

UNIQUENESS, MULTIPLE NORMALIZED SOLUTIONS FOR p -LAPLACIAN EQUATION WITH SINGULAR GROWTH

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Abstract. In this paper, we consider the existence of normalized solutions for a singular p -Laplacian equation. We derive a local minimizer and a mountain-pass type normalized solution using the concentration-compactness lemma and minimax theorems. Meanwhile, with the help of the truncation technique and genus theory, we establish the existence of infinitely many low-energy solutions. When the critical nonlinearity is replaced with an L^q -supercritical one, we further obtain infinitely many high-energy solutions. Additionally, we investigate the unique normalized solution to a p -Laplacian equation without a critical perturbation term.

Keywords. Critical exponent; Normalized solutions; p -Laplacian equation; Singularity.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we investigate the following p -Laplacian equation

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2} u = \frac{|u|^{p-2} u}{|x|^r} + |u|^{p^*-2} u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = c > 0, \end{cases} \quad (1.1)$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier, $1 < p < N$, $0 < r < p$, $p^* = \frac{Np}{N-p}$, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator.

When $p = 2$, the dynamic equation corresponding to (1.1) is

$$\begin{cases} i\partial_t \varphi + \Delta \varphi + \gamma |x|^{-\sigma} \varphi + |\varphi|^{2^*-2} \varphi = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \varphi(0, x) = \varphi_0(x) \in H^1(\mathbb{R}^N), & x \in \mathbb{R}^N, \end{cases} \quad (1.2)$$

where $N \geq 3$, $\varphi : \mathbb{R}^+ \times \mathbb{R}^N \rightarrow \mathbb{C}$ is a complex-valued function, $\gamma > 0$, $\sigma \in (0, 2)$, and $2^* = \frac{2N}{N-2}$.

In recent years, (1.2) has been widely studied. When the inverse-power potential disappears i.e., $\sigma = 0$, Soave in [37] and [38] carried out a detailed analysis of normalized ground-state solutions with composite nonlinearities. For the case where $\sigma = 1$, detailed content can be found in Lions [30] and Feng and Zhang [18]. When $\sigma = 2$, it corresponds to the Hardy potential, which has attracted significant attention in [5, 10, 27, 43]. Notably, Okazawa, Suzuki and Yokota in [36] established the local well-posedness of the Cauchy problem for this case. When

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we look for the stationary waves of the form $\varphi(t, x) = e^{i\lambda t}u(x)$ of equation (1.2), u satisfies the following elliptic equation

$$-\Delta u + \lambda u - \gamma|x|^{-\sigma}u - |u|^{2^*-2}u = 0, \text{ in } \mathbb{R}^N. \quad (1.3)$$

For equations of the form (1.3), some results have been obtained in recent years. In [22], the ground state solution without the critical term was obtained. In [31], Liao et al. studied a class of singular elliptic equations involving critical exponents. Under suitable assumptions on $Q(x)$, using constrained minimizers and perturbation methods, they established the existence of two positive solutions for sufficiently small $\lambda > 0$. Chen, Li and Li [11] applied variational methods to obtain positive and sign-changing solutions for a semilinear equation involving singular potential and critical exponents over an arbitrary bounded domain.

In view of the wide application potential of p -Laplacian type equations in elasticity, population models, ocean motion and plasma-related problems, which can offer mathematical support for modeling dynamic phenomena (see [16, 32] for more physical background), we generalize equation (1.3) to the following p -Laplacian equation

$$-\Delta_p u + \lambda|u|^{p-2}u - \gamma|x|^{-\sigma}|u|^{p-2}u - |u|^{p^*-2}u = 0, \text{ in } \mathbb{R}^N. \quad (1.4)$$

To our knowledge, few scholars explored equation (1.4) with both singular and critical terms simultaneously, though relevant studies on the equation with either term alone have been conducted in academic circles.

For analogues of (1.4) without singular terms, the authors in [15, 17] established the existence, multiplicity, and other properties of solutions to the equation by using the variational method, and the truncation technique. In [29], He, Liu and Rădulescu proved the existence of multiple normalized solutions for the Sobolev critical p -Laplacian Schrödinger-Poisson system by using the truncation technique, concentration-compactness principle, genus theory and the Pohožaev manifold. In [8], Chaves, Ercole and Miyagaki employed the mountain pass theorem combined with the Cerami compactness condition to prove the existence of a nonnegative non-trivial solution in $\mathcal{D}^{1,p}(\mathbb{R}^N) \cap \mathcal{D}^{1,q}(\mathbb{R}^N)$ for a (p, q) -Laplacian problem on $\mathbb{R}^N$ with weighted subcritical and critical nonlinear terms, where the classical Ambrosetti-Rabinowitz condition is not required. In [7], Chahma and Chen studied a p -Laplacian Kirchhoff-type equation with critical exponents on bounded domains. By employing the constraint variational method, topological degree and quantitative deformation lemma, they established the existence of a least energy sign-changing solution.

For analogues of (1.4) admitting singular terms but no critical nonlinearity, the existence of normalized solutions was proven in [23]. In [34], Li and Wang also proved the existence of solutions. In [42], the authors studied the existence of ground state normalized solutions via the mountain pass theorem for a class of N -Kirchhoff equations. In [28], Liu, Guo and Liu investigated a singular p -Laplacian equation on $\mathbb{R}^N$. They showed that there exists a critical value Λ such that the problem admits at least two positive solutions for $0 < \lambda < \Lambda$, one positive solution for $\lambda = \Lambda$, and no positive solution for $\lambda > \Lambda$. In [35], Lu and Xie studied a singular p -Laplacian boundary value problem on bounded domains. By critical point theories, they proved that the problem admits two positive weak solutions when λ is sufficiently small.

To better describe mass-preserving stationary states in nonlinear diffusive systems, motivated by the above literature, we extend the investigation to p -Laplacian equations with both singular interaction and critical growth. Let $E = W^{1,p}(\mathbb{R}^N)$ be the working space and denote by $|\cdot|_t$ the

usual norm of $L^t(\mathbb{R}^N)$ with $t \in [1, \infty)$. Moreover, we denote by $I : E \rightarrow \mathbb{R}$ the energy functional related to (1.1), given by

$$I(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx.$$

It is standard that I is of class C^1 in E , and any critical points of I are constrained to $S(c) = \{u \in E : |u|_p^p = c\}$. We aim to address the existence of ground-state solutions, with their definition specified hereafter.

Definition 1.1. $u \in S(c)$ is said to be a ground state solution of (1.1) if

$$(I|_{S(c)})'(u) = 0, \text{ and } I(u) = \inf\{I(v) : v \in S(c) \text{ with } (I|_{S(c)})'(v) = 0\}.$$

Next, we recall the generalized Sobolev estimate in [9] as follows.

Lemma 1.1. *Let $1 \leq p < \infty$. If $r < N$ such that $0 \leq r \leq p$. Then, for all $u \in E$, $\frac{|u(\cdot)|^p}{|\cdot|^r} \in L^1(\mathbb{R}^N)$. Moreover, the following inequality holds*

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx \leq \left(\frac{p}{N-r} \right)^r |u|_p^{p-r} \|\nabla u\|_p^r.$$

In order to obtain the ground state solution, we introduce the Pohožaev manifold

$$\mathcal{V}_c = \{u \in S(c) : P(u) = 0\},$$

where

$$P(u) = \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{r}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \int_{\mathbb{R}^N} |u|^{p^*} dx.$$

Now, we present our main results, with preliminary notations introduced first. Set

$$\alpha(N, p, r) = \frac{p^* - p}{(p^* - r)C_{N,r}^p} \left(\frac{(p-r)p^* S^{\frac{p^*}{p}}}{p(p^* - r)} \right)^{\frac{p-r}{p^* - p}},$$

where S is the best constant of the Sobolev embedding and $C_{N,r}^p = \left(\frac{p}{N-r}\right)^r$.

Theorem 1.1. *Assume that $1 < p < N$ and $0 < r < p$. Let $c > 0$ be such that*

$$c^{\frac{p-r}{p}} < \alpha(N, p, r). \quad (1.5)$$

Then,

- (i) problem (1.1) has a positive ground state normalized solution $u \in S(c)$ with $I(u) < 0$, and u is an interior local minimizer of I on $Z_{R_0}(c) = \{u \in S(c) : \|\nabla u\|_p < R_0\}$ for a suitable R_0 ;
- (ii) under the additional assumptions $N \geq p^2$ or $N < p^2 < 9$, the functional I restricted to $S(c)$ has a mountain-pass type critical point that is a normalized solution of (1.1).

Theorem 1.2. *Suppose $1 < p < N$ and $0 < r < p < p^*$. Then problem (1.1) has at least n pairs $(u_k, \lambda_k) \in E \times \mathbb{R}$ of weak solutions when $c \in (0, \alpha^{\frac{p}{p-r}})$. These solutions satisfy $\int_{\mathbb{R}^N} |u_k|^p dx = c$, $\lambda_k > 0$ and $I(u_k) \rightarrow 0^-$ as $k \rightarrow \infty$.*

Remark 1.1. In [15, 17], the authors established the existence of normalized solutions to the p -Laplacian equation with the critical Sobolev exponent in radial spaces. Extending this foundation, we consider the equation with singular and critical exponents terms in the whole space. However, since our work space is the whole space and the equation contains a singular term, these two factors pose significant challenges to our proof of the compactness of Palais-Smale sequences.

To address this challenge, we first establish the strong convergence of the critical term via a dichotomous strategy for the ball's interior and exterior: the interior case follows the first step of Lemma 4.5 in [17]; the exterior case draws on the second step of Lemma 5.3 in [44]. The strong convergence of the critical term implies that of the singular term, and thus leads to the strong convergence in the entire space. As for the compactness of the mountain-pass solution, it is proven by referring to the estimation method in [15]. Infinitely many low-energy solutions are derived using truncation techniques and the genus lemma, which is analogous to the approach in [15, 17].

When equation (1.1) has no critical perturbation term, we have the following result.

Theorem 1.3. *For $0 < r < p < N$, the normalized solution to the following equation is unique*

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2} u = \frac{|u|^{p-2} u}{|x|^r}, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = c. \end{cases} \quad (1.6)$$

This theorem considers the uniqueness of normalized solutions. However, when a L^q -supercritical term is added, the equation admits infinitely many high-energy solutions. Specifically, we consider the following problem

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2} u = \frac{|u|^{p-2} u}{|x|^r} + |u|^{q-2} u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = c, \end{cases} \quad (1.7)$$

where λ is frequency, and

$$q \in \left(p + \frac{p^2}{N}, p^* \right), \quad c \in \left(0, \left(\frac{1}{2C_{N,r}^p} \right)^{\frac{p}{p-r}} \right), \quad r \in \left(0, \frac{N(q-p)}{q} \right). \quad (1.8)$$

Next, we discuss infinitely many high-energy solutions in the radial space

$$D_r := \{u(x) \in E : u(x) = u(|x|)\}.$$

Theorem 1.4. *Assume (1.8). Then, there exists $k_0 \in \mathbb{N}^+$ such that, for arbitrary fixed $k \geq k_0$, problem (1.7) admits a sequence of couples of weak solutions $\{(v^{(k)}, \lambda^{(k)})\} \subseteq D_r \times \mathbb{R}^-$ with $|v^{(k)}|_p^p = c$. Additionally, $|v^{(k)}|_{D_r} \rightarrow +\infty$ and $J(v^{(k)}) \rightarrow +\infty$ as $k \rightarrow +\infty$, where*

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q dx.$$

Remark 1.2. Equation (1.6) has only a unique solution. However, when a perturbation term is added, equation (1.7) admits infinitely many high-energy solutions, which shows that the perturbation term has a significant impact on the existence of solutions. Then why do not we directly find infinitely many high-energy solutions of equation (1.1)? The root lies in the critical exponent p^* of the Sobolev embedding $E \hookrightarrow L^{p^*}(\mathbb{R}^N)$. Given the criticality of p^* the energy

functional I associated with equation (1.1) does not satisfy the Palais-Smale condition at relatively high energy levels. Since this condition is crucial for obtaining multiple high-energy solutions through variational methods, its violation renders deriving such solutions highly challenging. Readers interested can refer to [13] for in-depth understanding. Accordingly, we restrict the research scope to intervals satisfying the embedding and related favorable conditions for subsequent analysis. Since in the radial space, we can directly obtain the strong convergence of the sequence in $L^q(\mathbb{R}^N)$ for $p < q < p^*$. For the strong convergence of norm, we can use the inequality in [14]. Compared with [1], our difficulty lies in that we also need to separately handle the strong convergence of the singular term. But we can cleverly solve this difficulty using Theorem 1.1.

For $u \in E$ and $t \in \mathbb{R}$, we define

$$u \star t := e^{\frac{N}{p}t} u(e^t x), \text{ for a.e. } x \in \mathbb{R}^N.$$

It is straightforward to verify that $|u \star t|_p = |u|_p$ for all $t \in \mathbb{R}$, so $u \star t \in S(c)$ precisely when $u \in S(c)$. Furthermore, the properties of $\mathcal{V}_c$ are closely linked to how I behaves under such dilations, and for any $u \in S(c)$, we define the fiber map

$$\Psi_u(t) = I(u \star t) = \frac{e^{pt}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{e^{rt}}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \frac{e^{p^*t}}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx.$$

The monotonicity and convexity characteristics of Ψ_u will influence the structure of $\mathcal{V}_c$, and clearly, $u \in \mathcal{V}_c$ if and only if 0 is a critical point of the function Ψ_u . Following this line of thought, we will decompose $\mathcal{V}_c$ into three parts

$$\begin{aligned} \mathcal{V}_c^+ &= \{u \in S(c) : (\Psi_u)'(0) = 0, (\Psi_u)''(0) > 0\}, \\ \mathcal{V}_c^0 &= \{u \in S(c) : (\Psi_u)'(0) = 0, (\Psi_u)''(0) = 0\}, \\ \mathcal{V}_c^- &= \{u \in S(c) : (\Psi_u)'(0) = 0, (\Psi_u)''(0) < 0\}. \end{aligned}$$

Clearly, $\mathcal{V}_c = \mathcal{V}_c^+ \cup \mathcal{V}_c^0 \cup \mathcal{V}_c^-$. We will see from Lemma 2.2 that $\mathcal{V}_c^0 = \emptyset$.

The organization of this paper is as follows. Section 2 is divided into three subsections. Specifically, we first introduce the properties of the manifold, then obtain the first solution of Theorem 1.1, and finally derive the second solution of Theorem 1.1. In Section 3, we establish the existence of infinitely many low-energy solutions. Section 4 focuses on proving the uniqueness of the solution. Section 5 is dedicated to the existence of infinitely many high-energy solutions.

Notation. In this study, the symbol C is employed interchangeably to denote various positive constants.

2. PROOF OF THEOREM 1.1

2.1. Some properties of manifolds. By Lemma 1.1 and the Sobolev inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx \\ &\geq \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} C_{N,r}^p c^{\frac{p-r}{p}} \|\nabla u\|_p^r - \frac{1}{p^* S^{\frac{p^*}{p}}} \|\nabla u\|_p^{p^*}, \end{aligned} \quad (2.1)$$

for $u \in S(c)$. In the subsequent discussion, we examine the characteristics of the fiber map Ψ_u , for which the following technical lemma is needed.

Lemma 2.1. *The function*

$$f(t) = \frac{1}{p}t^p - \frac{1}{p}C_{N,r}^p c^{\frac{p-r}{p}} t^r - \frac{1}{p^* S^{\frac{p^*}{p}}} t^{p^*}, \quad t > 0,$$

has a strict negative local minimum and a strict positive global maximum, with two zeros $R_0 < R_1$ (depending on c) such that $f(t) > 0$ if and only if $t \in (R_0, R_1)$.

Proof. First, we take into account the auxiliary function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ given by

$$\varphi(t) = \frac{1}{p}t^{p-r} - \frac{1}{p^* S^{\frac{p^*}{p}}} t^{p^*-r} - \frac{1}{p}C_{N,r}^p c^{\frac{p-r}{p}}.$$

Through a straightforward calculation, we conclude that φ reaches its maximum at

$$\bar{t} = \left(\frac{(p-r)p^* S^{\frac{p^*}{p}}}{p(p^*-r)} \right)^{\frac{1}{p^*-p}},$$

and the maximum value of φ is

$$\varphi(\bar{t}) = \frac{p^*-p}{p(p^*-r)} \left(\frac{(p-r)p^* S^{\frac{p^*}{p}}}{p(p^*-r)} \right)^{\frac{p-r}{p^*-p}} - \frac{1}{p}C_{N,r}^p c^{\frac{p-r}{p}}.$$

Thus, recalling that

$$c^{\frac{p-r}{p}} < \alpha(N, p, r) = \frac{p^*-p}{p^*-r} \left(\frac{(p-r)p^* S^{\frac{p^*}{p}}}{p(p^*-r)} \right)^{\frac{p-r}{p^*-p}} \frac{1}{C_{N,r}^p},$$

we arrive at $f(\bar{t}) = (\bar{t})^r \varphi(\bar{t}) > 0$. Combined with the facts that $f(t) \rightarrow 0^-$ as $t \rightarrow 0^+$ and $f(t) \rightarrow -\infty$ as $t \rightarrow +\infty$, this indicates that f has at least two zeros satisfying $0 < R_0 < R_1$. It is evident that φ possesses only one critical point. Notably, since $f(t) = 0$ is equivalent to $\varphi(t) = 0$, f has precisely two zeros $0 < R_0 < R_1$, and $f(t) > 0$ if and only if $t \in (R_0, R_1)$.

As a result, f attains its global maximum with a positive value in (R_0, R_1) and a local minimum with a negative value in $(0, R_0)$. It should be noted that $f'(t) = 0$ is equivalent to

$$\zeta(t) = t^{p-r} - S^{-\frac{p^*}{p}} t^{p^*-r} - \frac{r}{p} C_{N,r}^p c^{\frac{p-r}{p}} = 0.$$

Given that ζ has only one critical point in $(0, +\infty)$, we can infer that f has exactly two critical points, which correspond to the strict local minimum point and the global maximum point identified earlier. $\square$

Lemma 2.2. $\mathcal{V}_c^0 = \emptyset$.

