

ON A SYSTEM OF FUZZY FRACTIONAL DIFFERENTIAL INCLUSION COUPLED WITH QUASI-VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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Abstract. In this paper, we investigate a coupled system (FFDIQVHVI) which consists of a fuzzy fractional differential inclusion and a quasi-variational-hemivariational inequality (QVHVI) with a variable constraint set in Banach spaces. First, we obtain a uniqueness and existence theorem for QVHVI under some appropriate assumptions. Then, we prove the nonemptiness of the mild solution set for FFDIQVHVI. Also, we show that the set of all mild trajectories of FFDIQVHVI is compact. Our proof is based on the fixed point theorem of condensing set-valued operators, fuzzy set theory, and theory of measure of noncompactness. Finally, we use a nonlinear quasistatic thermoelastic frictional contact problem with implicit obstacles to illustrate our abstract results.

Keywords. Coupled system; Fuzzy fractional differential inclusion; Thermoelastic frictional contact problem; Variational-hemivariational inequalities.

1. INTRODUCTION

Throughout the paper, we assume that E_i is a separable and reflexive Banach space with dual space E_i^* for each $i = 1, 2, 3$. The duality pairing between E_i^* and E_i is represented by $\langle \cdot, \cdot \rangle_{E_i^* \times E_i}$. Let $\mathbb{R} = (-\infty, +\infty)$, $I = [0, b]$, $\mathbb{R}^+ = (0, +\infty)$, and C be a convex, closed, and nonempty subset of E_2 . $\|\cdot\|_{E_i}$ ($\|\cdot\|_{E_i^*}$) stands for the norm in E_i (E_i^*). The symbol $\rightharpoonup$ ($\rightarrow$) indicates the convergence in the weak (strong) sense.

This paper is focused on investigating a new class of coupled system (FFDIQVHVI) which consists of a fuzzy fractional differential inclusion and a quasi-variational-hemivariational inequality (QVHVI) with a variable constraint set in Banach spaces as follows:

$$\begin{cases} {}^C D_{\vartheta}^q \theta(\vartheta) \in M\theta(\vartheta) + [G_{(\vartheta, \theta(\vartheta))}]_{\alpha} + E(\vartheta, \theta(\vartheta), u(\vartheta)), \text{ a.e. } \vartheta \in I, \\ u(\vartheta) \in SOL(C, A(\cdot) + B\theta(\vartheta), J, \varphi, K), \text{ a.e. } \vartheta \in I, \\ \theta(0) = \theta_0, \end{cases} \quad (1.1)$$

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where $\alpha \in [0, 1]$, ${}^C D_{\vartheta}^q$ denotes the Caputo fractional derivative of order $q \in (\frac{1}{2}, 1]$, $G : I \times E_1 \rightarrow F(E_1)$ is a fuzzy mapping where $F(E_1)$ represents the collection of all fuzzy sets of E_1 , $M : D(M) \subset E_1 \rightarrow E_1$ is an infinitesimal generator of C_0 -semigroup $T(\vartheta)$ ($\vartheta \geq 0$) on E_1 , the generalized directional derivative of a locally Lipschitz function $J : E_3 \rightarrow \mathbb{R}$ is denoted by J^0 , $E : I \times E_1 \times E_2 \rightarrow E_1$, $f : I \rightarrow E_2^*$, $A : E_2 \rightarrow E_2^*$, $K : C \rightarrow 2^C$, $B : E_1 \rightarrow E_2^*$, $\varphi : E_2 \times E_2 \rightarrow \mathbb{R}$, $\gamma : E_2 \rightarrow E_3$ are given functions, the set $SOL(C, A(\cdot) + B\theta(t), J, \varphi, K)$ denotes the solution set for the QVHVI described below: find $u : I \rightarrow C$ satisfying $u(\vartheta) \in K(u(\vartheta))$ and

$$\begin{aligned} \langle f(\vartheta), e - u(\vartheta) \rangle &\leq \langle Au(\vartheta) + B\theta(\vartheta), e - u(\vartheta) \rangle + J^0(\gamma u(\vartheta); \gamma(e - u(\vartheta))) + \varphi(u(\vartheta), e) \\ &\quad - \varphi(u(\vartheta), u(\vartheta)) \end{aligned}$$

for all $e \in K(u(\vartheta))$.

Variational inequality theory is a key brunch of optimization theory. It is often used as a mathematical programming tool to create models for various theoretical and applied problems (see, e.g., [9, 14, 16, 17, 25, 28]). The concept of quasi-variational inequalities, which incorporate both elliptic and parabolic characteristics, was initially introduced by Bensoussan and Lions [7]. In the early 1980s, inspired by various industrial processes and physical events, Panagiotopoulos [21, 22, 23] introduced the concept of hemivariational inequalities via Clarke's generalized subgradient (see [2]) and applied this concept to study systems with nonsmooth and nonconvex behavior. Recently, the theory of quasi-variational-hemivariational inequalities (QVHIVs) has emerged with many important new discoveries. Zeng et al. [40] examined a type of nonlinear evolutionary QVHIVs, and they demonstrated the existence of solutions for the QVHIVs via the Kluge fixed point theorem for set-valued operators. Migórski et al. [19] also studied a kind of QVHIVs. In addition, they used the Kakutani-Ky Fan fixed point theorem to prove that the set of solutions for QVHIVs is compact and nonempty. For more information about these topics, we refer to [4, 24, 39]. Moreover, for many real-life problems, the information in differential inclusions (equations) models is not complete and has some uncertainty. Fuzzy set theory, which was started by Zadeh [35], is widely recognized as a useful tool for showing uncertainty. Since fuzzy mathematical models can be used to keep information objective, researchers studied fuzzy set theory. For example, Hüllermeier et al. [10] considered a fuzzy differential inclusion (FDI) in finite dimensional Euclidean spaces and obtained the existence result for FDI. Note that fractional calculus is a more effective tool than integer calculus for describing physical processes and events, especially those with memory and inherited characteristics. In [1], Agarwal et al. were the first to combine fractional calculus with fuzzy differential equations. They introduced the notion of fuzzy fractional differential equations. In [33], Wu et al. examined a class of fuzzy fractional DVI (FFDVI) in Euclidean spaces and demonstrated the existence of solutions for FFDVI. In [31], Wu and Chen explored a class of fuzzy fractional delay differential hemivariational inequalities (FFDDHVI) in Banach spaces. They also demonstrated the compactness and nonemptiness of mild solution sets for the FFDDHVI by employing the fixed point theorem. For further details, we refer to ([11, 29, 32, 34, 37, 38]).

