

VOLTERRA INTEGRAL OPERATOR ON α -DIRICHLET-MORREY SPACES

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Abstract. In this paper, a new family of function spaces, referred to as α -Dirichlet-Morrey spaces and denoted by $\mathcal{D}_{\beta,\lambda}^\alpha$, is introduced. For this function space $\mathcal{D}_{\beta,\lambda}^\alpha$, several equivalent characterizations are established. Furthermore, a systematic and in-depth study is carried out on the Volterra integral operator T_g acting on $\mathcal{D}_{\beta,\lambda}^\alpha$, and detailed analyses of its boundedness, compactness, operator norm, essential norm, and strict singularity are included.

Keywords. Essential norm; Morrey space; Strict singularity; Volterra integral operator.

1. INTRODUCTION

The Morrey space, which was first introduced by Morrey Jr. in his foundational work [18] back in 1938, attracted the attention of numerous scholars in the field of analysis with great interest. This function space plays a pivotal and irreplaceable role in the in-depth study of partial differential equations and harmonic analysis. A typical and significant example is elliptic PDE solution analysis: critical solutions (e.g., Navier–Stokes weak solutions) do not belong to global L^p spaces but satisfy clear local integrability bounds, the very feature Morrey spaces describe. Due to this close connection with key PDE problems, Morrey spaces have been widely studied over eight decades, yielding abundant results and applications in PDE regularity theory.

In the open unit disc $\mathbb{D}$, the Morrey space $\mathcal{L}^{2,\lambda}(\mathbb{D})$ was first formally introduced and systematically studied by Wu and Xie [25], who established its basic structural properties and embedding relations. Later, scholars further explored related problems (see, e.g., [7, 11, 13, 23]). Based on $\mathcal{L}^{2,\lambda}(\mathbb{D})$, Yang and Liu [27] generalized it to the Bergman-Morrey space, rigorously defining it and examining the boundedness, compactness, and essential norm of two integral operators, with the results bridging Morrey and Bergman space theory. Notably, in recent decades, various Morrey-type spaces (e.g., Dirichlet–Morrey, Besov–Morrey) have been continuously introduced, attracting much interest in complex analysis. For further background and operator theory developments on these spaces, we refer to [9, 10, 23, 25, 28, 32] and the references therein.

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For a real parameter $\alpha > 0$, analytic function f , and holomorphic self-map $\phi : \mathbb{D} \rightarrow \mathbb{D}$, the weighted composition $f \circ_\alpha \phi$ (Zhao [29]) is defined as

$$f \circ_\alpha \phi(z) = \int_0^z f'(\phi(w)) (\phi'(w))^\alpha dw + f(\phi(0)) (\phi'(0))^{\alpha-1}, \quad z \in \mathbb{D},$$

where $(\phi'(0))^{\alpha-1}$ takes its principal value. Assume that $X \subset H(\mathbb{D})$ has norm/seminorm $\|\cdot\|_X$. X is an α -Möbius invariant space if there exists $C > 0$ (independent of f, ϕ) such that $f \circ_\alpha \phi \in X$ and $\|f \circ_\alpha \phi\|_X \leq C\|f\|_X$. Zhao [29] introduced this concept and showed the α -Bloch space $\mathcal{B}^\alpha$ is the largest such space; further results are in [3, 29, 1, 30]. When $\alpha = 1$, $f \circ_\alpha \phi$ reduces to the ordinary composition $f \circ \phi$, and X is called a Möbius invariant space. In complex function theory, these spaces are fundamental, including H^∞ , Bloch space $\mathcal{B}$, $BMOA$, $\mathcal{Q}_p$, $\mathcal{Q}_K$, and $F(p, p-2, s)$; they are indispensable in analytic function space theory; see, e.g., [1, 3, 15, 16, 24, 26, 29, 30] for more. In [33], inspired by the notion of α -Möbius invariant spaces and Bergman-Morrey space, Zhu and Yang introduced the α -Bergman-Morrey space $A^{p, \lambda, \alpha}$. The space $A^{p, \lambda, \alpha}$ generalizes several known function spaces in a natural way: For $\alpha = 1$, it recovers the Bergman-Morrey space $A^{p, \lambda}$ introduced in [27]; for $\lambda = 2$, it reduces to the α -Bloch space $\mathcal{B}^\alpha$; for $\lambda = 0$, it simply becomes the standard Bergman space A^p . Let $g \in H(\mathbb{D})$. The Volterra integral operator T_g is defined by $T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta$, $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. The operator T_g , first introduced by Pommerenke [22], constitutes one of the main objects in the operator theory of analytic function spaces. A key fact is that T_g is a natural and significant generalization of the classical Cesàro operator. Over the past four decades, some properties of T_g on various function spaces were extensively studied. For more results concerning T_g , we refer to [6, 7, 10, 11, 12, 13, 14, 17, 23, 27] and the references therein.

The purpose of this paper is to introduce the α -Dirichlet-Morrey space $\mathcal{D}_{\beta, \lambda}^\alpha$, and to investigate some fundamental properties of this space as well as the Volterra integral operator T_g acting on it. The organization of this article is as follows. In Section 2, we recall some notations and fundamental concepts and introduce the α -Dirichlet-Morrey space. Section 3 is dedicated to establishing various equivalent characterizations of the space $\mathcal{D}_{\beta, \lambda}^\alpha$, along with an analysis of the growth rate of functions belonging to this space. In Section 4, we characterize the boundedness, compactness, norm, essential norm and the strict singularity of the Volterra integral operator T_g acting on $\mathcal{D}_{\beta, \lambda}^\alpha$.

For convenience, we make some conventions on notation. We say that $h \lesssim g$ if there exists a constant C such that $h \leq Cg$. The symbol $h \asymp g$ means that $h \lesssim g \lesssim h$.

2. PRELIMINARIES

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $H(\mathbb{D})$ be the family of all analytic functions on $\mathbb{D}$. For $\beta > -1$ and $0 < p < \infty$, the weighted Dirichlet space $\mathcal{D}_\beta^p$ consists of those functions $f \in H(\mathbb{D})$ satisfying $\|f\|_{\mathcal{D}_\beta^p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\beta dA(z) < \infty$, where dA is the normalized area measure on $\mathbb{D}$. Two special cases are particularly worth noting: when $\beta = 1$ and $p = 2$, the space $\mathcal{D}_\beta^p$ coincides with the Hardy space H^2 ; if $\beta = p$, then $\mathcal{D}_\beta^p$ becomes the Bergman space A^p . For $f \in H(\mathbb{D})$ with $0 < p < \infty$, $-2 < q < \infty$, and $0 \leq s < \infty$, we say that f belongs to the space $F(p, q, s)$ if $\|f\|_{F(p, q, s)}^p = |f(0)|^p + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dA(z) < \infty$, where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius automorphism interchanging

0 and a . The space $F(p, q, s)$ was first introduced by Zhao in [29], and it is nontrivial provided $q + s > -1$. An important special case occurs when $q = p - 2$: If $s > 1$, then $F(p, p - 2, s)$ coincides with the classical Bloch space $\mathcal{B}$; When $p = 2$, it reduces to the Q_s space; For $p = 2$ and $s = 1$, it is exactly the $BMOA$ space.

Let $0 < \alpha < \infty$ and $-1 < \beta < 0 \leq \lambda \leq 1$. We say that an analytic function f belongs to $\mathcal{D}_{\beta, \lambda}^\alpha$ if

$$\|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha} = |f(0)| + \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \|f \circ \sigma_a - f \circ \sigma_a(0)\|_{\mathcal{D}_\beta^1} < \infty.$$

When $\lambda = 1$, this construction recovers the α -Möbius invariant space studied in [29, 30]. Note that when $\alpha = 1$, the space $\mathcal{D}_{\beta, \lambda}^\alpha$ coincides exactly with the $\mathcal{D}_{\beta, \lambda}$ space first introduced in [8]. In particular, when $\lambda = 1$ and $\alpha = 1$, the space $\mathcal{D}_{\beta, \lambda}^\alpha$ is just the space $F(1, -1, \beta + 1)$. We also define the little α -Dirichlet–Morrey space, denoted $\mathcal{D}_{\beta, \lambda}^{\alpha, 0}$, as the subspace of $H(\mathbb{D})$ consisting of those f for which $\lim_{|a| \rightarrow 1} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \|f \circ \sigma_a - f \circ \sigma_a(0)\|_{\mathcal{D}_\beta^1} = 0$. Let $s > 0$, and let μ denote a positive Borel measure defined on $\mathbb{D}$. The measure μ is said to be an s -Carleson measure if

$$\|\mu\|_s = \sup_{I \subset \partial \mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty.$$

Here, $S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1 - |I| \leq r < 1, e^{i\theta} \in I\}$. If $\lim_{|I| \rightarrow 0} \frac{\mu(S(I))}{|I|^s} = 0$, then μ is called a vanishing s -Carleson measure. In particular, the 1-Carleson measure coincides exactly with the classical Carleson measure, which plays a central role in both harmonic analysis and complex analysis (see [4, 5]).