Proof. Let us assume that there exists $u \in \mathcal{V}_c^0$. Then, combining $P(u) = 0$ and $(\Psi_u)''(0) = 0$, we deduce that

$$\left(r - \frac{r^2}{p} \right) \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx = (p^* - p) \int_{\mathbb{R}^N} |u|^{p^*} dx.$$

Using this equation and $P(u) = 0$, we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^p dx = \frac{p^* - r}{p - r} \int_{\mathbb{R}^N} |u|^{p^*} dx \leq \frac{S^{-\frac{p^*}{p}} (p^* - r)}{p - r} \int_{\mathbb{R}^N} |\nabla u|^{p^*} dx, \quad (2.2)$$

and

$$\int_{\mathbb{R}^N} |\nabla u|^p dx = \frac{p^* r - r^2}{(p^* - p)p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx \leq \frac{(p^* r - r^2)}{(p^* - p)p} C_{N,r}^p c^{\frac{p-r}{p}} \int_{\mathbb{R}^N} |\nabla u|^r dx. \quad (2.3)$$

From (2.2) and (2.3), we infer that

$$\left(\frac{p - r}{S^{-\frac{p^*}{p}} (p^* - r)} \right)^{\frac{1}{p^* - p}} \leq \left(\frac{(p^* r - r^2)}{(p^* - p)p} C_{N,r}^p c^{\frac{p-r}{p}} \right)^{\frac{1}{p-r}},$$

that is,

$$c^{\frac{p-r}{p}} \geq \left(\frac{p - r}{S^{-\frac{p^*}{p}} (p^* - r)} \right)^{\frac{p-r}{p^* - p}} \frac{p(p^* - p)}{(p^* r - r^2) C_{N,r}^p}. \quad (2.4)$$

It can be verified that this contradicts (1.5). To show this, it suffices to prove that the right-hand side of (1.5) is less than or equal to that of (2.4), and this is equivalent to

$$\left(\frac{p^*}{p} \right)^{p-r} \left(\frac{r}{p} \right)^{p^* - p} \leq 1. \quad (2.5)$$

Since $\phi(t) = \frac{\log t}{t-1}$ is monotonically decreasing over $(0, +\infty)$, we obtain $\phi(\frac{p^*}{p}) \leq \phi(\frac{r}{p})$, that is (2.5). $\square$

Lemma 2.3. *For every $u \in S(c)$, the function Ψ_u has exactly two critical points s_u, t_u and two zeros c_u, d_u with $s_u < c_u < t_u < d_u$. Moreover,*

- (a) $u \star s_u \in \mathcal{V}_c^+$ and $u \star t_u \in \mathcal{V}_c^-$;
- (b) $\|\nabla(u \star s)\|_p < R_0$ for $s < c_u$ and

$$I(u \star s_u) = \min\{I(u \star s) : s \in \mathbb{R}, \|\nabla(u \star s)\|_p < R_0\} < 0, \quad (2.6)$$

where R_0 is given in Lemma 2.1;

- (c) there holds

$$I(u \star t_u) = \max\{I(u \star s) : s \in \mathbb{R}\} > 0, \quad (2.7)$$

and if $t_u < 0$, then $P(u) < 0$;

- (d) the maps $u \in S(c) \mapsto s_u \in \mathbb{R}$ and $u \in S(c) \mapsto t_u \in \mathbb{R}$ are of class C^1 .

Proof. By (2.1), one has $\Psi_u(t) = I(u \star t) \geq f(\|\nabla(u \star t)\|_p) = f(e^t \|\nabla u\|_p)$. According to Lemma 2.1, Ψ_u is positive on $(\log(R_0/\|\nabla u\|_p), \log(R_1/\|\nabla u\|_p))$, and $\Psi_u(t) \rightarrow 0^-$ as $t \rightarrow -\infty$ and $\Psi_u(t) \rightarrow -\infty$ as $t \rightarrow +\infty$. Hence Ψ_u has at least two critical points $s_u < t_u$, where s_u is the local minimum point of Ψ_u on $(-\infty, \log(R_0/\|\nabla u\|_p))$ with $\Psi_u(s_u) < 0$, and t_u is the global maximum point of Ψ_u with $\Psi_u(t_u) > 0$. Indeed $\Psi'_u(t) = 0$ equals to

$$\psi(t) = e^{(p-r)t} \int_{\mathbb{R}^N} |\nabla u|^p dx - e^{(p^*-r)t} \int_{\mathbb{R}^N} |u|^{p^*} dx - \frac{r}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx = 0.$$

Moreover, it is evident that ψ possesses a single critical point. Consequently, this implies that Ψ_u has precisely two critical points, denoted as s_u and t_u . By applying a comparable line of

reasoning, one can establish that Ψ_u likewise contains exactly two zeros c_u and d_u , satisfying the order $s_u < c_u < t_u < d_u$. Recall that $u \star t \in \mathcal{V}_c$ if and only if $(\Psi_u)'(t) = 0$. Given s_u as the local minimum of Ψ_u and t_u as global maximum of Ψ_u , we have $\Psi_u'(s_u) = \Psi_u'(t_u) = 0$, along with $\Psi_u''(s_u) > 0 > \Psi_u''(t_u)$. By Lemma 2.2, it follows that $u \star s_u \in \mathcal{V}_c^+$ and $u \star t_u \in \mathcal{V}_c^-$. Clearly, $\|\nabla(u \star s)\|_p < \|\nabla(u \star c_u)\|_p \leq R_0$ when $s_u < c_u$ (this is what (2.6) conveys). The reason lies in s_u being a local minimum of Ψ_u over $(-\infty, \log(R_0/\|\nabla u\|_p))$ with $\Psi_u(s_u) < 0$. For (2.7), it stems from t_u being the global maximum of Ψ_u on the real line, where $\Psi_u(t_u) > 0$. Notice that Ψ_u' has precisely two zeros s_u and t_u . Moreover, $\Psi_u'(t) < 0$ for $t > t_u$, indicating that Ψ_u is decreasing on $(t_u, +\infty)$. Specifically, if $t_u < 0$, then $P(u) = \Psi_u'(0) < 0$.

Next, we aim to demonstrate $u \mapsto t_u$ is of class C^1 . To this end, we define $\Upsilon : S(c) \times \mathbb{R} \rightarrow \mathbb{R}$ by $\Upsilon(u, t) = \Psi_u'(t)$. Clearly, Υ is C^1 , $\Upsilon(u, t_u) = 0$, and $\partial_t \Upsilon(u, t_u) = \Psi_u''(t_u) < 0$. By the Implicit Function Theorem, the map $u \mapsto t_u$ is C^1 . The same argument proves that $u \mapsto s_u$ is of class C^1 . $\square$

2.2. Proof of Theorem 1.1 (Part I). It is easy to check that $R_0 < \bar{t} < S^{\frac{p^*}{p(p^*-p)}}$, where $\bar{t}$ is given in Lemma 2.1. For $k > 0$, set $Z_k(c) = \{u \in S(c) : \|\nabla u\|_p < k\}$. We prove the existence of positive ground states for $I|_{S(c)}$ in $Z_{R_0}(c)$ by minimizing I . Consider the following minimization problem $\inf_{u \in Z_{R_0}(c)} I(u)$.

Lemma 2.4.

$$m(c) = \inf_{u \in \mathcal{V}_c^+} I(u) = \inf_{u \in Z_{R_0}(c)} I(u) = \inf_{u \in Z_{\bar{t}}(c)} I(u) < \inf_{u \in Z_{R_0}(c) \setminus Z_{R_0-\beta}(c)} I(u),$$

for $\beta > 0$ small and $m(c) \in (-\infty, 0)$.

Proof. From Lemma 2.3, we deduce that $I(u) < 0$ for $u \in \mathcal{V}_c^+$ and $I(u) > 0$ for $u \in \mathcal{V}_c^-$. Together with Lemma 2.2, this implies $\inf_{u \in \mathcal{V}_c^+} I(u) = \inf_{u \in \mathcal{V}_c} I(u) = m(c)$. Since $\mathcal{V}_c^+ \subset Z_{R_0}(c)$, we have $\inf_{u \in Z_{R_0}(c)} I(u) \leq m(c)$. By Lemma 2.3 again, for any $u \in Z_{R_0}(c)$, we have $u \star s_u \in \mathcal{V}_c^+$ and

$$I(u \star s_u) = \min\{I(u \star s) : s \in \mathbb{R}, \|\nabla(u \star s)\|_p < R_0\} \leq I(u),$$

which yields $m(c) \leq \inf_{u \in Z_{R_0}(c)} I(u)$. Thus, $m(c) = \inf_{u \in Z_{R_0}(c)} I(u)$. From (2.1), for $u \in Z_{R_0}(c)$, we have

$$I(u) \geq f(\|\nabla u\|_p) \geq \min_{t \in [0, R_0]} f(t) > -\infty.$$

Lemma 2.3 indicates that if $u \in S(c)$, then $u \star s_u \in Z_{R_0}(c)$ and $I(u \star s_u) < 0$. Therefore, $-\infty < m(c) < 0$. By Lemma 2.1, $I(u) \geq 0$ for $u \in Z_{\bar{t}}(c) \setminus Z_{R_0}(c)$, which implies that $m(c) = \inf_{u \in Z_{\bar{t}}(c)} I(u)$. Since f is continuous and $f(R_0) = 0$, we have $\inf_{t \in [R_0-\beta, R_0]} f(t) > m(c)$ for small $\beta > 0$. Note that $I(u) \geq f(\|\nabla u\|_p)$ for $u \in S(c)$. Hence,

$$\inf_{u \in Z_{R_0}(c) \setminus Z_{R_0-\beta}(c)} I(u) \geq \inf_{t \in [R_0-\beta, R_0]} f(t) > m(c).$$

$\square$

Lemma 2.5. Let $1 < p < N$ and $1 < r < p$. Assume that $\{u_n\} \subset S(c)$ is a Palais-Smale sequence for $I|_{S(c)}$ at level $m(c) \in \mathbb{R}$ and $P(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Then $\{u_n\}$ is bounded in E .

Proof. The fact that $P(u_n) = o(1)$, Lemma 1.1 and $0 < r < p$ lead to

$$\begin{aligned} C &\geq I(u_n) - \frac{1}{p^*} P(u_n) \\ &= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u_n|^p dx - \left(\frac{1}{p} - \frac{r}{p^* p} \right) \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx \\ &\geq \frac{1}{N} \|\nabla u_n\|_p^p - \left(\frac{1}{p} - \frac{r}{p^* p} \right) C_{N,r}^p c^{\frac{p-r}{p}} \|\nabla u_n\|_p^r, \end{aligned}$$

given that n is large enough, we can deduce that $\{u_n\}$ is bounded in E . $\square$

By Lemma 2.5 and the Ekeland variational principle, there exists a sequence of Lagrange multipliers $\{\lambda_n\} \subset \mathbb{R}^N$ such that $I(u_n) \rightarrow m(c)$, and

$$-\Delta_p u_n + \lambda_n |u_n|^{p-2} u_n = \frac{|u_n|^{p-2} u_n}{|x|^r} + |u_n|^{p^*-2} u_n + o(1), \text{ in } \mathbb{R}^N. \quad (2.8)$$

It follows from (2.8) that

$$|\lambda_n|c = \left| \int_{\mathbb{R}^N} |\nabla u_n|^p dx - \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx - \int_{\mathbb{R}^N} |u_n|^{p^*} dx \right| \leq C < +\infty.$$

This implies that $\{\lambda_n\}$ is bounded, and if $P(u_n) = 0$, then

$$\begin{aligned} \lambda_n c &= \lambda_n |u_n|_p^p \\ &= - \int_{\mathbb{R}^N} |\nabla u_n|^p dx + \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx + \int_{\mathbb{R}^N} |u_n|^{p^*} dx \\ &= \left(1 - \frac{r}{p} \right) \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx > 0. \end{aligned} \quad (2.9)$$

In the following, we denote $\lambda_n \rightarrow \lambda_* > 0$ as $n \rightarrow \infty$.

Next, we discuss the compactness of the Palais-Smale sequence.

Lemma 2.6. ([12]) *Let $\{v_k\}$ be a sequence in $D^{1,p}(\mathbb{R}^N)$ and be such that $v_k \rightharpoonup v$ weakly in $D^{1,p}(\mathbb{R}^N)$. Assume that, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} \varphi (|\nabla v_k|^{p-2} \nabla v_k - |\nabla v|^{p-2} \nabla v) \cdot \nabla (T(v_k - v)) dx = 0,$$

where the truncation function $T : \mathbb{R} \rightarrow [0, 1]$ is given by

$$T(t) = \begin{cases} t, & \text{if } |t| \leq 1, \\ \frac{t}{|t|}, & \text{if } |t| \geq 1. \end{cases} \quad (2.10)$$

Then, after passing to a subsequence, $\nabla v_k \rightarrow \nabla v$ a.e. in $\mathbb{R}^N$.

Lemma 2.7. *If $\{u_k\}$ satisfies (2.8) and $P(u_n) = 0$, then $\nabla u_n \rightarrow \nabla u$ a.e. in $\mathbb{R}^N$.*

Proof. If $\{u_k\}$ satisfies (2.8) and $P(u_n) = 0$. Then, similar to [12], we take $\varphi \in C_0^\infty(\mathbb{R}^N)$. Egorov's Theorem implies that, for any $\rho > 0$, there exists $E_\rho \subset \text{supp}(\varphi)$ with $|E_\rho| < \rho$ such

that $u_n \rightarrow u$ uniformly on $\text{supp}(\varphi) \setminus E_\rho$. Thus, for all $x \in \text{supp}(\varphi) \setminus E_\rho$ and sufficiently large n , we have $|u_n(x) - u(x)| \leq 1$. Then, from (2.10), we can state that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla (T(u_n - u)) dx \right| \\ & \leq \left| \int_{\mathbb{R}^N \setminus E_\rho} \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla (u_n - u) dx \right| + \left| \int_{E_\rho} \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla (T(u_n - u)) dx \right| \\ & \leq o(1) + C\rho. \end{aligned}$$

Thus, we infer that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla (T(u_n - u)) dx = 0. \quad (2.11)$$

On the other hand, using the Hölder's inequality and (2.8), we can verify that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \varphi |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (T(u_n - u)) dx \right| \\ & \leq \left| \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (\varphi T(u_n - u)) dx \right| + \left| \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \cdot (T(u_n - u)) \nabla \varphi dx \right| \\ & \leq \left| \int_{\mathbb{R}^N} |u_n|^{p^*-2} u_n (\varphi T(u_n - u)) dx \right| + \left| \int_{\mathbb{R}^N} \frac{|u_n|^{p-2}}{|x|^r} u_n (\varphi T(u_n - u)) dx \right| \\ & \quad + \left| \int_{\mathbb{R}^N} \lambda_* |u_n|^{p-2} u_n (\varphi T(u_n - u)) dx \right| + \left| \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \cdot (T(u_n - u)) \nabla \varphi dx \right| + o(1) \\ & \leq C \left(\int_{\mathbb{R}^N} |\varphi T(u_n - u)|^{p^*} dx \right)^{\frac{1}{p^*}} + C \left(\int_{\mathbb{R}^N} |\varphi T(u_n - u)|^p dx \right)^{\frac{1}{p}} \\ & \quad + C \left(\int_{\mathbb{R}^N} |\varphi T(u_n - u)|^p dx \right)^{\frac{1}{p}} + C \left(\int_{\mathbb{R}^N} |T(u_n - u) \nabla \varphi|^p dx \right)^{\frac{1}{p}} + o(1) = o(1). \end{aligned}$$

This means $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \varphi |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (T(u_n - u)) dx = 0$, which together with (2.11) and Lemma 2.6 entails that this Lemma holds. $\square$

Lemma 2.8. *If $\{u_n\} \subset \mathcal{V}_c^+$ is a minimizing sequence of E , then $u_n \rightarrow u \in L^{p^*}(\mathbb{R}^N)$.*

Proof. Since $u_n \rightharpoonup u$ in $D^{1,p}(\mathbb{R}^N)$, using the second concentration compactness lemma of Lions [19], there exist at most a countable index set L , a family $\{x_i : i \in L\} \subset \mathbb{R}^N$, and two families of positive numbers $\{\sigma_i : i \in L\}$, $\{\alpha_i : i \in L\}$ satisfying

$$\|\nabla u_n\|^p \rightharpoonup \sigma \geq \|\nabla u\|^p + \sum_{i \in L} \sigma_i \beta_{x_i}, \quad |u_n|^{p^*} \rightharpoonup \alpha = |u|^{p^*} + \sum_{i \in L} \alpha_i \beta_{x_i}, \quad (2.12)$$

where the convergence is in the weak sense of measures. Here, β_{x_i} denotes the Dirac mass concentrated at x_i , and it holds that

$$\sum_{i \in L} \alpha_i^{\frac{p}{p^*}} < \infty; \quad S \alpha_i^{\frac{p}{p^*}} \leq \sigma_i, \quad i \in L. \quad (2.13)$$

Take $x_i \in \mathbb{R}^N$ in the support of the singular parts of ω, α, ξ . For any $\varepsilon > 0$, define $\chi_\varepsilon(x) := \bar{\chi}_\varepsilon(x - x_i)$, where $\bar{\chi}_\varepsilon \in C_0^\infty(\mathbb{R}^N, [0, 1])$ satisfies

$$\bar{\chi}_\varepsilon \equiv \begin{cases} 1, & \text{on } B_\varepsilon(0), \\ 0, & \text{on } \mathbb{R}^N \setminus B_{2\varepsilon}(0), \end{cases}$$

and $\|\nabla \tilde{\chi}_\varepsilon\| \in [0, \frac{2}{\varepsilon}]$. The proof is now divided into three steps.

Step 1. For any $i \in L$, $\sigma_i \leq \alpha_i$. It is clear that the sequence $\{\chi_\varepsilon u_n\}$ is bounded in E . Moreover, $\{u_n\} \subset \mathcal{V}_c^+$ means $P(u_n) = 0$ and $\{u_n\} \subset Z_{R_0}(c)$. Then, from (2.8)

$$\int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \chi_\varepsilon dx = \int_{\mathbb{R}^N} (-\lambda_* |u_n|^p + \frac{|u_n|^p}{|x|^r} + |u_n|^{p^*} - |\nabla u_n|^p) \chi_\varepsilon dx + o(1).$$

By applying the Hölder inequality, the following limit expression is derived

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \chi_\varepsilon dx \right| \\ & \leq \limsup_{n \rightarrow \infty} \left(\int_{B_{2\varepsilon}(x_i)} |u_n \nabla \chi_\varepsilon|^p dx \right)^{\frac{1}{p}} \left(\int_{B_{2\varepsilon}(x_i)} |\nabla u_n|^p dx \right)^{\frac{p-1}{p}} \\ & \leq C \left(\int_{B_{2\varepsilon}(x_i)} |u|^p |\nabla \chi_\varepsilon|^p dx \right)^{\frac{1}{p}} \\ & \leq C \left(\int_{B_{2\varepsilon}(x_i)} |u|^{p^*} dx \right)^{\frac{1}{p^*}} \left(\int_{B_{2\varepsilon}(x_i)} |\nabla \chi_\varepsilon|^{\frac{p^* p}{p^*-p}} dx \right)^{\frac{p^*-p}{p^* p}} \\ & = C \left(\int_{B_{2\varepsilon}(x_i)} |u|^{p^*} dx \right)^{\frac{1}{p^*}} \rightarrow 0, \end{aligned}$$

as $\varepsilon \rightarrow 0$. In addition, since χ_ε is compactly supported, we have that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p \chi_\varepsilon dx \geq \int_{\mathbb{R}^N} |\nabla u|^p \chi_\varepsilon dx + \left\langle \sum_{i \in L} \sigma_i \beta_{x_i}, \chi_\varepsilon \right\rangle, \quad (2.14)$$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{p^*} \chi_\varepsilon dx = \int_{\mathbb{R}^N} |u|^{p^*} \chi_\varepsilon dx + \left\langle \sum_{i \in L} \alpha_i \beta_{x_i}, \chi_\varepsilon \right\rangle. \quad (2.15)$$

Moreover, it follows from (2.14) and (2.15) that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p-2} |\nabla u_n| \nabla \chi_\varepsilon dx \leq - \int_{\mathbb{R}^N} |\nabla u|^p \chi_\varepsilon dx + \int_{\mathbb{R}^N} |u|^{p^*} \chi_\varepsilon dx \\ & + \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} \chi_\varepsilon dx - \lambda_* \int_{\mathbb{R}^N} |u|^p \chi_\varepsilon dx - \left\langle \sum_{i \in L} \sigma_i \beta_{x_i}, \chi_\varepsilon \right\rangle + \left\langle \sum_{i \in L} \alpha_i \beta_{x_i}, \chi_\varepsilon \right\rangle. \end{aligned} \quad (2.16)$$

Thus, taking the limit in (2.16) as $\varepsilon \rightarrow 0$, we have

$$\sigma_i \leq \alpha_i. \quad (2.17)$$

We claim that $\sigma_i = 0$ for all $i \in L$ and $L = \emptyset$. Assume that there exists an $i_0 \in L$ such that $\sigma_{i_0} > 0$. Using (2.13) and (2.17), we obtain

$$\sigma_{i_0} \geq S^{\frac{p^*}{p^*-p}}. \quad (2.18)$$

Since $\{u_n\} \subset Z_{R_0}(c)$, (2.14) and (2.18) yield

$$R_0^p \geq \limsup_{n \rightarrow \infty} \|\nabla u_n\|_p^p \geq \sigma_{i_0} \geq S^{\frac{p^*}{p^*-p}},$$

which contradicts $R_0 < S^{\frac{p^*}{p(p^*-p)}}$. Then $L = \emptyset$, and $u_n \rightarrow u$ in $L_{loc}^{p^*}(\mathbb{R}^N)$.