Drawing motivation and inspiration from FFDVI and QVHVI, we study a new class of FFDIQVHVI in Banach spaces. It is important to state that the FFDIQVHVI has real-world practical value. For instance, Raghunathan et al. [26] proposed that the process of fermentation could be described by using a suitable DVI model, where the differential equations showed

how extracellular metabolites change, and the variational inequalities explained the relationship between cell metabolism and its environment. However, in real situations, the metabolic processes of microorganisms can be affected by uncertain environmental factors, such as the unpredictable availability of nutrients in the culture medium and small changes in temperature. Additionally, Toledo-Hernandez et al. [30] stated that fermentations show behavior with memory. Therefore, FFDIQVHVI seems to be a better model than DVI for describing fermentation dynamics. Consequently, it is both important and interesting to explore FFDIQVHVI in the Banach space.

The goal of this paper is to show that the set of mild solutions for FFDIQVHVI is nonempty and that the set of all mild trajectories of FFDIQVHVI is compact. We also apply our abstract results to a frictional contact problem as an example. The novelty of this paper manifests in three distinct aspects. First, we prove that the set of mild solutions for FFDIQVHVI is nonempty and show that the set of all mild trajectories is compact. Our approach is based on the fixed point theorem of condensing set-valued operators, fuzzy set theory, and theory of measure of non-compactness. Compared to other literature, this model has some new features: (i) the set of unilateral constraints K depends on the unknown solution; (ii) the φ in (1.1) involves two variables which is able to deal with a wider range of convex law. Second, we establish the existence and uniqueness theorem for QVHVI under suitable assumptions. Finally, we use the general theoretical results about FFDIQVHVI to study a new quasistatic thermoelastic contact problem with friction and implicit obstacles. For this problem, we confirm the existence of solutions. In practical contexts, considering factors such as external environmental conditions, thermal conductivity, and surface roughness of the contact area can cause uncertainty in estimating and measuring volume heat source density. Therefore, we examine the fuzzy fractional differential inclusions that come from the heat conduction equation and fuzzy systems in the contact problem. Moreover, as far as we know, FFDIQVHVI has not been studied. The paper follows this structure. Section 2 offers definitions and fundamental tools. The existence and uniqueness theorem for QVHVI is discussed in Section 3. Section 4 establishes the main results for FFDIQVHVI (1.1). Section 5 illustrates the applicability of our abstract results in the study of a quasistatic thermoelastic frictional contact problem with implicit obstacles. The conclusion of the paper can be found in Section 6.

2. PRELIMINARIES

In this section, let us review some concepts, lemmas, and theorems.

We denote $C(I; E_1)$ the Banach space of all continuous functions mapping from I to E_1 with the norm $\|z\|_{C(I; E_1)} = \max\{\|z(\vartheta)\|_{E_1} : \vartheta \in I\}$. $L^p(I; E_1)$ denotes the Banach space comprising all Bochner integrable functions from I into E_1 , defined by the norm $\|z\|_{L^p(I; E_1)} = \int_I \|z(\vartheta)\|_{E_1} d\vartheta$. Subsequently, we denote the set of all nonempty subsets of E_1 by 2^{E_1} and set

$$C_b(E_1) := \{W \in 2^{E_1} : W \text{ is bounded}\}, \quad C_{cc}(E_1) := \{Z \in 2^{E_1} : Z \text{ is compact and convex}\},$$

$$C_c(E_1) := \{Y \in 2^{E_1} : Y \text{ is compact}\}, \quad C_{clc}(E_1) := \{X \in 2^{E_1} : X \text{ is closed and convex}\},$$

$$C_{cl}(E_1) := \{W \in 2^{E_1} : W \text{ is closed}\}.$$

Next, let us revisit the notion of fractional calculus. Let $u \in L^1(I; E_1)$. The Riemann-Liouville fractional integral of order p with the lower limit zero of u [15] is given by

$$I_{\vartheta}^p u(\vartheta) = \frac{1}{\Gamma(p)} \int_0^{\vartheta} (\vartheta - s)^{p-1} u(s) ds, \quad \vartheta > 0, p > 0,$$

where Γ represents the Gamma function.

The Caputo fractional derivative for $u \in C([0, +\infty); E_1)$ [15] is expressed as

$${}^C D_{\vartheta}^p u(\vartheta) = \frac{1}{\Gamma(1-p)} \frac{d}{d\vartheta} \int_0^{\vartheta} (\vartheta - s)^{-p} [u(s) - u(0)] ds, \quad \vartheta > 0, p \in (0, 1].$$

If $u \in C^1([0, +\infty); E_1)$, then ${}^C D_{\vartheta}^p u(\vartheta) = \frac{1}{\Gamma(1-p)} \int_0^{\vartheta} (\vartheta - s)^{-p} u'(s) ds = I_{\vartheta}^{1-p} u'(\vartheta)$. According to [41], we introduce the concept of the mild solution for FFDIQVHVI (1.1).