Lemma 2.1. [21] *Let μ be a positive Borel measure on $\mathbb{D}$ and $s > 0$. Then μ is an s -Carleson measure if and only if $\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^\eta}{|1 - \bar{a}w|^{\eta+s}} d\mu(w) < \infty$ for every $\eta \in (0, \infty)$. The measure μ is a vanishing s -Carleson measure if and only if $\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} \frac{(1 - |a|^2)^\eta}{|1 - \bar{a}w|^{\eta+s}} d\mu(w) = 0$ for every $\eta \in (0, \infty)$.*

3. BASIC PROPERTIES OF α -DIRICHLET-MORREY SPACES

In this section, we turn our attention to a few fundamental properties that are inherent to the α -Dirichlet–Morrey spaces $\mathcal{D}_{\beta, \lambda}^\alpha$. These properties form the foundational basis for further analysis of such function spaces. To this end, we first state several auxiliary lemmas, which will be indispensable in our arguments. The following two integral estimates are of great importance in our proof.

Lemma 3.1. [31, Lemma 3.10] *Suppose that $a \in \mathbb{D}$, c is real, $t > -1$, and*

$$I_{c,t}(a) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^t}{|1 - \bar{a}w|^{2+t+c}} dA(w).$$

(a) *If $c < 0$, then as a function of a , $I_{c,t}(a)$ is bounded from above and bounded from below on $\mathbb{D}$.*

(b) *If $c > 0$, then $I_{c,t}(a) \asymp \frac{1}{(1 - |a|^2)^c}$, $|a| \rightarrow 1^-$.*

(c) *If $c = 0$, then $I_{0,t}(a) \asymp \log \frac{1}{1 - |a|^2}$, $|a| \rightarrow 1^-$.*

Lemma 3.2. [19, Lemma 2.5] *Let $s > -1$, $r, t > 0$. If $r < s + 2 < t$, then*

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^s}{|1 - \bar{a}z|^r |1 - \bar{b}z|^t} dA(z) \lesssim \frac{1}{|1 - \bar{a}b|^r (1 - |b|^2)^{t-s-2}}$$

where $a, b \in \mathbb{D}$.

First, we give several characterizations for functions in $\mathcal{D}_{\beta, \lambda}^\alpha$.

Theorem 3.1. *Let $f \in H(\mathbb{D})$, $-1 < \beta < 0 < \lambda \leq 1$ and $0 < \alpha < \infty$. Then the following statements are equivalent.*

- (i) $f \in \mathcal{D}_{\beta, \lambda}^\alpha$.
- (ii) $M_1 = \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(w)| (1 - |w|^2)^\beta |\sigma'_a(w)|^{\beta+2-\alpha} dA(w) < \infty$.
- (iii) $M_2 = \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(w)| (1 - |w|^2)^{\alpha-2} (1 - |\sigma_a(w)|^2)^{\beta+2-\alpha} dA(w) < \infty$.

Moreover, $\|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha} - |f(0)| = M_1 = M_2$.

Proof. To establish the equivalence relation (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), making the change of variable $w = \sigma_a(z)$, we can rigorously derive the following sequence of equivalent expressions

$$\begin{aligned} \|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha} - |f(0)| &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \|f \circ \sigma_a - f \circ \sigma_a(0)\|_{\mathcal{D}_\beta^1} \\ &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(\sigma_a(z))| (1 - |z|^2)^{\beta-\alpha} (1 - |\sigma_a(z)|^2)^\alpha dA(z) \\ &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(w)| (1 - |\sigma_a(w)|^2)^{\beta-\alpha} (1 - |w|^2)^\alpha |\sigma'_a(w)|^2 dA(w) \\ &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(w)| (1 - |w|^2)^{\alpha-2} (1 - |\sigma_a(w)|^2)^{\beta+2-\alpha} dA(w). \end{aligned}$$

The desired result follows. □

If we further impose the condition that $\alpha < 1 + \lambda(\beta + 1)$, we can get more characterizations as follows.

Theorem 3.2. *Let $f \in H(\mathbb{D})$, $-1 < \beta < 0 < \lambda \leq 1$ and $0 < \alpha < 1 + \lambda(\beta + 1)$. Then the following statements are equivalent.*

- (i) $f \in \mathcal{D}_{\beta, \lambda}^\alpha$.
- (ii) $M_3 = \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f'(w)| (1 - |w|^2)^\beta dA(w) < \infty$.
- (iii) $M_4 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(w)| (1 - |w|^2)^{\beta+\alpha-1-\beta\lambda-\lambda} (1 - |\sigma_a(w)|^2)^{1+\beta\lambda+\lambda-\alpha} dA(w) < \infty$.

Moreover, $\|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha} - |f(0)| \asymp M_3 \asymp M_4$.

Proof. (i) $\Rightarrow$ (ii). Let I be an arbitrary arc on the boundary $\partial \mathbb{D}$ of the unit disk $\mathbb{D}$. Let us first denote by ξ the center point of this arc I , and then define the point $b \in \mathbb{D}$ by setting

$b = (1 - |I|)\xi$. In view of $|1 - \bar{b}w| \asymp 1 - |b|^2 \asymp |I|$, $w \in S(I)$, we have

$$\begin{aligned} \|f\|_{\mathcal{D}_{\beta,\lambda}^\alpha} - |f(0)| &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(w)|(1 - |w|^2)^\beta |\sigma'_a(w)|^{\beta+2-\alpha} dA(w) \\ &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(w)|(1 - |w|^2)^\beta \frac{(1 - |a|^2)^{\beta+2-\alpha}}{|1 - \bar{a}w|^{2(\beta+2-\alpha)}} dA(w) \\ &\gtrsim \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f'(w)|(1 - |w|^2)^\beta dA(w), \end{aligned}$$

which implies the desired result.

(ii) $\Rightarrow$ (i). Let $d\mu_f(w) = |f'(w)|(1 - |w|^2)^\beta dA(w)$. We obtain

$$\sup_{I \subset \partial \mathbb{D}} \frac{\mu_f(S(I))}{|I|^{1+\beta\lambda+\lambda-\alpha}} = \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f'(w)|(1 - |w|^2)^\beta dA(w) < \infty.$$

By the definition of s -Carleson measures, we see that μ_f is $(1 + \beta\lambda + \lambda - \alpha)$ -Carleson measure.

By Lemma 2.1 we have

$$\begin{aligned} &\sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(w)|(1 - |w|^2)^\beta |\sigma'_a(w)|^{\beta+2-\alpha} dA(w) \\ &= \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} \frac{(1 - |a|^2)^{\beta+2-\alpha}}{|1 - \bar{a}w|^{2(\beta+2-\alpha)}} d\mu_f(w) \\ &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^{2\beta+3-\beta\lambda-\lambda-\alpha}}{|1 - \bar{a}w|^{2\beta+3-\beta\lambda-\lambda-\alpha+(1+\beta\lambda+\lambda-\alpha)}} d\mu_f(w) \\ &< \infty. \end{aligned}$$

(ii) $\Leftrightarrow$ (iii). Let $d\mu_f(z) = |f'(w)|(1 - |w|^2)^\beta dA(w)$. By Lemma 2.1, we obtain that μ_f is a $(1 + \beta\lambda + \lambda - \alpha)$ -Carleson measure if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(w)|(1 - |w|^2)^{\beta+\alpha-1-\beta\lambda-\lambda} (1 - |\sigma_a(w)|^2)^{1+\beta\lambda+\lambda-\alpha} dA(w) < \infty.$$

The proof is complete. $\square$

Remark 3.1. From the proof of Theorem 3.1, we see that a function $f \in \mathcal{D}_{\beta,\lambda}^\alpha$ if and only if

$$\sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \|f \circ \sigma_a\|_{\mathcal{D}_\beta^1} < \infty.$$

From the proof of Theorems 3.1, 3.2 and Lemma 2.1, we have the following characterization of the space $\mathcal{D}_{\beta,\lambda}^{\alpha,0}$. We omit the detail of the proof.

