) \preceq_{\mathcal{D}}^u F(\bar{x})$. Then $F(\bar{x}) \sim_{\mathcal{D}}^u F(\bar{x})$. Assume that $\bar{x} \notin \text{Min}(F, S, \mathcal{D})$. Then, there exists $x_0 \in S \setminus \{\bar{x}\}$ such that $F(x_0) \preceq_{\mathcal{D}}^u F(\bar{x})$ and $F(x_0) \approx_{\mathcal{D}}^u F(\bar{x})$. Since $T(\cdot)$ is increasing on $\mathcal{F}(S)$, we see that $T(F(x_0)) \leq T(F(\bar{x})) = 0$. Hence, $T(F(x_0)) \leq 0$, which contradicts (ii). Therefore, $\bar{x} \in \text{Min}(F, S, \mathcal{D})$. $\square$

The following example shows that the condition “ $T(F(x)) > 0$ for all $x \in S \setminus \{\bar{x}\}$ with $F(x) \approx_{\mathcal{D}}^u F(\bar{x})$ ” in Theorem 4.2 can not be omitted.

Example 4.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [0, 1]$, $k_0 = (1, 1)$. The set-valued mappings $F : X \rightrightarrows Y$ and $\mathcal{D} : Y \times Y \rightrightarrows Y$ are defined, respectively, as

$$F(x) := \begin{cases} \{(1, 1)\}, & x = 0, \\ \{(0, 0)\}, & x \neq 0, \end{cases}$$

and

$$\mathcal{D}(y_1, y_2) := \begin{cases} \mathbb{R}_+^2, & y_1 = y_2, \\ (1, 1) + \mathbb{R}_+^2, & y_1 \neq y_2. \end{cases}$$

where $y_1, y_2 \in Y$. Take $\bar{x} = 0 \in S$. Then, $\mathbf{0} \in \mathcal{D}(y, y)$ for all $y \in F(\bar{x})$. Let the function $T : \mathcal{P}_0(Y) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be defined as

$$T(Q) = h(Q, F(0)) = \inf \{t \in \mathbb{R} | Q \preceq_{\mathcal{D}}^u tk_0 + F(0)\}, \quad \text{for all } Q \in \mathcal{P}_0(Y).$$

Then, $T(\cdot)$ is increasing on $\mathcal{P}_0(Y)$. A direct calculation gives that $h(F(0), F(0)) = 0$. For all $x \in (0, 1]$, $F(x) \approx_{\mathcal{D}}^u F(0)$ and $h(F(x), F(0)) = -1 < 0$. We can directly verify that $\bar{x} \notin \text{Min}(F, S, \mathcal{D})$.

Theorem 4.3. Assume that $\bar{x} \in S$ and $F(\bar{x})$ is compact, (H_1) , (H_2) and (H_3) hold, $\mathcal{D}$ is closed on $F(S) \times Y$, $\mathbf{0} \in \mathcal{D}(y, y)$ for any $y \in F(\bar{x})$. Let $\mathcal{D}(y_1, y_2) + \mathcal{D}(y_2, y_3) \subseteq \mathcal{D}(y_1, y_3)$ for any $y_1, y_2 \in Y$ and $y_3 \in F(\bar{x}) + (-\infty, +\infty)k_0$. If there exists $\bar{y} \in F(\bar{x})$ such that, for any $y \in F(\bar{x})$, $y - \bar{y} \notin \text{int}\mathcal{D}(\bar{y}, y)$, then $\bar{x} \in H\text{-Min}(F, S, \mathcal{D})$ implies that there exists a function $T : \mathcal{P}_0(Y) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ which is increasing on $\mathcal{P}_0(Y)$ and

- (i) $T(F(\bar{x})) = 0$,
- (ii) $T(F(x) + H) > 0$, for all $x \in S$ with $F(x) + H \approx_{\mathcal{D}}^u F(\bar{x})$.

Proof. It is similar to the proof of Theorem 4.1. □

We now illustrate Theorem 4.3 by the following example.

Example 4.3. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [0, 2]$, $k_0 = (1, 1)$, and $H = [0, 1] \times [0, 1]$. The set-valued mappings $F : X \rightrightarrows Y$ and $\mathcal{D} : Y \times Y \rightrightarrows Y$ are defined as

$$F(x) := [0, |x|] \times [0, |x|], \quad \text{for all } x \in \mathbb{R}$$

and

$$\mathcal{D}(y_1, y_2) := \begin{cases} \mathbb{R}_+^2, & y_1 \in V \text{ or } y_2 \in V, \\ \{(s, r) \in \mathbb{R}_+^2 : r \geq -s + 2\}, & \text{others,} \end{cases}$$

respectively, where $V := \{(\alpha, \beta) \in Y : \alpha - 2 \leq \beta \leq \alpha + 2\}$.

We can easily verify that the conditions of Theorem 4.3 are satisfied. We calculate that $H\text{-Min}(F, S, \mathcal{D}) = [0, 1]$. Let $T(\cdot) = h(\cdot, F(0))$. Then, $T(\cdot)$ is increasing on $\mathcal{P}_0(Y)$, $h(F(0), F(0)) = 0$ and for any $x \in [0, 2]$, $F(x) + H \not\approx_{\mathcal{D}}^u F(0)$, and $h(F(x) + H, F(0)) = (x + 1)\sqrt{2} > 0$.

Theorem 4.4. Assume that $\bar{x} \in S$. If there exists a function $T : \mathcal{P}_0(Y) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ which is increasing on $\mathcal{F}(S)$ and satisfies the following conditions:

- (i) $T(F(\bar{x})) = 0$,
- (ii) $T(F(x) + H) > 0$ for all $x \in S$ with $F(x) + H \approx_{\mathcal{D}}^u F(\bar{x})$,

then $\bar{x} \in H\text{-Min}(F, S, \mathcal{D})$.

Proof. It is similar to the proof of Theorem 4.2. □

Theorem 4.5. Assume that $\bar{x} \in S$, $F(\bar{x})$ is compact, $F(S)$ is bounded, (H_1) – (H_4) hold, $k_0 \in \text{int}\bar{\mathcal{D}}[F(S), F(S)]$ and $\mathcal{D}$ is closed on $F(S) \times Y$. For any $y \in F(\bar{x})$, $\mathbf{0} \in \mathcal{D}(y, y)$. For any $y_1, y_2 \in Y$, $y_3 \in F(\bar{x}) + (-\infty, +\infty)k_0$, $\mathcal{D}(y_1, y_2)$ is a coradiant set and $\mathcal{D}(y_1, y_2) + \mathcal{D}(y_2, y_3) \subseteq \mathcal{D}(y_1, y_3)$. If there exists $\bar{y} \in F(\bar{x})$ such that, for any $y \in F(\bar{x})$, $y - \bar{y} \notin \text{int}\mathcal{D}(\bar{y}, y)$, then $\bar{x} \in W\text{Min}(F, S, \mathcal{D})$ implies that there exists a function $T : \mathcal{P}_0(Y) \rightarrow \mathbb{R} \cup \{\pm\infty\}$, which is strictly increasing on $\mathcal{F}(S)$ and

- (i) $T(F(\bar{x})) = 0$,
- (ii) $T(F(x)) \geq 0$ for all $x \in S \setminus \{\bar{x}\}$.

Proof. Let $T(\cdot) = h(\cdot, F(\bar{x}))$. From Proposition 3.8, we know that $T(F(\bar{x})) = 0$. Since $\bar{x} \in W\text{Min}(F, S, \mathcal{D})$, we conclude that $F(x) \not\approx_{\mathcal{D}}^u F(\bar{x})$ for any $x \in S \setminus \{\bar{x}\}$. Then, by applying Corollary 3.2(a), we obtain $T(F(x)) \geq 0$. Therefore, for any $x \in S$, $T(F(x)) \geq 0 > -\infty$. Similar to the proof of Proposition 3.7, we can prove that $T(\cdot)$ is strictly increasing on $\mathcal{F}(S)$. □

The following example illustrates Theorem 4.5.

Example 4.4. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [0, \frac{1}{2}]$ and $k_0 = (0, 1)$. The set-valued mappings $F : X \rightrightarrows Y$ and $\mathcal{D} : Y \times Y \rightrightarrows Y$ are defined as

$$F(x) := [0, |x|] \times [0, |x|], \quad \text{for all } x \in \mathbb{R},$$

and

$$\mathcal{D}(y_1, y_2) := \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha + \beta \geq |s_1 - s_2|, \beta \geq 0\},$$

respectively, where $y_1 = (s_1, r_1)$, $y_2 = (s_2, r_2) \in Y$. Then, we can easily verify the conditions of Theorem 4.5 are satisfied. We calculate that $\text{WMin}(F, S, \mathcal{D}) = \{0\}$. Let $T(\cdot) = h(\cdot, F(0))$. Then $T(\cdot)$ is strictly increasing on $\mathcal{P}_0(Y)$ and $h(F(0), F(0)) = 0$. Next, we verify that $h(F(x), F(0)) > 0$ for any $x \in (0, \frac{1}{2}]$. Taking $y_0 = (x, x) \in F(x)$, we have

$$\{t \in \mathbb{R} : F(x) \preceq_{\mathcal{D}}^u F(0)\} \subseteq \{t \in \mathbb{R} : y_0 \in tk_0 - \mathcal{D}(y_0, \mathbf{0})\} \subseteq \{t \in \mathbb{R} : t \geq x\}.$$

Then, $h(F(x), F(0)) \geq \inf \{t \in \mathbb{R} : y_0 \in tk_0 - \mathcal{D}(y_0, \mathbf{0})\} \geq x > 0$.