Step 2. $\sigma_\infty \leq \alpha_\infty$. By the well-known concentration-compactness principle due to Lions [33], let $\{u_n\}$ be a sequence weakly converging to u in $D^{1,p}(\mathbb{R}^N)$ and define

$$\alpha_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^{p^*} dx, \quad \sigma_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla u_n|^p dx. \quad (2.19)$$

Then the quantities α_∞ and σ_∞ are well defined and satisfy

$$\alpha_\infty + \int_{\mathbb{R}^N} d\alpha = \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^{p^*} dx, \quad \sigma_\infty + \int_{\mathbb{R}^N} d\sigma = \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla u_n|^p dx, \quad (2.20)$$

and

$$S\alpha_\infty^{p/p^*} \leq \sigma_\infty, \quad (2.21)$$

where σ and α are defined in (2.12). For any $R > 0$, let $\varphi_R : \mathbb{R}^N \rightarrow [0, 1]$ be a smooth function satisfying

$$\varphi_R = \begin{cases} 0, & \text{if } |x| \leq R, \\ 1, & \text{if } |x| \geq 2R, \end{cases}$$

and $|\nabla \varphi_R| \leq 2/R$. It is easy to see that $\{u_n \varphi_R\}$ is bounded in E , similarly, from (2.8) we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-1} \nabla (\varphi_R \cdot u_n) dx + \lambda_* \int_{\mathbb{R}^N} |u_n|^p \varphi_R dx = \int_{\mathbb{R}^N} |u_n|^{p^*} \varphi_R dx + \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} \varphi_R dx + o(1). \quad (2.22)$$

Note that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| \geq R} \frac{|u_n|^p}{|x|^r} \varphi_R dx \leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^p}{|R|^r} \varphi_R dx \rightarrow 0. \quad (2.23)$$

On one hand, by (2.19) and (2.23), we have

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |u_n|^{p^*} \varphi_R dx + \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} \varphi_R dx \right) = \alpha_\infty. \quad (2.24)$$

On the other hand, by (2.19), we also have

$$\begin{aligned} & \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-1} \nabla (\varphi_R \cdot u_n) dx \\ &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^p \varphi_R dx + \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-1} \nabla \varphi_R \cdot u_n dx \\ &= \sigma_\infty + \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-1} \nabla \varphi_R \cdot u_n dx, \end{aligned} \quad (2.25)$$

and

$$\begin{aligned} & \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-1} \nabla \varphi_R \cdot u_n dx \\ & \leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |\nabla \varphi_R \cdot u_n|^p dx \right)^{\frac{1}{p}} \\ & \leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |u_n|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |\nabla \varphi_R|^q dx \right)^{\frac{1}{q}} \\ & \leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{C}{R^{\frac{q-N}{q}}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^p dx \right)^{\frac{p-1}{p}} \|u_n\|_p = 0, \end{aligned} \quad (2.26)$$

where $q > N$ and $\frac{1}{p} + \frac{1}{q} = 1$. Combining (2.22), (2.24), (2.25), (2.26), we have $\sigma_\infty \leq \alpha_\infty$.

Step 3. $\alpha_i = 0$ for any $i \in L$ and $\alpha_\infty = 0$.

Suppose that there exists a $i_0 \in L$ such that $\alpha_{i_0} > 0$ or $\alpha_\infty > 0$. Using (2.21) and $\sigma_\infty \leq \alpha_\infty$, we obtain $S^{p^*} \alpha_{i_0}^p \leq \sigma_{i_0}^{p^*} = \alpha_{i_0}^{p^*}$, $S^{p^*} \alpha_\infty^p \leq \sigma_\infty^{p^*} = \alpha_\infty^{p^*}$, which implies that

$$\alpha_{i_0} \geq S^{\frac{p^*}{p^*-p}}, \quad \alpha_\infty \geq S^{\frac{p^*}{p^*-p}}. \quad (2.27)$$

By Lemma 2.1, we have $\|\nabla u_n\|_p \leq S^{\frac{p^*}{p(p^*-p)}}$. Moreover, using (2.27), (2.12), the fact that $P(u_n) = o(1)$, $c^{\frac{p-r}{p}} < \alpha(N, p, r)$, Lemma 1.1, and $0 < r < p < p^*$, we obtain

$$\begin{aligned} I(u_n) - \frac{1}{p}P(u_n) &= \frac{p^*-p}{pp^*} \int_{\mathbb{R}^N} |u_n|^{p^*} dx - \frac{p-r}{p^2} \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx + o(1) \\ &\geq \frac{p^*-p}{pp^*} \int_{\mathbb{R}^N} |u_n|^{p^*} \chi_\varepsilon dx - \frac{p-r}{p^2} C_{N,r}^p c^{\frac{p-r}{p}} \|\nabla u_n\|_p^r + o(1) \\ &\geq \frac{p^*-p}{pp^*} \alpha_{i_0} - \frac{p-r}{p^2} C_{N,r}^p \frac{p^*-p}{(p^*-r)C_{N,r}^p} \left(\frac{(p-r)p^*S^{\frac{p^*}{p}}}{p(p^*-r)} \right)^{\frac{p-r}{p^*-p}} S^{\frac{rp^*}{p(p^*-p)}} \\ &\geq \frac{1}{N} S^{\frac{N}{p}} - \frac{p-r}{p^2} C_{N,r}^p \frac{p^*-p}{(p^*-r)C_{N,r}^p} \left(\frac{(p-r)p^*S^{\frac{p^*}{p}}}{p(p^*-r)} \right)^{\frac{p-r}{p^*-p}} S^{\frac{rp^*}{p(p^*-p)}} \\ &= S^{\frac{N}{p}} \left(\frac{1}{N} - \frac{p^*-p}{p^2} \left(\frac{p-r}{p^*-r} \right)^{\frac{p^*-r}{p^*-p}} \left(\frac{p^*}{p} \right)^{\frac{p-r}{p^*-p}} \right) > 0, \end{aligned}$$

which is a contradiction since $m(c) < 0$. Similarly, if $\alpha_\infty > 0$, we can obtain

$$\begin{aligned} I(u_n) - \frac{1}{p}P(u_n) &= \frac{p^*-p}{pp^*} \int_{\mathbb{R}^N} |u_n|^{p^*} dx - \frac{p-r}{p^2} \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx + o(1) \\ &\geq \frac{p^*-p}{pp^*} \int_{\mathbb{R}^N} |u_n|^{p^*} \chi_\varepsilon dx - \frac{p-r}{p^2} C_{N,r}^p c^{\frac{p-r}{p}} \|\nabla u_n\|_p^r + o(1) \\ &\geq \frac{p^*-p}{pp^*} \alpha_\infty - \frac{p-r}{p^2} C_{N,r}^p \frac{p^*-p}{(p^*-r)C_{N,r}^p} \left(\frac{(p-r)p^*S^{\frac{p^*}{p}}}{p(p^*-r)} \right)^{\frac{p-r}{p^*-p}} S^{\frac{rp^*}{p(p^*-p)}} \\ &\geq \frac{1}{N} S^{\frac{N}{p}} - \frac{p-r}{p^2} C_{N,r}^p \frac{p^*-p}{(p^*-r)C_{N,r}^p} \left(\frac{(p-r)p^*S^{\frac{p^*}{p}}}{p(p^*-r)} \right)^{\frac{p-r}{p^*-p}} S^{\frac{rp^*}{p(p^*-p)}} \\ &= S^{\frac{N}{p}} \left(\frac{1}{N} - \frac{p^*-p}{p^2} \left(\frac{p-r}{p^*-r} \right)^{\frac{p^*-r}{p^*-p}} \left(\frac{p^*}{p} \right)^{\frac{p-r}{p^*-p}} \right) > 0, \end{aligned}$$

which is also a contradiction. Therefore, $\alpha_i = 0$ for any $i \in L$ and $\alpha_\infty = 0$. By the conclusion of Step 3 and (2.12) and (2.20), we have $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{p^*} dx = \int_{\mathbb{R}^N} |u|^{p^*} dx$. Using the Brezis-Lieb Lemma, the following holds immediately $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n - u|^{p^*} dx = 0$. This completes the proof. $\square$

For $0 < R_0 < R_1 < \infty$, let $\tau : \mathbb{R}^+ \rightarrow [0, 1]$ be a nonincreasing and C^∞ function satisfying $\tau(x) = 1, x \in [0, R_0]$ and $\tau(x) = 0, x \geq R_1$. In what follows, we consider the truncated functional

$$\begin{aligned} I_T(u) &= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{\tau(|\nabla u|_p)}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx - \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx \\ &\geq \frac{1}{p} \|\nabla u\|_p^p - \frac{S^{-\frac{p^*}{p}} \tau(|\nabla u|_p)}{p^*} \|\nabla u\|_p^{p^*} - \frac{1}{p} C_{N,r}^p c^{\frac{p-r}{p}} \|\nabla u\|_p^r. \end{aligned}$$

A straightforward calculation shows $I_T(u) \geq \tilde{f}(\|\nabla u\|_p)$, where $\tilde{f} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is given by

$$\tilde{f}(t) = \frac{1}{p} t^p - \frac{S^{-\frac{p^*}{p}} \tau(t)}{p^*} t^{p^*} - \frac{1}{p} C_{N,r}^p c^{\frac{p-r}{p}} t^r.$$

It is noted that $\tilde{f} < 0$ over interval $(0, R_0)$ and $\tilde{f}(t) > 0$ over $(R_0, +\infty)$ and hence

$$m(c) = \inf_{u \in Z_{R_0}(c)} I(u) = \inf_{u \in Z_{R_0}(c)} I_T(u) = \inf_{u \in S(c)} I_T(u).$$

Lemma 2.9. (i) $I_T \in C^1(E, \mathbb{R})$. If $I_T(u) < 0$, then $\|\nabla u\|_p < R_0$ and $I(v) = I_T(v)$ holds for all v within a sufficiently small neighborhood of u .

(ii) I_T verifies a local $(PS)_{m(c)}$ condition on $S(c)$ for the level $m(c) < 0$.

Proof. Conclusion (i) is an immediate consequence of the definition of I_T .

(ii) Let $\{u_n\}$ be a $(PS)_{m(c)}$ sequence of I_T restricted to $S(c)$ with $m(c) < 0$. Then, by the definition of I_T , for sufficiently large n , we have $\|\nabla u_n\|_p < R_0 < t_0$. Consequently, $\{u_n\}$ is also a $(PS)_{m(c)}$ sequence of I restricted to $S(c)$ with $m(c) < 0$. Given that $\{u_n\} \subset E$ is bounded, we can assume $u_n \rightharpoonup u$ weakly in E . By Lemma 2.8 and $0 < r < p$, it is easy for us to obtain that

$$\int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx \rightarrow \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx. \quad (2.28)$$

In fact

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx &= \int_{|x| \geq R} \frac{|u_n|^p}{|x|^r} dx + \int_{|x| \leq R} \frac{|u_n|^p}{|x|^r} dx \\ &\leq \frac{1}{R^r} \int_{|x| \geq R} |u_n|^p dx + \left(\int_{|x| \leq R} |u_n|^{p^*} dx \right)^{\frac{p}{p^*}} \left(\int_{|x| \leq R} \frac{1}{|x|^{\frac{rp^*}{p^*-p}}} dx \right)^{\frac{p^*-p}{p^*}} \\ &\leq \frac{1}{R^r} \int_{|x| \geq R} |u_n|^p dx + \left(\int_{\mathbb{R}^N} |u_n|^{p^*} dx \right)^{\frac{p}{p^*}} \left(\int_{|x| \leq R} \frac{1}{|x|^{\frac{rp^*}{p^*-p}}} dx \right)^{\frac{p^*-p}{p^*}} \\ &\leq C \left(\int_{\mathbb{R}^N} |u_n|^{p^*} dx \right)^{\frac{p}{p^*}}, \end{aligned}$$

by letting first $n \rightarrow \infty$ and then $R \rightarrow \infty$. Since

$$\left(\int_{|x| \leq R} \frac{1}{|x|^{\frac{rp^*}{p^*-p}}} dx \right)^{\frac{p^*-p}{p^*}} = \left(\int_0^R \frac{t^{N-1}}{|t|^{\frac{rp^*}{p^*-p}}} dt \right)^{\frac{p^*-p}{p^*}} \leq C,$$

where $\frac{rp^*}{p^*-p} - N + 1 < 1$. From (2.8), we infer the existence of a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that for $\phi \in E$,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla \phi dx &= \int_{\mathbb{R}^N} |u_n|^{p^*-2} u_n \phi dx + \int_{\mathbb{R}^N} \frac{|u_n|^{p-2}}{|x|^r} u_n \phi dx \\ &\quad - \lambda_* \int_{\mathbb{R}^N} |u_n|^{p-2} u_n \phi dx + o(1) \|\phi\|, \end{aligned} \quad (2.29)$$

as $n \rightarrow \infty$. If $u = 0$, then $\int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx \rightarrow 0$ in $L^p(\mathbb{R}^N)$, and by the condition that $P(u_n) = 0$, we can get

$$\begin{aligned} I(u_n) &= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |u_n|^{p^*} dx \\ &= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u_n|^p dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |u_n|^{p^*} dx \\ &= \frac{1}{N} \|\nabla u_n\|_p^p, \end{aligned}$$

this contradicts $m(c) < 0$. Therefore, $u \neq 0$. By Lemma 2.7, we have $\nabla u_n \rightarrow \nabla u$ a.e. in $\mathbb{R}^N$. Therefore, when taking the limit in (2.29) as $n \rightarrow \infty$, we have

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \phi dx = \int_{\mathbb{R}^N} |u|^{p^*-2} u \phi dx + \int_{\mathbb{R}^N} \frac{|u|^{p-2}}{|x|^r} u \phi dx - \lambda_* \int_{\mathbb{R}^N} |u|^{p-2} u \phi dx. \quad (2.30)$$

This signifies that u is a weak solution of the equation

$$-\Delta_p u + \lambda_* |u|^{p-2} u = \frac{|u|^{p-2} u}{|x|^r} + |u|^{p^*-2} u, \text{ in } \mathbb{R}^N.$$

Let $\phi = u_n - u$, by Lemma 2.8, (2.28) and (2.29), (2.30) we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) (\nabla u_n - \nabla u) dx + \lambda_* \int_{\mathbb{R}^N} (|u_n|^{p-2} u_n - |u|^{p-2} u) (u_n - u) dx \\ &= \int_{\mathbb{R}^N} \frac{(|u_n|^{p-1} - |u|^{p-1})}{|x|^r} (u_n - u) dx + \int_{\mathbb{R}^N} (|u_n|^{p^*-2} u_n - |u|^{p^*-2} u) (u_n - u) dx + o(1) \\ &= o(1). \end{aligned} \quad (2.31)$$

Using a well known inequality found in Lemma A 0.5 of [14], we know that

$$(|\rho|^{p-2} \rho - |\zeta|^{p-2} \zeta) \cdot (\rho - \zeta) \geq \begin{cases} C|\rho - \zeta|^p, & \text{if } p \geq 2, \\ C(|\zeta| + |\rho|)^{p-2} |\rho - \zeta|^2, & \text{if } p \in (1, 2). \end{cases} \quad (2.32)$$

We need to discuss this in two cases now. If $p \geq 2$

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) (\nabla u_n - \nabla u) dx \geq C_1 \int_{\mathbb{R}^N} |\nabla(u_n - u)|^p dx,$$

and

$$\int_{\mathbb{R}^N} (|u_n|^{p-2} u_n - |u|^{p-2} u) (u_n - u) dx \geq C_2 \int_{\mathbb{R}^N} |u_n - u|^p dx,$$

from (2.31), we have

$$C_1 \|\nabla(u_n - u)\|_p^p + \lambda_* C_2 \|u_n - u\|_p^p = o(1).$$

If $1 < p < 2$, we deduce that

$$C_3 |\rho - \zeta|^p \leq \left| (|\rho|^{p-2} \rho - |\zeta|^{p-2} \zeta) \cdot (\rho - \zeta) \right|^{\frac{p}{2}} (|\rho| + |\zeta|)^{\frac{p(2-p)}{2}}.$$

By this inequality, Hölder's inequality and $(a+b)^w \leq 2^{w-1}(a^w + b^w)$ for $w \geq 1$, we have

$$\begin{aligned} C_3 \int_{\mathbb{R}^N} |\nabla(u_n - u)|^p dx &\leq \int_{\mathbb{R}^N} |M(u_n, u)|^{\frac{p}{2}} (\nabla u_n + \nabla u)^{\frac{p(2-p)}{2}} dx \\ &\leq \left(\int_{\mathbb{R}^N} |M(u_n, u)| dx \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^N} (\nabla u_n + \nabla u)^p dx \right)^{\frac{2-p}{2}} \\ &\leq \left(\int_{\mathbb{R}^N} |M(u_n, u)| dx \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^N} 2^{p-1} (|\nabla u_n|^p + |\nabla u|^p) dx \right)^{\frac{2-p}{2}}, \end{aligned}$$

where $M(u_n, u) = (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u)(\nabla u_n - \nabla u)$. As $\{u_n\}$ is bounded in E , we have

$$\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u)(\nabla u_n - \nabla u) dx \geq C_3 \left(\int_{\mathbb{R}^N} |\nabla(u_n - u)|^p dx \right)^{\frac{2}{p}},$$

similarly

$$\int_{\mathbb{R}^N} (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) dx \geq C_4 \left(\int_{\mathbb{R}^N} |u_n - u|^p dx \right)^{\frac{2}{p}}.$$

From (2.31), we also obtain

$$C_3 \|\nabla(u_n - u)\|_p^2 + \lambda_* C_4 |u_n - u|_p^2 = o(1).$$

Note that $\lambda_* > 0$, we immediately obtain $u_n \rightarrow u$ in E . $\square$

Proof of Theorem 1.1 (Part I). Recall that I_T is bounded below on $S(c)$. According to the definition of $m(c)$ for each positive integer n , there exists $\{v_n\} \subset S(c)$ with $v_n \geq 0$ and

$$I_T(v_n) < m(c) + \frac{1}{n}.$$

Applying [41] guarantees the existence of $\{u_n\} \subset S(c)$ satisfying

$$m(c) \leq I_T(u_n) \leq m(c) + \frac{2}{n}, \quad \min_{\lambda \in \mathbb{R}} \|I'_T(u_n) + \lambda |u_n|^{p-2} u_n\| \leq \frac{4}{\sqrt{n}}, \quad \|u_n - v_n\| \leq \frac{1}{\sqrt{n}}.$$

Thus, $\{u_n\}$ is a Palais-Smale sequence for $I_T|_{S(c)}$ at the level $m(c)$ with $|u_n^-|_p \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 2.9, $u_n \rightarrow u$ in E for some u , which implies $I_T(u) = m(c)$. Furthermore, from Lemma 2.4 and Lemma 2.9, it follows that $u \in Z_{R_0}(c)$. Consequently, u is also a critical point of I on $S(c)$ with $I(u) = m(c)$ and solves (1.1) for some $\lambda_* > 0$. Since $|u_n^-|_p \rightarrow 0$ as $n \rightarrow \infty$, we have $u \geq 0$. In addition, following the proof in [39], it is straightforward to conclude that $u(x) > 0$ for $x \in \mathbb{R}^N$.