Theorem 3.3. Let $f \in H(\mathbb{D})$, $-1 < \beta < 0 < \lambda \leq 1$ and $0 < \alpha < \infty$. Then the following statements are equivalent.

- (i) $f \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$.
- (ii) $\lim_{|a| \rightarrow 1} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(w)|(1 - |w|^2)^\beta |\sigma'_a(w)|^{\beta+2-\alpha} dA(w) = 0$.
- (iii) $\lim_{|a| \rightarrow 1} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(w)|(1 - |w|^2)^{\alpha-2} (1 - |\sigma_a(w)|^2)^{\beta+2-\alpha} dA(w) = 0$.

Theorem 3.4. Let $f \in H(\mathbb{D})$, $-1 < \beta < 0 < \lambda \leq 1$ and $0 < \alpha < 1 + \lambda(\beta + 1)$. Then the following statements are equivalent.

- (i) $f \in \mathcal{D}_{\beta, \lambda}^{\alpha, 0}$.
- (ii) $\lim_{|I| \rightarrow 0} \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f'(w)|(1-|w|^2)^\beta dA(w) = 0$.
- (iii) $\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |f'(w)|(1-|w|^2)^{\beta+\alpha-1-\beta\lambda-\lambda} (1-|\sigma_a(w)|^2)^{1+\beta\lambda+\lambda-\alpha} dA(w) = 0$.

Proposition 3.1. *Let $-1 < \beta < 0 < \lambda \leq 1$ and $0 < \alpha < \infty$. Then $\mathcal{D}_{\beta, \lambda}^\alpha \subseteq \mathcal{D}_\beta^1$.*

Proof. Let $f \in \mathcal{D}_{\beta, \lambda}^\alpha$. By Theorem 3.1, we have

$$\begin{aligned} \infty &> \sup_{a \in \mathbb{D}} (1-|a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(w)|(1-|w|^2)^{\alpha-2} (1-|\sigma_a(w)|^2)^{\beta+2-\alpha} dA(w) \\ &\geq \int_{\mathbb{D}} |f'(w)|(1-|w|^2)^\beta dA(w). \end{aligned}$$

Consequently, $f \in \mathcal{D}_\beta^1$. This establishes the inclusion $\mathcal{D}_{\beta, \lambda}^\alpha \subseteq \mathcal{D}_\beta^1$. This completes the proof. $\square$

Proposition 3.2. *Let $-1 < \beta < 0 < \lambda \leq 1$ and $\max\left\{\frac{\beta}{2} + 1, (\beta+1)\lambda\right\} < \alpha \leq 1 + \lambda(\beta+1)$. Then, for $b \in \mathbb{D}$, $f_b(z) = \frac{(1-|b|^2)}{(1-\bar{b}z)^{(\beta+1)(1-\lambda)+\alpha}}$ for all $z \in \mathbb{D}$, belongs to $\mathcal{D}_{\beta, \lambda}^\alpha$. Moreover, $\|f_b\|_{\mathcal{D}_{\beta, \lambda}^\alpha} \lesssim 1$.*

Proof. Using Theorem 3.1 and Lemma 3.2, we obtain

$$\begin{aligned} &\|f_b\|_{\mathcal{D}_{\beta, \lambda}^\alpha} - |f_b(0)| \\ &= \sup_{a \in \mathbb{D}} (1-|a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'_b(w)|(1-|w|^2)^{\alpha-2} (1-|\sigma_a(w)|^2)^{\beta+2-\alpha} dA(w) \\ &\lesssim \sup_{a \in \mathbb{D}} (1-|a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} (1-|b|^2) \int_{\mathbb{D}} \frac{(1-|w|^2)^\beta dA(w)}{|1-\bar{b}w|^{(\beta+1)(1-\lambda)+\alpha+1} |1-\bar{a}w|^{2(\beta+2-\alpha)}} \\ &\lesssim \sup_{a \in \mathbb{D}} \frac{(1-|a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} (1-|b|^2)}{|1-\bar{a}b|^{2\beta+4-2\alpha} (1-|b|^2)^{\alpha-\beta\lambda-\lambda}} \\ &\lesssim 1. \end{aligned}$$

The proof is complete. $\square$

Proposition 3.3. *Let $-1 < \beta < 0 < \lambda \leq 1$ and $0 < \alpha < \infty$. If $f \in \mathcal{D}_{\beta, \lambda}^\alpha$, then, for $z \in \mathbb{D}$,*

$$|f(z)| \lesssim \begin{cases} \frac{\|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha}}{(1-|z|^2)^{(\beta+1)(1-\lambda)+\alpha-1}}, & \alpha > 1 - (\beta+1)(1-\lambda), \\ \log \frac{2}{1-|z|^2} \|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha}, & \alpha = 1 - (\beta+1)(1-\lambda), \\ \|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha}, & \alpha < 1 - (\beta+1)(1-\lambda). \end{cases}$$

Proof. For $a \in \mathbb{D}$ and $r > 0$, set $\mathbb{D}(a, r) = \{w \in \mathbb{D} : \beta(a, w) < r\}$, where $\beta(\cdot, \cdot)$ denotes the Bergman metric. Note that (see [31]) $\frac{(1-|a|^2)^2}{|1-\bar{a}w|^4} \asymp \frac{1}{(1-|w|^2)^2} \asymp \frac{1}{(1-|a|^2)^2}$, $w \in \mathbb{D}(a, r)$, we have

$$\begin{aligned} |f'(a)| &\lesssim \frac{1}{(1-|a|^2)^\alpha} \int_{\mathbb{D}(a, r)} |f'(w)|(1-|w|^2)^{\alpha-2} dA(w) \\ &\lesssim \frac{1}{(1-|a|^2)^\alpha} \int_{\mathbb{D}(a, r)} |f'(w)|(1-|w|^2)^{\alpha-2} (1-|\sigma_a(w)|^2)^{\beta+2-\alpha} dA(w) \\ &\lesssim \frac{1}{(1-|a|^2)^{(\beta+1)(1-\lambda)+\alpha}} \|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha}. \end{aligned}$$

When $\alpha > 1 - (\beta + 1)(1 - \lambda)$, we have

$$\begin{aligned} |f(a) - f(0)| &= \left| a \int_0^1 f'(at) dt \right| \lesssim |a| \int_0^1 |f'(at)| dt \\ &\lesssim \|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha} \int_0^1 \frac{|a| dt}{(1-|at|^2)^{(\beta+1)(1-\lambda)+\alpha}} \lesssim \frac{\|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha}}{(1-|a|^2)^{(\beta+1)(1-\lambda)+\alpha-1}}. \end{aligned}$$

Because the point a is arbitrary, we obtain

$$|f(z)| \lesssim \frac{\|f\|_{\mathcal{D}_{\beta, \lambda}^\alpha}}{(1-|z|^2)^{(\beta+1)(1-\lambda)+\alpha-1}}, \quad z \in \mathbb{D}.$$

Other cases can be proven similarly. The proof is complete. $\square$

The following two propositions are devoted to illustrate the relationship between polynomials and $\mathcal{D}_{\beta, \lambda}^{\alpha, 0}$. We show that $\mathcal{D}_{\beta, \lambda}^{\alpha, 0}$ is the closure of all polynomials in $\mathcal{D}_{\beta, \lambda}^\alpha$.

Proposition 3.4. *Let $-1 < \beta < 0 < \lambda \leq 1$. If $0 < \alpha < 1 + \lambda(\beta + 1)$, then $\mathcal{D}_{\beta, \lambda}^{\alpha, 0}$ contains all polynomials.*

Proof. Let g be a polynomial. Then it is clear that there exists a positive constant M such that $\sup_{w \in \mathbb{D}} |g'(w)| \leq M < \infty$. We have

$$\begin{aligned} &\lim_{|a| \rightarrow 1} (1-|a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |g'(w)|(1-|w|^2)^{\alpha-2} (1-|\sigma_a(w)|^2)^{\beta+2-\alpha} dA(w) \\ &\leq M \lim_{|a| \rightarrow 1} (1-|a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} \int_{\mathbb{D}} \frac{(1-|w|^2)^\beta}{|1-\bar{a}w|^{2(\beta+2-\alpha)}} dA(w). \end{aligned}$$

When $1 + \frac{\beta}{2} < \alpha < 1 + \lambda(\beta + 1)$, we find by Lemma 3.1(a) that

$$M \lim_{|a| \rightarrow 1} (1-|a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} \int_{\mathbb{D}} \frac{(1-|w|^2)^\beta}{|1-\bar{a}w|^{2(\beta+2-\alpha)}} dA(w) = 0.$$

By Theorem 3.3 we conclude that $g \in \mathcal{D}_{\beta, \lambda}^{\alpha, 0}$. When $\alpha < 1 + \frac{\beta}{2}$ and $\alpha = 1 + \frac{\beta}{2}$, we obtain

$$M \lim_{|a| \rightarrow 1} (1-|a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} \int_{\mathbb{D}} \frac{(1-|w|^2)^\beta}{|1-\bar{a}w|^{2(\beta+2-\alpha)}} dA(w) = 0,$$

by applying Lemma 3.1(b) and Lemma 3.1(c), respectively. The proof is complete. $\square$

Much like the proof given for Proposition 2.15 in [29], we arrive at the proposition below. We omit the full details of its proof here.