Theorem 4.6. Assume that $\bar{x} \in S$. If there exists a function $T : \mathcal{P}_0(Y) \rightarrow \mathbb{R} \cup \{\pm\infty\}$, which is strictly increasing on $\mathcal{F}(S)$ and satisfies the following conditions

- (i) $T(F(\bar{x})) = 0$,
- (ii) $T(F(x)) \geq 0$ for all $x \in S \setminus \{\bar{x}\}$,

then $\bar{x} \in \text{WMin}(F, S, \mathcal{D})$.

Proof. Assume that $\bar{x} \notin \text{WMin}(F, S, \mathcal{D})$. Then, there exists $x_0 \in S \setminus \{\bar{x}\}$ such that $F(x_0) \prec_{\mathcal{D}}^u F(\bar{x})$. Since $T(\cdot)$ is strictly increasing on $\mathcal{F}(S)$, we obtain $T(F(x_0)) < T(F(\bar{x})) = 0$, and hence, $T(F(x_0)) < 0$, a contradiction to (ii). $\square$

5. SEPARATION FUNCTIONS AND OPTIMALITY CONDITIONS FOR CONSTRAINED SET OPTIMIZATION PROBLEMS WITH A VARIABLE SET RELATION

In this section, we investigate optimality conditions of the constrained set optimization problem by using the nonlinear scalarization function formulated in Section 3 and image space analysis (ISA).

Let $G : X \rightrightarrows Z$ be a set-valued mapping with nonempty values, $C \subseteq Z$, $q_0 \in Z \setminus \{\mathbf{0}\}$ be such that $[0, +\infty)q_0 + C \subseteq C$, and $\Omega := \{x \in X : G(x) \cap (-C) \neq \emptyset\}$. Let $S = \Omega$. The set optimization problem as defined in Section 4 is given by

$$\text{Minimize } F(x) \quad \text{subject to } x \in \Omega. \quad (\text{SOPC})$$

Let $\bar{x} \in X$. The set-valued mapping $\mathcal{A}_{\bar{x}}^j : X \rightarrow (\mathbb{R} \cup \{\pm\infty\}) \times 2^Z$ is defined by

$$\mathcal{A}_{\bar{x}}^j(x) := (-h(F(x) + jH, F(\bar{x})), G(x)),$$

where $j = 0$ or $j = 1$ and H is a given nonempty subset of Y .

The image of (SOP) at $\bar{x}$ is denoted by

$$\mathcal{H}(\bar{x}) := \{(u, v) \in (\mathbb{R} \cup \{\pm\infty\}) \times Z : u = -h(F(x), F(\bar{x})), v \in G(x), x \in X\}.$$

Let

$$\mathcal{H} := \{(u, v) \in (\mathbb{R} \cup \{+\infty\}) \times Z : u \geq 0, v \in -C\}$$

and

$$\mathcal{H}' := \{(u, v) \in (\mathbb{R} \cup \{+\infty\}) \times Z : u > 0, v \in -C\}.$$

We establish optimality conditions for (SOPC) via ISA.

Theorem 5.1. Let $\bar{x} \in \Omega$. If $\mathcal{K}(\bar{x}) \cap \mathcal{H} = \emptyset$, then $\bar{x} \in \text{Min}(F, \Omega, \mathcal{D})$.

Proof. Assume that $\bar{x} \notin \text{Min}(F, \Omega, \mathcal{D})$. Then there exists $x_0 \in \Omega$ such that $F(x_0) \preceq_{\mathcal{D}}^u F(\bar{x})$, but $F(\bar{x}) \not\preceq_{\mathcal{D}}^u F(x_0)$. By combining $F(x_0) \preceq_{\mathcal{D}}^u F(\bar{x})$ with Corollary 3.1(a), we conclude that $h(F(x_0), F(\bar{x})) \leq 0$. Let $u_0 = -h(F(x_0), F(\bar{x}))$. Since $x_0 \in \Omega$, we have $G(x_0) \cap (-C) \neq \emptyset$. Taking $v_0 \in G(x_0) \cap (-C)$, one has $(u_0, v_0) \in \mathcal{K}(\bar{x}) \cap \mathcal{H}$. This contradicts $\mathcal{K}(\bar{x}) \cap \mathcal{H} = \emptyset$. $\square$

Theorem 5.2. Let $\bar{x} \in \Omega$ and (H_1) hold. Assume that $\mathcal{D}(y_1, y_2) \subseteq \mathcal{D}(y_1, y_2 + \lambda k_0)$ for all $y_1 \in F(\Omega)$, $y_2 \in F(\bar{x})$ and $\lambda \in \mathbb{R}$. Assume further that there exists $\bar{y} \in F(\bar{x})$ such that $y - \bar{y} \notin \text{int}\mathcal{D}(\bar{y}, y)$ for all $y \in F(\bar{x})$. Then, $\bar{x} \in \text{WMin}(F, \Omega, \mathcal{D})$ if and only if $\mathcal{K}(\bar{x}) \cap \mathcal{H}' = \emptyset$.

Proof. NECESSITY. Assume that $\bar{x} \in \text{WMin}(F, \Omega, \mathcal{D})$. We prove that $\mathcal{K}(\bar{x}) \cap \mathcal{H}' = \emptyset$. For any $(u, v) \in \mathcal{K}(\bar{x})$, there exists $\hat{x} \in X$ such that $u = -h(F(\hat{x}), F(\bar{x}))$ and $v \in G(\hat{x})$. There are two cases to be considered:

CASE 1. If $v \notin -C$, then $(u, v) \notin \mathcal{H}'$.

CASE 2. If $v \in -C$, then $v \in G(\hat{x}) \cap (-C)$, which implies that $\hat{x} \in \Omega$. Due to $\bar{x} \in \text{WMin}(F, \Omega, \mathcal{D})$, we have $F(x) \not\preceq_{\mathcal{D}}^u F(\bar{x})$ for all $x \in \Omega \setminus \{\bar{x}\}$. Since $y - \bar{y} \notin \text{int}\mathcal{D}(\bar{y}, y)$ for any $y \in F(\bar{x})$, we have $F(\bar{x}) \not\preceq_{\mathcal{D}}^u F(\bar{x})$. Thus $F(x) \not\preceq_{\mathcal{D}}^u F(\bar{x})$ for all $x \in \Omega$. From Corollary 3.2(a), we have $h(F(x), F(\bar{x})) \geq 0$ for all $x \in \Omega$. Hence, $u \leq 0$, and so, $(u, v) \notin \mathcal{H}'$. By synthesizing the above two cases, we can see that $\mathcal{K}(\bar{x}) \cap \mathcal{H}' = \emptyset$.

SUFFICIENCY. Assume to the contrary that $\bar{x} \notin \text{WMin}(F, \Omega, \mathcal{D})$. Then, there exists $x_0 \in \Omega \setminus \{\bar{x}\}$ such that $F(x_0) \prec_{\mathcal{D}}^u F(\bar{x})$. From Corollary 3.2(b), we have $h(F(x_0), F(\bar{x})) < 0$. Let $u_0 = -h(F(x_0), F(\bar{x}))$. Then, $u_0 > 0$. Since $x_0 \in \Omega$, we obtain $G(x_0) \cap (-C) \neq \emptyset$. Let $v_0 \in G(x_0) \cap (-C)$. Then $(u_0, v_0) \in \mathcal{H}' \cap \mathcal{K}(\bar{x})$. Therefore, $\mathcal{K}(\bar{x}) \cap \mathcal{H}' \neq \emptyset$. This is a contradiction. $\square$

The following example illustrates Theorem 5.2.