Definition 2.1. A pair of functions (θ, u) , with $\theta \in C(I; E_1)$ and $u : I \rightarrow C$ measurable, is a mild solution of FFDIQVHVI (1.1) if there exists $g \in L^2(I; E_1)$ satisfying $g(\vartheta) \in [G_{(\vartheta, \theta(\vartheta))}]_{\alpha}$, a.e. $\vartheta \in I$ and

$$\theta(\vartheta) = S_q(\vartheta)\theta_0 + \int_0^{\vartheta} (\vartheta - s)^{q-1} T_q(\vartheta - s)[E(s, \theta(s), u(s)) + g(s)] ds, \quad \vartheta \in I,$$

where $u(\vartheta) \in SOL(C, A(\cdot) + B\theta(\vartheta), J, \varphi, K)$ and

$$S_q(\vartheta) = \int_0^{\infty} \phi_q(s) T(\vartheta^q s) ds, \quad T_q(\vartheta) = q \int_0^{\infty} s \phi_q(s) T(\vartheta^q s) ds,$$

$$\phi_q(s) = \frac{1}{q} s^{-1-\frac{1}{q}} \omega_q(s^{-\frac{1}{q}}), \quad \omega_q(s) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} s^{-nq-1} \frac{\Gamma(nq+1)}{n!} \sin(nq\pi), \quad s \in (0, \infty).$$

If (θ, u) is a mild solution of FFDIQVHVI, then θ is said to be a mild trajectory and u is called a variational control trajectory.

Lemma 2.1. [41, Lemma 3.2] *The operators $S_q(\eta)$ and $T_q(\eta)$ in Definition 2.1 exhibit the following characteristics:*

- (1) $S_q(\eta)$ and $T_q(\eta)$ are linear operators for any $\eta \geq 0$. Furthermore, if $\sup_{\eta \geq 0} \|T(\eta)\| \leq a_M$, then $\|S_q(\eta)\| \leq a_M$, and $\|T_q(\eta)\| \leq a_M$.
- (2) $S_q(\eta)$ and $T_q(\eta)$ are strongly continuous for each $\eta \geq 0$.
- (3) $S_q(\eta)$ and $T_q(\eta)$ are compact if $T(\eta)$ is compact for each $\eta > 0$.

At the point $e \in E_1$, the directional derivative of a locally Lipschitz function $J : E_1 \rightarrow \mathbb{R}$ in the direction $m \in E_1$ is expressed as

$$J^0(e; m) = \limsup_{s \rightarrow e, \vartheta \downarrow 0} \frac{J(s + \vartheta m) - J(s)}{\vartheta}.$$

The generalized gradient of J at the point $m \in E_1$, represented as $\partial J(m)$, is specified by

$$\partial J(m) = \{\zeta \in U^* \mid J^0(m; \xi) \geq \langle \zeta, \xi \rangle, \text{ for all } \xi \in E_1\}.$$

The subdifferential set of a convex function $\varpi : E_1 \rightarrow \mathbb{R}$ at a point ϑ is represented as

$$\partial_c \varpi(\vartheta) = \{z \in U^* \mid \varpi(\theta) \geq \varpi(\vartheta) + \langle z, \theta - \vartheta \rangle, \text{ for all } \theta \in E_1\}.$$

Next, we review the convergence of sets in the sense of Mosco.

Definition 2.2. [18] Consider K_n as convex and closed subsets of E_1 . The sequence K_n converges in the Mosco sense to K , represented by $K_n \xrightarrow{M} K$ as $n \rightarrow \infty$, if

- (1) for each $\xi_n \in K_n$ with $\xi_n \rightarrow \xi$ in E_1 , up to a subsequence, $\xi \in K$,
- (2) for every $\xi \in K$, there exists $\xi_n \in K_n$ with $\xi_n \rightarrow \xi$ in E_1 .

Recall that an operator $A : E_1 \rightarrow 2^{E_1^*}$ is said to be

- (1) monotone if, for all $\eta, \xi \in E_1$, $\langle A\eta - A\xi, \eta - \xi \rangle \geq 0$;
- (2) strongly monotone if there exists $a_A > 0$ satisfying $\langle A\eta - A\xi, \eta - \xi \rangle \geq a_A \|\eta - \xi\|^2$, for all $\eta, \xi \in E_1$;
- (3) bounded if A map bounded sets of E_1 into bounded sets of E_1^* ;
- (4) pseudomonotone if it is bounded and $\eta_n \rightarrow \eta$ in E_1 with $\limsup_{n \rightarrow +\infty} \langle A\eta_n, \eta_n - \eta \rangle \leq 0$ implies $\liminf_{n \rightarrow +\infty} \langle A\eta_n, \eta_n - \xi \rangle \geq \langle A\eta, \eta - \xi \rangle$ for all $\xi \in E_1$.