Proposition 3.5. *Let $-1 < \beta < 0 < \lambda \leq 1$ and $0 < \alpha < 1 + \lambda(\beta + 1)$. Then $\lim_{r \rightarrow 1} \|g_r - g\|_{\mathcal{D}_{\beta,\lambda}^\alpha} = 0$, where $g_r(z) = g(rz)$. In particular, $\mathcal{D}_{\beta,\lambda}^{\alpha,0}$ is the closure of all polynomials in $\mathcal{D}_{\beta,\lambda}^\alpha$.*

4. VOLTERRA INTEGRAL OPERATOR ON $\mathcal{D}_{\beta,\lambda}^\alpha$

In this section, we establish the boundedness, compactness, norm, essential norm and strictly singular characteristics of the Volterra integral operator $T_g : \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha$.

To state and prove our main results in this section, we need the following lemmas.

Lemma 4.1. [2] *Let $0 < p < \infty, s > -1, t \geq 0$ such that $t < 2 + s$. If $f \in H(\mathbb{D})$, then*

$$\int_{\mathbb{D}} |f(w) - f(0)|^p \frac{(1 - |w|^2)^s}{|1 - \bar{a}w|^t} dA(w) \lesssim \int_{\mathbb{D}} |f'(w)|^p \frac{(1 - |w|^2)^{s+p}}{|1 - \bar{a}w|^t} dA(w), \quad a \in \mathbb{D}.$$

Lemma 4.2. [15, Theorem 9] *Let $-1 < \beta < 0$ and μ be a positive Borel measure on $\mathbb{D}$. Then μ is a $(\beta + 1)$ -Carleson measure if and only if, for all functions $f \in \mathcal{D}_\beta^1$, $\int_{\mathbb{D}} |f(w)| d\mu(w) \lesssim \|f\|_{\mathcal{D}_\beta^1}$.*

Now we are in a position to state and prove the main result in this section.

Theorem 4.1. *Let $g \in H(\mathbb{D})$, $-1 < \beta < 0 < \lambda \leq 1$ and*

$$\max \left\{ \frac{\beta}{2} + 1, \lambda(\beta + 1) - \beta \right\} < \alpha < \min \left\{ 1 + \lambda(\beta + 1), \frac{\beta}{2} + \frac{3}{2} \right\}.$$

Then $T_g : \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha$ is bounded if and only if $g \in F(1, -1, \beta + 1)$. Moreover, $\|T_g\| \asymp \|g\|_{F(1, -1, \beta + 1)}$.

Proof. Suppose that $T_g : \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha$ is bounded. For any $I \subseteq \partial\mathbb{D}$, let $a = (1 - |I|)\zeta \in \mathbb{D}$, where ζ is the center of I . Set

$$f_a(w) = \frac{(1 - |a|^2)}{(1 - \bar{a}w)^{(\beta+1)(1-\lambda)+\alpha}}, \quad w \in \mathbb{D}.$$

By Proposition 3.2, $f_a \in \mathcal{D}_{\beta,\lambda}^\alpha$ and $\|f_a\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \lesssim 1$. Thus $\|T_g f_a\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \lesssim \|T_g\| \|f_a\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \lesssim \|T_g\|$. By Lemma 3.2 and the fact that $|I| = 1 - |a| \asymp |1 - \bar{a}w|$, $w \in S(I)$, we have

$$\begin{aligned} \frac{1}{|I|^{\beta+1}} \int_{S(I)} |g'(w)|(1 - |w|^2)^\beta dA(w) &\lesssim \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f_a(w)| |g'(w)|(1 - |w|^2)^\beta dA(w) \\ &= \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |(T_g f_a)'(w)|(1 - |w|^2)^\beta dA(w) \\ &\lesssim \|T_g f_a\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \lesssim \|T_g\| < \infty. \end{aligned}$$

By the arbitrariness of I , we see that $g \in F(1, -1, \beta + 1)$. Moreover, $\|g\|_{F(1, -1, \beta + 1)} \lesssim \|T_g\|$.

Conversely, assume that $g \in F(1, -1, \beta + 1)$. From [29], we have

$$\|g\|_{F(1, -1, \beta + 1)} \asymp \sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^{\beta+1}} \int_{S(I)} |g'(z)|(1 - |z|^2)^\beta dA(z) \asymp \sup_{I \subset \partial\mathbb{D}} \frac{\mu_g(S(I))}{|I|^{\beta+1}},$$

where $d\mu_g = |g'(z)|(1 - |z|^2)^\beta dA(z)$. This means that μ_g is a $(\beta + 1)$ -Carleson measure. For any $I \subset \partial\mathbb{D}$, let ξ be the center of I and $a = (1 - |I|)\xi$. For any $f \in \mathcal{D}_{\beta,\lambda}^\alpha$,

$$\begin{aligned} & \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f(z)| d\mu_g(z) \\ & \lesssim \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f(a)| d\mu_g(z) + \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f(z) - f(a)| d\mu_g(z) \\ & = G_1 + G_2. \end{aligned}$$

By Proposition 3.3, one has

$$\begin{aligned} G_1 &= \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f(a)| d\mu_g(z) \\ &\lesssim \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} \frac{\|f\|_{\mathcal{D}_{\beta,\lambda}^\alpha}}{(1 - |a|^2)^{(\beta+1)(1-\lambda)+\alpha-1}} d\mu_g(z) \\ &\lesssim \|f\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \frac{\mu_g(S(I))}{|I|^{\beta+1}} \lesssim \sup_{I \subset \partial\mathbb{D}} \frac{\mu_g(S(I))}{|I|^{\beta+1}} \|f\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \\ &\asymp \|g\|_{F(1,-1,\beta+1)} \|f\|_{\mathcal{D}_{\beta,\lambda}^\alpha}. \end{aligned}$$

Since μ_g is a $(\beta + 1)$ -Carleson measure, from Lemma 4.2 we see that $I_d : \mathcal{D}_\beta^1 \rightarrow L^1(\mu_g)$ is bounded. Using the fact that

$$\frac{d}{dz} \frac{f(z) - f(a)}{(1 - \bar{a}z)^{2(\beta+2-\alpha)}} = \frac{f'(z)(1 - \bar{a}z) + 2(\beta + 2 - \alpha)\bar{a}(f(z) - f(a))}{(1 - \bar{a}z)^{2(\beta+2-\alpha)+1}},$$

we have

$$\begin{aligned} G_2 &= \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f(z) - f(a)| d\mu_g(z) \\ &\lesssim (1 - |a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} \int_{S(I)} \frac{|f(z) - f(a)|}{|1 - \bar{a}z|^{2(\beta+2-\alpha)}} d\mu_g(z) \\ &\lesssim (1 - |a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} \int_{\mathbb{D}} \frac{|f(z) - f(a)|}{|1 - \bar{a}z|^{2(\beta+2-\alpha)}} d\mu_g(z) \\ &\lesssim \sup_{I \subset \partial\mathbb{D}} \frac{\mu_g(S(I))}{|I|^{\beta+1}} (1 - |a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} \int_{\mathbb{D}} \frac{|f'(z)|(1 - |z|^2)^\beta}{|1 - \bar{a}z|^{2(\beta+2-\alpha)}} dA(z) \\ &\quad + \sup_{I \subset \partial\mathbb{D}} \frac{\mu_g(S(I))}{|I|^{\beta+1}} (1 - |a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} \int_{\mathbb{D}} \frac{|f(z) - f(a)|(1 - |z|^2)^\beta}{|1 - \bar{a}z|^{2(\beta+2-\alpha)+1}} dA(z) \\ &=: G_{21} + G_{22}. \end{aligned}$$

On one hand,

$$\begin{aligned} G_{21} &\lesssim \|g\|_{F(1,-1,\beta+1)} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(z)|(1 - |z|^2)^{\alpha-2} (|1 - |\sigma_a(z)|^2|)^{(\beta+2-\alpha)} dA(z) \\ &\lesssim \|g\|_{F(1,-1,\beta+1)} \|f\|_{\mathcal{D}_{\beta,\lambda}^\alpha}. \end{aligned}$$