Example 5.1. Let $X = \ell^\infty$, $Y = \mathbb{R}^2$, $Z = \mathbb{R}$, $C = \mathbb{R}_+$, $k_0 = (1, 1)$, $q_0 = 1$, and the set-valued mappings $G : X \rightrightarrows Z$, $F : X \rightrightarrows Y$ and $\mathcal{D} : Y \times Y \rightrightarrows Y$ be defined as

$$G(x) := [\|x\|_\infty^2 - 4, +\infty),$$

$$F(x) := \begin{cases} \{(1, 1)\}, & x = \mathbf{0}, \\ [0, 1] \times [0, 1], & \|x\|_\infty \leq 1 \text{ and } x \neq \mathbf{0}, \\ [0, \|x\|_\infty] \times [0, \|x\|_\infty], & \text{otherwise,} \end{cases}$$

and

$$\mathcal{D}(y_1, y_2) := \begin{cases} \mathbb{R}_+^2, & y_1 \in V \text{ or } y_2 \in V, \\ \{(s, r) \in \mathbb{R}_+^2 : r \geq -s + 2\}, & \text{otherwise,} \end{cases}$$

respectively, where $V := \{(\alpha, \beta) \in Y \mid \alpha - 2 \leq \beta \leq \alpha + 2\}$. It can be seen that $\Omega = \{x \in X : \|x\|_\infty \leq 2\}$, $[0, +\infty)q_0 + C \subseteq C$. Let $\bar{x} = \mathbf{1} = (1, 1, \dots, 1, \dots)$. Then, $\bar{x} \in \text{WMin}(F, \Omega, \mathcal{D})$ and

$$\mathcal{K}(\mathbf{1}) = \{(u, v) \in (\mathbb{R} \cup \{\pm\infty\}) \times Z : u = -h(F(x), F(\mathbf{1})), v \in G(x), x \in \ell^\infty\}.$$

For any $(u, v) \in \mathcal{K}(\mathbf{1})$, there exists $\hat{x} \in X$ such that $u = -h(F(\hat{x}), F(\mathbf{1}))$ and $v \in G(\hat{x})$. If $v \notin -C$, then $(u, v) \notin \mathcal{H}'$; if $v \in -C$, then $v \in G(\hat{x}) \cap (-C)$, which implies that $\hat{x} \in \Omega$. Then,

$$\begin{aligned} h(F(\hat{x}), F(\mathbf{1})) &= \inf \{t \in \mathbb{R} : F(\hat{x}) \subseteq F(\mathbf{1}) + tk_0 - \mathbb{R}_+^2\} \\ &= \max \{0, (\|\hat{x}\|_\infty - 1)\sqrt{2}\} \geq 0. \end{aligned}$$

Thus, $u \leq 0$, and therefore, $(u, v) \notin \mathcal{H}'$. Hence, $\mathcal{K}(\mathbf{1}) \cap \mathcal{H}' = \emptyset$.

On the other hand, let $\bar{x} = \mathbf{0}$. Then,

$$\mathcal{H}(\mathbf{0}) = \{(u, v) \in (\mathbb{R} \cup \{\pm\infty\}) \times Z : u = -h(F(x), F(\mathbf{0})), v \in G(x), x \in \ell^\infty\}.$$

For any $(u, v) \in \mathcal{H}(\mathbf{0})$, there exists $\tilde{x} \in X$ such that $u = -h(F(\tilde{x}), F(\mathbf{0}))$ and $v \in G(\tilde{x})$. If $v \notin -C$, then $(u, v) \notin \mathcal{H}'$; if $v \in -C$, then $v \in G(\tilde{x}) \cap (-C)$, which implies that

$$u = -\max \left\{ 0, (\|\tilde{x}\|_\infty - 1)\sqrt{2} \right\} \leq 0.$$

Hence, $\mathcal{H}(\mathbf{0}) \cap \mathcal{H}' = \emptyset$. We can directly verify that $\mathbf{0} \in \text{WMin}(F, \Omega, \mathcal{D})$.

Let $\bar{x} \in \Omega$ and $H \in \mathcal{P}_0(Y)$ be a given set. We next present a sufficient condition for $\bar{x} \in H\text{-Min}(F, \Omega, \mathcal{D})$.

Theorem 5.3. *Let $\mathbf{0} \in H$ and $\mathcal{H}(\bar{x}) \cap \mathcal{H} = \emptyset$. Then, $\bar{x} \in H\text{-Min}(F, \Omega, \mathcal{D})$.*

Proof. Suppose to the contrary that $\bar{x} \notin H\text{-Min}(F, \Omega, \mathcal{D})$. Then there exists $x_0 \in \Omega$ such that $F(x_0) + H \preceq_{\mathcal{D}}^u F(\bar{x})$, but $F(\bar{x}) \not\preceq_{\mathcal{D}}^u F(x_0) + H$. From Corollary 3.1(a), we obtain $h(F(x_0) + H, F(\bar{x})) \leq 0$. There are two cases to be considered:

CASE 1. If $h(F(x_0) + H, F(\bar{x})) = t_0 \leq 0$, then there exists $\varepsilon_n \geq 0$ with $\varepsilon_n \rightarrow 0$ such that $F(x_0) + H \preceq_{\mathcal{D}}^u F(\bar{x}) + (t_0 + \varepsilon_n)k_0$. Since $\mathbf{0} \in H$, we conclude that, for any $y \in F(x_0)$, there exists $y_1 \in F(\bar{x})$ such that $y + \mathbf{0} \in y_1 + (t_0 + \varepsilon_n)k_0 - \mathcal{D}(y, y_1 + (t_0 + \varepsilon_n)k_0)$, which implies $F(x_0) \preceq_{\mathcal{D}}^u F(\bar{x}) + (t_0 + \varepsilon_n)k_0$. From Corollary 3.1(a), we have $h(F(x_0), F(\bar{x})) \leq t_0 + \varepsilon_n$. Thus $h(F(x_0), F(\bar{x})) \leq t_0 \leq 0$.

CASE 2. If $h(F(x_0) + H, F(\bar{x})) = -\infty$, then there exists $t_n \rightarrow -\infty$ such that $F(x_0) + H \preceq_{\mathcal{D}}^u F(\bar{x}) + t_n k_0$. Similarly, we can show that $F(x_0) \preceq_{\mathcal{D}}^u F(\bar{x}) + t_n k_0$, which implies $h(F(x_0), F(\bar{x})) \leq t_n \rightarrow -\infty$. Let $u_0 = -h(F(x_0), F(\bar{x}))$. Then, $u_0 \geq 0$. Since $x_0 \in \Omega$, we have $G(x_0) \cap (-C) \neq \emptyset$. Take $v_0 \in G(x_0) \cap (-C)$. Then, $(u_0, v_0) \in \mathcal{H} \cap \mathcal{H}(\bar{x})$, which is a contradiction. $\square$

Consider a function $\omega : (\mathbb{R} \times Z) \times \Pi \rightarrow \mathbb{R} \cup \{\pm\infty\}$, where Π is a set of parameters to be specified case by case. For each $\pi \in \Pi$, the sets

$$\text{lev}_{\geq 0}\omega(\cdot, \pi) := \{(u, v) \in \mathbb{R} \times Z : \omega(u, v, \pi) \geq 0\}$$

and

$$\text{lev}_{> 0}\omega(\cdot, \pi) := \{(u, v) \in \mathbb{R} \times Z : \omega(u, v, \pi) > 0\}$$

are called the nonnegative and positive level sets of the function ω , respectively.

Definition 5.1. The class of all functions $\omega : (\mathbb{R} \times Z) \times \Pi \rightarrow \mathbb{R} \cup \{\pm\infty\}$ such that

- (i) $\mathcal{H} \subseteq \text{lev}_{\geq 0}\omega(\cdot, \pi), \forall \pi \in \Pi$,
- (ii) $\bigcap_{\pi \in \Pi} \text{lev}_{> 0}\omega(\cdot, \pi) \subseteq \mathcal{H}$,

is called the class of weak separation functions.

Definition 5.2. The class of all functions $\omega : (\mathbb{R} \times Z) \times \Pi \rightarrow \mathbb{R} \cup \{\pm\infty\}$ such that

$$\mathcal{H}' = \bigcap_{\pi \in \Pi} \text{lev}_{> 0}\omega(\cdot, \pi),$$

is called the class of regular weak separation functions.

Now we introduce the following nonlinear functions for a set of parameters $\Pi \subseteq \mathbb{R}^2$:

$$\omega_1(u, v, \theta, \lambda) = \theta u - \lambda z^{C, q_0}(v), \quad (5.1)$$

and

$$\omega_2(u, v, \theta, \lambda) = \theta u - \lambda \triangle_{-C}(v), \quad (5.2)$$

for all $(u, v, \theta, \lambda) \in (\mathbb{R} \times Z) \times \Pi$, where

$$z^{C, q_0}(v) = \inf \{t \in \mathbb{R} : v \in tq_0 - C\}, \quad \triangle_{-C}(v) = d(v, -C) - d(v, Z \setminus (-C))$$

and

$$d(v, -C) = \inf_{c \in -C} \|v - c\|.$$

We follow the convention that $0 \times \{\pm\infty\} = 0$. Next, we show some properties of ω_1 and ω_2 .

Proposition 5.1. *Let $\Pi = \mathbb{R}_+^2 \setminus \{0\}$. Then the nonlinear functions ω_1 and ω_2 defined respectively by (5.1) and (5.2) are both weak separation functions.*

Proof. For any $(\theta, \lambda) \in \mathbb{R}_+^2 \setminus \{0\}$, we prove $\mathcal{H} \subseteq \text{lev}_{\geq 0} \omega_1(\cdot, \theta, \lambda)$.