Next, we will review some concepts and features related to the Hausdorff metric and fuzzy set. Suppose that $F(E_1)$ is the collection of all fuzzy sets of E_1 and $\mu : E_1 \rightarrow [0, 1]$ is a fuzzy set of E_1 . $G_{(\vartheta, m)}$ represents the value of a fuzzy mapping $G : I \times E_1 \rightarrow F(E_1)$ at $(\vartheta, m) \in I \times E_1$. $G_{(\vartheta, m)}(z)$ represents the membership grade of the element z in $G_{(\vartheta, m)}$. For $\delta \in [0, 1]$, $[G_{(\vartheta, m)}]_\delta = \{\eta \in E_1 : G_{(\vartheta, m)}(\eta) \geq \delta\}$ is referred to as the δ -closed cut set of $G_{(\vartheta, m)}$. Suppose $\phi : E_2 \rightarrow [0, 1]$. A fuzzy mapping $\rho : E_2 \rightarrow F(E_1)$ is designated as topologically closed [3] if, for any open set $X \subset E_2$ and for each $z_0 \in E_2$ with $\rho_{z_0}(v) \geq \phi(z_0) \Rightarrow v \in X$, there exists a neighborhood Y_0 of z_0 satisfying $\rho_\delta(v) \geq \phi(\delta) \Rightarrow v \in X$, for all $\delta \in Y_0$.

Remark 2.1. It is evident that if a fuzzy mapping ϖ is topologically closed, then $\Upsilon : E_2 \rightarrow 2^{E_1}$ defined by $\Upsilon(e) := [\varpi(e)]_{\beta(e)}$ is upper semicontinuous.

A fuzzy mapping $\rho : I \times E_1 \rightarrow F(E_1)$ is classified as convex [42] if, for any $(\vartheta, \xi) \in I \times E_1$, and for all $z, e \in E_1$, along with $\varepsilon \in [0, 1]$, $\rho_{(\vartheta, \xi)}(\varepsilon z + (1 - \varepsilon)e) \geq \min\{\rho_{(\vartheta, \xi)}(z), \rho_{(\vartheta, \xi)}(e)\}$. A fuzzy mapping $\phi : I \rightarrow F(E_1)$ is said to be strongly measurable if, for each $\iota \in [0, 1]$, the mapping $[\phi(\cdot)]_\iota : I \rightarrow 2^{E_1}$ is measurable and has convex, compact, nonempty values.

Lemma 2.2. [6, Proposition 3.4 (b)] Let $\phi : I \rightarrow 2^{E_1}$ be a strongly measurable mapping. Then, ϕ possesses a strongly measurable selection.

Lemma 2.3. [5] If $\rho : E_1 \rightarrow [0, 1]$ is a function and $\phi : I \times E_1 \rightarrow F(E_1)$ is a fuzzy mapping with convex values, then, for every $\vartheta \in I$, the function $\Psi(\vartheta, \cdot) : E_1 \rightarrow 2^{E_1}$, defined as $\Psi(\vartheta, e) := [\phi(\vartheta, e)]_{\rho(e)}$, exhibits convex values.

Lemma 2.4. [6, Proposition 3.5 (a)] If $\Psi : I \times E_1 \rightarrow 2^{E_1}$ is a set-valued mapping with nonempty compact values, and for every $(\vartheta, m) \in I \times E_1$, $\Psi(\cdot, m)$ has a strongly measurable selection and $\Psi(\vartheta, \cdot)$ is upper semicontinuous, then, for $\mu \in C(I; E_1)$, there exists a strongly measurable selection $e(\cdot) \in \Psi(\cdot, \mu(\cdot))$.

Next, we review the concept of measurability for set-valued mapping. Consider $(\mathfrak{R}, \leq)$ as a partially ordered set. $\delta : C_b(E_1) \rightarrow \mathfrak{R}$ is called a measure of noncompactness (MNC) in E_1 [13] if $\delta(\overline{\text{co}}X) = \delta(X)$, for each $X \in C_b(E_1)$, where $\overline{\text{co}}(X)$ represents the closure of the convex hull of X .

In particular, a measure of noncompactness δ is said to be

- (1) monotone if $X_1, X_2 \in C_b(E_1)$, $X_1 \subseteq X_2$ implies $\delta(X_1) \leq \delta(X_2)$;
- (2) nonsingular if $\delta(\{a\} \cup X) = \delta(X)$ for each $a \in E_1$, $X \in C_b(E_1)$;

(3) regular if $\delta(X) = 0$ is equivalent to the relative compactness of X .

An essential instance of MNC is the following Hausdorff MNC on the space $C(I; E_1)$:

$$\chi_T(X) = \frac{1}{2} \limsup_{\beta \rightarrow 0} \sup_{x \in X} \max_{|\vartheta_1 - \vartheta_2| \leq \beta} \|x(\vartheta_1) - x(\vartheta_2)\|_{E_1},$$

which is utilized in Section 4.

Recall that a set-valued mapping $F : X \subseteq E_1 \rightarrow C_c(E_1)$ is said to be condensing with respect to a MNC δ (or δ -condensing) [13] if, for each $Y \subset X$ which is not relatively compact, $\delta(F(Y)) \not\leq \delta(Y)$.

The results below are essential for establishing the existence of solutions for FFDIQVHVI.

Lemma 2.5. [13] *Let $X \in C_{clc}(E_1)$ and $Q : X \rightarrow C_{cc}(X)$ be a closed δ -condensing set-valued mapping, where δ represents a nonsingular MNC defined on subsets of X . Then, the set of fixed points of Q , represented as $\text{Fix } Q := \{e \in X : e \in Q(e)\}$, is nonempty.*

Lemma 2.6. [13] *Assume $X \in C_{cl}(E_1)$ and $F : X \rightarrow C_c(E_1)$ is a closed set-valued mapping, which exhibits δ -condensing behavior on every bounded subset of X , with δ being a monotone MNC in E_1 . If $\text{Fix } F$ is bounded, then it is compact.*

3. PROPERTIES OF SOLUTION SET FOR QVHVI

This section is devoted to establishing the uniqueness and existence of solutions for QVHVI with a variable constraint set. Now, we examine the problem QVHVI: find $\eta : I \rightarrow C$ satisfying $\eta(\vartheta) \in K(\eta(\vartheta))$, a.e. $\vartheta \in I$, and

$$\begin{aligned} \langle f(\vartheta), \xi - \eta(\vartheta) \rangle &\leq \langle A\eta(\vartheta) + B\theta(\vartheta), \xi - \eta(\vartheta) \rangle + J^0(\gamma\eta(\vartheta); \gamma(\xi - \eta(\vartheta))) + \varphi(\eta(\vartheta), \xi) \\ &\quad - \varphi(\eta(\vartheta), \eta(\vartheta)), \text{ for all } \xi \in K(\eta(\vartheta)). \end{aligned} \quad (3.1)$$

To achieve this goal, we need to make the following assumptions.