On the other hand, making the change of variable $z = \sigma_a(w)$, by Lemma 4.1 we obtain

$$\begin{aligned}
G_{22} &\lesssim \|g\|_{F(1,-1,\beta+1)} (1-|a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} \int_{\mathbb{D}} \frac{|f(z)-f(a)|(1-|z|^2)^\beta}{|1-\bar{a}z|^{2(\beta+2-\alpha)+1}} dA(z) \\
&\lesssim (1-|a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} \int_{\mathbb{D}} \frac{|f \circ \sigma_a(w) - f \circ \sigma_a(0)|(1-|\sigma_a(w)|^2)^\beta}{|1-\bar{a}\sigma_a(w)|^{2(\beta+2-\alpha)+1}} |\sigma'_a(w)|^2 dA(w) \\
&= (1-|a|^2)^{\beta+\alpha-\lambda-\beta\lambda} \int_{\mathbb{D}} \frac{|f \circ \sigma_a(w) - f \circ \sigma_a(0)|(1-|w|^2)^\beta}{|1-\bar{a}w|^{2\alpha-1}} dA(w) \\
&\lesssim (1-|a|^2)^{\beta+\alpha-\lambda-\beta\lambda} \int_{\mathbb{D}} \frac{|(f \circ \sigma_a)'(w)|(1-|w|^2)^{\beta+1}}{|1-\bar{a}w|^{2\alpha-1}} dA(w) \\
&= (1-|a|^2)^{\beta+\alpha-\lambda-\beta\lambda} \int_{\mathbb{D}} \frac{|f'(\sigma_a(w))|(1-|\sigma_a(w)|^2)(1-|w|^2)^\beta}{|1-\bar{a}w|^{2\alpha-1}} dA(w) \\
&= (1-|a|^2)^{\beta+\alpha-\lambda-\beta\lambda} \int_{\mathbb{D}} |f'(z)|(1-|z|^2) \frac{(1-|\sigma_a(z)|^2)^\beta}{|1-\bar{a}\sigma_a(z)|^{2\alpha-1}} |\sigma'_a(z)|^2 dA(z) \\
&= (1-|a|^2)^{\beta+\alpha-\lambda-\beta\lambda} \int_{\mathbb{D}} |f'(z)| \frac{(1-|z|^2)^{\beta+1}(1-|a|^2)^{\beta+3-2\alpha}}{|1-\bar{a}z|^{2(\beta+2-\alpha)+1}} dA(z) \\
&\lesssim (1-|a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(z)|(1-|z|^2)^{\alpha-2}(1-|\sigma_a(z)|^2)^{\beta+2-\alpha} dA(z) \\
&\lesssim \|f\|_{\mathcal{D}_{\beta,\lambda}^\alpha}.
\end{aligned}$$

Therefore, for any $f \in \mathcal{D}_{\beta,\lambda}^\alpha$,

$$\sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f(w)| d\mu_g(w) \lesssim \sup_{I \subset \partial \mathbb{D}} (G_1 + G_2) \lesssim \|g\|_{F(1,-1,\beta+1)} \|f\|_{\mathcal{D}_{\beta,\lambda}^\alpha}.$$

Let $f \in \mathcal{D}_{\beta,\lambda}^\alpha$. It follows that

$$\begin{aligned}
\|T_g f\|_{\mathcal{D}_{\beta,\lambda}^\alpha} &\asymp \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |(T_g f)'(w)|(1-|w|^2)^\beta dA(w) \\
&\asymp \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f(w)| |g'(w)|(1-|w|^2)^\beta dA(w) \\
&\asymp \sup_{I \subset \partial \mathbb{D}} \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f(w)| d\mu_g(w) \\
&\lesssim \|g\|_{F(1,-1,\beta+1)} \|f\|_{\mathcal{D}_{\beta,\lambda}^\alpha} < \infty.
\end{aligned}$$

Therefore $T_g : \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha$ is bounded. Thus $\|T_g\| \asymp \|g\|_{F(1,-1,\beta+1)}$. The proof is complete. $\square$

Next, we study the essential norm of the operator T_g acting on the space $\mathcal{D}_{\beta,\lambda}^\alpha$. Recall that the essential norm of a bounded linear operator $T : X \rightarrow X$, denoted by $\|T\|_{e,X \rightarrow X}$, is precisely defined as the infimum of the set $\{\|T - K\|\}$, where K ranges over all compact operators on Banach space X .

Lemma 4.3. [8, Lemma 5] *Let $-1 < \beta < 0$. If $g \in F(1, -1, \beta + 1)$, then*

$$\limsup_{|a| \rightarrow 1} \int_{\mathbb{D}} |g'(z)| (1 - |z|^2)^{-1} (1 - |\sigma_a(z)|^2)^{\beta+1} dA(z) \\ \asymp \text{dist}_{F(1, -1, \beta+1)}(g, F_0(1, -1, \beta + 1)) \asymp \limsup_{r \rightarrow 1^-} \|g - g_r\|_{F(1, -1, \beta+1)}.$$

Here $g_r(z) = g(rz)$, $0 < r < 1$, $z \in \mathbb{D}$.

Similar to the proof of Lemma 5.2 in [27], we have the following result.

Lemma 4.4. *Let $g \in H(\mathbb{D})$, $-1 < \beta < 0 < \lambda \leq 1$ and*

$$\max \left\{ \frac{\beta}{2} + 1, \lambda(\beta + 1) - \beta \right\} < \alpha < \min \left\{ 1 + \lambda(\beta + 1), \frac{\beta}{2} + \frac{3}{2} \right\}.$$

If $0 < r < 1$ and $g \in F(1, -1, \beta + 1)$, then $T_{g_r} : \mathcal{D}_{\beta, \lambda}^{\alpha} \rightarrow \mathcal{D}_{\beta, \lambda}^{\alpha}$ is compact.

Lemma 4.5. [24] *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that*

- (1) *the point evaluation functionals on X are continuous;*
- (2) *the closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets;*
- (3) *$T : X \rightarrow Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if, given a bounded sequence $\{f_n\}$ in X such that $f_n \rightarrow 0$ uniformly on compact sets, the sequence $\{Tf_n\}$ converges to zero in the norm of Y .

Theorem 4.2. *Let $g \in H(\mathbb{D})$, $-1 < \beta < 0 < \lambda \leq 1$ and*

$$\max \left\{ \frac{\beta}{2} + 1, \lambda(\beta + 1) - \beta \right\} < \alpha < \min \left\{ 1 + \lambda(\beta + 1), \frac{\beta}{2} + \frac{3}{2} \right\}.$$

If $T_g : \mathcal{D}_{\beta, \lambda}^{\alpha} \rightarrow \mathcal{D}_{\beta, \lambda}^{\alpha}$ is bounded, then

$$\|T_g\|_{e, \mathcal{D}_{\beta, \lambda}^{\alpha} \rightarrow \mathcal{D}_{\beta, \lambda}^{\alpha}} \asymp \limsup_{|a| \rightarrow 1} \|g \circ \sigma_a - g(a)\|_{\mathcal{D}_{\beta}^1} \asymp \text{dist}_{F(1, -1, \beta+1)}(g, F_0(1, -1, \beta + 1)).$$

Proof. Let $\{I_j\}$ denote a sequence of arcs, with the length $|I_j|$ of each arc I_j tending to 0 as $j \rightarrow \infty$. For each j , let ξ_j denote the center of I_j and define $d_j = (1 - |I_j|)\xi_j$. Set

$$f_j(w) = \frac{1 - |d_j|^2}{(1 - \bar{d}_j w)^{(\beta+1)(1-\lambda)+\alpha}}, \quad w \in \mathbb{D}.$$

A direct verification reveals that each f_j lies in $\mathcal{D}_{\beta, \lambda}^{\alpha}$. In addition, the sequence $\{f_j\}$ converges uniformly to the zero over every compact subset of $\mathbb{D}$.