Let $(u, v) \in \mathcal{H}$. Then $u \geq 0$ and $v \in -C$. Therefore, $\theta u \geq 0$ and $\inf \{t \in \mathbb{R} : v \in tq_0 - C\} \leq 0$, which together with $\lambda \geq 0$ implies $\lambda z^{C, q_0}(v) \leq 0$. Consequently, $\theta u - \lambda z^{C, q_0}(v) \geq 0$. Therefore, $\mathcal{H} \subseteq \text{lev}_{\geq 0} \omega_1(\cdot, \theta, \lambda)$.

Next, we show that $\bigcap_{(\theta, \lambda) \in \mathbb{R}_+^2 \setminus \{0\}} \text{lev}_{> 0} \omega_1(\cdot, \theta, \lambda) \subseteq \mathcal{H}$. Assume that there exists $(\bar{u}, \bar{v}) \in \bigcap_{(\theta, \lambda) \in \mathbb{R}_+^2 \setminus \{0\}} \text{lev}_{> 0} \omega_1(\cdot, \theta, \lambda)$, but $(\bar{u}, \bar{v}) \notin \mathcal{H}$. We consider the following two cases:

CASE 1. $\bar{u} < 0$. Let $(\theta, \lambda) = (1, 0)$. Then, $\omega_1(\bar{u}, \bar{v}, 1, 0) = \bar{u} < 0$, this contradicts $(\bar{u}, \bar{v}) \in \bigcap_{(\theta, \lambda) \in \mathbb{R}_+^2 \setminus \{0\}} \text{lev}_{> 0} \omega_1(\cdot, \theta, \lambda)$.

CASE 2. $\bar{u} \geq 0, \bar{v} \notin -C$. Take $(\theta, \lambda) = (0, 1)$. Then,

$$\omega_1(\bar{u}, \bar{v}, 0, 1) = -z^{C, q_0}(\bar{v}) = -\inf \{t \in \mathbb{R} : \bar{v} \in tq_0 - C\}.$$

As $\bar{v} \notin -C$, we see that $\bar{v} \notin 0q_0 - C$. Since $(-\infty, 0]q_0 - C \subseteq -C$, we obtain $\bar{v} \notin tq_0 - C$ for any $t \leq 0$. Hence, $z^{C, q_0}(\bar{v}) \geq 0$, and therefore, $\omega_1(\bar{u}, \bar{v}, 0, 1) \leq 0$, which contradicts $(\bar{u}, \bar{v}) \in \bigcap_{(\theta, \lambda) \in \mathbb{R}_+^2 \setminus \{0\}} \text{lev}_{> 0} \omega_1(\cdot, \theta, \lambda)$. Similarly, we can prove that ω_2 is also a weak separation function. $\square$

Proposition 5.2. *Let $\Pi = \mathbb{R}_{++} \times \mathbb{R}_+$ and C be closed. Then, ω_1 and ω_2 are both regular weak separation functions.*

Proof. We first prove that $\bigcap_{(\theta, \lambda) \in \Pi} \text{lev}_{> 0} \omega_1(\cdot, \theta, \lambda) \subseteq \mathcal{H}'$. Suppose to the contrary that there exists $(\bar{u}, \bar{v}) \in \bigcap_{(\theta, \lambda) \in \Pi} \text{lev}_{> 0} \omega_1(\cdot, \theta, \lambda)$, but $(\bar{u}, \bar{v}) \notin \mathcal{H}'$. We consider the following two cases:

CASE 1. $\bar{u} \leq 0$. Taking $(\theta, \lambda) = (1, 0)$, we get $\omega_1(\bar{u}, \bar{v}, 1, 0) = \bar{u} \leq 0$. This contradicts $(\bar{u}, \bar{v}) \in \text{lev}_{> 0} \omega_1(\cdot, 1, 0)$.

CASE 2. $\bar{u} > 0, \bar{v} \notin -C$. Then, by using the similar argument as in CASE 2 of Proposition 5.1, we have $z^{C, q_0}(\bar{v}) \geq 0$. Since C is closed, we have $z^{C, q_0}(\bar{v}) > 0$. Let $(\hat{\theta}, \hat{\lambda}) = \left(\frac{z^{C, q_0}(\bar{v})}{\bar{u}}, 1\right)$.

Then, $(\hat{\theta}, \hat{\lambda}) \in \Pi$ and

$$\omega_1(\bar{u}, \bar{v}, \hat{\theta}, \hat{\lambda}) = z^{C, q_0}(\bar{v}) - z^{C, q_0}(\bar{v}) = 0,$$

which contradicts $(\bar{u}, \bar{v}) \in \text{lev}_{> 0} \omega_1\left(\cdot, \frac{z^{C, q_0}(\bar{v})}{\bar{u}}, 1\right)$.

We next prove that $\mathcal{H}' \subseteq \bigcap_{(\theta, \lambda) \in \Pi} \text{lev}_{>0} \omega_1(\cdot, \theta, \lambda)$. Let $(\bar{u}, \bar{v}) \in \mathcal{H}'$. Then, $\bar{u} > 0$ and $\bar{v} \in -C$, and therefore, $\inf\{t \in \mathbb{R} : \bar{v} \in tq_0 - C\} \leq 0$, which implies that $\omega(\bar{u}, \bar{v}, \theta, \lambda) = \theta\bar{u} - \lambda z^{C, q_0}(\bar{v}) > 0$ for all $\theta > 0$ and for all $\lambda \geq 0$.

Similarly, we can prove that ω_2 is also a regular weak separation function. $\square$

Definition 5.3. We say that the sets $\mathcal{K}(\bar{x})$ and $\mathcal{H}'$ admit a nonlinear separation with respect to ω_j if there exists $(\bar{\theta}, \bar{\lambda}) \in \mathbb{R}_{++} \times \mathbb{R}_+$ such that $\omega_j(u, v, \bar{\theta}, \bar{\lambda}) \leq 0$ for all $(u, v) \in \mathcal{K}(\bar{x})$, where

$$\omega_j(u, v, \bar{\theta}, \bar{\lambda}) := \begin{cases} \bar{\theta}u - \bar{\lambda}z^{C, q_0}(v), & j = 1, \\ \bar{\theta}u - \bar{\lambda} \triangle_{-C}(v), & j = 2. \end{cases}$$

The following two theorems establish sufficient optimality conditions of the weakly optimal solution of (SOP) with respect to the variable upper set less relation.

Theorem 5.4. Let $\bar{x} \in \Omega$ and $\mathcal{D}(y_1, y_2 + \lambda k_0) \subseteq \mathcal{D}(y_1, y_2)$ for all $\lambda \geq 0, y_1, y_2 \in F(\Omega)$. If $\mathcal{K}(\bar{x})$ and $\mathcal{H}'$ admit a nonlinear separation with respect to ω_1 , then $\bar{x} \in \text{WMin}(F, \Omega, \mathcal{D})$.

Proof. Since $\mathcal{K}(\bar{x})$ and $\mathcal{H}'$ admit a nonlinear separation with respect to ω_1 , there exists $(\bar{\theta}, \bar{\lambda}) \in \mathbb{R}_{++} \times \mathbb{R}_+$ such that for any $(u, v) \in \mathcal{K}(\bar{x})$, $\omega_1(u, v, \bar{\theta}, \bar{\lambda}) = \bar{\theta}u - \bar{\lambda}z^{C, q_0}(v) \leq 0$.

We next prove that $\mathcal{K}(\bar{x}) \cap \mathcal{H}' = \emptyset$. If there exists $(\bar{u}, \bar{v}) \in \mathcal{K}(\bar{x}) \cap \mathcal{H}'$, then $\omega_1(\bar{u}, \bar{v}, \bar{\theta}, \bar{\lambda}) = \bar{\theta}\bar{u} - \bar{\lambda}z^{C, q_0}(\bar{v}) \leq 0$. Since $(\bar{u}, \bar{v}) \in \mathcal{H}'$, we have $\bar{u} > 0$ and $\bar{v} \in -C$. Therefore, $z^{C, q_0}(\bar{v}) = \inf\{t \in \mathbb{R} : \bar{v} \in tq_0 - C\} \leq 0$, which implies that $\bar{\theta}\bar{u} - \bar{\lambda}z^{C, q_0}(\bar{v}) > 0$, a contradiction. Thus, $\mathcal{K}(\bar{x}) \cap \mathcal{H}' = \emptyset$. Similar to the proof of the sufficiency in Theorem 5.2, we can verify that $\bar{x} \in \text{WMin}(F, \Omega, \mathcal{D})$. $\square$

Theorem 5.5. Let $\bar{x} \in \Omega$ and $\mathcal{D}(y_1, y_2 + \lambda k_0) \subseteq \mathcal{D}(y_1, y_2)$ for all $\lambda \geq 0, y_1, y_2 \in F(\Omega)$. If $\mathcal{K}(\bar{x})$ and $\mathcal{H}'$ admit a nonlinear separation with respect to ω_2 , then $\bar{x} \in \text{WMin}(F, \Omega, \mathcal{D})$.

Proof. It is similar to that of Theorem 5.4. $\square$

Recently, the relations between saddle points of the Lagrangian set-valued mapping and the optimal solutions of set optimization problems are discussed in the literature. For example, Zhou et al. [40] used the improvement set and the oriented distance function to introduce two nonlinear separation functions, and established Lagrange-type optimality conditions for constrained set-valued optimization problems. Based on two nonlinear regular weak separation functions, Xu et al. [36] obtained sufficient optimality conditions for saddle points.