$\overline{H(A)}$: $A : E_2 \rightarrow E_2^*$ is a map such that

(1) A is pseudomonotone;

(2) A is strongly monotone, i.e., there exists $a_A > 0$ such that $\langle Ae - Az, e - z \rangle \geq a_A \|e - z\|_{E_2}^2$, for all $e, z \in E_2$.

$\overline{H(B)}$: $B : E_1 \rightarrow E_2^*$ is a map such that

(1) there exists $a_B \geq 0$ satisfying $\|Bx\| \leq a_B$, for all $x \in E_1$.

$\overline{H(J)}$: $J : E_3 \rightarrow \mathbb{R}$ is a function such that

(1) J is a locally Lipschitz;

(2) there exist $d_0, d_1 \geq 0$ such that $\|\partial J(z)\| := \sup\{\|\xi\|_{E_3^*} : \xi \in \partial J(z)\} \leq d_0 + d_1 \|z\|$, for all $z \in E_3$;

(3) there exists $a_J > 0$ with $J^0(w; s - w) + J^0(s; w - s) \leq a_J \|w - s\|^2$, for all $w, s \in E_3$.

$\overline{H(\varphi)}$: $\varphi : E_2 \times E_2 \rightarrow \mathbb{R}$ is a function such that

(1) $\varphi(\eta, \cdot) : E_2 \rightarrow \mathbb{R}$ is lower semicontinuous and convex for all $\eta \in E_2$;

(2) there exists $a_\varphi > 0$ such that

$$\varphi(s_1, \xi_2) - \varphi(s_1, \xi_1) + \varphi(s_2, \xi_1) - \varphi(s_2, \xi_2) \leq a_\varphi \|s_1 - s_2\| \|\xi_1 - \xi_2\|,$$

for all $s_1, s_2, \xi_1, \xi_2 \in E_2$;

(3) there exists a continuous function $c_\varphi : [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\varphi(u, s_1) - \varphi(u, s_2) \leq c_\varphi(\|u\|) \|s_1 - s_2\|, \text{ for all } u, s_1, s_2 \in E_2;$$

(4) for all $\{\eta_n\}, \{\xi_n\}, \{z_n\} \subset E_2$, such that $\eta_n \rightarrow \eta$, $\xi_n \rightarrow \xi$ and $w_n \rightarrow w$ in E_2 ,

$$\limsup_{n \rightarrow +\infty} (\varphi(\eta_n, \xi_n) - \varphi(\eta_n, w_n)) \leq \varphi(\eta, \xi) - \varphi(\eta, w).$$

$H(K)$: $K : C \rightarrow 2^C$ is a map such that

(1) K has convex, closed and nonempty values;

(2) K exhibits weak Mosco continuity, meaning that for any sequence $\{\xi_n\} \subset C$ satisfying ξ_n converges to ξ in C , it follows that $K(\xi_n) \xrightarrow{M} K(\xi)$;

(3) there exist $r > 0$ and $e_0 \in C$ satisfying $B(e_0, r) \subset \cap_{\xi \in C} K(\xi)$, where $B(e_0, r)$ denotes a closed ball centered at e_0 with radius r .

$H(\gamma)$: γ is a compact, linear and continuous operator.

H_0 : $a_A > \max\{d_1, a_J\} \|\gamma\|^2 + a_\varphi$.

$H(f)$: $f \in L^\infty(I; E^*)$.

Theorem 3.1. Assuming that $H(\varphi)$, $H(J)$, $H(A)$, $H(K)$, $H(B)$, $H(f)$, H_0 , and $H(\gamma)$ are all valid. If r and e_0 fulfill the following condition

$$\frac{a_B + (d_0 + d_1 \|\gamma e_0\|) \|\gamma\| + c_\varphi(\|e_0\|) + \|f\|_{L^\infty(I; E_2^*)} + \|Ae_0\|}{a_A - a_\varphi - a_J \|\gamma\|^2} \leq r, \quad (3.2)$$

then, for each $\theta \in C(I; E_1)$, QVHVI (3.1) possesses a singular solution $\eta \in L^2(I; E_2)$. Furthermore, it holds

$$\|\eta(\vartheta) - e_0\| \leq \frac{a_B + (d_0 + d_1 \|\gamma e_0\|) \|\gamma\| + c_\varphi(\|e_0\|) + \|f\|_{L^\infty(I; E_2^*)} + \|Ae_0\|}{a_A - a_\varphi - a_J \|\gamma\|^2}. \quad (3.3)$$

Proof. Given a fixed $\theta \in C(I; E_1)$, to establish Theorem 3.1, we follow a three-step process.

Step 1. We demonstrate that there exists a nonempty set of solutions to QVHVI (3.1) in $L^2(I; E_2)$.

[12, Theorem 16] implies that the solution set of QVHVI (3.1) is nonempty in $L^2(I; E_2)$.

Step 2. We prove the validity of inequality (3.3).