Now, we suppose that $K : \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha$ is compact on the space $\mathcal{D}_{\beta,\lambda}^\alpha$. By Lemma 4.5, we conclude that $\lim_{j \rightarrow \infty} \|Kf_j\|_{\mathcal{D}_{\beta,\lambda}^\alpha} = 0$. Then,

$$\begin{aligned} \|T_g - K\| &\gtrsim \limsup_{j \rightarrow \infty} \|(T_g - K)f_j\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \\ &\gtrsim \limsup_{j \rightarrow \infty} \left(\|T_g f_j\|_{\mathcal{D}_{\beta,\lambda}^\alpha} - \|Kf_j\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \right) = \limsup_{j \rightarrow \infty} \|T_g f_j\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \\ &\gtrsim \limsup_{j \rightarrow \infty} \int_{\mathbb{D}} |f_j(w)| |g'(w)| (1 - |w|^2)^{\beta+\alpha-1-\beta\lambda-\lambda} (1 - |\sigma_{d_j}(w)|^2)^{1+\beta\lambda+\lambda-\alpha} dA(w) \\ &= \limsup_{j \rightarrow \infty} \int_{\mathbb{D}} \frac{(1 - |d_j|^2) |g'(w)|}{|1 - \overline{d_j} w|^{(\beta+1)(1-\lambda)+\alpha}} (1 - |w|^2)^{\beta+\alpha-1-\beta\lambda-\lambda} (1 - |\sigma_{d_j}(w)|^2)^{1+\beta\lambda+\lambda-\alpha} dA(w) \\ &\gtrsim \limsup_{j \rightarrow \infty} \frac{1}{|I_j|^{\beta+1}} \int_{S(I_j)} |g'(w)| (1 - |w|^2)^\beta dA(w), \end{aligned}$$

which implies that

$$\|T_g\|_{e, \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha} \gtrsim \limsup_{|I| \rightarrow 0} \frac{1}{|I|^{\beta+1}} \int_{S(I)} |g'(w)| (1 - |w|^2)^\beta dA(w).$$

From Lemma 4.3 and Theorem 2.5 in [29], we see that, for $g \in F(1, -1, \beta + 1)$,

$$\|T_g\|_{e, \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha} \gtrsim \limsup_{|a| \rightarrow 1} \|g \circ \sigma_a - g(a)\|_{\mathcal{D}_\beta^1}.$$

On the other hand, for $0 < r < 1$, we see that $T_{g_r} : \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha$ is compact by Lemma 4.4. Applying Theorem 4.1, we have

$$\|T_g\|_{e, \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha} \leq \|T_g - T_{g_r}\| = \|T_{g-g_r}\| \asymp \|g - g_r\|_{F(1, -1, \beta+1)}.$$

Using Lemma 4.3, we have

$$\|T_g\|_{e, \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha} \lesssim \limsup_{r \rightarrow 1^-} \|g - g_r\|_{F(1, -1, \beta+1)} \asymp \limsup_{|a| \rightarrow 1} \|g \circ \sigma_a - g(a)\|_{\mathcal{D}_\beta^1}.$$

Since we have derived both the lower bound and the upper bound, the proof for the desired result regarding the essential norm of the operator T_g is completed. $\square$

Corollary 4.1. *Let $g \in H(\mathbb{D})$, $-1 < \beta < 0 < \lambda \leq 1$ and*

$$\max \left\{ \frac{\beta}{2} + 1, \lambda(\beta + 1) - \beta \right\} < \alpha < \min \left\{ 1 + \lambda(\beta + 1), \frac{\beta}{2} + \frac{3}{2} \right\}.$$

Then $T_g : \mathcal{D}_{\beta,\lambda}^\alpha \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha$ is compact if and only if $g \in F_0(1, -1, \beta + 1)$.

Remark 4.1. The result established in Theorem 4.1 lies within the scope of Theorem 5.1 (iii) in [20], but we adopt a different proof method herein. Furthermore, we have investigated the essential norm of the operator T_g .

Theorem 4.3. *Let $g \in H(\mathbb{D})$, $-1 < \beta < 0 < \lambda \leq 1$ and*

$$\max \left\{ \frac{\beta}{2} + 1, \lambda(\beta + 1) - \beta \right\} < \alpha < \min \left\{ 1 + \lambda(\beta + 1), \frac{\beta}{2} + \frac{3}{2} \right\}.$$

Then $T_g : \mathcal{D}_{\beta,\lambda}^{\alpha,0} \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ is bounded if and only if $g \in F(1, -1, \beta + 1)$.

Proof. Suppose that $T_g : \mathcal{D}_{\beta,\lambda}^{\alpha,0} \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ is bounded. It is clear that $T_g : \mathcal{D}_{\beta,\lambda}^{\alpha,0} \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha}$ is bounded. Let $I \subset \partial\mathbb{D}$, $e^{i\theta}$ be the center of I and $b = (1 - |I|)e^{i\theta}$. Let

$$f_b(w) = \frac{(1 - |b|^2)}{(1 - \bar{b}w)^{(\beta+1)(1-\lambda)+\alpha}}, \quad w \in \mathbb{D}.$$

By the proof of Proposition 3.2, we see that

$$\begin{aligned} & \lim_{|a| \rightarrow 1} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'_b(w)| (1 - |w|^2)^{\alpha-2} (1 - |\sigma_a(w)|^2)^{\beta+2-\alpha} dA(w) \\ & \lesssim \lim_{|a| \rightarrow 1} \frac{(1 - |a|^2)^{(\beta+1)(1-\lambda)+\beta+2-\alpha} (1 - |b|^2)}{|1 - \bar{a}b|^{2\beta+4-2\alpha} (1 - |b|^2)^{\alpha-\beta\lambda-\lambda}} = 0. \end{aligned}$$

Hence $f_b \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$. Then the rest proof is similar to Theorem 4.1.

Conversely, suppose $g \in F(1, -1, \beta + 1)$. We see that $T_g : \mathcal{D}_{\beta,\lambda}^{\alpha} \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha}$ is bounded. Let $f \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$. For any $\varepsilon > 0$, by Proposition 3.5, there exists a polynomial P such that

$$\|f - P\|_{\mathcal{D}_{\beta,\lambda}^{\alpha}} < \varepsilon. \quad (4.1)$$

Let $I \subset \partial\mathbb{D}$. From the proof of Theorem 3.1, we have

$$\begin{aligned} & \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |(T_g f)'(w)| (1 - |w|^2)^{\beta} dA(w) \\ & = \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f(w)| |g'(w)| (1 - |w|^2)^{\beta} dA(w) \\ & \lesssim \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |f(w) - P(w)| |g'(w)| (1 - |w|^2)^{\beta} dA(w) \\ & \quad + \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |P(w)| |g'(w)| (1 - |w|^2)^{\beta} dA(w) \\ & \lesssim \|f - P\|_{\mathcal{D}_{\beta,\lambda}^{\alpha}} \|g\|_{F(1,-1,\beta+1)} + \|P\|_{H^{\infty}} |I|^{\alpha+\beta-\lambda(\beta+1)} \|g\|_{F(1,-1,\beta+1)}, \end{aligned}$$

which implies that

$$\lim_{|I| \rightarrow 0} \frac{1}{|I|^{1+\beta\lambda+\lambda-\alpha}} \int_{S(I)} |(T_g f)'(w)| (1 - |w|^2)^{\beta} dA(w) = 0.$$

Hence $T_g f \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$. The proof is complete. $\square$

Finally, we show that a non-compact operator T_g is not strictly singular, i.e., there exists a subspace $M \subset \mathcal{D}_{\beta,\lambda}^{\alpha}$ such that $T_g|_M$ is bounded below on M which yields the compactness and strict singularity are equivalent for T_g on $\mathcal{D}_{\beta,\lambda}^{\alpha}$. In order to prove this result, we need the following lemma; see [8, Theorem 1 and Corollary 1].