We next discuss a generalized saddle point for (SOP).

Let $\bar{x} \in \Omega$. Define the generalized Lagrangian function $\mathcal{L}_{\bar{x}} : X \times (\mathbb{R}_{++} \times \mathbb{R}_+) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ by

$$\mathcal{L}_{\bar{x}}(x, (\theta, \lambda)) := \theta h(F(x), F(\bar{x})) + \lambda \inf_{v \in G(x)} z^{C, q_0}(v).$$

Definition 5.4. A pair $(\bar{x}, (\bar{\theta}, \bar{\lambda})) \in X \times (\mathbb{R}_{++} \times \mathbb{R}_+)$ is called a saddle point of $\mathcal{L}_{\bar{x}}(\cdot, \cdot)$ on $X \times (\mathbb{R}_{++} \times \mathbb{R}_+)$ if, for all $x \in X, (\theta, \lambda) \in \mathbb{R}_{++} \times \mathbb{R}_+$,

$$\mathcal{L}_{\bar{x}}(\bar{x}, (\theta, \lambda)) \leq \mathcal{L}_{\bar{x}}(\bar{x}, (\bar{\theta}, \bar{\lambda})) \leq \mathcal{L}_{\bar{x}}(x, (\bar{\theta}, \bar{\lambda})).$$

The following theorem presents a sufficient condition for a saddle point.

Theorem 5.6. Let $\bar{x} \in \Omega$ and $\mathbf{0} \in \mathcal{D}(y, y)$ for all $y \in F(\bar{x})$. If $\mathcal{K}(\bar{x})$ and $\mathcal{H}'$ admit a nonlinear separation with respect to ω_1 , then there exists $(\bar{\theta}, \bar{\lambda}) \in \mathbb{R}_{++} \times \mathbb{R}_+$ such that $(\bar{x}, (\bar{\theta}, \bar{\lambda}))$ is a saddle point of $\mathcal{L}_{\bar{x}}(\cdot, \cdot)$ on $X \times (\mathbb{R}_{++} \times \mathbb{R}_+)$.

Proof. Since $\mathcal{K}(\bar{x})$ and $\mathcal{H}'$ admit a nonlinear separation with respect to ω_1 , we deduce that there exists $(\bar{\theta}, \bar{\lambda}) \in \mathbb{R}_{++} \times \mathbb{R}_+$ such that, for all $(u, v) \in \mathcal{K}(\bar{x})$,

$$\omega_1(u, v, \bar{\theta}, \bar{\lambda}) = \bar{\theta}u - \bar{\lambda}z^{C, q_0}(v) \leq 0. \quad (5.3)$$

For all $x \in X$, let $u = -h(F(x), F(\bar{x}))$ and $v \in G(x)$. Then, $(u, v) \in \mathcal{K}(\bar{x})$, and therefore from (5.3), we obtain $-\bar{\theta}h(F(x), F(\bar{x})) - \bar{\lambda}z^{C, q_0}(v) \leq 0$ for all $x \in X$ and $v \in G(x)$. This implies that

$$\bar{\theta}h(F(x), F(\bar{x})) + \bar{\lambda}z^{C, q_0}(v) \geq 0, \quad \text{for all } x \in X, v \in G(x). \quad (5.4)$$

Taking $x = \bar{x}$ in (5.4), for any $v \in G(\bar{x})$, we have $\bar{\theta}h(F(\bar{x}), F(\bar{x})) + \bar{\lambda}z^{C, q_0}(v) \geq 0$, and therefore,

$$\bar{\theta}h(F(\bar{x}), F(\bar{x})) + \bar{\lambda} \inf_{v \in G(\bar{x})} z^{C, q_0}(v) \geq 0. \quad (5.5)$$

Since $G(\bar{x}) \cap (-C) \neq \emptyset$, we see that there exists $v_0 \in G(\bar{x}) \cap (-C)$ such that $z^{C, q_0}(v_0) \leq 0$. Thus, $\inf_{v \in G(\bar{x})} z^{C, q_0}(v) \leq 0$. Due to $\mathbf{0} \in \mathcal{D}(y, y)$ for all $y \in F(\bar{x})$, we obtain that $F(\bar{x}) \preceq_{\mathcal{D}}^u F(\bar{x})$, therefore, $h(F(\bar{x}), F(\bar{x})) \leq 0$. Hence,

$$\bar{\theta}h(F(\bar{x}), F(\bar{x})) + \bar{\lambda} \inf_{v \in G(\bar{x})} z^{C, q_0}(v) \leq 0. \quad (5.6)$$

Combining (5.5) and (5.6), we arrive at $\bar{\theta}h(F(\bar{x}), F(\bar{x})) + \bar{\lambda} \inf_{v \in G(\bar{x})} z^{C, q_0}(v) = 0$, which implies that $\mathcal{L}_{\bar{x}}(\bar{x}, (\bar{\theta}, \bar{\lambda})) = 0$. Since $h(F(\bar{x}), F(\bar{x})) \leq 0$ and $\inf_{v \in G(\bar{x})} z^{C, q_0}(v) \leq 0$, it follows that, for any $(\theta, \lambda) \in \mathbb{R}_{++} \times \mathbb{R}_+$, $\theta h(F(\bar{x}), F(\bar{x})) + \lambda \inf_{v \in G(\bar{x})} z^{C, q_0}(v) \leq 0$. This implies that $\mathcal{L}_{\bar{x}}(\bar{x}, (\theta, \lambda)) \leq \mathcal{L}_{\bar{x}}(\bar{x}, (\bar{\theta}, \bar{\lambda}))$. From (5.3), we obtain $\bar{\theta}h(F(x), F(\bar{x})) + \bar{\lambda} \inf_{v \in G(x)} z^{C, q_0}(v) \geq 0$, that is, $\mathcal{L}_{\bar{x}}(\bar{x}, (\bar{\theta}, \bar{\lambda})) \leq \mathcal{L}_{\bar{x}}(x, (\bar{\theta}, \bar{\lambda}))$ for all $x \in X$. $\square$

The following example illustrates Theorem 5.6.

Example 5.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $Z = \mathbb{R}^2$, $C = \mathbb{R}_+^2$, $q_0 = (1, 1)$ and $k_0 = (0, 1)$. The set-valued mappings $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$ and $\mathcal{D} : Y \times Y \rightrightarrows Y$ are defined, respectively, by

$$F(x) := [0, |x|] \times [0, |x|],$$

$$G(x) := \begin{cases} [1, 1-x] \times [1, 1-x], & x < 0, \\ \{(-1, -1)\}, & x = 0, \\ [0, x] \times [0, x], & x \in (0, 1], \\ [1, x] \times [1, x], & x > 1, \end{cases}$$

and

$$\mathcal{D}(y_1, y_2) := \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha + \beta \geq s_1 + s_2, \beta \geq 0\},$$

where $y_1 = (s_1, r_1)$, $y_2 = (s_2, r_2) \in Y$. It can be seen that $\Omega = [0, 1]$. Let $\bar{x} = 0$. A direct calculation gives that $h(F(0), F(0)) = 0$ and

$$\inf_{v \in G(x)} \inf \{t \in \mathbb{R} : v \in tq_0 - C\} = \begin{cases} -1, & x = 0, \\ 0, & x \in (0, 1], \\ 1, & x \notin [0, 1]. \end{cases}$$

For any $x \neq 0$, taking $y_0 = (|x|, |x|) \in F(x)$, we obtain from Proposition 3.9 that

$$h(F(x), F(0)) = \inf \{t \in \mathbb{R} : F(x) \preceq_{\mathcal{D}}^u F(0)\} \geq \inf \{t \in \mathbb{R} : y_0 \in tk_0 - \mathcal{D}(y_0, \mathbf{0})\} \geq |x|.$$

Let $\bar{\theta} > 0$ and $\bar{\lambda} = 0$. It can be seen that $\mathcal{L}_0(0, (\bar{\theta}, \bar{\lambda})) = 0$. For any $(\theta, \lambda) \in \mathbb{R}_{++} \times \mathbb{R}_+$, $\mathcal{L}_0(0, (\theta, \lambda)) \leq 0$. For any $x \in X$, $\mathcal{L}_0(x, (\bar{\theta}, \bar{\lambda})) \geq 0$. Then, $(0, (\bar{\theta}, \bar{\lambda}))$ is a saddle point of $\mathcal{L}_{\bar{x}}$ on $X \times (\mathbb{R}_{++} \times \mathbb{R}_+)$.

The following theorem presents a necessary condition for a saddle point.