Consider $\eta \in L^2(I; E_2)$ as a solution to QVHVI (3.1). Hence, there exist $\eta(\vartheta) \in K(\eta(\vartheta))$ and

$$\begin{aligned} \langle f(\vartheta), \xi - \eta(\vartheta) \rangle &\leq \langle A\eta(\vartheta) + B\theta(\vartheta), \xi - \eta(\vartheta) \rangle + J^0(\gamma\eta(\vartheta); \gamma(\xi - \eta(\vartheta))) + \varphi(\eta(\vartheta), \xi) \\ &\quad - \varphi(\eta(\vartheta), \eta(\vartheta)), \text{ for all } \xi \in K(\eta(\vartheta)). \end{aligned} \quad (3.4)$$

From $H(K)$ (3), we can opt for $\xi = e_0$ in (3.4), thereby yielding

$$\begin{aligned} \langle f(\vartheta), e_0 - \eta(\vartheta) \rangle &\leq \langle A\eta(\vartheta) + B\theta(\vartheta), e_0 - \eta(\vartheta) \rangle + J^0(\gamma\eta(\vartheta); \gamma(e_0 - \eta(\vartheta))) + \varphi(\eta(\vartheta), e_0) \\ &\quad - \varphi(\eta(\vartheta), \eta(\vartheta)). \end{aligned} \quad (3.5)$$

According to $H(\varphi)$ (2) and (3), it can be concluded that

$$\begin{aligned} \varphi(\eta(\vartheta), e_0) - \varphi(\eta(\vartheta), \eta(\vartheta)) &= \varphi(\eta(\vartheta), e_0) - \varphi(\eta(\vartheta), \eta(\vartheta)) + \varphi(e_0, \eta(\vartheta)) - \varphi(e_0, e_0) \\ &\quad + \varphi(e_0, e_0) - \varphi(e_0, \eta(\vartheta)) \\ &\leq a_\varphi \|\eta(\vartheta) - e_0\|^2 + c_\varphi(\|e_0\|) \|\eta(\vartheta) - e_0\|. \end{aligned} \quad (3.6)$$

By [20, Proposition 3.23(iii)], $H(J)$ (2) and (3), one sees that

$$\begin{aligned} J^0(\gamma\eta(\vartheta); \gamma(e_0 - \eta(\vartheta))) &= J^0(\gamma\eta(\vartheta); \gamma(e_0 - \eta(\vartheta))) + J^0(\gamma e_0; \gamma(\eta(\vartheta) - e_0)) \\ &\quad - J^0(\gamma e_0; \gamma(\eta(\vartheta) - e_0)) \\ &\leq a_J \|\gamma\|^2 \|\eta(\vartheta) - e_0\|^2 + (d_0 + d_1 \|\gamma e_0\|) \|\gamma\| \|\eta(\vartheta) - e_0\|. \end{aligned} \quad (3.7)$$

Hence, combining (3.5), (3.6), (3.7) and $H(A)$ (1), $H(B)$, $H(f)$, one obtains

$$\begin{aligned} a_A \|\eta(\vartheta) - e_0\|^2 &\leq \langle A\eta(\vartheta) - Ae_0, \eta(\vartheta) - e_0 \rangle \\ &= \langle A\eta(\vartheta), \eta(\vartheta) - e_0 \rangle - \langle Ae_0, \eta(\vartheta) - e_0 \rangle \\ &\leq \langle B\theta(\vartheta), e_0 - \eta(\vartheta) \rangle + J^0(\gamma\eta(\vartheta); \gamma(e_0 - \eta(\vartheta))) + \varphi(\eta(\vartheta), e_0) \\ &\quad - \varphi(\eta(\vartheta), \eta(\vartheta)) - \langle f(\vartheta), e_0 - \eta(\vartheta) \rangle - \langle Ae_0, \eta(\vartheta) - e_0 \rangle \\ &\leq a_B \|\eta(\vartheta) - e_0\| + a_J \|\gamma\|^2 \|\eta(\vartheta) - e_0\|^2 + (d_0 + d_1 \|\gamma e_0\|) \|\gamma\| \|\eta(\vartheta) - e_0\| \\ &\quad + a_\varphi \|\eta(\vartheta) - e_0\|^2 + c_\varphi (\|e_0\|) \|\eta(\vartheta) - e_0\| + \|f\|_{L^\infty(I; E_2^*)} \|\eta(\vartheta) - e_0\| \\ &\quad + \|Ae_0\| \|\eta(\vartheta) - e_0\|. \end{aligned}$$

According to H_0 , one has

$$\|\eta(\vartheta) - e_0\| \leq \frac{a_B + (d_0 + d_1 \|\gamma e_0\|) \|\gamma\| + c_\varphi (\|e_0\|) + \|f\|_{L^\infty(I; E_2^*)} + \|Ae_0\|}{a_A - a_\varphi - a_J \|\gamma\|^2}.$$

Thus (3.3) is satisfied.

Step 3. We demonstrate that the problem QVHVI (3.1) possesses a unique solution.