2.3. Proof of Theorem 1.1 (Part II). In this subsection, we will need the following lemma, see Lemma 3.6 of [3].

Lemma 2.10. For $u \in S(c)$ and $s \in \mathbb{R}$, the map $\psi \rightarrow \psi \star s$ from $T_u S(c)$ to $T_{u \star s} S(c)$ is a linear isomorphism with inverse, $\psi \rightarrow \psi \star (-s)$, where $T_u S(c) := \{\psi \in S(c) : \int_{\mathbb{R}^N} |u|^{p-2} u \psi dx = 0\}$.

Proof. We follow the approach in Lemma 3.6 of [3]. For $\varphi \in T_u S(c)$ and $t > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |(u \star t)|^{p-2} (u \star t) (\psi \star t) dx &= \int_{\mathbb{R}^N} e^{Nt} |u(e^t x)|^{p-2} u(e^t x) \psi(e^t x) dx \\ &= \int_{\mathbb{R}^N} |u(y)|^{p-2} u(y) \psi(y) dy = 0, \end{aligned}$$

which implies that $\psi \star t \in T_{u \star t} S(c)$, and the map is well-defined. Clearly, it is linear. Taking into account that, for every $t, s > 0$ and $g \in E$,

$$(g \star t) \star s = e^{\frac{N}{p}(t+s)} g(e^{t+s}x) = g \star (t+s), \quad g \star 0 = g.$$

The result follows immediately. $\square$

Lemma 2.11. *The map $(u, t) \in E \times \mathbb{R} \rightarrow (u \star t) \in E$ is continuous.*

Proof. Similar to [17, Lemma 2.3], we assume that $u_n \rightarrow u$ in E and $t_n \rightarrow t$ in $\mathbb{R}$ as $n \rightarrow \infty$. Observed, for $\phi \in C_0^\infty(\mathbb{R}^N)$, that

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla(u_n \star t_n)|^{p-2} \nabla(u_n \star t_n) \nabla \phi dx + \int_{\mathbb{R}^N} |u_n \star t_n|^{p-2} (u_n \star t_n) \phi dx \\ &= e^{\frac{p^2-p-N}{p} t_n} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla \phi(e^{-t_n} x) dx + e^{-\frac{N t_n}{p}} \int_{\mathbb{R}^N} |u_n|^{p-2} u_n \phi(e^{-t_n} x) dx. \end{aligned}$$

By Lebesgue dominated convergence theorem, we conclude that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla \phi(e^{-t_n} x) dx - \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \phi(e^{-t_n} x) dx \right| \\ &+ \left| \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \phi(e^{-t_n} x) dx - \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \phi(e^{-t} x) dx \right| \\ &\leq e^{\frac{N}{p} t_n} \left| |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right|_{\frac{p}{p-1}} |\nabla \phi|_p \\ &+ \left| \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \phi(e^{-t_n} x) dx - \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \phi(e^{-t} x) dx \right| \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. In the same way, we can show that

$$\int_{\mathbb{R}^N} |u_n|^{p-2} u_n \phi(e^{-t_n} x) dx \rightarrow \int_{\mathbb{R}^N} |u|^{p-2} u \phi(e^{-t} x) dx$$

as $n \rightarrow \infty$. As a consequence, for $\phi \in E$,

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla(u_n \star t_n)|^{p-2} \nabla(u_n \star t_n) \nabla \phi dx + \int_{\mathbb{R}^N} |(u_n \star t_n)|^{p-2} (u_n \star t_n) \phi dx \\ &\rightarrow \int_{\mathbb{R}^N} |\nabla(u \star t)|^{p-2} \nabla(u \star t) \nabla \phi dx + \int_{\mathbb{R}^N} |(u \star t)|^{p-2} (u \star t) \phi dx \end{aligned}$$

as $n \rightarrow \infty$, by employing the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in E . Now, it is easy to check that

$$\|u_n \star t_n\|^p = e^{p t_n} |\nabla u_n|_p^p + |u_n|_p^p \rightarrow e^{p t} |\nabla u|_p^p + |u|_p^p = \|u \star t\|^p$$

as $n \rightarrow \infty$. Thereby, we infer from (2.32) that $u_n \star t_n \rightarrow u \star t$ in E . $\square$

Next, we give a version of linking theorem, see the Section 5 of [21].

Definition 2.1. Let X be a topological space and B be a closed subset of X . We say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with extended boundary B if, for any set A in $\mathcal{F}$ and any $\eta \in C([0, 1] \times X; X)$ satisfying $\eta(t, x) = x$ for all $(t, x) \in (\{0\} \times X) \cup ([0, 1] \times B)$, $\eta(\{1\} \times A) \in \mathcal{F}$.

Lemma 2.12. (See [21]) *Let ϕ be a C^1 -functional on a complete connected C^1 -Finsler manifold X and consider a homotopy-stable family $\mathcal{F}$ with an extended closed boundary B . Set $m = m(\phi, \mathcal{F})$ and let F be a closed subset of X satisfying*

- (i) $(A \cap F) \setminus B \neq \emptyset$ for every $A \in \mathcal{F}$;
(ii) $\sup \phi(B) \leq m \leq \inf \phi(F)$.

Then, for any sequence of sets $(A_n)_n$ in $\mathcal{F}$ such that $\lim_{n \rightarrow \infty} \sup_{A_n} \phi = m$, there exists a sequence $(x_n)_n$ in X such that

$$\lim_{n \rightarrow \infty} \phi(x_n) = m, \quad \lim_{n \rightarrow \infty} \|d\phi(x_n)\| = 0, \quad \lim_{n \rightarrow \infty} \text{dist}(x_n, F) = 0, \quad \lim_{n \rightarrow \infty} \text{dist}(x_n, A_n) = 0.$$

Lemma 2.13. For every $u \in S(c)$ with $I(u) < m(c)$, the value t_u given in Lemma 2.3 is negative.

Proof. Let $u \in S(c)$ satisfy $I(u) < m(c)$, and let $s_u < c_u < t_u < d_u$ be as given in Lemma 2.3. We aim to show that $d_u \leq 0$, which implies that $t_u < 0$. Suppose, for contradiction, that $d_u > 0$. Since $\Phi_u(0) = I(u) < m(c) < 0$, it follows that $c_u > 0$. Applying Lemma 2.3 then yields

$$\begin{aligned} m(c) &> I(u) = \Phi_u(0) \geq \min_{s \in (-\infty, c_u)} \Phi_u(s) \\ &\geq \min \{I(u \star s) : s \in \mathbb{R}, \|\nabla(u \star s)\|_p < R_0\} = I(u \star s_u) \geq m(c), \end{aligned}$$

which is absurd. $\square$

The first part of Theorem 1.1 asserts that there exists $u_* \in \mathcal{V}_c^+$ with $I(u_*) = m(c) < 0$. It is also evident that $I(u_* \star t) < 2m(c)$ for sufficiently large $t > 0$. By Lemma 2.11, this implies that the set

$$\Gamma := \Gamma(c) = \left\{ \gamma \in C([0, 1], S(c)) : \gamma(0) \in \mathcal{V}_c^+, \gamma(1) \in I^{2m(c)} \right\},$$

is not empty, where $I^{m(c)} = \{u \in S(c) : I(u) \leq 2m(c)\}$. Now let us consider the associated min-max value

$$d = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} I(\gamma(t)).$$

Lemma 2.14. There holds $d = \inf_{u \in \mathcal{V}_c^-} I(u) > 0$.

Proof. An argument analogous to that in the proof of [37] shows that $\inf_{u \in \mathcal{V}_c^-} I(u) > 0$. For any $\gamma \in \Gamma$, we have $0 = s_{\gamma(0)} < t_{\gamma(0)}$, and by Lemma 2.13, $t_{\gamma(1)} < 0$. With the help of Lemma 2.3, there exists $t_0 = t_0(\gamma) \in (0, 1)$ such that $t_{\gamma(t_0)} = 0$, which implies $\gamma(t_0) \in \mathcal{V}_c^-$ and necessarily

$$\sup_{t \in [0, 1]} I(\gamma(t)) \geq I(\gamma(t_0)) \geq \inf_{u \in \mathcal{V}_c^-} I(u). \quad (2.33)$$

Clearly, $d \geq \inf_{u \in \mathcal{V}_c^-} I(u)$. For any $u \in \mathcal{V}_c^-$, we know $s_u < 0$ and there exists $t_1 = t_1(u) > 0$ such that $I(u \star t_1) < 2m(c)$. If we define $\gamma : [0, 1] \rightarrow S(c)$ by $\gamma(t) = u \star ((1-t)s_u + tt_1)$, then $\gamma \in \Gamma$ from 2.11, and, by (2.33), $I(u) = \sup_{t \in [0, 1]} I(\gamma(t)) \geq d$. Hence, $\inf_{u \in \mathcal{V}_c^-} I(u) \geq d$. $\square$

To obtain the compactness when $\{u_n\} \subset \mathcal{V}_c^-$, we need the following estimate.

Lemma 2.15. Let $N \geq p^2$. Then, $d < m(c) + \frac{1}{N} S^{\frac{N}{p}}$.

Proof. By Appendix A in [15], we have $\|\nabla u_\varepsilon\|_p^p = S^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p-1}})$, $\|u_\varepsilon\|_{p^*}^{p^*} = S^{\frac{N}{p}} + O(\varepsilon^{\frac{N}{p-1}})$, and

$$|u_\varepsilon|_{r'}^{r'} = \begin{cases} C\varepsilon^{N - \frac{(N-p)r'}{p}} + O(\varepsilon^{\frac{(N-p)r'}{p(p-1)}}), & N > p^2 \text{ or } p < r' < p^*, \\ C\varepsilon^p |\log \varepsilon| + O(\varepsilon^p), & N = p^2 \text{ and } r' = p, \end{cases}$$

where $p \leq r' \leq p^*$. Let $W_{\varepsilon, \tau} = u_c^+ + \tau u_\varepsilon(\cdot - x_\varepsilon)$, where u_c^+ is obtained in Theorem 1.1 (Part I), and

$$G_{\varepsilon, \tau}(x) = (c^{-\frac{1}{p}} \|W_{\varepsilon, \tau}\|_p)^{\frac{N-p}{p}} W_{\varepsilon, \tau} (c^{-\frac{1}{p}} \|W_{\varepsilon, \tau}\|_p x), \quad (2.34)$$

where $|x_\varepsilon| = \varepsilon^{-1}$. Then, $\|G_{\varepsilon, \tau}\|_p^p = c$, $\|\nabla G_{\varepsilon, \tau}\|_p^p = \|\nabla W_{\varepsilon, \tau}\|_p^p$, $\|G_{\varepsilon, \tau}\|_{p^*}^{p^*} = \|W_{\varepsilon, \tau}\|_{p^*}^{p^*}$, and

$$\int_{\mathbb{R}^N} \frac{|G_{\varepsilon, \tau}|^p}{|x|^r} dx = (c^{-\frac{1}{p}} \|W_{\varepsilon, \tau}\|_p)^{r-p} \int_{\mathbb{R}^N} \frac{|W_{\varepsilon, \tau}|^p}{|x|^r} dx.$$

Thus, there exists a unique $t_{\varepsilon, \tau} \in \mathbb{R}$ satisfying $t_{\varepsilon, \tau} \star G_{\varepsilon, \tau} \in \mathcal{V}_c^-$, i.e.,

$$e^{pt_{\varepsilon, \tau}} \int_{\mathbb{R}^N} |\nabla G_{\varepsilon, \tau}|^p dx = \frac{re^{rt_{\varepsilon, \tau}}}{p} \int_{\mathbb{R}^N} \frac{|G_{\varepsilon, \tau}|^p}{|x|^r} dx + e^{p^* t_{\varepsilon, \tau}} \int_{\mathbb{R}^N} |G_{\varepsilon, \tau}|^{p^*} dx. \quad (2.35)$$

Since $u_c^+ \in \mathcal{V}_c^+$, we see $t_{\varepsilon, 0} > 0$. By (2.35),

$$e^{(p^*-p)t_{\varepsilon, \tau}} < \frac{\|\nabla G_{\varepsilon, \tau}\|_p^p}{\|G_{\varepsilon, \tau}\|_{p^*}^{p^*}} = \frac{\|\nabla W_{\varepsilon, \tau}\|_p^p}{\|W_{\varepsilon, \tau}\|_{p^*}^{p^*}},$$

which implies $t_{\varepsilon, \tau} \rightarrow -\infty$ as $\tau \rightarrow +\infty$. By Lemma 2.3, $t_{\varepsilon, \tau}$ is continuous with respect to τ , so we can select an appropriate $\tau = \tau_\varepsilon > 0$ satisfying $t_{\varepsilon, \tau_\varepsilon} = 0$. It thus follows that

$$\begin{aligned} d &\leq I(G_{\varepsilon, t_\varepsilon}) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla G_{\varepsilon, t_\varepsilon}|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} \frac{|G_{\varepsilon, t_\varepsilon}|^p}{|x|^r} dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |G_{\varepsilon, t_\varepsilon}|^{p^*} dx \\ &= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla W_{\varepsilon, t_\varepsilon}|^p dx - \frac{1}{p} (c^{-\frac{1}{p}} \|W_{\varepsilon, t_\varepsilon}\|_p)^{r-p} \int_{\mathbb{R}^N} \frac{|W_{\varepsilon, t_\varepsilon}|^p}{|x|^r} dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |W_{\varepsilon, t_\varepsilon}|^{p^*} dx. \end{aligned} \quad (2.36)$$

If $\liminf_{\varepsilon \rightarrow 0} \tau_\varepsilon = 0$ or $\limsup_{\varepsilon \rightarrow 0} \tau_\varepsilon = +\infty$, then $d \leq \liminf_{\varepsilon \rightarrow 0} I(G_{\varepsilon, \tau}) \leq I(u_c^+) = m(c)$, which contradicts Lemma 2.4 and Lemma 2.14. Therefore, there exist $t_2 > t_1 > 0$ independent of ε such that $\tau_\varepsilon \in [t_1, t_2]$.

Now, we estimate $\|\nabla W_{\varepsilon, \tau_\varepsilon}\|_p^p$. By $(a_1 + a_2)^p \leq a_1^p + a_2^p + C(a_1^{p-1}a_2 + a_1a_2^{p-1})$, $\forall a_1, a_2 \geq 0$, we have

$$\begin{aligned} \|\nabla W_{\varepsilon, \tau_\varepsilon}\|_p^p &\leq \|\nabla u_c^+\|_p^p + \tau_\varepsilon^p \|\nabla u_\varepsilon\|_p^p \\ &\quad + C \int_{\mathbb{R}^N} |\nabla u_c^+|^{p-1} |\nabla u_\varepsilon| dx + C \int_{\mathbb{R}^N} |\nabla u_c^+| \cdot |\nabla u_\varepsilon|^{p-1} dx \\ &= \|\nabla u_c^+\|_p^p + S^{\frac{N}{p}} \tau_\varepsilon^p \\ &\quad + C \int_{\mathbb{R}^N} |\nabla u_c^+|^{p-1} |\nabla u_\varepsilon| dx + C \int_{\mathbb{R}^N} |\nabla u_c^+| \cdot |\nabla u_\varepsilon|^{p-1} dx + O(\varepsilon^{\frac{N-p}{p-1}}). \end{aligned} \quad (2.37)$$

By the Hölder inequality,

$$\int_{\mathbb{R}^N} |\nabla u_c^+|^{p-1} |\nabla u_\varepsilon| dx = \int_{B_2(x_\varepsilon)} |\nabla u_c^+|^{p-1} |\nabla u_\varepsilon| dx \leq \|\nabla u_c^+\|_{L^p(B_2(x_\varepsilon))}^{p-1} \|\nabla u_\varepsilon\|_p \leq C \|\nabla u_c^+\|_{L^p(B_2(x_\varepsilon))}^{p-1}.$$

We know there exists $\lambda_*^+ > 0$ such that

$$-\Delta_p u_c^+(x) + \lambda_*^+ |u_c^+(x)|^{p-1} = \frac{|u_c^+(x)|^{p-1}}{|x|^r} + |u_c^+(x)|^{p^*-1}.$$

Then, according to [20], there exist $a, b > 0$ satisfying $\|\nabla u_c^+(x)\| \leq ae^{-b|x|}$ for $|x|$ sufficiently large. This means $\|\nabla u_c^+(x)\|_{L^p(B_2(x_\varepsilon))} \leq Ce^{-b\varepsilon^{-1}}$, and

$$\int_{\mathbb{R}^N} |\nabla u_c^+|^{p-1} |\nabla u_\varepsilon| dx \leq Ce^{-b(p-1)\varepsilon^{-1}}, \quad \int_{\mathbb{R}^N} |\nabla u_c^+| \cdot |\nabla u_\varepsilon|^{p-1} dx \leq Ce^{-b\varepsilon^{-1}}.$$

Therefore, by (2.37), we obtain

$$\|\nabla W_{\varepsilon, \tau_\varepsilon}\|_p^p \leq \|\nabla u_c^+\|_p^p + S^{\frac{N}{p}} \tau_\varepsilon^p + O(\varepsilon^{\frac{N-p}{p-1}}). \quad (2.38)$$

Next, we estimate $\|W_{\varepsilon, \tau_\varepsilon}\|_p^p$. We have

$$\begin{aligned} \int_{\mathbb{R}^N} |u_c^+ + \tau_\varepsilon u_\varepsilon|^p dx &= \int_{B_2(x_\varepsilon)} |u_c^+ + \tau_\varepsilon u_\varepsilon|^p dx + \int_{B_2^c(x_\varepsilon)} |u_c^+|^p dx \\ &\leq c + \int_{B_2(x_\varepsilon)} |u_c^+ + \tau_\varepsilon u_\varepsilon|^p dx \\ &\leq c + C \int_{B_2(x_\varepsilon)} (|u_c^+|^p + |u_\varepsilon|^p) dx \\ &\leq c + C \left(e^{-pb\varepsilon^{-1}} + \varepsilon^p |\log \varepsilon| \right) \\ &= c + O(\varepsilon^p |\log \varepsilon|). \end{aligned}$$

Thus

$$\left(c^{-\frac{1}{p}} \|\nabla W_{\varepsilon, \tau}\|_p \right)^{r-p} \geq 1 + O(\varepsilon^p |\log \varepsilon|). \quad (2.39)$$

In view of $u_\varepsilon(x) = \varepsilon^{\frac{N-p}{p(p-1)}} \left(\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}} \right)^{\frac{p-N}{p}}$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|u_\varepsilon|^p}{|x|^r} dx &= \varepsilon^{\frac{N-p}{p-1}} \int_{B_R(0)} \frac{1}{\left(\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}} \right)^{N-p} |x|^r} dx \\ &= \varepsilon^{\frac{N-p}{p-1}} \int_0^R \frac{t^{N-1}}{\left(\varepsilon^{\frac{p}{p-1}} + |t|^{\frac{p}{p-1}} \right)^{N-p} |t|^r} dt \\ &= \varepsilon^{\frac{N-p}{p-1}} \int_0^{\frac{R}{\varepsilon}} \frac{\varepsilon^{N-1} w^{N-1} \varepsilon}{\varepsilon^{\frac{p(N-p)}{p-1}} \left(1 + |w|^{\frac{p}{p-1}} \right)^{N-p} \varepsilon^r w^r} dw \\ &= \varepsilon^{p-r} \int_0^1 \frac{w^{N-1-r}}{\left(1 + |w|^{\frac{p}{p-1}} \right)^{N-p}} dw + \varepsilon^{p-r} \int_1^{\frac{R}{\varepsilon}} \frac{w^{N-1-r}}{\left(1 + |w|^{\frac{p}{p-1}} \right)^{N-p}} dw \\ &\geq C \varepsilon^{p-r} \int_0^1 w^{N-1-r} dw \\ &= C \varepsilon^{p-r}. \end{aligned}$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|W_{\varepsilon, \tau}|^p}{|x|^r} dx &\geq \int_{\mathbb{R}^N} \frac{|u_c^+|^p}{|x|^r} dx + \tau_\varepsilon^p \int_{\mathbb{R}^N} \frac{|u_\varepsilon|^p}{|x|^r} dx \\ &\geq \int_{\mathbb{R}^N} \frac{|u_c^+|^p}{|x|^r} dx + C \varepsilon^{p-r}. \end{aligned} \quad (2.40)$$