Lemma 4.6. *Let $-1 < \beta < 0$ and $g \in F(1, -1, \beta + 1)$. Then T_g is bounded on $\mathcal{D}_{\beta}^1$.*

In the rest of this paper, we use $\mathcal{C}_0$ to denote the Banach space of complex sequences converging to zero endowed with the supremum norm $\|\cdot\|_{l^{\infty}}$, and set

$$\|f\|_{*, \mathcal{D}_{\beta,\lambda}^{\alpha}} = \sup_{a \in \mathbb{D}} (1 - |a|^2)^{(\beta+1)(1-\lambda)} \|f \circ \sigma_a - f \circ \sigma_a(0)\|_{\mathcal{D}_{\beta}^1}.$$

Proposition 4.1. *Let $-1 < \beta < 0 < \lambda \leq 1$ and*

$$\max \left\{ \frac{\beta}{2} + 1, \lambda(\beta + 1) - \beta \right\} < \alpha < \min \left\{ 1 + \lambda(\beta + 1), \frac{\beta}{2} + \frac{3}{2} \right\}.$$

Let $\{f_k\}_{k=1}^\infty$ be a sequence in $\mathcal{D}_{\beta,\lambda}^{\alpha,0}$ with $\|f_k\|_{,\mathcal{D}_{\beta,\lambda}^\alpha} \asymp 1$ and $\lim_{k \rightarrow \infty} \|f_k\|_{\mathcal{D}_\beta^1} = 0$. Then there exists a subsequence $\{f_{k_j}\} \subseteq \{f_k\}_{k=1}^\infty$ such that the operator $T : \mathcal{C}_0 \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ is an isomorphism onto its range, where $T(\eta) = \sum_{j=1}^\infty \eta_j f_{k_j}$, $\eta = \{\eta_j\} \subset \mathcal{C}_0$.*

Proof. Since $f_k \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$, for each $k \geq 1$, we have

$$\lim_{|a| \rightarrow 1} \Psi(f_k, a) = 0. \quad (4.2)$$

Here and henceforth

$$\Psi(f_k, a) = (1 - |a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'_k(w)| (1 - |w|^2)^{\alpha-2} (1 - |\sigma_a(w)|^2)^{\beta+2-\alpha} dA(w).$$

If $|a| \leq t_0$ for some $t_0 \in (0, 1)$, we find from the assumption that $\lim_{k \rightarrow \infty} \|f_k\|_{\mathcal{D}_\beta^1} = 0$ that

$$\Psi(f_k, a) \lesssim \int_{\mathbb{D}} |f'_k(w)| (1 - |w|^2)^\beta dA(w) \lesssim \|f_k\|_{\mathcal{D}_\beta^1} \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Therefore, for any $t \in (0, 1)$,

$$\lim_{k \rightarrow \infty} \sup_{|a| \leq t} \Psi(f_k, a) = 0. \quad (4.3)$$

From $\|f_k\|_{*,\mathcal{D}_{\beta,\lambda}^\alpha} \asymp 1$, there exists $\Gamma_1, \Gamma_2 > 0$ such that

$$\Gamma_1 \leq \|f_k\|_{*,\mathcal{D}_{\beta,\lambda}^\alpha} \leq \Gamma_2 \quad (4.4)$$

for any $k \geq 1$. Combining (4.2), (4.3) with $\lim_{k \rightarrow \infty} \|f_k\|_{\mathcal{D}_\beta^1} = 0$, there exists a subsequence $\{f_{k_j}\}$ and an increasing sequence $\{t_j\}$ with $t_j \in (0, 1)$ such that, for any $j \geq 1$, $\sup_{|a| \leq t_j} \Psi(f_{k_j}, a) < 3^{-j-1}\Gamma_1$, $\sup_{|a| > t_{j+1}} \Psi(f_{k_j}, a) < 3^{-j-1}\Gamma_1$ and $\|f_{k_j}\|_{\mathcal{D}_\beta^1} < 3^{-j-1}\Gamma_1$. Since a belongs to at most one circular region $t_j < |a| \leq t_{j+1}$ within $\mathbb{D}$, for any $a \in \mathbb{D}$, we see that $\Psi(f_{k_j}, a) < 3^{-j-1}\Gamma_1$ for all but at most one index j . For the exceptional index j , $t_j < |a| \leq t_{j+1}$, the inequality $\Psi(f_{k_j}, a) \leq \Gamma_2$ is satisfied. In addition, from (4.4), for any $j \geq 1$, there exists $a_j \in \mathbb{D}$ such that $\Psi(f_{k_j}, a_j) \geq \frac{2\Gamma_1}{3}$. By the triangle inequality, for any $\eta = \{\eta_j\} \in \mathcal{C}_0$, we obtain

$$\begin{aligned} \|T(\eta)\|_{\mathcal{D}_{\beta,\lambda}^\alpha} &= \sup_{a \in \mathbb{D}} \Psi(T(\eta), a) + \left| \sum_{j=1}^\infty \eta_j f_{k_j}(0) \right| \\ &\leq \sup_{a \in \mathbb{D}} \sum_{j=1}^\infty |\eta_j| \Psi(f_{k_j}, a) + \|\eta\|_{l^\infty} \sum_{j=1}^\infty \|f_{k_j}\|_{\mathcal{D}_\beta^1} \\ &< \|\eta\|_{l^\infty} \left(\sum_{j=1}^\infty 3^{-j-1}\Gamma_1 + \Gamma_2 \right) + \|\eta\|_{l^\infty} \sum_{j=1}^\infty 3^{-j-1}\Gamma_1 \\ &< (\Gamma_1 + \Gamma_2) \|\eta\|_{l^\infty}, \end{aligned}$$

which implies that $T(\eta) \in \mathcal{D}_{\beta,\lambda}^\alpha$ and $T : \mathcal{C}_0 \rightarrow \mathcal{D}_{\beta,\lambda}^\alpha$ is bounded.

Next, we prove that $T(\eta) \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$. In fact, for any $a \in \mathbb{D}$ and $N > 1$,

$$\begin{aligned} \Psi(T(\eta), a) &\leq \sum_{j=1}^{\infty} |\eta_j| \Psi(f_{k_j}, a) \\ &\leq \sup_{j>N} |\eta_j| \sum_{j=N+1}^{\infty} \Psi(f_{k_j}, a) + \|\eta\|_{l^\infty} \sum_{j=1}^N \Psi(f_{k_j}, a) \\ &\lesssim \sup_{j>N} |\eta_j| + \|\eta\|_{l^\infty} \sum_{j=1}^N \Psi(f_{k_j}, a). \end{aligned}$$

From the fact that $f_{k_j} \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ for each $j \geq 1$, we have $\overline{\lim}_{|a| \rightarrow 1} \Psi(T(\eta), a) \lesssim \sup_{j>N} |\eta_j|$. Letting $N \rightarrow \infty$, we have $\overline{\lim}_{|a| \rightarrow 1} \Psi(T(\eta), a) = 0$ and hence $T(\eta) \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$. Therefore, $T : \mathcal{C}_0 \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ is bounded.

Finally, we show that $T : \mathcal{C}_0 \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ is bounded below. For any $\eta = \{\eta_j\}$ and $j \geq 1$, we obtain

$$\begin{aligned} \|T(\eta)\|_{\mathcal{D}_{\beta,\lambda}^{\alpha,0}} &\geq \Psi(T(\eta), a_j) \\ &\geq |\eta_j| \Psi(f_{k_j}, a_j) - \sum_{t \neq j} |\eta_t| \Psi(f_{k_t}, a_j) \\ &\geq \frac{2\Gamma_1}{3} |\eta_j| - \sum_{t \neq j} |\eta_t| 3^{-t-1} \Gamma_1 \\ &\geq \frac{2\Gamma_1}{3} |\eta_j| - \frac{\Gamma_1}{6} \|\eta\|_{l^\infty}. \end{aligned}$$

By the arbitrariness of j , we see that $\|T(\eta)\|_{\mathcal{D}_{\beta,\lambda}^{\alpha,0}} \gtrsim \frac{\Gamma_1}{2} \|\eta\|_{l^\infty}$. Therefore $T : \mathcal{C}_0 \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ is an isomorphism onto its range. $\square$

Theorem 4.4. Let $-1 < \beta < 0 < \lambda \leq 1$ and

$$\max \left\{ \frac{\beta}{2} + 1, \lambda(\beta + 1) - \beta \right\} < \alpha < \min \left\{ 1 + \lambda(\beta + 1), \frac{\beta}{2} + \frac{3}{2} \right\}.$$

Suppose $g \in F(1, -1, \beta + 1) \setminus F_0(1, -1, \beta + 1)$. Then there exists a subspace $M \subset \mathcal{D}_{\beta,\lambda}^{\alpha}$ isomorphic to $\mathcal{C}_0$ such that $T_g|_M : M \rightarrow T_g(M)$ is an isomorphism. In particular, T_g is not strictly singular.