To demonstrate uniqueness, suppose that the problem QVHVI (3.1) has two solutions $\eta_1, \eta_2 \in L^2(I; E_2)$. Thus, we have $\eta_i(\vartheta) \in K(\eta_i(\vartheta))$ ($i = 1, 2$), and

$$\begin{aligned} \langle f(\vartheta), \xi - \eta_i(\vartheta) \rangle &\leq \langle A\eta_i(\vartheta) + B\theta(\vartheta), \xi - \eta_i(\vartheta) \rangle + J^0(\gamma\eta_i(\vartheta); \gamma(\xi - \eta_i(\vartheta))) + \varphi(\eta_i(\vartheta), \xi) \\ &\quad - \varphi(\eta_i(\vartheta), \eta_i(\vartheta)), \text{ for all } \xi \in K(\eta_i(\vartheta)). \end{aligned} \quad (3.8)$$

(3.2) and (3.3) imply that $\|\eta_i(\vartheta) - e_0\| \leq r$, for $i = 1, 2$. Thus, $\eta_i(\vartheta) \in B(e_0, r) \subset \cap_{\xi \in C} K(\xi)$, for $i = 1, 2$. Now, we take in (3.8) $\xi = \eta_2(\vartheta)$ for $i = 1$, and $\xi = \eta_1(\vartheta)$ for $i = 2$. Summing up the resulting inequalities, we obtain

$$\begin{aligned} \langle A\eta_1(\vartheta) - A\eta_2(\vartheta), \eta_1(\vartheta) - \eta_2(\vartheta) \rangle &\leq J^0(\gamma\eta_1(\vartheta); \gamma(\eta_2(\vartheta) - \eta_1(\vartheta))) \\ &\quad + J^0(\gamma\eta_2(\vartheta); \gamma(\eta_1(\vartheta) - \eta_2(\vartheta))) \\ &\quad + \varphi(\eta_1(\vartheta), \eta_2(\vartheta)) - \varphi(\eta_1(\vartheta), \eta_1(\vartheta)) \\ &\quad + \varphi(\eta_2(\vartheta), \eta_1(\vartheta)) - \varphi(\eta_2(\vartheta), \eta_2(\vartheta)). \end{aligned}$$

According to $H(A)$ (2), $H(J)$ (3) and $H(\varphi)$ (2), it follows that $(a_A - a_\varphi - a_J \|\gamma\|^2) \|\eta_1(\vartheta) - \eta_2(\vartheta)\| \leq 0$, which together with H_0 entails $\eta_1(\vartheta) = \eta_2(\vartheta)$, a.e. $\vartheta \in I$. Therefore, $\eta_1 = \eta_2$ in $L^2(I; E_2)$. $\square$

Remark 3.1. According to Theorem 3.1, it follows that, for every $\theta \in C(I; E_1)$, problem QVHVI (3.1) possesses a unique solution $\eta_\theta \in L^2(I; E_2)$. Consequently, we introduce a mapping $Q : C(I; E_1) \rightarrow L^2(I; E_2)$ as follows

$$Q(\theta) = \eta_\theta, \text{ for all } \theta \in C(I; E_1), \quad (3.9)$$

where $u_\theta \in C(I; E_1)$ denotes the singular solution to the problem QVHVI (3.1) corresponding to $\theta \in C(I; E_1)$.

4. MAIN RESULTS FOR FFDIQVHVI

In this section, we demonstrate the nonemptiness of the set of mild solutions for FFDIQVHVI (1.1) and the compactness of the set of all mild trajectories of FFDIQVHVI (1.1). In this respect, we require some assumptions:

$\underline{H(M)}$: $\{T(\vartheta)\}_{\vartheta > 0}$ is compact, and there exists a constant $a_M \geq 0$ satisfying $\sup_{\vartheta \geq 0} \|T(\vartheta)\| \leq a_M$.

$\underline{H(G)}$: $G : I \times E_1 \rightarrow F(E_1)$ is a fuzzy mapping such that:

- (1) $\tilde{G}$ is topologically closed and convex;
- (2) $\tilde{G} : I \times E_1 \rightarrow 2^{E_1}$, defined as $\tilde{G}_{(\vartheta, \theta)} = [G_{(\vartheta, \theta)}]_\alpha$, exhibits convex, nonempty, and compact values for all $(\vartheta, \theta) \in I \times E_1$;
- (3) $\tilde{G}_{(\cdot, \theta)}$ is strongly measurable for all $\theta \in E_1$;
- (4) for every $\eta > 0$, there is $a_\eta \in L^2(I; \mathbb{R}^+)$ satisfying $\|G_{(\vartheta, \theta)}\|_{E_1} \leq a_\eta(\vartheta)$, a.e. $\vartheta \in I$ and all $\theta \in E_1$ with $\|\theta\|_{E_1} \leq \eta$, where $\|G\| = H(G, \tilde{0})$, and H stands for the Hausdorff metric, $\tilde{0}$ represents the fuzzy set defined as $\tilde{0}(v) = 0$ when $v \neq 0$ and $\tilde{0}(v) = 1$ when $v = 0$.

$\underline{H(E)}$: $E : I \times E_1 \times E_2 \rightarrow E_1$ is a map such that:

- (1) the mapping $\vartheta \mapsto E(\vartheta, \theta, v)$ is measurable for each $(\theta, v) \in E_1 \times E_2$;
- (2) the mapping $(\theta, \xi) \mapsto E(\vartheta, \theta, \xi)$ is continuous, a.e. $\vartheta \in I$;
- (3) there exists $a_E \in L^2(I; \mathbb{R}^+)$ such that $\|E(\vartheta, \theta, \xi)\| \leq a_E(\vartheta)$, a.e. $\vartheta \in I$ and for all $(\theta, \xi) \in E_1 \times E_2$.

$\underline{H(Q)}$: Q is continuous on $C(I; E_1)$.

According to $H(G)$ (1), Lemma 2.2 and Remark 2.1, it can be inferred that $\tilde{G}_{(\vartheta, \cdot)}$ exhibits upper semicontinuous with convex values for all $\vartheta \in I$. Also, by utilizing Lemma 2.3, Lemma 2.4, $H(G)$ (2) and (3), we deduce that, for $z \in C(I; E_1)$, $\tilde{G}_{(\vartheta, z(\vartheta))}$ has a strongly measurable selection $w \in L^2(I; E_1)$. Thus, the set-valued mapping $P_{\tilde{G}} : C(I; E_1) \rightarrow 2^{L^2(I; E_1)}$ defined as

$$P_{\tilde{G}}(z) = \{h \in L^2(I; E_1) : h(\vartheta) \in \tilde{G}_{(\vartheta, z(\vartheta))}, \text{ a.e. } \vartheta \in I\},$$

for all $z \in C(I; E_1)$, is well-defined.