It is easy to know that

$$\|W_{\varepsilon, \tau_\varepsilon}\|_{p^*}^{p^*} \geq \|u_c^+\|_{p^*}^{p^*} + \tau_\varepsilon^{p^*} \|u_\varepsilon\|_{p^*}^{p^*} = \|u_c^+\|_{p^*}^{p^*} + S^{\frac{N}{p}} \tau_\varepsilon^{p^*} + O(\varepsilon^{\frac{N}{p-1}}). \quad (2.41)$$

Combining (2.36), (2.38), (2.39) (2.40), and (2.41), we obtain

$$d \leq m(c) + S^{\frac{N}{p}} \left(\frac{1}{p} \tau_\varepsilon^p - \frac{1}{p^*} \tau_\varepsilon^{p^*} \right) - C \varepsilon^{p-r} + O(\varepsilon^p |\log \varepsilon|) < m(c) + \frac{1}{N} S^{\frac{N}{p}},$$

by taking ε sufficiently small. $\square$

Lemma 2.16. *Let $N < p^2 < 9$. Then, $d < m(c) + \frac{1}{N}S^{\frac{N}{p}}$.*

Proof. We set $W_{\varepsilon, \tau_\varepsilon} = u_c^+ + \tau u_\varepsilon$ and $G_{\varepsilon, \tau_\varepsilon}$ is defined as in (2.34). We can then select $\tau = \tau_\varepsilon > 0$ such that $G_{\varepsilon, \tau_\varepsilon} \in \mathcal{V}_c^-$. Additionally, there exist $t_2 > t_1 > 0$ independent of ε with $\tau_\varepsilon \in [t_1, t_2]$. Therefore

$$d \leq \frac{1}{p} \int_{\mathbb{R}^N} |\nabla W_{\varepsilon, \tau_\varepsilon}|^p dx - \frac{1}{p} (c^{-\frac{1}{p}} \|W_{\varepsilon, \tau_\varepsilon}\|_p)^{r-p} \int_{\mathbb{R}^N} \frac{|W_{\varepsilon, \tau_\varepsilon}|^p}{|x|^r} dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |W_{\varepsilon, \tau_\varepsilon}|^{p^*} dx. \quad (2.42)$$

Now, we estimate $\|\nabla W_{\varepsilon, \tau_\varepsilon}\|_p^p$ and $\|W_{\varepsilon, \tau_\varepsilon}\|_p^p$. We can prove that, for $a_1, a_2 \geq 0$, there exists

$$(a_1^2 + a_2^2 + 2a_1 a_2 \cos \alpha)^{\frac{p}{2}} \leq a_1^p + a_2^p + p a_1^{p-1} a_2 \cos \alpha + C a_1^{\frac{p-1}{2}} a_2^{\frac{p+1}{2}},$$

uniformly in α . Thus

$$\begin{aligned} \|\nabla W_{\varepsilon, \tau_\varepsilon}\|_p^p &= \int_{\mathbb{R}^N} (|\nabla u_c^+|^2 + \tau_\varepsilon^2 |\nabla u_\varepsilon|^2 + 2\tau_\varepsilon \nabla u_c^+ \cdot \nabla u_\varepsilon)^{\frac{p}{2}} dx \\ &\leq \|\nabla u_c^+\|_p^p + \tau_\varepsilon^p \|\nabla u_\varepsilon\|_p^p \\ &\quad + p\tau_\varepsilon \int_{\mathbb{R}^N} |\nabla u_c^+|^{p-2} \nabla u_c^+ \cdot \nabla u_\varepsilon dx + C \int_{\mathbb{R}^N} |\nabla u_c^+|^{\frac{p-1}{2}} |\nabla u_\varepsilon|^{\frac{p+1}{2}} dx. \end{aligned} \quad (2.43)$$

By [4], ∇u_c^+ is local Hölder continuous. Hence,

$$\int_{\mathbb{R}^N} |\nabla u_c^+|^{\frac{p-1}{2}} |\nabla u_\varepsilon|^{\frac{p+1}{2}} dx \leq C \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^{\frac{p+1}{2}} dx = O\left(\varepsilon^{\frac{(N-p)(p+1)}{2p(p-1)}}\right). \quad (2.44)$$

By (1.1), we have

$$\int_{\mathbb{R}^N} |\nabla u_c^+|^{p-2} \nabla u_c^+ \cdot \nabla u_\varepsilon dx + \lambda_*^+ \int_{\mathbb{R}^N} |u_c^+|^{p-1} u_\varepsilon dx = \int_{\mathbb{R}^N} \frac{|u_c^+|^{p-1}}{|x|^r} u_\varepsilon dx + \int_{\mathbb{R}^N} |u_c^+|^{p^*-1} u_\varepsilon dx. \quad (2.45)$$

From (2.43), (2.44) and (2.45), we obtain

$$\begin{aligned} \|\nabla W_{\varepsilon, \tau_\varepsilon}\|_p^p &\leq \|\nabla u_c^+\|_p^p + S^{\frac{N}{p}} \tau_\varepsilon^p - p\tau_\varepsilon \lambda_*^+ \int_{\mathbb{R}^N} |u_c^+|^{p-1} u_\varepsilon dx \\ &\quad + p\tau_\varepsilon \int_{\mathbb{R}^N} \frac{|u_c^+|^{p-1}}{|x|^r} u_\varepsilon dx + p\tau_\varepsilon \int_{\mathbb{R}^N} |u_c^+|^{p^*-1} u_\varepsilon dx + O\left(\varepsilon^{\frac{(N-p)(p+1)}{2p(p-1)}}\right). \end{aligned} \quad (2.46)$$

Similarly, we obtain

$$\begin{aligned} \|W_{\varepsilon, \tau_\varepsilon}\|_p^p &\leq \|u_c^+\|_p^p + \tau_\varepsilon^p \|u_\varepsilon\|_p^p + p\tau_\varepsilon \int_{\mathbb{R}^N} |u_c^+|^{p-1} u_\varepsilon dx + C \int_{\mathbb{R}^N} |u_c^+|^{\frac{p-1}{2}} |u_\varepsilon|^{\frac{p+1}{2}} dx \\ &\leq c + p\tau_\varepsilon \int_{\mathbb{R}^N} |u_c^+|^{p-1} u_\varepsilon dx + O\left(\varepsilon^{\frac{(N-p)(p+1)}{2p(p-1)}}\right), \end{aligned}$$

which implies

$$\left(c^{-\frac{1}{p}} \|W_{\varepsilon, \tau_\varepsilon}\|_p\right)^{r-p} \geq 1 - \frac{(p-r)\tau_\varepsilon}{c} \int_{\mathbb{R}^N} |u_c^+|^{p-1} u_\varepsilon dx + O\left(\varepsilon^{\frac{(N-p)(p+1)}{2p(p-1)}}\right). \quad (2.47)$$

Next, we estimate $\int_{\mathbb{R}^N} \frac{|W_{\varepsilon, \tau_\varepsilon}|^p}{|x|^r} dx$ and $\|W_{\varepsilon, \tau_\varepsilon}\|_{p^*}^{p^*}$. For every $a, b \geq 0$, we have

$$(a+b)^{r'} \geq \begin{cases} a^{r'} + b^{r'} + r'(a^{r'-1}b + ab^{r'-1}), & \forall r' \geq 3, \\ a^{r'} + r'a^{r'-1}b, & \forall r' \geq 1. \end{cases}$$

Thus, we have

$$\int_{\mathbb{R}^N} \frac{|W_{\varepsilon, \tau_\varepsilon}|^p}{|x|^r} \geq \int_{\mathbb{R}^N} \frac{|u_c^+|^p}{|x|^r} dx + p\tau_\varepsilon \int_{\mathbb{R}^N} \frac{|u_c^+|^{p-1}}{|x|^r} u_\varepsilon dx, \quad (2.48)$$

and

$$\begin{aligned} \|W_{\varepsilon, \tau_\varepsilon}\|_{p^*}^{p^*} &\geq \|u_c^+\|_{p^*}^{p^*} + \tau_\varepsilon^{p^*} \|u_\varepsilon\|_{p^*}^{p^*} + p^* \tau_\varepsilon \int_{\mathbb{R}^N} |u_c^+|^{p^*-1} u_\varepsilon dx + p^* \tau_\varepsilon^{p^*-1} \int_{\mathbb{R}^N} u_c^+ |u_\varepsilon|^{p^*-1} dx \\ &= \|u_c^+\|_{p^*}^{p^*} + S^{\frac{N}{p}} \tau_\varepsilon^{p^*} + p^* \tau_\varepsilon \int_{\mathbb{R}^N} |u_c^+|^{p^*-1} u_\varepsilon dx + p^* \tau_\varepsilon^{p^*-1} \int_{\mathbb{R}^N} u_c^+ |u_\varepsilon|^{p^*-1} dx + O(\varepsilon^{\frac{N}{p-1}}). \end{aligned} \quad (2.49)$$

Combining (2.42), (2.46), (2.47), (2.48), (2.49) and using $c\lambda_*^+ = (p-r) \int_{\mathbb{R}^N} \frac{|u_c^+|^p}{|x|^r} dx$, we obtain

$$\begin{aligned} d &\leq m(c) + S^{\frac{N}{p}} \left(\frac{1}{p} \tau_\varepsilon^p - \frac{1}{p^*} \tau_\varepsilon^{p^*} \right) - \tau_\varepsilon^{p^*-1} \int_{\mathbb{R}^N} u_c^+ |u_\varepsilon|^{p^*-1} dx + C\varepsilon^{\frac{(N-p)(p+1)}{2p(p-1)}} \\ &\leq m(c) + \frac{1}{N} S^{\frac{N}{p}} - C\varepsilon^{\frac{N-p}{p}} + C\varepsilon^{\frac{(N-p)(p+1)}{2p(p-1)}} \\ &< m(c) + \frac{1}{N} S^{\frac{N}{p}}, \end{aligned}$$

by taking ε sufficiently small. $\square$

Lemma 2.17. Suppose that $1 < p < N$ and $c^{\frac{p-r}{p}} < \alpha(N, p, r)$. Then there exists a Palais-Smale sequence $\{u_n\} \subset S(c)$ for the constrained functional $I|_{S(c)}$ at the level d , satisfying $|u_n^-|_p \rightarrow 0$ and $P(u_n) \rightarrow 0$ as $n \rightarrow \infty$, where $u^- = \min\{u, 0\}$.

Proof. Following the approach of [40], we define a rescaled functional $\tilde{I} : E \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$\tilde{I}(u, t) = I(u \star t) = \frac{e^{pt}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{e^{tr}}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \frac{e^{p^*t}}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx.$$

Let us consider the set

$$\tilde{\Gamma} = \left\{ \tilde{\gamma} \in C([0, 1], S(c) \times \mathbb{R}) : \tilde{\gamma}(0) \in \mathcal{V}_c^+ \times \{0\}, \tilde{\gamma}(1) \in I^{2m(c)} \times \{0\} \right\}.$$

Clearly, if $\gamma \in \Gamma$, then $\tilde{\gamma} := (\gamma, 0) \in \tilde{\Gamma}$ satisfies $\tilde{I}(\tilde{\gamma}(t)) = I(\gamma(t))$ for all $t \in [0, 1]$. Conversely, given any $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2) \in \tilde{\Gamma}$, Lemma 2.11 implies that $\gamma(\cdot) := \tilde{\gamma}_1 \star \tilde{\gamma}_2(\cdot)$ belongs to Γ and $I(\gamma(t)) = \tilde{I}(\tilde{\gamma}(t))$ for $t \in [0, 1]$. Therefore, $d = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \sup_{t \in [0, 1]} \tilde{I}(\tilde{\gamma}(t))$. For $\varepsilon_n = \frac{1}{n^2}$, there exists $\gamma_n \in \Gamma$ satisfying $\sup_{t \in [0, 1]} I(\gamma_n(t)) \leq d + \frac{1}{n^2}$. By replacing γ_n with $|\gamma_n|$ if necessary, we may assume $\gamma_n(t) \geq 0$ in $\mathbb{R}^N$ for all $t \in [0, 1]$. Defining $\tilde{\gamma}_n = (\gamma_n, 0) \in \tilde{\Gamma}$, we obtain $\sup_{t \in [0, 1]} \tilde{I}(\tilde{\gamma}_n(t)) \leq d + \frac{1}{n^2}$. For any $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2) \in \tilde{\Gamma}$, we look at the function $P_{\tilde{\gamma}} : [0, 1] \rightarrow \mathbb{R}$ defined as

$$P_{\tilde{\gamma}}(\tau) = P(\tilde{\gamma}_1(\tau) \star \tilde{\gamma}_2(\tau)).$$

Since $\tilde{\gamma}(0) = (\tilde{\gamma}_1(0), \tilde{\gamma}_2(0)) \in (\mathcal{V}_c^+, 0)$, Lemma 2.3 implies that $t_{\tilde{\gamma}_1(0)} > s_{\tilde{\gamma}_1(0)} = 0$. Furthermore, from the inequality $I(\tilde{\gamma}_1(1)) = \tilde{I}(\tilde{\gamma}(1)) < 2m(c)$, Lemma 2.13 establishes that $t_{\tilde{\gamma}_1(1)} < 0$. By the continuity of the mapping $\tau \mapsto \tilde{\gamma}_1(\tau) \star \tilde{\gamma}_2(\tau)$ from $[0, 1]$ to E (Lemma 2.11), there exists $\tau_{\tilde{\gamma}} \in (0, 1)$ such that $P_{\tilde{\gamma}}(\tau_{\tilde{\gamma}}) = 0$.

$$\max_{\tilde{\gamma} \in [0, 1]} \tilde{I} \geq \tilde{I}(\tilde{\gamma}(\tau_{\tilde{\gamma}})) \geq \inf_{\mathcal{V}_c^-} I(u) = d. \quad (2.50)$$

It follows from Lemma 2.3 that

$$d > 0 \geq \sup_{\mathcal{V}_c^+ \cup I^{2m(c)}} I(u) = \sup_{(\mathcal{V}_c^+, 0) \cup (I^{2m(c)}, 0)} \tilde{I}(u, t). \quad (2.51)$$

To establish our result, we will apply Lemma 2.12. For this purpose, let

$$X = S(c) \times \mathbb{R}, \mathcal{F} = \{\tilde{\gamma}([0, 1]) : \tilde{\gamma} \in \tilde{\Gamma}\}, B = (\mathcal{V}_c^+, 0) \cup (I^{2m(c)}, 0),$$

$$F = \{(u, s) \in S(c) \times \mathbb{R} | \tilde{I}(u, s) \geq d\}, A = \tilde{\gamma}([0, 1]), A_n = \tilde{\gamma}_n([0, 1]) = \gamma_n([0, 1]) \times \{0\}.$$

We verify that $\mathcal{F}$ is a homotopy-stable family of compact subsets of X with extended closed boundary B , and that F satisfies conditions (1) and (2) of Lemma 2.12. Indeed, for any $\tilde{\gamma} \in \tilde{\Gamma}$, since $\tilde{\gamma}(0) \in (\mathcal{V}_c^+, 0)$ and $\tilde{\gamma}(1) \in (I^{2m(c)}, 0)$, it follows that $\tilde{\gamma}(0), \tilde{\gamma}(1) \in B$. Moreover, for any $A \in \mathcal{F}$ and any $\eta \in C([0, 1] \times X, X)$ such that $\eta(t, x) = x$ on $(\{0\} \times X) \cup ([0, 1] \times B)$, we have $\eta(1, \tilde{\gamma}(0)) = \tilde{\gamma}(0)$ and $\eta(1, \tilde{\gamma}(1)) = \tilde{\gamma}(1)$. This ensures $\eta(\{1\} \times A) \in \mathcal{F}$. By (2.50), $A \cap F \neq \emptyset$, while (2.51) implies $F \cap B = \emptyset$. Thus, conditions (1) and (2) of Lemma 2.12 are satisfied.

Consequently, Lemma 2.12 guarantees the existence of a sequence $\{(v_n, t_n)\} \subset S(c) \times \mathbb{R}$ satisfying, as $n \rightarrow \infty$

$$|t_n| + \text{dist}(v_n, \gamma_n([0, 1])) \rightarrow 0, \quad (2.52)$$

$$\tilde{I}(v_n, t_n) \rightarrow d, \quad (2.53)$$

$$(\tilde{I}|_{S(c) \times \mathbb{R}})'(v_n, t_n) \rightarrow 0. \quad (2.54)$$

Furthermore, we obtain the relation $\tilde{I}(v_n, t_n) = \tilde{I}(v_n \star t_n, 0)$, and

$$\left\langle (\tilde{I}|_{S(c) \times \mathbb{R}})'(v_n, t_n), (\phi, s) \right\rangle = \left\langle (\tilde{I}|_{S(c) \times \mathbb{R}})'(v_n \star t_n, 0), (\phi \star t_n, s) \right\rangle, \quad (2.55)$$

For $(\phi, s) \in E \times \mathbb{R}$ satisfying $\int_{\mathbb{R}^N} v_n \phi = 0$, and recalling that $u_n = (v_n \star t_n) \in S(c)$, we see from (2.53) that

$$I(u_n) = \tilde{I}(v_n \star t_n, 0) = \tilde{I}(v_n, t_n) \rightarrow d, \text{ as } n \rightarrow \infty.$$

Taking $(0, 1)$ as a test function in (2.55), we deduce from (2.54) that $P(u_n) = \partial_t \tilde{I}(u_n, 0) \rightarrow 0$ as $n \rightarrow \infty$. For any $\mu \in E$ satisfying $\int_{\mathbb{R}^N} (v_n \star t_n) \mu = 0$, we may take $(\mu \star (-t_n), 0)$ as a test function in (2.55). Combining this with (2.52), (2.54) and Lemma 2.10, we conclude that $(I|_{S(c)})'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Furthermore, from (2.52) we obtain

$$|u_n^-|_p^p = |(v_n^-) \star t_n|_p^p = |v_n^-|_p^p \rightarrow 0,$$

as $n \rightarrow \infty$. □

Lemma 2.18. Assume the conditions of Lemma 2.15 or Lemma 2.16 hold. Then $u_n \rightarrow u$ in E .

Proof. If $\{u_n\} \subset \mathcal{V}_c^-$, let $w_n = u_n - u$. Then $\{w_n\}$ is also bounded in E and $w_n \rightharpoonup 0$ weakly in E . Moreover, it is obvious that, for some $l \geq 0$, $\|w_n\| \rightharpoonup l$. Similar to [26], we consider the following two cases:

If $l = 0$, our conclusion is proved.

If $l > 0$, up to a subsequence, by the concentration compactness principle, we may assume that $\{w_n\}$ satisfies the following alternative:

(i) Vanishing: $\limsup_{n \rightarrow \infty} \int_{B_r(y)} |w_n|^p dx = 0$ for all $r > 0$.