Proof. Set

$$\Theta := \limsup_{|I| \rightarrow 0} \frac{1}{|I|^{1+\beta}} \int_{S(I)} |g'(w)| (1 - |w|^2)^\beta dA(w).$$

By the assumption that $g \in F(1, -1, \beta + 1) \setminus F_0(1, -1, \beta + 1)$, there exists $\{I_k\} \subset \partial\mathbb{D}$ with $\lim_{k \rightarrow \infty} |I_k| = 0$ such that

$$\Theta = \lim_{k \rightarrow \infty} \frac{1}{|I_k|^{1+\beta}} \int_{S(I_k)} |g'(w)| (1 - |w|^2)^\beta dA(w) > 0. \quad (4.5)$$

Let η_k denote the center of I_k and take $\gamma_k = (1 - |I_k|)\eta_k$. Set $f_k(w) = \frac{(1 - |\gamma_k|^2)}{(1 - \bar{\gamma}_k w)^{(\beta+1)(1-\lambda)+\alpha}}$ $w \in \mathbb{D}$. Then $f_k \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ and $\|f_k\|_{\mathcal{D}_{\beta,\lambda}^{\alpha,0}} \lesssim 1$ for any $k \geq 1$. Using the fact that $|1 - \bar{\gamma}_k w| \asymp 1 - |\gamma_k|^2 \asymp$

$|I_k|$, $w \in S(I_k)$, we have

$$|f_k(w)| \gtrsim \frac{1}{|I_k|^{(\beta+1)(1-\lambda)+\alpha-1}}, \quad w \in S(I_k). \quad (4.6)$$

Using (4.5) and (4.6), we see that there exist some constant $\delta > 0$ such that

$$\frac{1}{|I_k|^{1+\lambda(\beta+1)-\alpha}} \int_{S(I_k)} |f_k(w)| |g'(w)| (1-|w|^2)^\beta dA(w) \geq \frac{\Theta\delta}{3}$$

for k larger than some fixed positive number $N \geq 1$. Since $g \in F(1, -1, \beta+1)$, we see that $T_g f_k \in \mathcal{D}_{\beta,\lambda}^\alpha$ and $\|T_g f_k\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \lesssim 1$. Moreover, $T_g : \mathcal{D}_{\beta,\lambda}^{\alpha,0} \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ is also bounded and then $T_g f_k \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ by Theorem 4.3. Thus, passing to a further subsequence if necessary, we assume that

$$\frac{1}{|I_{k+1}|^{1+\lambda(\beta+1)-\alpha}} \int_{S(I_{k+1})} |f_k(w)| |g'(w)| (1-|w|^2)^\beta dA(w) \leq \frac{\Theta\delta}{4}$$

for any $k \geq N$. Set $F_k = f_{k+1} - f_k$, $k \geq N$. Then $F_k \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ and $\|F_k\|_{\mathcal{D}_{\beta,\lambda}^\alpha} \lesssim 1$. Thus

$$\begin{aligned} & \frac{1}{|I_{k+1}|^{1+\lambda(\beta+1)-\alpha}} \int_{S(I_{k+1})} |F_k(z)| |g'(w)| (1-|w|^2)^\beta dA(w) \\ & \geq \frac{1}{|I_{k+1}|^{1+\lambda(\beta+1)-\alpha}} \int_{S(I_{k+1})} |f_{k+1}(w)| |g'(w)| (1-|w|^2)^\beta dA(w) \\ & \quad - \frac{1}{|I_{k+1}|^{1+\lambda(\beta+1)-\alpha}} \int_{S(I_{k+1})} |f_k(w)| |g'(w)| (1-|w|^2)^\beta dA(w) \geq \frac{\Theta\delta}{12}. \end{aligned}$$

Therefore, for any $k \geq N$,

$$\begin{aligned} \frac{\Theta\delta}{12} & \leq \frac{1}{|I_{k+1}|^{1+\lambda(\beta+1)-\alpha}} \int_{S(I_{k+1})} |F_k(w)| |g'(w)| (1-|w|^2)^\beta dA(w) \lesssim \|T_g F_k\|_{*, \mathcal{D}_{\beta,\lambda}^\alpha} \\ & \lesssim \|F_k\|_{*, \mathcal{D}_{\beta,\lambda}^\alpha} \lesssim 1. \end{aligned}$$

Thus $T_g F_k \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ and $\|T_g F_k\|_{*, \mathcal{D}_{\beta,\lambda}^\alpha} \asymp \|F_k\|_{*, \mathcal{D}_{\beta,\lambda}^\alpha} \asymp 1$ for any $k \geq N$. For simplicity of notation, we set $\vartheta = (1-\lambda)(\beta+1) + \alpha + 1$. Since $F_k \in \mathcal{D}_{\beta,\lambda}^{\alpha,0}$, we have that $F_k \in \mathcal{D}_\beta^1$ and

$$\begin{aligned} & \int_{\mathbb{D}} |F'_k(w)| (1-|w|^2)^\beta dA(w) = \int_{\mathbb{D}} |f'_{k+1}(w) - f'_k(w)| (1-|w|^2)^\beta dA(w) \\ & \lesssim \int_{\mathbb{D}} \frac{|(1-|\gamma_{k+1}|^2)\gamma_{k+1}(1-\overline{\gamma_k}w)^\vartheta - (1-|\gamma_k|^2)\gamma_k(1-\overline{\gamma_{k+1}}w)^\vartheta|}{|1-\overline{\gamma_{k+1}}w|^\vartheta |1-\overline{\gamma_k}w|^\vartheta} (1-|w|^2)^\beta dA(w) \\ & \lesssim \int_{\mathbb{D}} \frac{(1-|\gamma_{k+1}|^2) |(1-\overline{\gamma_k}w)^\vartheta - (1-\overline{\gamma_{k+1}}w)^\vartheta|}{|1-\overline{\gamma_{k+1}}w|^\vartheta |1-\overline{\gamma_k}w|^\vartheta} (1-|w|^2)^\beta dA(w) \\ & \quad + \int_{\mathbb{D}} \frac{|(1-|\gamma_{k+1}|^2)\gamma_{k+1} - (1-|\gamma_k|^2)\gamma_k| |1-\overline{\gamma_{k+1}}w|^\vartheta}{|1-\overline{\gamma_{k+1}}w|^\vartheta |1-\overline{\gamma_k}w|^\vartheta} (1-|w|^2)^\beta dA(w) \\ & \lesssim \int_{\mathbb{D}} \frac{(1-|\gamma_{k+1}|^2)(1-|w|^2)^\beta}{|1-\overline{\gamma_{k+1}}w|^\vartheta} dA(w) + \int_{\mathbb{D}} \frac{(1-|\gamma_{k+1}|^2)(1-|w|^2)^\beta}{|1-\overline{\gamma_k}w|^\vartheta} dA(w) + \\ & \quad \int_{\mathbb{D}} \frac{(1-|\gamma_k|^2)(1-|w|^2)^\beta}{|1-\overline{\gamma_k}w|^\vartheta} dA(w). \end{aligned} \quad (4.7)$$

By Lemma 3.1,

$$\int_{\mathbb{D}} \frac{(1 - |\gamma_k|^2)(1 - |w|^2)^\beta}{|1 - \overline{\gamma_k}w|^\vartheta} dA(w) \lesssim (1 - |\gamma_k|^2)^{1+\lambda(\beta+1)-\alpha} \rightarrow 0 \quad (4.8)$$

as $k \rightarrow \infty$. Similarly,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{D}} \frac{(1 - |\gamma_{k+1}|^2)(1 - |w|^2)^\beta}{|1 - \overline{\gamma_{k+1}}w|^\vartheta} dA(w) = 0 \quad (4.9)$$

and

$$\lim_{k \rightarrow \infty} \int_{\mathbb{D}} \frac{(1 - |\gamma_{k+1}|^2)(1 - |w|^2)^\beta}{|1 - \overline{\gamma_k}w|^\vartheta} dA(w) = 0. \quad (4.10)$$

Combining (4.7), (4.8), (4.9) and (4.10), we obtain that $\lim_{k \rightarrow \infty} \|F_k\|_{\mathcal{D}_\beta^1} = 0$. By Lemma 4.6, T_g is bounded on $\mathcal{D}_\beta^1$. Hence $\lim_{k \rightarrow \infty} \|T_g F_k\|_{\mathcal{D}_\beta^1} = 0$. Using Proposition 4.1, we are able to construct a subsequence $\{F_{k_j}\}$ of the sequence $\{F_k\}$ such that $T : \mathcal{C}_0 \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ and $V : \mathcal{C}_0 \rightarrow \mathcal{D}_{\beta,\lambda}^{\alpha,0}$ are isomorphisms onto their respective ranges, where $T(\eta) = \sum_{j=1}^\infty \eta_j F_{k_j}$, $\eta = \{\eta_j\} \subset \mathcal{C}_0$ and $V(\eta) = \sum_{j=1}^\infty \eta_j T_g F_{k_j}$, $\eta = \{\eta_j\} \subset \mathcal{C}_0$. Let M be the closure of span $\{F_{k_j}\}$ in $\mathcal{D}_{\beta,\lambda}^{\alpha,0}$. Then M is isomorphic to $\mathcal{C}_0$ and $T_g|_M : M \rightarrow T_g(M)$ is an isomorphism. The proof is complete. $\square$

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