Theorem 5.7. *Let $(\bar{x}, (\bar{\theta}, \bar{\lambda}))$ be a saddle point of $\mathcal{L}_{\bar{x}}(\cdot, \cdot)$ on $X \times (\mathbb{R}_{++} \times \mathbb{R}_+)$, (H_1) and (H_3) hold. If there exists $\bar{y} \in F(\bar{x})$ such that $y - \bar{y} \notin \text{int}\mathcal{D}(\bar{y}, y)$ for all $y \in F(\bar{x})$, then $\mathcal{K}(\bar{x})$ and $\mathcal{H}'$ admit a nonlinear separation with respect to ω_1 .*

Proof. Since $(\bar{x}, (\bar{\theta}, \bar{\lambda}))$ is a saddle point of $\mathcal{L}_{\bar{x}}(\cdot, (\cdot, \cdot))$ on $X \times (\mathbb{R}_{++} \times \mathbb{R}_+)$, for any $x \in X$ and $(\theta, \lambda) \in \mathbb{R}_{++} \times \mathbb{R}_+$, we have

$$\theta h(F(\bar{x}), F(\bar{x})) + \lambda \inf_{v \in G(\bar{x})} z^{C, q_0}(v) \leq \bar{\theta} h(F(\bar{x}), F(\bar{x})) + \bar{\lambda} \inf_{v \in G(\bar{x})} z^{C, q_0}(v) \quad (5.7)$$

and

$$\bar{\theta} h(F(\bar{x}), F(\bar{x})) + \bar{\lambda} \inf_{v \in G(\bar{x})} z^{C, q_0}(v) \leq \bar{\theta} h(F(x), F(\bar{x})) + \bar{\lambda} \inf_{v \in G(x)} z^{C, q_0}(v). \quad (5.8)$$

Since $y - \bar{y} \notin \text{int}\mathcal{D}(\bar{y}, y)$ for all $y \in F(\bar{x})$, it can be concluded that $F(\bar{x}) \not\prec_{\mathcal{D}}^u F(\bar{x})$. From Corollary 3.2 (a), we obtain $h(F(\bar{x}), F(\bar{x})) \geq 0$. Taking $\lambda = 0$ and $\theta \rightarrow 0^+$ in (5.7), we obtain

$$\bar{\theta} h(F(\bar{x}), F(\bar{x})) + \bar{\lambda} \inf_{v \in G(\bar{x})} z^{C, q_0}(v) \geq 0.$$

This together with (5.8) gives $\bar{\theta} h(F(x), F(\bar{x})) + \bar{\lambda} \inf_{v \in G(x)} z^{C, q_0}(v) \geq 0$. Thus

$$\bar{\theta} h(F(x), F(\bar{x})) + \bar{\lambda} z^{C, q_0}(v) \geq 0, \quad \text{for all } v \in G(x), x \in X.$$

Taking $u = -h(F(x), F(\bar{x}))$, we obtain $\bar{\theta} u - \bar{\lambda} z^{C, q_0}(v) \leq 0$ for all $(u, v) \in \mathcal{K}(\bar{x})$. In other words, $\omega_1(u, v, \bar{\theta}, \bar{\lambda}) \leq 0$ for all $(u, v) \in \mathcal{K}(\bar{x})$. Then, $\mathcal{K}(\bar{x})$ and $\mathcal{H}'$ admit a nonlinear separation with respect to ω_1 . $\square$

Remark 5.1. Example 5.2 can also be used to illustrate Theorem 5.7.

Remark 5.2. If we replace ω_1 by ω_2 , then Theorems 5.6 and 5.7 still hold. The proofs are analogous to those of Theorems 5.6 and 5.7 in the case of ω_1 .

6. APPLICATIONS

6.1. An application to traffic network problems with interval-valued cost functions. As described in [36], we consider the traffic network problem where the sets of nodes and directed arcs are denoted by N and E , respectively, the capacity of arc $e \in E$ is denoted by $c_e > 0$, and the capacity vector by $C = (c_e)_{e \in E}$. We denote by W the set of origin-destination (O-D) pairs, and by $D = (d_w)_{w \in W}$ the demand vector, where $d_w > 0$ is the demand of traffic flow on O-D pair w .

Let P_w denote the set of available paths connecting O-D pair $w \in W$. Set $\sum_{w \in W} |P_w| = m$ and $P = \bigcup_{w \in W} P_w$. For a given path $k \in P_w$, x_k is called the traffic flow, and $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ is called a path flow.

The arc flow needs to meet the capacity constraints $0 \leq v_e \leq c_e$ for all $e \in E$, where $v_e = \sum_{w \in W} \sum_{k \in P_w} \delta_{ek} x_k$ and

$$\delta_{ek} = \begin{cases} 1, & \text{if path } k \text{ includes arc } e, \\ 0, & \text{otherwise.} \end{cases}$$

The path flow vector x needs to satisfy the demand constraints $\sum_{k \in P_w} x_k = d_w$.

Let $v = (v_e)$, $e \in E$ and $g^e(v) = (g_1(v), g_2(v), \dots, g_s(v))$ be the cost function for each arc e , where $g_i(v) = [g_i^L(v), g_i^R(v)]$ ($i = 1, 2, \dots, s$) denotes an interval-valued cost function. It could be a time interval-valued cost function or a money interval-valued cost function, etc.

For each path k , let $F^k(x) = (f_1^k(x), \dots, f_s^k(x))$ be a cost function, where $f_i^k(x) = [f_i^{kL}(v), f_i^{kR}(v)]$ ($i = 1, 2, \dots, s$) is an interval-valued cost function. Then, $F^k(x) = \sum_{e \in E} \delta_{ek} g^e(v)$. We consider the case where $s = 2$ and the following minimum cost flow problem for the traffic network equilibrium:

$$\begin{aligned} & \text{minimize } \sum_{w \in W} \sum_{k \in P_w} F^k(x) = (f_1^k(x), f_2^k(x)) \\ & \text{subject to } \begin{cases} \sum_{k \in P_w} x_k = d_w, & \text{for all } w \in W, \\ c_e \geq \sum_{w \in W} \sum_{k \in P_w} \delta_{ek} x_k \geq 0, & \text{for all } e \in E, \\ x_k \in \mathbb{Z}, & \text{for all } k \in P_w, w \in W. \end{cases} \end{aligned}$$

For the above set optimization problem, Xu et al. [36] used a fixed order cone to compare the outcome of two interval-valued functions. However, in practical applications, decision-makers select different preferences to compare the outcome of the set-valued mappings. For example, for the cost vector $(s, t) \in \mathbb{R}_+^2$, where s denotes the monetary cost and t represents the time cost, we define the set-valued mapping to characterize the decision-maker's preferences:

$$\mathcal{D}(\alpha, \beta) = \begin{cases} \mathbb{R}_+^2, & \text{for } \Delta s \leq M_1 \text{ and } \Delta t \leq M_2, \\ (s_2 - s_1, 0) + \mathbb{R}_+^2, & \text{for } \Delta s \leq M_1 \text{ and } \Delta t > M_2, \\ (0, t_2 - t_1) + \mathbb{R}_+^2, & \text{for } \Delta s > M_1 \text{ and } \Delta t \leq M_2, \\ (M_1, M_2) + \mathbb{R}_+^2, & \text{for } \Delta s > M_1 \text{ and } \Delta t > M_2, \end{cases}$$

where $\alpha = (s_1, t_1) \in \mathbb{R}_+^2$, $\beta = (s_2, t_2) \in \mathbb{R}_+^2$, $\Delta s = |s_2 - s_1|$ and $\Delta t = |t_2 - t_1|$, and $M_1 > 0$ and $M_2 > 0$ are the preset acceptable thresholds for cost differences by the decision-maker.

If the differences in monetary costs and time costs are both within the acceptable range (that is, $\Delta s \leq M_1$ and $\Delta t \leq M_2$), then $\mathcal{D}(\alpha, \beta) = \mathbb{R}_+^2$ is used for a comprehensive comparison. If $\Delta s \leq M_1$ but $\Delta t > M_2$, then the ordered set $\mathcal{D}(\alpha, \beta) = (s_2 - s_1, 0) + \mathbb{R}_+^2$ is used to eliminate the impact of monetary costs, focusing primarily on comparing time costs. If $\Delta s > M_1$ and $\Delta t \leq M_2$, then the ordered set $\mathcal{D}(\alpha, \beta) = (0, t_2 - t_1) + \mathbb{R}_+^2$ is used to compare the monetary costs. If both cost differences exceed the acceptable ranges (that is, $\Delta s > M_1$ and $\Delta t > M_2$), then the ordered set $\mathcal{D}(\alpha, \beta) = (M_1, M_2) + \mathbb{R}_+^2$ is used to compare the two given elements.

According to the given ordered set, if $\alpha \in \beta - \mathcal{D}(\alpha, \beta)$, we argue that α is superior to β .

Inspired by Example 7.1 in [36], where a fixed ordered set is considered to defined the set order relation, we present a specific example to illustrate the above set optimization problem where the decision-maker's preferences change according to practical circumstances.