(ii) Nonvanishing: There exist $\alpha > 0$, $r \in (0, +\infty)$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \rightarrow \infty} \int_{B_r(y_n)} |w_n|^p dx = \alpha.$$

Claim(i). Vanishing cannot occur.

Otherwise, by the vanishing lemma of Lions in Lemma 1.21 of [41], we see that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |w_n|^q dx = 0, \quad \forall q \in (p, p^*). \quad (2.56)$$

It follows from (2.9) that

$$\int_{\mathbb{R}^N} |\nabla w_n|^p dx + \lambda_* \int_{\mathbb{R}^N} |w_n|^p dx - \int_{\mathbb{R}^N} \frac{|w_n|^p}{|x|^r} dx - \int_{\mathbb{R}^N} |w_n|^{p^*} dx = o(1). \quad (2.57)$$

Note that (2.56) holds for $\frac{Np}{N-r} < q < p^*$ and that $\|w_n\| \leq C$ for some $C > 0$. We deduce that, for any $R > 0$

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|w_n|^p}{|x|^r} dx &= \int_{|x| \geq R} \frac{|w_n|^p}{|x|^r} dx + \int_{|x| \leq R} \frac{|w_n|^p}{|x|^r} dx \\ &\leq \frac{1}{R^r} \int_{\mathbb{R}^N} |w_n|^p dx + \left(\int_{|x| \leq R} \frac{1}{|x|^{\frac{rq}{q-p}}} dx \right)^{\frac{q-p}{q}} \left(\int_{|x| \leq R} |w_n|^q dx \right)^{\frac{p}{q}} \\ &\leq \frac{C}{R^r} + C \left(\int_{\mathbb{R}^N} |w_n|^q dx \right)^{\frac{p}{q}} \left(\int_0^R \frac{dt}{t^{\frac{rq}{q-p} - N + 1}} \right)^{\frac{q-p}{q}} \\ &\rightarrow 0, \end{aligned}$$

by letting first $n \rightarrow \infty$ and then $R \rightarrow \infty$. Therefore, this and (2.57) show that

$$\int_{\mathbb{R}^N} |\nabla w_n|^p dx + \lambda_* \int_{\mathbb{R}^N} |w_n|^p dx - \int_{\mathbb{R}^N} |w_n|^{p^*} dx = o(1). \quad (2.58)$$

Noting that u solves the equation

$$-\Delta_p u + \lambda_* |u|^{p-2} u = \frac{|u|^{p-2} u}{|x|^r} + |u|^{p^*-2} u, \quad \text{in } \mathbb{R}^N,$$

there holds

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla u|^p dx - \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \int_{\mathbb{R}^N} |u|^{p^*} dx + \lambda_* \int_{\mathbb{R}^N} |u|^p dx = 0, \\ \frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{N-r}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \frac{N-p}{p} \int_{\mathbb{R}^N} |u|^{p^*} dx + \lambda_* \frac{N}{p} \int_{\mathbb{R}^N} |u|^p dx = 0. \end{cases}$$

It follows that

$$P(u) = \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{r}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \int_{\mathbb{R}^N} |u|^{p^*} dx = 0.$$

Consequently,

$$\begin{aligned} 0 &= P(u_n) = \int_{\mathbb{R}^N} |\nabla u_n|^p dx - \frac{r}{p} \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^r} dx - \int_{\mathbb{R}^N} |u_n|^{p^*} dx \\ &= P(w_n) + P(u) + o(1) \\ &= \int_{\mathbb{R}^N} |\nabla w_n|^p dx - \int_{\mathbb{R}^N} |w_n|^{p^*} dx + o(1). \end{aligned}$$

Combining with (2.58), we obtain $\int_{\mathbb{R}^N} |w_n|^p dx = o(1)$. This implies that $u_n \rightarrow u$ strongly in $L^p(\mathbb{R}^N)$ and $u \neq 0$, and then $u \in S(c)$ and $u \in \mathcal{V}_c^+$ or $u \in \mathcal{V}_c^-$ by $P(u) = 0$. Moreover, we have

$\int_{\mathbb{R}^N} |\nabla w_n|^p dx \geq S^{\frac{N}{p}}$, and so

$$\begin{aligned} d &= \lim_{n \rightarrow \infty} I(u_n) \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{p} \int_{\mathbb{R}^N} |\nabla w_n|^p dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |w_n|^{p^*} dx \right) + I(u) \\ &= \frac{1}{N} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla w_n|^p dx + I(u) \\ &\geq \frac{1}{N} S^{\frac{N}{p}} + I(u). \end{aligned}$$

If $u \in \mathcal{V}_c^+$, then $I(u) \geq m(c)$. By condition $d < \frac{1}{N} S^{\frac{N}{p}} + m(c)$, it holds that $\|w_n\| \rightarrow 0$. This contradicts $l > 0$. If $u \in \mathcal{V}_c^-$, then $I(u) > 0$. We are done.

Claim(ii). Nonvanishing cannot occur.

In this case, there exists $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_r(y_n)} |w_n|^p dx > \frac{\alpha}{2}$. If there exists a constant $R_0 > 0$ such that $\sup_n |y_n| < R_0$, then $\int_{B_{R_0+r}(0)} |w_n|^p dx > \frac{\alpha}{2}$. By the compactness of embedding $E \hookrightarrow L_{loc}^q(\mathbb{R}^N)$ with $q \in (p, p^*)$, we see that (2.52) leads to a contradiction since $w_n \rightharpoonup 0$ weakly in E . Therefore, $|y_n| \rightarrow +\infty$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi_n(x) = \varphi(x - y_n)$. Then $\|\varphi_n\| = \|\varphi\|$. By (2.8), we have

$$0 \leftarrow \int_{\mathbb{R}^N} |\nabla w_n|^{p-2} \nabla w_n \nabla \varphi_n dx + \lambda_* \int_{\mathbb{R}^N} |w_n|^{p-1} \varphi_n dx - \int_{\mathbb{R}^N} \frac{|w_n|^{p-1} \varphi_n}{|x|^r} dx - \int_{\mathbb{R}^N} |w_n|^{p^*-1} \varphi_n dx.$$

Denoting $\tilde{w}_n(x) = w_n(x + y_n)$, we have

$$\int_{\mathbb{R}^N} |\nabla \tilde{w}_n|^{p-2} \nabla \tilde{w}_n \nabla \varphi dx + \lambda_* \int_{\mathbb{R}^N} |\tilde{w}_n|^{p-1} \varphi dx - \int_{\mathbb{R}^N} \frac{|\tilde{w}_n|^{p-1} \varphi}{|x + y_n|^r} dx - \int_{\mathbb{R}^N} |\tilde{w}_n|^{p^*-1} \varphi dx \rightarrow 0.$$

Since $\{w_n\}$ is bounded in E , $\{\tilde{w}_n\}$ is also bounded in E and $\|w_n\| = \|\tilde{w}_n\|$. Hence, passing to a subsequence, there exists $\tilde{w} \in E$ such that $\tilde{w}_n \rightharpoonup \tilde{w}$ weakly in E , and $\tilde{w} \neq 0$ since

$$\lim_{n \rightarrow \infty} \int_{B_r(0)} |\tilde{w}_n|^p dx = \lim_{n \rightarrow \infty} \int_{B_r(y_n)} |w_n|^p dx \geq \frac{\alpha}{2}.$$

In view of $|y_n| \rightarrow +\infty$, we have

$$\int_{\mathbb{R}^N} \frac{|\tilde{w}_n|^{p-1} \varphi}{|x + y_n|^r} dx = \int_{\text{supp } \varphi} \frac{|\tilde{w}_n|^{p-1} \varphi}{|x + y_n|^r} dx \rightarrow 0,$$

as $n \rightarrow \infty$. This and (2.8) imply that

$$\int_{\mathbb{R}^N} |\nabla \tilde{w}|^{p-2} \nabla \tilde{w} \nabla \varphi dx + \lambda_* \int_{\mathbb{R}^N} |\tilde{w}|^{p-1} \varphi dx - \int_{\mathbb{R}^N} |\tilde{w}|^{p^*-1} \varphi dx = 0.$$

That is, $\tilde{w} \in E$ is a nontrivial solution of the following equation

$$\begin{cases} -\Delta_p u + \lambda_* |u|^{p-2} u = |u|^{p^*-2} u, & \text{in } \mathbb{R}^N, \\ u \in E, \end{cases} \quad (2.59)$$

which is impossible since equation (2.59) has no nontrivial solution with $\lambda_* > 0$. Thus, neither Vanishing nor Nonvanishing can occur. So, $l = 0$ and we complete the proof of the lemma. $\square$

Proof of Theorem 1.1 (Part II). Lemma 2.17 guarantees the existence of a sequence $\{u_n\} \subset S(c)$ satisfying

$$I(u_n) \rightarrow d, \quad (I|_{S(c)})'(u_n) \rightarrow 0, \quad |u_n^-|_p \rightarrow 0, \quad P(u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

By Lemma 2.18, we get $u_n \rightarrow u$ strongly in E , where u is a ground state solution of (1.1) for some $\lambda_* > 0$. Furthermore, since $|u_n^-|_p \rightarrow 0$ as $n \rightarrow \infty$, an argument analogous to the proof of Theorem 1.1 (Part I) shows that u is positive.

3. PROOF OF THEOREM 1.2

To establish the existence of infinitely many solutions, we employ a minimax theorem for constrained even functionals, as developed by [25]. First, we recall the notion of genus for symmetric sets in Banach spaces.

Let X be a Banach space and $A \subset X$. The set A is called symmetric if $u \in A$ implies $-u \in A$.

$$\Sigma := \{A \subset X \setminus \{0\} : A \text{ is closed and symmetric with respect to the origin}\}.$$

$$\text{For } A \in \Sigma, \text{ set } \mathcal{U}(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ \inf\{k \in \mathbb{N} : \exists \text{ an odd } \varphi \in C(A, \mathbb{R}^k \setminus \{0\})\}, & \\ +\infty, & \text{if no such odd map,} \end{cases}$$

$$\text{and } \Sigma_k := \{A \in \Sigma(S(c)) : \mathcal{U}(A) \geq k\}.$$

Next, we introduce the minimax theorem required for our analysis. Since it is a special case of a result in [25], we omit the proof.

Lemma 3.1. *Let $I : E \rightarrow \mathbb{R}$ be an even functional of class C^1 . Assume that $I|_{S(c)}$ is bounded from below and satisfies the $(PS)_m$ condition for all $m < 0$, and that $\Sigma_k \neq \emptyset$ for each $k = 1, 2, \dots, n$. Then the minimax values $-\infty < m_1 \leq m_2 \leq \dots \leq m_n$ can be defined as $m_k := \inf_{A \in \Sigma_k} \sup_{u \in A} I(u)$, $k = 1, 2, \dots, n$, and the following statements hold.*

(i) m_k is a critical value of $I|_{S(c)}$ provided $m_k < 0$.

(ii) Denote by K^m the set of critical points of $I|_{S(c)}$ at a level $m \in \mathbb{R}$. If $m_k = m_{k+1} = \dots = m_{k+l-1} = m < 0$ for some $k, l \geq 1$, then $\mathcal{U}(K^m) \geq l$. Hence, $I|_{S(c)}$ has at least n critical points at the level m if $l \geq 2$.

(iii) If $m_k < 0$ for all $k \geq 1$, then $m_k \rightarrow 0^-$ as $k \rightarrow \infty$.

Now, for $\zeta > 0$, we define the set $B_\zeta = \{u \in S(c) : I_T(u) \leq -\zeta\} \subset E$, which is a closed symmetric subset of $S(c)$, because I_T is even and continuous.

Lemma 3.2. *Given $n \in \mathbb{N}^*$, there exists $\zeta = \zeta(n) > 0$ such that $\mathcal{U}(B_\zeta) \geq n$, for any $0 < \zeta \leq \zeta(n)$.*

Proof. For each $n \in \mathbb{N}^*$, there exist n linearly independent functions $u_1, \dots, u_n \in C_0^\infty(\mathbb{R}^N)$ with pairwise disjoint supports, i.e., $\text{supp } u_i \cap \text{supp } u_j = \emptyset$ for all $i \neq j$. The n -dimensional subspace $E_n := \text{span}\{u_1, \dots, u_n\}$, equipped with the norm of E , satisfies $\|\nabla u_j\|_p = \rho < R_0$ and $|u_j|_p = c^{\frac{1}{p}}$ for each j , which implies $\|u_j\| = (\rho^p + c)^{1/p}$. For any $t \in \mathbb{R}$, we define the corresponding set

$$\Upsilon_n(t) = \{l_1(u_1 \star t) + l_2(u_2 \star t) + \dots + l_n(u_n \star t) : l_1 \star p + l_2 \star p + \dots + l_n \star p = 1\}.$$

One can readily verify the existence of a continuous homomorphism between $\Upsilon_n(t)$ and the sphere

$$\mathcal{D}(t) = \{(a_1, a_2, \dots, a_n) \in \mathbb{R}^N : a_1 \star p + a_2 \star p + \dots + a_n \star p = e^{pt} \rho^p + c\},$$

in $\mathbb{R}^N$ for $t \in \mathbb{R}$. As a consequence, by the properties of genus, one has $\mathcal{U}(\Upsilon_n(t)) = n$. Let

$$l_1 \star p + l_2 \star p + \dots + l_n \star p = 1, \quad v = l_1(u_1 \star t) + l_2(u_2 \star t) + \dots + l_n(u_n \star t) \in \Upsilon_n(t),$$

with $t < 0$, there holds $\|\nabla v\|_p = e^t \rho < R_0$. At this stage of the argument, we note that

$$I_T(v) = I(v) = \frac{e^{pt}}{p} \rho^p - \frac{e^{p^*t}}{p^*} \rho^{p^*} \int_{\mathbb{R}^N} \left| \frac{u}{\rho} \right|^{p^*} dx - \frac{e^{tr}}{p} \rho^p \int_{\mathbb{R}^N} \left| \frac{u}{\rho} \right|^p \frac{1}{|x|^r} dx,$$

where $u = l_1 u_1 + l_2 u_2 + \cdots + l_n u_n$. Let us define

$$\alpha_n = \inf \left\{ \int_{\mathbb{R}^N} |V|^{p^*} dx : V \in E_n, \|\nabla V\|_p = 1 \right\} > 0,$$

and

$$\beta_n = \inf \left\{ \int_{\mathbb{R}^N} \frac{|V|^p}{|x|^r} dx : V \in E_n, \|\nabla V\|_p = 1 \right\} > 0.$$

Thus, we have

$$I_T(v) \leq \frac{e^{pt}}{p} \rho^p - \frac{e^{p^*t}}{p^*} \rho^{p^*} \alpha_n - \frac{e^{tr}}{p} \rho^p \beta_n.$$

Note that $0 < r < p < p^*$. There exist $\zeta(n) > 0$, $t_n < 0$ such that, for $0 < \zeta \leq \zeta(n)$, $I_T(v) \leq -\zeta$ for all $v \in \Upsilon_n(t_n)$. Hence, $\Upsilon_n(t_n) \subset B_\zeta$ and $\mathcal{U}(B_\zeta) \geq \mathcal{U}(\Upsilon_n(t_n)) = n$. $\square$

Proof of Theorem 1.2. Set $\Sigma_k = \{A \in \Sigma(S(c)) : \mathcal{U}(A) \geq k\}$ for $k \geq 1$. By Lemma 3.2, there exists $\zeta(k) > 0$ such that $B_{\zeta(k)} \in \Sigma_k$, where $B_{\zeta(k)} = \{u \in S(c) : I_T(u) \leq -\zeta(k)\}$. Then $\Sigma_k \neq \emptyset$. By Lemma 2.9, the functional I_T satisfies the $(PS)_{m(c)}$ condition on $S(c)$ for any level $m(c) < 0$, and it is bounded from below. We therefore introduce the minimax values

$$m_k = \inf_{A \in \Sigma_k} \sup_{u \in A} I_T(u), \quad \text{for } k \geq 1.$$

Then $-\infty < m_1 \leq m_2 \leq \cdots \leq m_k \leq \cdots < 0$. Lemma 3.1 guarantees that $I_T|_{S_c}$ admits critical points at levels m_k with $m_k \rightarrow 0^-$ as $k \rightarrow \infty$. Furthermore, since $I_T = I$ in a neighborhood of any u satisfying $I_T(u) < 0$, these critical points of $I_T|_{S(c)}$ coincide with critical points of $I|_{S(c)}$.

4. PROOF OF THEOREM 1.3

By a weak solution of (1.3), we mean any $u \in E$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx + \lambda \int_{\mathbb{R}^N} |u|^{p-2} u \phi \, dx = \int_{\mathbb{R}^N} \frac{|u|^{p-2} u}{|x|^r} \phi \, dx, \quad (4.1)$$

is satisfied for all $\phi \in E$.

Proof of Theorem 1.3. Let u_1 and u_2 be solutions to equation (1.3), and $u_1 \neq u_2$. We consider the following two functions,

$$u_1 - \frac{u_2^p}{u_1^{p-1}} \quad \text{and} \quad u_2 - \frac{u_1^p}{u_2^{p-1}},$$

as test functions in (4.1). Then

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u_1|^{p-2} \nabla u_1 \nabla \left(u_1 - \frac{u_2^p}{u_1^{p-1}} \right) dx + \int_{\mathbb{R}^N} |\nabla u_2|^{p-2} \nabla u_2 \nabla \left(u_2 - \frac{u_1^p}{u_2^{p-1}} \right) dx \\ & + \lambda \int_{\mathbb{R}^N} |u_1|^{p-2} u_1 \left(u_1 - \frac{u_2^p}{u_1^{p-1}} \right) dx + \lambda \int_{\mathbb{R}^N} |u_2|^{p-2} u_2 \left(u_2 - \frac{u_1^p}{u_2^{p-1}} \right) dx \\ & = \int_{\mathbb{R}^N} \frac{|u_1|^{p-2} u_1}{|x|^r} \left(u_1 - \frac{u_2^p}{u_1^{p-1}} \right) dx + \int_{\mathbb{R}^N} \frac{|u_2|^{p-2} u_2}{|x|^r} \left(u_2 - \frac{u_1^p}{u_2^{p-1}} \right) dx. \end{aligned}$$

Obviously, a simple calculation shows that

$$\begin{aligned} & \lambda \int_{\mathbb{R}^N} |u_1|^{p-2} u_1 \left(u_1 - \frac{u_2^p}{u_1^{p-1}} \right) dx + \lambda \int_{\mathbb{R}^N} |u_2|^{p-2} u_2 \left(u_2 - \frac{u_1^p}{u_2^{p-1}} \right) dx = 0, \\ & \int_{\mathbb{R}^N} \frac{|u_1|^{p-2} u_1}{|x|^r} \left(u_1 - \frac{u_2^p}{u_1^{p-1}} \right) dx + \int_{\mathbb{R}^N} \frac{|u_2|^{p-2} u_2}{|x|^r} \left(u_2 - \frac{u_1^p}{u_2^{p-1}} \right) dx = 0, \end{aligned}$$

and

$$\nabla \left(u_1 - \frac{u_2^p}{u_1^{p-1}} \right) = \left(1 + (p-1) \left(\frac{u_2}{u_1} \right)^p \right) \nabla u_1 - p \left(\frac{u_2}{u_1} \right)^{p-1} \nabla u_2.$$

Similarly, for $\nabla \left(u_2 - \frac{u_1^p}{u_2^{p-1}} \right)$ by interchanging u_1 and u_2 , we obtain

$$\begin{aligned} T &= \int_{\mathbb{R}^N} \left[\left(1 + (p-1) \left(\frac{u_2}{u_1} \right)^p \right) |\nabla u_1|^p + \left(1 + (p-1) \left(\frac{u_1}{u_2} \right)^p \right) |\nabla u_2|^p \right] dx \\ & - \int_{\mathbb{R}^N} \left[p \left(\frac{u_2}{u_1} \right)^{p-1} |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla u_2 + p \left(\frac{u_1}{u_2} \right)^{p-1} |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla u_1 \right] dx = 0. \end{aligned}$$

Let $\eta_1 = u_1^p$ and $\eta_2 = u_2^p$. Then

$$\begin{aligned} T &= \int_{\mathbb{R}^N} (\eta_1 - \eta_2) (|\nabla \log u_1|^p - |\nabla \log u_2|^p) dx \\ & - \int_{\mathbb{R}^N} p \eta_2 |\nabla \log u_1|^{p-2} \nabla \log u_1 \cdot (\nabla \log u_2 - \nabla \log u_1) dx \\ & - \int_{\mathbb{R}^N} p \eta_1 |\nabla \log u_2|^{p-2} \nabla \log u_2 \cdot (\nabla \log u_1 - \nabla \log u_2) dx. \end{aligned} \quad (4.2)$$

By using the inequality $|b|^r \geq |a|^r + p|a|^{r-2}a(b-a)$ for $r \geq 1$, we deduce that

$$\int_{\mathbb{R}^N} p \eta_2 |\nabla \log u_1|^{p-2} \nabla \log u_1 \cdot (\nabla \log u_2 - \nabla \log u_1) dx \leq \int_{\mathbb{R}^N} \eta_2 |\nabla \log u_2|^p - \eta_2 |\nabla \log u_1|^p dx, \quad (4.3)$$

and

$$\int_{\mathbb{R}^N} p \eta_1 |\nabla \log u_2|^{p-2} \nabla \log u_2 \cdot (\nabla \log u_1 - \nabla \log u_2) dx \leq \int_{\mathbb{R}^N} \eta_1 |\nabla \log u_1|^p - \eta_1 |\nabla \log u_2|^p dx. \quad (4.4)$$

Combining (4.2), (4.3) and (4.4), we have

$$\begin{aligned}
T &= \int_{\mathbb{R}^N} (\eta_1 - \eta_2) (|\nabla \log u_1|^p - |\nabla \log u_2|^p) dx \\
&\quad - \int_{\mathbb{R}^N} p\eta_2 |\nabla \log u_1|^{p-2} \nabla \log u_1 \cdot (\nabla \log u_2 - \nabla \log u_1) dx \\
&\quad - \int_{\mathbb{R}^N} p\eta_1 |\nabla \log u_2|^{p-2} \nabla \log u_2 \cdot (\nabla \log u_1 - \nabla \log u_2) dx \\
&\geq \int_{\mathbb{R}^N} (\eta_1 - \eta_2) (|\nabla \log u_1|^p - |\nabla \log u_2|^p) dx \\
&\quad - \int_{\mathbb{R}^N} \eta_2 |\nabla \log u_2|^p + \eta_2 |\nabla \log u_1|^p dx \\
&\quad - \int_{\mathbb{R}^N} \eta_1 |\nabla \log u_1|^p + \eta_1 |\nabla \log u_2|^p dx \\
&= \int_{\mathbb{R}^N} (\eta_1 - \eta_2) (|\nabla \log u_1|^p - |\nabla \log u_2|^p) dx \\
&\quad + \int_{\mathbb{R}^N} (\eta_2 - \eta_1) (|\nabla \log u_1|^p - |\nabla \log u_2|^p) dx \\
&= 0.
\end{aligned}$$

Based on the above derivation, we can obtain that $0 \leq T = 0$, clearly $u_1 \equiv u_2$. The proof is complete.

5. PROOF OF THEOREM 1.4

Recall $D_r = \{u(x) \in E : u(x) = u(|x|)\}$. Let $\{V_k\} \subset D_r$ be a strictly ascending sequence of finite-dimensional linear subspaces within D_r , where $\bigcup_k V_k$ is dense in D_r . Denote $V_k^\perp$ as the orthogonal complement of V_k in D_r .

Lemma 5.1. (See [4]) Under the assumptions of (1.8), the following relation holds

$$J_k := \inf_{u \in V_{k-1}^\perp} \frac{(\|\nabla u\|_p + |u|_p)^p}{|u|_q^p} = \inf_{u \in V_{k-1}^\perp} \frac{|u|_D^p}{\|u\|_q^p} \rightarrow +\infty, \text{ as } k \rightarrow +\infty.$$

From this point onward, fix c that satisfies (1.8) and define

$$S_{r,c} := \{u \in S(c) : u(x) = u(|x|)\}.$$

For each $n \in \mathbb{N}^+$ with $n \geq 3$, we set

$$\alpha_k := \left(\frac{qJ_k^{q/p}}{2Lp} \right)^{\frac{1}{q-p}}, \text{ where } L = \max_{x>0} \frac{(x+c)^q}{x^q + c^q},$$

and

$$A_k := \{u \in V_{k-1}^\perp \cap S_{r,c} : \|\nabla u\|_p = \alpha_k\}. \quad (5.1)$$

Also, define

$$\mu_k := \inf_{u \in A_k} J(u). \quad (5.2)$$

In what follows, we employ a refined analysis to show that $\mu_k \rightarrow +\infty$ as $k \rightarrow +\infty$.

Lemma 5.2. Assume (1.8) holds, then $\mu_k \rightarrow +\infty$ as $k \rightarrow +\infty$.

Proof. For any fixed $u \in A_k$, we have $\|\nabla u\|_p = \alpha_k$ and $0 < r < p$

$$\begin{aligned}
J(u) &= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q dx \\
&\geq \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} C_{N,r}^p c^{\frac{p-r}{p}} \|\nabla u\|_p^p - \frac{1}{q J_k^{q/p}} (\|\nabla u\|_p + c^{\frac{1}{p}})^q \\
&\geq \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} C_{N,r}^p c^{\frac{p-r}{p}} \|\nabla u\|_p^p - \frac{L}{q J_k^{q/p}} \|\nabla u\|_p^q - \frac{L}{q J_k^{q/p}} c^{\frac{q}{p}} \\
&\geq \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} C_{N,r}^p c^{\frac{p-r}{p}} \|\nabla u\|_p^p - \frac{L}{q J_k^{q/p}} \|\nabla u\|_p^p \cdot \|\nabla u\|_p^{q-p} - \frac{L}{q J_k^{q/p}} c^{\frac{q}{p}} \\
&= \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} C_{N,r}^p c^{\frac{p-r}{p}} \|\nabla u\|_p^p - \frac{L}{q J_k^{q/p}} \|\nabla u\|_p^p \cdot \frac{q J_k^{q/p}}{2Lp} - \frac{L}{q J_k^{q/p}} c^{\frac{q}{p}} \\
&= \frac{1}{p} \left(\frac{1}{2} - C_{N,r}^p c^{\frac{p-r}{p}} \right) \|\nabla u\|_p^p - \frac{L}{q J_k^{q/p}} c^{\frac{q}{p}}.
\end{aligned}$$

Leveraging Lemma 5.1, it follows that $J_k \rightarrow +\infty$ as $k \rightarrow +\infty$. Given $q > p$, from the definition of α_k , we can deduce $\alpha_k \rightarrow +\infty$ as $k \rightarrow +\infty$. Recall that, for any $u \in A_k$, $\|\nabla u\|_p = \alpha_k$. Thus, as $k \rightarrow +\infty$, $\|\nabla u\|_p$ must tend to $+\infty$. So, for any fixed $u \in A_k$,

$$J(u) \geq \frac{1}{p} \left(\frac{1}{2} - C_{N,r}^p c^{\frac{p-r}{p}} \right) \|\nabla u\|_p^p - \frac{L}{q J_k^{q/p}} c^{\frac{q}{p}} \rightarrow +\infty,$$

as $k \rightarrow +\infty$. Finally, taking the infimum over $u \in A_k$, we have $\mu_k := \inf_{u \in A_k} J(u) \rightarrow +\infty$. $\square$

Next, we define a map

$$\rho : D_r \times \mathbb{R} \rightarrow D_r, (u, t) \mapsto \rho(u, t) = (u \star t) := e^{\frac{N}{p}t} u(e^t x). \quad (5.3)$$

For any fixed $u \in S_{r,c}$, we have $u \star t \in S_{r,c}$ for all $t \in \mathbb{R}$ and

$$J(u \star t) = \frac{e^{pt}}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{e^{rt}}{p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^r} dx - \frac{e^{qr_q t}}{q} \int_{\mathbb{R}^N} |u|^q dx,$$

where $r_q = \frac{N(q-p)}{qp}$. Since $q > \frac{p^2+Np}{N}$, we have $qr_q > p$, so that

$$\begin{cases} \|\nabla(u \star t)\|_p \rightarrow 0^+, J(u \star t) \rightarrow 0^+, \text{ as } t \rightarrow -\infty, \\ \|\nabla(u \star t)\|_p \rightarrow +\infty, J(u \star t) \rightarrow -\infty, \text{ as } t \rightarrow +\infty. \end{cases}$$

Hence, there exists $t_k > 0$ sufficiently large such that

$$\begin{cases} \|\nabla u \star (-t_k)\|_p < \alpha_k < \|\nabla(u \star t_k)\|_p, \\ \max\{J(u \star (-t_k)), J(u \star t_k)\} < \mu_k. \end{cases} \quad (5.4)$$

Let $\Gamma_k := \{\gamma \in C([0, 1] \times (S_{r,c} \cap V_k), S_{r,c}) \mid \gamma \text{ is odd in } u, \gamma(0, u) = u \star (-t_k), \gamma(1, u) = u \star t_k\}$. Then $\Gamma_k \neq \emptyset$ as $u \star (2f-1)t_k \in \Gamma_k$. So we can introduce a sequence of min-max values

$$\varepsilon_k(c) = \inf_{\gamma \in \Gamma_k} \max_{f \in [0, 1], u \in S_{r,c} \cap V_k} J(\gamma(f, u)). \quad (5.5)$$

Lemma 5.3. (See [2]) Let L_1 and L be two finite-dimensional normed vector spaces, where $L_1 \subseteq L$ and $L_1 \neq L$. For $\beta \in \mathbb{R}$ and $u \in S = \{u \in L : \|u\|_p^p = c > 0\}$, if $\phi = (\phi_1, \phi_2) \in C([0, 1] \times S, \mathbb{R} \times L_1)$ satisfies $\phi_1(f, u) = \phi_1(f, -u)$, $\phi_2(f, u) = -\phi_2(f, -u)$, and $\phi_1(0, u) < \beta < \phi_1(1, u)$, then there exists $(f, u) \in [0, 1] \times S$ such that $\phi(f, u) = (\beta, 0)$.

We utilize Lemma 5.3 to establish the crucial intersection lemma.

Lemma 5.4. Given any $\gamma \in \Gamma_k$ there exists $(f, u) \in [0, 1] \times (S_{r,c} \cap V_k)$ such that $\gamma(f, u) \in A_k$, where A_k is defined in (5.1). That is, for any $\gamma \in \Gamma_k$ it holds that $\gamma([0, 1] \times (S_{r,c} \cap V_k)) \cap A_k \neq \emptyset$.

Proof. Set $L = V_k$ and $L_1 = V_{k-1}$ equipped with the $L^p(\mathbb{R}^N)$ norm. Let $S = S_{r,c} \cap V_k$ and $\beta = \alpha_k$ in Lemma 5.3. Denote P_{k-1} as the orthogonal projection from D_r to V_{k-1} and define

$$G_k : S_{r,c} \rightarrow \mathbb{R} \times V_{k-1}, \quad u \mapsto (\|\nabla u\|_p, P_{k-1}u).$$

For $\gamma \in \Gamma_k$, define

$$\phi = (\phi_1, \phi_2) = G_k \circ \gamma : [0, 1] \times (S_{r,c} \cap V_k) \rightarrow \mathbb{R} \times V_{k-1}, \quad (f, u) \mapsto G_k(\gamma(f, u)).$$

It is evident that

$$G_k(u) = (\alpha_k, 0) \Leftrightarrow u \in A_k := \{v \in V_{k-1}^\perp \cap S_{r,c} : \|\nabla v\|_p = \alpha_k\}.$$

We can verify via (5.4) that all the conditions of Lemma 5.3 are satisfied. Thus there exists $(f, u) \in [0, 1] \times (S_{r,c} \cap V_k)$ such that $\phi(f, u) = (\alpha_k, 0)$. Since $\phi(f, u) = G_k(\gamma(f, u))$, we conclude $\gamma(f, u) \in A_k$. $\square$

Lemma 5.5. Let μ_k and $\varepsilon_k(c)$ stand for the quantities defined in (5.2) and (5.5), respectively. Then $\varepsilon_k(c) \geq \mu_k$.

Proof. From Lemma 5.4, it can be observed that, for each $\gamma \in \Gamma_k$, the intersection $\gamma([0, 1] \times (S_{r,c} \cap V_k)) \cap A_k$ is non-empty. Consequently, we derive that

$$\varepsilon_k(c) := \inf_{\gamma \in \Gamma_k} \max_{0 \leq f \leq 1, u \in S_{r,c} \cap V_k} J(\gamma(f, u)) \geq \mu_k := \inf_{u \in A_k} J(u).$$

$\square$

According to Lemma 5.2 and Lemma 5.5, there exists $k_0 \in \mathbb{N}$ such that

$$\varepsilon_k(c) \geq \mu_k > 1, \quad \text{for } k \geq k_0. \quad (5.6)$$

For any fixed $k \geq k_0$, we aim to demonstrate that $\{\varepsilon_k(c)\} (k \geq k_0)$ actually constitutes a sequence of critical values for $J|_{S_{r,c}}$. To achieve this, we construct a bounded Palais-Smale sequence at the level $\varepsilon_k(c)$. First, we introduce the stretched functional

$$\tilde{J} : S_{r,c} \times \mathbb{R} \rightarrow \mathbb{R}, \quad (u, t) \mapsto J(u \star t),$$

and a novel min-max value

$$\tilde{\varepsilon}_k(c) := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_k} \max_{0 \leq f \leq 1, u \in S_{r,c} \cap V_k} \tilde{J}(\tilde{\gamma}(f, u)), \quad (5.7)$$

where the operation $u \star t$ is specified in (5.3), and the min-max class is given by

$$\tilde{\Gamma}_k := \{\tilde{\gamma} \in C([0, 1] \times (S_{r,c} \cap V_k), S_{r,c} \times \mathbb{R}) \mid \tilde{\gamma} \text{ is odd in } u \text{ and } \rho \circ \tilde{\gamma} \in \Gamma_k\}.$$

Evidently, for any $\gamma \in \Gamma_k$, setting $\tilde{\gamma} := (\gamma, 0)$, places $\tilde{\gamma}$ in $\tilde{\Gamma}_k$.

Lemma 5.6. Let $\varepsilon_k(c)$ and $\tilde{\varepsilon}_k(c)$ be defined in (5.5) and (5.7) respectively. Then $\tilde{\varepsilon}_k(c) = \varepsilon_k(c)$.

Proof. Consider the maps

$$\psi : \Gamma_k \rightarrow \tilde{\Gamma}_k, \gamma \mapsto \psi(\gamma) := (\gamma, 0),$$

and

$$\xi : \tilde{\Gamma}_k \rightarrow \Gamma_k, \tilde{\gamma} \mapsto \xi(\tilde{\gamma}) := \rho \circ \tilde{\gamma},$$

These maps satisfy $\tilde{J}(\psi(\gamma)) = J(\gamma)$ and $J(\xi(\tilde{\gamma})) = \tilde{J}(\tilde{\gamma})$. Thus, $\tilde{\varepsilon}_k(c) = \varepsilon_k(c)$ follows directly. $\square$

Let $E := D_r \times \mathbb{R}$ be endowed with the norm $\|\cdot\|_E = \|\nabla \cdot\|_p + |\cdot|_p + |\cdot|_{\mathbb{R}}$, and let E^{-1} denote its dual space. Using Ekeland's variational principle, we derive the following Lemma.

Lemma 5.7. (See [24]) Let $\tau > 0$. If $\tilde{\gamma} \in \tilde{\Gamma}_k$ satisfies $\max_{0 \leq f \leq 1, u \in S_{r,c} \cap V_k} \tilde{J}(\tilde{\gamma}(f, u)) \leq \tilde{\varepsilon}_k(c) + \tau$, then there exists $(u, t) \in S_{r,c} \times \mathbb{R}$ such that

- (a) $\tilde{J}(u, t) \in [\tilde{\varepsilon}_k(c) - \tau, \tilde{\varepsilon}_k(c) + \tau]$;
- (b) $\min_{0 \leq f \leq 1, u \in S_{r,c} \cap V_k} \|(u, t) - \tilde{\gamma}(f, u)\|_E \leq \sqrt{\tau}$;
- (c) For all $(\eta, z) \in \tilde{T}_{(u,t)} := \{(\eta, z) \in E : \langle u, \eta \rangle_{L^p} = 0\}$, it holds that

$$\|(\tilde{J}|_{S_{r,c} \times \mathbb{R}})'(u, t)\|_{E^{-1}} \leq 2\sqrt{\tau}, \text{ i.e., } |\langle \tilde{J}'(u, t), (\eta, z) \rangle_{E^{-1} \times E}| \leq 2\sqrt{\tau} \|(\eta, z)\|_E.$$

Lemma 5.8. Assume (1.8). For any fixed $k \geq k_0$, where k_0 is defined earlier to satisfy (5.6), there exists a sequence $\{v_s^{(k)}\} \subset S_{r,c}$ such that

$$J(v_s^{(k)}) \rightarrow \varepsilon_k(c), (J|_{S_{r,c}})'(v_s^{(k)}) \rightarrow 0, P(v_s^{(k)}) \rightarrow 0 \text{ as } s \rightarrow +\infty.$$

Proof. For each $s \in \mathbb{N}^+$, there exists $\gamma_s \in \Gamma_k$ satisfying $\max_{0 \leq f \leq 1, u \in S_{r,c} \cap V_k} J(\gamma_s(f, u)) \leq \varepsilon_k(c) + \frac{1}{s}$. In view of $\tilde{\varepsilon}_k(c) = \varepsilon_k(c)$ and $\tilde{\gamma}_s = (\gamma_s, 0) \in \tilde{\Gamma}_k$, we obtain

$$\max_{0 \leq f \leq 1, u \in S_{r,c} \cap V_k} \tilde{J}(\tilde{\gamma}_s(f, u)) \leq \tilde{\varepsilon}_k(c) + \frac{1}{s}.$$

Setting $\tau = \frac{1}{s}$ and $\tilde{\gamma} = \tilde{\gamma}_s \in \tilde{\Gamma}_k$ in Lemma 5.7, we obtain a sequence $\{(u_s^{(k)}, t_s^{(k)})\} \subset S_{r,c} \times \mathbb{R}$ such that

- (i) $\tilde{J}(u_s^{(k)}, t_s^{(k)}) \in [\varepsilon_k(c) - \frac{1}{s}, \varepsilon_k(c) + \frac{1}{s}]$;
- (ii) $\min_{0 \leq f \leq 1, u \in S_{r,c} \cap V_k} \|(u_s^{(k)}, t_s^{(k)}) - (\gamma_s(f, u), 0)\|_E \leq \frac{1}{\sqrt{s}}$;
- (iii) For all $(\eta, z) \in \tilde{T}_{(u_s^{(k)}, t_s^{(k)})} := \{(\eta, z) \in E : \langle u_s^{(k)}, \eta \rangle_{L^p} = 0\}$, it holds that

$$\|(\tilde{J}|_{S_{r,c} \times \mathbb{R}})'(u_s^{(k)}, t_s^{(k)})\|_{E^{-1}} \leq \frac{2}{\sqrt{s}}, \text{ i.e., } |\langle \tilde{J}'(u_s^{(k)}, t_s^{(k)}), (\eta, z) \rangle_{E^{-1} \times E}| \leq \frac{2}{\sqrt{s}} \|(\eta, z)\|_E.$$