Example 6.1. Consider the network problem depicted in Figure 1. $N = \{1, 2, 3, 4\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$, $C = (3, 2, 1, 4, 3)$, $W = \{(1, 3), (4, 3)\}$, $D = (3, 4)$. Let $v_i = v_{e_i}$, where v_{e_i}

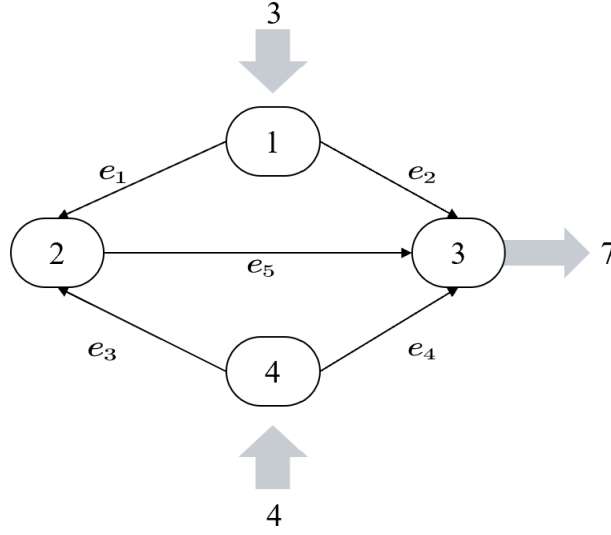


FIGURE 1. Network topology for Example 6.1

represents the flow of arc e_i . The cost functions of arcs from $\mathbb{R}^5$ to $2^{\mathbb{R}}$ are defined as

$$\begin{aligned} g_1^{e_1}(v) &= [v_1, 1.5v_1], & g_2^{e_1}(v) &= [v_1, 2v_1], \\ g_1^{e_2}(v) &= [0.5v_2, 2.5v_2], & g_2^{e_2}(v) &= [2 - (v_2 - 2)^2, 4 - 2(v_2 - 2)^2], \\ g_1^{e_3}(v) &= [1.5v_3, 2v_3], & g_2^{e_3}(v) &= [v_3, 2v_3], \\ g_1^{e_4}(v) &= [v_4, 2v_4], & g_2^{e_4}(v) &= [v_4, 2v_4], \\ g_1^{e_5}(v) &= [v_1 + v_5, 2v_1 + 2v_5], & g_2^{e_5}(v) &= [v_5, 2v_5], \end{aligned}$$

where $v = (v_1, v_2, v_3, v_4, v_5)$. Then,

$$\begin{aligned} f_1^1(x) &= g_1^{e_1}(v) + g_1^{e_5}(v) = [3v_1 + v_5, 4.5v_1 + 2v_5], & f_2^1(x) &= g_2^{e_1}(v) + g_2^{e_5}(v) = [2v_1 + v_3, 4v_1 + 2v_3], \\ f_1^2(x) &= g_1^{e_2}(v) = [0.5v_2, 2.5v_2], & f_2^2(x) &= g_2^{e_2}(v) = [2 - (v_2 - 2)^2, 4 - 2(v_2 - 2)^2], \\ f_1^3(x) &= g_1^{e_3}(v) + g_1^{e_5}(v) = [2v_1 + 2.5v_3, 3v_1 + 4v_3], & f_2^3(x) &= g_2^{e_3}(v) + g_2^{e_5}(v) = [v_1 + 2v_3, 2v_1 + 4v_3], \\ f_1^4(x) &= g_1^{e_4}(v) = [v_4, 2v_4], & f_2^4(x) &= g_2^{e_4}(v) = [v_4, 2v_4], \end{aligned}$$

where $x = (x_1, x_2, x_3, x_4)$ and $x_1 = v_1, x_2 = v_2, x_3 = v_3, x_4 = v_4, v_5 = v_1 + v_3$. Let $y_1 = (s_1, t_1)$, $y_2 = (s_2, t_2) \in \mathbb{R}^2$, $k_0 = (1, 1)$ and

$$\mathcal{D}(y_1, y_2) = \begin{cases} \mathbb{R}_+^2, & \text{for } \Delta s \leq 2 \text{ and } \Delta t \leq 4, \\ (s_2 - s_1, 0) + \mathbb{R}_+^2, & \text{for } \Delta s \leq 2 \text{ and } \Delta t > 4, \\ (0, t_2 - t_1) + \mathbb{R}_+^2, & \text{for } \Delta s > 2 \text{ and } \Delta t \leq 4, \\ (2, 4) + \mathbb{R}_+^2, & \text{for } \Delta s > 2 \text{ and } \Delta t > 4, \end{cases}$$

where $\Delta s = |s_1 - s_2|$ and $\Delta t = |t_1 - t_2|$. Then, the minimum cost flow problem of this example can be expressed as follows:

$$\text{Minimize } \Phi(x) = (\Phi_1(x), \Phi_2(x)) = f^1(x) + f^2(x) + f^3(x) + f^4(x) \quad (\text{SOP1})$$

subject to

$$\begin{cases} 0 \leq x_1 \leq 3, \\ 0 \leq x_2 \leq 2, \\ 0 \leq x_3 \leq 1, \\ 0 \leq x_4 \leq 4, \\ 0 \leq x_1 + x_3 \leq 3, \\ x_1 + x_2 = 3, \\ x_3 + x_4 = 4, \\ x_i \in \mathbb{Z}, i = 1, 2, 3, 4. \end{cases}$$

where $x = (x_1, x_2, x_3, x_4)$.

Let S denote the feasible region. Then,

$$S = \{(3, 0, 0, 4), (2, 1, 0, 4), (2, 1, 1, 3), (1, 2, 0, 4), (1, 2, 1, 3)\}.$$

We obtain

$$\begin{aligned} \Phi_1(x) &= f_1^1(x) + f_1^2(x) + f_1^3(x) + f_1^4(x) \\ &= [6v_1 + 0.5v_2 + 3.5v_3 + v_4, 9.5v_1 + 2.5v_2 + 6v_3 + 2v_4] \end{aligned}$$

and

$$\begin{aligned} \Phi_2(x) &= f_2^1(x) + f_2^2(x) + f_2^3(x) + f_2^4(x) \\ &= [3v_1 + 2 - (v_2 - 2)^2 + 3v_3 + v_4, 6v_1 + 4 - 2(v_2 - 2)^2 + 6v_3 + 2v_4]. \end{aligned}$$

Thus,

$$\Phi_1(x) = [6x_1 + 0.5x_2 + 3.5x_3 + x_4, 9.5x_1 + 2.5x_2 + 6x_3 + 2x_4]$$

and

$$\Phi_2(x) = [3x_1 + 2 - (x_2 - 2)^2 + 3x_3 + x_4, 6x_1 + 4 - 2(x_2 - 2)^2 + 6x_3 + 2x_4].$$

Let $\bar{x} = (1, 2, 0, 4)$. Through a simple calculation, we obtain that for any $x \in S \setminus \{(1, 2, 0, 4)\}$,

$$h(\Phi(x), \Phi(\bar{x})) = \inf \{ \alpha \in \mathbb{R} : \Phi(x) \preceq_{\mathcal{D}}^u \Phi(\bar{x}) + \alpha k_0 \} \geq 4 > 0.$$

Then, it follows from Theorem 5.1 that $(1, 2, 0, 4) \in \text{Min}(\Phi, S, \mathcal{D})$. Actually, we can verify that $\text{Min}(\Phi, S, \mathcal{D}) = \{(1, 2, 0, 4)\}$. In fact, if $\bar{x} = (1, 2, 0, 4)$, we observe that $\Phi(\bar{x}) = [11, 22.5] \times [9, 18]$ and for any $x \in S \setminus \{\bar{x}\}$, $\Phi(x) \not\preceq_{\mathcal{D}}^u \Phi(\bar{x})$; if $\bar{x} \in S \setminus \{(1, 2, 0, 4)\}$, taking $\hat{x} = (1, 2, 0, 4)$, we observe that $\Phi(\hat{x}) \preceq_{\mathcal{D}}^u \Phi(\bar{x})$ but $\Phi(\bar{x}) \not\preceq_{\mathcal{D}}^u \Phi(\hat{x})$. Then, $\text{Min}(\Phi, S, \mathcal{D}) = \{(1, 2, 0, 4)\}$.

6.2. An application in medical image registration. The problem of image registration aims to find an optimal spatial transformation that aligns two given data sets (images) such that their corresponding structural or functional information is spatially consistent.

Let T denote the set of admissible transformations (e.g., rigid transformations, affine transformations, non-rigid deformations). Let $H \subseteq \mathbb{R}^2$ and $J \subseteq \mathbb{R}^2$ (for 2D images, such as X-ray or angiographic images) represent the two images to be registered.

As described in [11], to quantify the alignment quality between the transformed image $H \circ t$ (where $\circ$ denotes the composition of the image H with transformation t) and the reference image J , we introduce a set of set-valued similarity measures: $f_i : T \times 2^{\mathbb{R}^2} \times 2^{\mathbb{R}^2} \rightrightarrows \mathbb{R}$, here $f_i(t, H, J) \subseteq \mathbb{R}$ denotes the collection of all possible values of the i -th similarity metric (e.g., sum of squared differences, mutual information, or normalized cross-correlation) under transformation t .

Let $F(t) = (f_1(t, H, J), f_2(t, H, J), \dots, f_m(t, H, J))$. For $m = 2$, we consider the following image registration problem:

$$\text{Minimize } F(t) := (f_1(t, H, J), f_2(t, H, J)), \quad \text{subject to } t \in T, \quad (\text{SOP2})$$

where $f_1(t, H, J)$ represents the sum of squared differences and $f_2(t, H, J)$ represents normalized cross-correlation.

Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2) \in \cup_{t \in T} F(t)$. Using a fixed order cone to compare α and β is indeed intuitive, but it only applies to the ideal scenario where “the priorities of the two objectives are completely consistent in all situations”.

Actually, for the images of cancer patients, the “tumor region” focuses on structural matching and needs to be measured using the normalized cross-correlation metric, while the “background region” focuses on grayscale consistency. If α and β are both the alignment quality of tumor region, then we mainly focus on comparing α_2 and β_2 . Therefore, a variable ordered set can give a better cope for actual situations. For example, consider $\mathcal{D}(\alpha, \beta) = (\beta_1 - \alpha_1, 0) + \mathbb{R}_+^2$, if β is the alignment result in the tumor area; $\mathcal{D}(\alpha, \beta) = (0, \beta_2 - \alpha_2) + \mathbb{R}_+^2$, if neither α nor β is alignment result in the tumor area; $\mathcal{D}(\alpha, \beta) = \mathbb{R}_+^2$, if β is not the alignment result in the tumor area but α is the alignment result in the tumor area.

We now use the variable ordered set introduced in [11] to illustrate the result of this paper.

The set-valued mapping $\mathcal{D}_1 : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined as

$$\mathcal{D}_1(y) := \left\{ d \in \mathbb{R}^2 : \sum_{i=1}^2 \text{sign}(d_i) \omega_i(y) \geq 0 \right\}, \quad \text{for all } y \in \mathbb{R}^2,$$

where $\omega(y) = (\omega_1(y), \omega_2(y)) \in \mathbb{R}_+^2$ is a non-negative weight vector which is assigned by the decision-maker and $\text{sign}(d_i) = \begin{cases} 1, & \text{if } d_i > 0, \\ 0, & \text{if } d_i = 0, \\ -1, & \text{if } d_i < 0. \end{cases}$

For a given $\bar{t} \in T$, we use the set relation $\prec_{\mathcal{D}}^u$ to study the problem (SOP2). $\bar{t}$ is called a weakly optimal solution of problem (SOP2) if $\bar{t} \in \text{WMin}(F, T, \mathcal{D}_1)$.

Define the mapping $\mathcal{A}_{\bar{t}} : T \rightarrow \mathbb{R}$ by $\mathcal{A}_{\bar{t}}(t) := h(F(t), F(\bar{t}))$ for all $t \in T$. Let

$$\begin{aligned} \mathcal{K}(\bar{t}) &:= \{u \in \mathbb{R} : u = -h(F(t), F(\bar{t})), t \in T\}, \\ \mathcal{H} &:= \{u \in \mathbb{R} : u \geq 0\}, \\ \mathcal{H}' &:= \{u \in \mathbb{R} : u > 0\}. \end{aligned}$$

Proposition 6.1. Let $\bar{t} \in T$, $k_0 = (k_{01}, k_{02}) \in \mathbb{R}_{++}^2$. Assume that there exists $\bar{y} \in F(\bar{t})$ such that for any $y \in F(\bar{t})$, $y - \bar{y} \notin \text{int} \mathcal{D}_1(\bar{y})$, then $\bar{t} \in \text{WMin}(F, T, \mathcal{D}_1)$ if and only if $\mathcal{K}(\bar{t}) \cap \mathcal{H}' = \emptyset$, that is, $h(F(t), F(\bar{t})) \geq 0$ for all $t \in T$, where

$$h(F(t), F(\bar{t})) = \sup_{y \in F(t)} \inf_{\bar{y} \in F(\bar{t})} \inf \{ \alpha \in \mathbb{R} : y \in \bar{y} + \alpha k_0 - \mathcal{D}_1(y) \},$$

and

$$\begin{aligned} &\inf \{ \alpha \in \mathbb{R} : y \in \bar{y} + \alpha k_0 - \mathcal{D}_1(y) \} \\ &= \begin{cases} \frac{y_1 - \bar{y}_1}{k_{01}}, & \text{if } (\omega_1(y) > 0, \omega_2(y) = 0) \text{ or } \omega_1(y) \geq \omega_2(y) > 0, \\ \frac{y_2 - \bar{y}_2}{k_{02}}, & \text{if } (\omega_1(y) = 0, \omega_2(y) > 0) \text{ or } \omega_2(y) \geq \omega_1(y) > 0. \end{cases} \end{aligned}$$

Proof. As in [11], for each $\omega = (\omega_1, \omega_2) \in \mathbb{R}_+^2$ and for each $y \in \mathbb{R}^2$, we have

$$\mathcal{D}_1(y) = \begin{cases} \{d \in \mathbb{R}^2 : d_1 \geq 0, d_2 \in \mathbb{R}\}, & \text{if } \omega_1(y) > 0, \omega_2(y) = 0, \\ \{d \in \mathbb{R}^2 : d_1 \in \mathbb{R}, d_2 \geq 0\}, & \text{if } \omega_1(y) = 0, \omega_2(y) > 0, \\ \{d \in \mathbb{R}^2 : (d_1 \geq 0, d_2 \geq 0) \text{ or } (d_1 < 0, d_2 > 0)\}, & \text{if } \omega_2(y) \geq \omega_1(y) > 0, \\ \{d \in \mathbb{R}^2 : (d_1 \geq 0, d_2 \geq 0) \text{ or } (d_1 > 0, d_2 < 0)\}, & \text{if } \omega_1(y) \geq \omega_2(y) > 0. \end{cases}$$

Since $k_0 \in \mathbb{R}_{++}^2$, (H_1) holds. In the view of Theorem 5.2, we obtain $\bar{t} \in \text{WMin}(F, T, \mathcal{D}_1)$ if and only if $\mathcal{H}(\bar{t}) \cap \mathcal{H}' = \emptyset$. Then, from the definition of $\mathcal{H}(\bar{t})$ and $\mathcal{H}'$, $\bar{t} \in \text{WMin}(F, T, \mathcal{D}_1)$ if and only if for any $t \in T$, $h(F(t), F(\bar{t})) \geq 0$. In the view of Corollary 3.3, we have

$$h(F(t), F(\bar{t})) = \sup_{y \in F(t)} \inf_{\bar{y} \in F(\bar{t})} \inf \{\alpha \in \mathbb{R} : y \in \bar{y} + \alpha k_0 - \mathcal{D}_1(y)\}.$$

Observe that

$$\inf \{\alpha \in \mathbb{R} : y \in \bar{y} + \alpha k_0 - \mathcal{D}_1(y)\} = \inf \left\{ \alpha \in \mathbb{R} : \sum_{i=1}^2 \text{sign}(\bar{y} + \alpha k_0 - y)_i \omega_i(y) \geq 0 \right\}.$$

From Proposition 5 in [29], we have

$$\inf \{\alpha \in \mathbb{R} : y \in \bar{y} + \alpha k_0 - \mathcal{D}_1(y)\} = \begin{cases} \frac{y_1 - \bar{y}_1}{k_{01}}, & \text{if } (\omega_1(y) > 0, \omega_2(y) = 0) \text{ or } \omega_1(y) \geq \omega_2(y) > 0, \\ \frac{y_2 - \bar{y}_2}{k_{02}}, & \text{if } (\omega_1(y) = 0, \omega_2(y) > 0) \text{ or } \omega_2(y) \geq \omega_1(y) > 0. \end{cases}$$

□

Remark 6.1. To solve the medical image registration problem, in [11, Section 2.2], it is necessary to resort to the dual cone of the order cone $D(y)$. Under the convex cone structure assumption, the linear function $l \in D(y)^*$ (the dual of the order cone) is adopted to characterize non-dominated elements. Building on Proposition 6.1 and in conjunction with Theorem 5.2, we propose a solution method for the medical image registration problem based on nonlinear scalarization. This method directly embeds the order set $D_1(y)$ into the scalarization process: it neither require to pre-compute the order set $D_1(y)$, nor any convexity.

7. CONCLUSIONS

In this paper, we established optimality conditions for set optimization problems with a variable structure, which are different from the existing results in the literature in the following aspects:

- We introduced a scalarizing function with respect to a binary variable structure and derived its properties, whereas existing scalarization functions are typically induced by fixed ordering structures (see [6, 7]). As an application, we established necessary and sufficient optimality conditions for set optimization problems.

- By utilizing ISA and the introduced scalarization function with respect to a variable structure, we showed the separation between two suitable sets in image space to obtain the optimality conditions for the constrained set optimization problem, while the suitable sets are usually defined by fixed ordering structures, see [21, 40]. We also showed the relationship between the existence of a nonlinear separation and a saddle point of a generalized Lagrangian set-valued function. As applications, we investigated traffic network problems with interval-valued cost functions and a medical image registration problem.

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