Define $v_s^{(k)} = \rho(u_s^{(k)}, t_s^{(k)}) = u_s^{(k)} \star t_s^{(k)} = e^{\frac{N}{p} t_s^{(k)}} u_s^{(k)}(e^{t_s^{(k)}} x)$, for any fixed $u_s^{(k)} \in S_{r,c}$, we have $u_s^{(k)} \star t_s^{(k)} \in S_{r,c}$ for all $t_s^{(k)} \in \mathbb{R}$. By (i) and $J(v_s^{(k)}) = J(\rho(u_s^{(k)}, t_s^{(k)})) = \tilde{J}(u_s^{(k)}, t_s^{(k)})$, we have $J(v_s^{(k)}) \rightarrow \varepsilon_k(c)$ as $s \rightarrow \infty$. Next, we use (iii) to show $P(v_s^{(k)}) \rightarrow 0$ for $s \rightarrow \infty$. For all $(\eta, z) \in \tilde{T}_{(u_s^{(k)}, t_s^{(k)})} =$

$\{(\eta, z) \in E : \langle u_s^{(k)}, \eta \rangle_{L^p} = 0\}$, condition (iii) implies

$$\begin{aligned}
& \left\langle \tilde{J}' \left(u_s^{(k)}, t_s^{(k)} \right), (\eta, z) \right\rangle \\
&= \left\langle \tilde{J}' \left(u_s^{(k)}, t_s^{(k)} \right), (\eta, 0) \right\rangle + \left\langle \tilde{J}' \left(u_s^{(k)}, t_s^{(k)} \right), (0, z) \right\rangle \\
&= e^{pt_s^{(k)}} \int_{\mathbb{R}^N} |\nabla u_s^{(k)}|^{p-2} \nabla u_s^{(k)} \cdot \nabla \eta dx - e^{rt_s^{(k)}} \int_{\mathbb{R}^N} \frac{|u_s^{(k)}|^{p-2} u_s^{(k)}}{|x|^r} \eta dx - e^{qr_q t_s^{(k)}} \int_{\mathbb{R}^N} |u_s^{(k)}|^{q-2} u_s^{(k)} \cdot \eta dx \\
&+ z \left(e^{pt_s^{(k)}} \int_{\mathbb{R}^N} |\nabla u_s^{(k)}|^p dx - \frac{re^{rt_s^{(k)}}}{p} \int_{\mathbb{R}^N} \frac{|u_s^{(k)}|^p}{|x|^r} dx - r_q e^{qr_q t_s^{(k)}} \int_{\mathbb{R}^N} |u_s^{(k)}|^q dx \right) \\
&= \left\langle \tilde{J}'((u_s^{(k)} \star t_s^{(k)}), \eta \star t_s^{(k)}) \right\rangle + z P(u_s^{(k)} \star t_s^{(k)}).
\end{aligned}$$

In particular, taking $(\eta, z) = (0, 1)$ and substituting into the equality above, we have

$$\begin{aligned}
o_s(1) &= \left\langle \tilde{J}' \left(u_s^{(k)}, t_s^{(k)} \right), (0, 1) \right\rangle \\
&= e^{pt_s^{(k)}} \int_{\mathbb{R}^N} |\nabla u_s^{(k)}|^p dx - \frac{re^{rt_s^{(k)}}}{p} \int_{\mathbb{R}^N} \frac{|u_s^{(k)}|^p}{|x|^r} dx - r_q e^{qr_q t_s^{(k)}} \int_{\mathbb{R}^N} |u_s^{(k)}|^q dx \\
&= \int_{\mathbb{R}^N} |\nabla v_s^{(k)}|^p dx - \frac{r}{p} \int_{\mathbb{R}^N} \frac{|v_s^{(k)}|^p}{|x|^r} dx - r_q \int_{\mathbb{R}^N} |v_s^{(k)}|^q dx \\
&= P(u_s^{(k)}).
\end{aligned}$$

Thus, $P(v_s^{(k)}) \rightarrow 0$ as $s \rightarrow \infty$, which confirms the first claim. For the final assertion, we aim to show $(J|_{S_{r,c}})'(v_s^{(k)}) \rightarrow 0$ as $s \rightarrow \infty$. It suffices to verify that

$$\left| \langle J'(v_s^{(k)}), \eta \rangle \right| \leq \frac{4}{\sqrt{s}} \|\eta\|_{D_r}, \quad \forall \eta \in T_{v_s^{(k)}} := \left\{ \eta \in D_r : \langle v_s^{(k)}, \eta \rangle_{L^p} = 0 \right\}.$$

Indeed, taking $\tilde{\eta}(x) = e^{-\frac{Nt_s^{(k)}}{p}} \eta(e^{-t_s^{(k)}} x)$ for $\eta \in T_{v_s^{(k)}}$, we obtain

$$\begin{aligned}
& \left\langle J'(u_s^{(k)}), \eta \right\rangle \\
&= \int_{\mathbb{R}^N} |\nabla v_s^{(k)}|^{p-2} \nabla v_s^{(k)} \cdot \nabla \eta dx - \int_{\mathbb{R}^N} \frac{|v_s^{(k)}|^{p-2} v_s^{(k)}}{|x|^r} \eta dx - \int_{\mathbb{R}^N} |v_s^{(k)}|^{q-2} v_s^{(k)} \cdot \eta dx \\
&= e^{pt_s^{(k)}} \int_{\mathbb{R}^N} |\nabla u_s^{(k)}|^{p-2} \nabla u_s^{(k)} \cdot \nabla \tilde{\eta} dx - e^{rt_s^{(k)}} \int_{\mathbb{R}^N} \frac{|u_s^{(k)}|^{p-2} u_s^{(k)}}{|x|^r} \tilde{\eta} dx \\
&\quad - e^{qr_q t_s^{(k)}} \int_{\mathbb{R}^N} |u_s^{(k)}|^{q-2} u_s^{(k)} \cdot \tilde{\eta} dx \\
&= \left\langle \tilde{J}'(u_s^{(k)}, t_s^{(k)}), (\tilde{\eta}, 0) \right\rangle.
\end{aligned}$$

As $\int_{\mathbb{R}^N} |u_s^{(k)}|^{p-2} u_s^{(k)} \tilde{\eta} dx = \int_{\mathbb{R}^N} |v_s^{(k)}|^{p-2} v_s^{(k)} \eta dx$, it follows that

$$\eta \in T_{v_s^{(k)}} \iff (\tilde{\eta}, 0) \in \tilde{T}_{(u_s^{(k)}, t_s^{(k)})}.$$

Hence, by (ii), it follows that

$$|t_s^{(k)}| = |t_s^{(k)} - 0| \leq \min_{0 \leq f \leq 1, u \in S_{r,c} \cap V_k} \|(u_s^{(k)}, t_s^{(k)}) - (\gamma_s(f, u), 0)\|_E \leq \frac{1}{\sqrt{s}}.$$

The above estimate on $t_s^{(k)}$ indicates that

$$\|(\tilde{\eta}, 0)\|_E = \|\tilde{\eta}\|_{D_r} = e^{-t_s^{(k)}} \|\nabla \eta\|_p + |\eta|_p \leq 2\|\eta\|_{D_r}.$$

Thus, we derive

$$|\langle J'(v_s^{(k)}), \eta \rangle| = \langle \tilde{J}(u_s^{(k)}, t_s^{(k)}), (\tilde{\eta}, 0) \rangle \leq \frac{2}{\sqrt{s}} \|(\tilde{\eta}, 0)\|_E \leq \frac{4}{\sqrt{s}} \|\eta\|_{D_r}.$$

Consequently, we obtain

$$\|(J|_{S_{r,c}})'(v_s^{(k)})\| = \sup_{\eta \in T_{v_s^{(k)}}, \|\eta\| \leq 1} |\langle J'(v_s^{(k)}), \eta \rangle| \leq \frac{4}{\sqrt{s}} \rightarrow 0 \text{ as } s \rightarrow \infty.$$

□

To establish compactness of the Palais-Smale sequence $\{v_s^{(k)}\}$ from Lemma 5.8, we introduce the following Lemma.

Lemma 5.9. (See [6]) *Let $J \in C^1(D_r, \mathbb{R})$ and let $\{u_s\} \subset S_{r,c}$ be bounded in D_r . There exists $\lambda_s = -\frac{\langle J'(u_s), u_s \rangle}{c}$, such that*

$$(J|_{S_{r,c}})'(u_s) \rightarrow 0 \text{ in } D_r^{-1} \iff J'(u_s) + \lambda_s |u_s|^{p-1} \rightarrow 0 \text{ in } D_r^{-1} \text{ as } s \rightarrow +\infty.$$

Lemma 5.10. *Assume (1.8) holds. Let $\{v_s^{(k)}\} \subset S_{r,c}$ be the Palais-Smale sequence satisfying Lemma 5.8. Then, there exist $v^{(k)} \in D_r$ and $\{\lambda_s^{(k)}\} \subset \mathbb{R}$ such that, up to a subsequence,*

- (i) $v_s^{(k)} \rightharpoonup v^{(k)} \not\equiv 0$ in D_r as $s \rightarrow +\infty$;
- (ii) $\lambda_s^{(k)} \rightarrow \lambda^{(k)} > 0$ for some $\lambda^{(k)} \in \mathbb{R}$ as $s \rightarrow +\infty$;
- (iii) $-\Delta_p v_s^{(k)} + \lambda |v_s^{(k)}|^{p-2} v_s^{(k)} - \frac{|v_s^{(k)}|^{p-2}}{|x|^r} v_s^{(k)} - |v_s^{(k)}|^{q-2} v_s^{(k)} \rightarrow 0$ in D_r^{-1} as $s \rightarrow +\infty$;
- (iv) $-\Delta_p v^{(k)} + \lambda |v^{(k)}|^{p-2} v^{(k)} - \frac{|v^{(k)}|^{p-2}}{|x|^r} v^{(k)} - |v^{(k)}|^{q-2} v^{(k)} = 0$ in D_r^{-1} ;
- (v) $v_s^{(k)} \rightarrow v^{(k)}$ in D_r as $s \rightarrow +\infty$.

Proof. Using $P(v_s^{(k)}) \rightarrow 0$ as $s \rightarrow \infty$, we have

$$\begin{aligned} \varepsilon_k(c) + 1 \geq J(v_s^{(k)}) &= \left(\frac{1}{p} - \frac{1}{qr_q}\right) \int_{\mathbb{R}^n} |\nabla v_s^{(k)}| dx - \left(\frac{1}{p} - \frac{r}{qr_q p}\right) \int_{\mathbb{R}^n} \frac{|v_s^{(k)}|^p}{|x|^r} dx \\ &\geq \left(\frac{1}{p} - \frac{1}{qr_q}\right) \|\nabla v_s^{(k)}\|_p^p - \left(\frac{1}{p} - \frac{r}{qr_q p}\right) C_{N,r}^p c^{\frac{p-r}{p}} \|\nabla v_s^{(k)}\|_p^r. \end{aligned}$$

In view of $q > \frac{Np+p^2}{N}$, we have $\frac{1}{p} - \frac{1}{qr_q} > 0$. Thus $\{v_s^{(k)}\}$ is bounded in D_r . There exists $v^{(k)} \in D_r$ such that

$$v_s^{(k)} \rightharpoonup v^{(k)} \text{ in } D_r, \quad v_s^{(k)} \rightarrow v^{(k)} \text{ in } L_{loc}^q(\mathbb{R}^N), \quad v_s^{(k)} \rightarrow v^{(k)} \text{ a.e. on } \mathbb{R}^N,$$

up to a subsequence. We assert $v^{(k)} \not\equiv 0$. If $\|v_s^{(k)}\|_q \rightarrow 0$, it then follows from $P(v_s^{(k)}) = o_s(1)$ that

$$\int_{\mathbb{R}^N} |\nabla v_s^{(k)}|^p dx - \frac{r}{p} \int_{\mathbb{R}^N} \frac{|v_s^{(k)}|^p}{|x|^r} dx = r_q \int_{\mathbb{R}^N} |v_s^{(k)}|^q dx + o_s(1) = o_s(1),$$

which implies $J(v_s^{(k)}) = o_s(1)$, contradicting $\varepsilon_k(c) \geq \mu_k > 1$. Hence, claim (i) holds. By Lemma 5.9, there exists $\lambda_s^{(k)} \in \mathbb{R}$ such that $J'(v_s^{(k)}) + \lambda_s^{(k)} |v_s^{(k)}|^{p-1} \rightarrow 0$ in D_r^{-1} , as $s \rightarrow \infty$, where

$$\lambda_s^{(k)} = -\frac{\langle J'(v_s^{(k)}), v_s^{(k)} \rangle}{c} = \frac{-\int_{\mathbb{R}^N} |\nabla v_s^{(k)}|^p dx + \int_{\mathbb{R}^N} \frac{|v_s^{(k)}|^p}{|x|^r} dx + \int_{\mathbb{R}^N} |v_s^{(k)}|^q dx}{c}. \quad (5.8)$$

In other words, for any $\eta \in D_r$, we have

$$\begin{aligned} \langle J'(v_s^{(k)}) + \lambda_s^{(k)} v_s^{(k)}, \eta \rangle &= \int_{\mathbb{R}^N} |\nabla v_s^{(k)}|^{p-2} \nabla v_s^{(k)} \cdot \nabla \eta dx - \int_{\mathbb{R}^N} \frac{|v_s^{(k)}|^{p-2} v_s^{(k)}}{|x|^r} \eta dx \\ &\quad - \int_{\mathbb{R}^N} |v_s^{(k)}|^{q-2} v_s^{(k)} \eta dx + \lambda_s^{(k)} \int_{\mathbb{R}^N} |v_s^{(k)}|^{p-2} v_s^{(k)} \eta dx \\ &= o_s(1) \|\eta\|_{D_r}. \end{aligned} \quad (5.9)$$

Thus, (iii) is valid. The boundedness of $\{v_s^{(k)}\}$ in D_r and $P(v_s^{(k)}) = o_s(1)$ imply $\{\lambda_s^{(k)}\}$ is bounded. Then, there exists $\lambda^{(k)} \in \mathbb{R}$ such that $\lambda_s^{(k)} \rightarrow \lambda^{(k)}$ as $s \rightarrow +\infty$ (up to a subsequence). Moreover, by (5.8), we have $P(v_s^{(k)}) = o_s(1)$, $r_q < 1$ and (1.8) shows that

$$\begin{aligned} \lambda^k &= \lim_{s \rightarrow \infty} \lambda_s^{(k)} = \lim_{s \rightarrow \infty} \frac{1}{c} \left(-\int_{\mathbb{R}^N} |\nabla v_s^{(k)}|^p dx + \int_{\mathbb{R}^N} \frac{|v_s^{(k)}|^p}{|x|^r} dx + \int_{\mathbb{R}^N} |v_s^{(k)}|^q dx \right) \\ &= \lim_{s \rightarrow \infty} \frac{1}{c} \left(\left(\frac{1}{r_q} - 1 \right) \int_{\mathbb{R}^N} |\nabla v_s^{(k)}|^p dx + \left(1 - \frac{r}{r_q p} \right) \int_{\mathbb{R}^N} \frac{|v_s^{(k)}|^p}{|x|^r} dx \right), \end{aligned} \quad (5.10)$$

so $\lambda^{(k)} \geq 0$. It must be $\lambda^{(k)} > 0$. Otherwise, if $\lambda^{(k)} = 0$, then (5.10) gives

$$\int_{\mathbb{R}^N} |\nabla v_s^{(k)}|^p dx = \int_{\mathbb{R}^N} \frac{|v_s^{(k)}|^p}{|x|^r} dx = 0,$$

but this contradicts $\varepsilon_k(c) > 1$. Hence, (ii) holds. From (ii) – (iii), $v^{(k)}$ satisfies

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v^{(k)}|^{p-2} \nabla v^{(k)} \cdot \nabla \eta dx - \int_{\mathbb{R}^N} \frac{|v^{(k)}|^{p-2} v^{(k)}}{|x|^r} \eta dx \\ &\quad - \int_{\mathbb{R}^N} |v^{(k)}|^{q-2} v^{(k)} \eta dx + \lambda^{(k)} \int_{\mathbb{R}^N} |v^{(k)}|^{p-2} v^{(k)} \eta = 0, \end{aligned} \quad (5.11)$$

for any $\eta \in D_r$. That is equivalent to (iv). Test (5.9) and (5.11) with $\eta = v_s^{(k)} - v^{(k)}$ and use $v_s^{(k)} \rightarrow v^{(k)}$ in $L^q(\mathbb{R}^N)$, Then, by a proof similar to that of (2.28), it can be obtained

$$\int_{\mathbb{R}^N} \frac{|v_s^{(k)}|^p}{|x|^r} dx \rightarrow \int_{\mathbb{R}^N} \frac{|v^{(k)}|^p}{|x|^r} dx, \text{ in } L^p(\mathbb{R}^N).$$

It follows that

$$\begin{aligned}
 & \int_{\mathbb{R}^N} (|\nabla v_s^{(k)}|^{p-2} \nabla v_s^{(k)} - |\nabla v^{(k)}|^{p-2} \nabla v^{(k)}) (\nabla v_s^{(k)} - \nabla v^{(k)}) dx \\
 & \quad + \lambda^{(k)} \int_{\mathbb{R}^N} (|v_s^{(k)}|^{p-2} v_s^{(k)} - |v^{(k)}|^{p-2} v^{(k)}) (v_s^{(k)} - v^{(k)}) \\
 & = \int_{\mathbb{R}^N} \frac{(|v_s^{(k)}|^{p-1} - |v^{(k)}|^{p-1})}{|x|^r} (v_s^{(k)} - v^{(k)}) dx \\
 & \quad + \int_{\mathbb{R}^N} (|v_s^{(k)}|^{q-2} v_s^{(k)} - |v^{(k)}|^{q-2} v^{(k)}) (v_s^{(k)} - v^{(k)}) dx + o_s(1) \\
 & = o_s(1).
 \end{aligned}$$

Since the strong convergence of the norm here is very similar to Lemma 2.9, it can be easily obtained $v_s^{(k)} \rightarrow v^{(k)}$ in D_r as $s \rightarrow \infty$. Thus (v) is proved. $\square$

Proof of Theorem 1.4. In the case of (1.8), when $k \geq k_0$, Lemma 5.10 implies that (1.7) admits a sequence of weak solutions $\{(v^{(k)}, \lambda^{(k)})\} \subseteq D_r \times \mathbb{R}^-$ with $\|v^{(k)}\|_{L^p}^p = c$. Since $P(v^k) = 0$, from Lemma 5.5, we have

$$\left(\frac{1}{p} - \frac{1}{qr_q} \right) \int_{\mathbb{R}^n} |\nabla v^{(k)}|^p dx - \left(\frac{1}{p} - \frac{r}{qr_q p} \right) \int_{\mathbb{R}^n} \frac{|v^{(k)}|^p}{|x|^r} dx = J(v^{(k)}) = \varepsilon_k(c) \geq \mu_k \rightarrow \infty,$$

as $k \rightarrow +\infty$. Consequently, we have $J(v^{(k)}) \rightarrow +\infty$ and $\|v^{(k)}\|_{D_r} \rightarrow +\infty$.

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