Lemma 4.1. [13, Lemma 5.1.1] Assume that $H(G)$ (2), (3), and (4) hold. If $\{\theta_n\} \subset C(I; E_1)$ and $\{g_n\} \subset L^2(I; E_1)$, with $g_n \in P_{\tilde{G}}(\theta_n)$ for each $n \geq 1$ satisfy $\theta_n \rightarrow \theta$ and $g_n \rightarrow g$, then it holds $g \in P_{\tilde{G}}(\theta)$.

Theorem 4.1. Let all the assumptions of Theorem 3.1 be valid. If conditions $H(E)$, $H(M)$, $H(Q)$, and $H(G)$ are satisfied, then the set of mild solutions for FFDIQVHVI (1.1) is nonempty and the set of all mild trajectories for FFDIQVHVI (1.1) is compact.

Proof. According to the definition of mild solutions for FFDIQVHVI (1.1), we define the set-valued mapping $\Phi : C(I; E_1) \rightarrow 2^{C(I; E_1)}$ by

$$\Phi(\theta) = \{y \in C(I; E_1) : y(\vartheta) = S_q(\vartheta)\theta_0 + \int_0^\vartheta (\vartheta - s)^{q-1} T_q(\vartheta - s)[E(s, \theta(s), (Q\theta)(s)) + g(s)]ds\},$$

where $g \in P_{\tilde{G}}(\theta)$ and Q is defined in (3.9). In fact, to establish the existence of solutions for FFDIQVHVI (1.1), it suffices to demonstrate that Φ possesses fixed points in $C(I; E_1)$. Next, we proceed with the following steps.

Step 1. $\Phi(\theta) \in C_{cc}(C(I; E_1))$ for each $\theta \in C(I; E_1)$.

Clearly, for each $\theta \in C(I; E_1)$, $\Phi(\theta)$ is convex due to the convexity of $P_{\tilde{G}}(\theta)$. Next, we establish the compactness of $\Phi(\theta)$ for $\theta \in C(I; E_1)$. For each $y \in \Phi(\theta)$, there exists $g \in P_{\tilde{G}}(\theta)$ satisfying

$$y(\vartheta) = S_q(\vartheta)\theta_0 + \int_0^{\vartheta} (\vartheta - s)^{q-1} T_q(\vartheta - s)[E(s, \theta(s), (Q\theta)(s)) + g(s)]ds, \quad \vartheta \in I.$$

According to the Hölder inequality and Lemma 2.1 (1), one has

$$\begin{aligned} \|y(\vartheta)\|_{E_1} &\leq \|S_q(\vartheta)\| \|\theta_0\|_{E_1} \\ &\quad + \int_0^{\vartheta} (\vartheta - s)^{q-1} \|T_q(\vartheta - s)\| [\|E(s, \theta(s), (Q\theta)(s))\|_{E_1} + \|g(s)\|_{E_1}] ds \\ &\leq a_M \|\theta_0\|_{E_1} + \frac{a_M}{\Gamma(q)} \int_0^{\vartheta} (\vartheta - s)^{q-1} (a_E(s) + a_\eta(s)) ds \\ &\leq a_M \|\theta_0\|_{E_1} + \frac{a_M b^{q-\frac{1}{2}}}{\Gamma(q)\sqrt{2q-1}} (\|a_E\|_{L^2(I; \mathbb{R}^+)} + \|a_\eta\|_{L^2(I; \mathbb{R}^+)}). \end{aligned}$$

Hence, $\Phi(\theta)$ is bounded in $C(I; E_1)$ for $\theta \in C(I; E_1)$. Additionally, we aim to demonstrate that $\Phi(\theta)$ constitutes a family of equicontinuous functions for each $\theta \in C(I; E_1)$. To achieve this, we tackle two scenarios in the following manner.

Case 1. If $\vartheta_1 = 0$ and $0 < \vartheta_2 \leq \varepsilon_0$, where ε_0 is sufficiently small, then we obtain by employing the compactness of $T(\vartheta)$ and the Hölder inequality that

$$\begin{aligned} \|y(\vartheta_2) - y(\vartheta_1)\|_{E_1} &= \|(S_q(\vartheta_2) - S_q(0))\theta_0 \\ &\quad + \int_0^{\vartheta_2} (\vartheta_2 - s)^{q-1} T_q(\vartheta_2 - s)[E(s, \theta(s), (Q\theta)(s)) + g(s)]ds\|_{E_1} \\ &\leq \|S_q(\vartheta_2) - S_q(0)\| \|\theta_0\|_{E_1} + \frac{a_M}{\Gamma(q)} \int_0^{\vartheta_2} (\vartheta_2 - s)^{q-1} (a_E(s) + a_\eta(s))ds \\ &\leq \|S_q(\vartheta_2) - S_q(0)\| \|\theta_0\|_{E_1} \\ &\quad + \frac{a_M}{\Gamma(q)\sqrt{2q-1}} (\|a_E\|_{L^2(I; \mathbb{R}^+)} + \|a_\eta\|_{L^2(I; \mathbb{R}^+)}) \vartheta_2^{q-\frac{1}{2}} \\ &\rightarrow 0, \quad \text{as } 0 < \vartheta_2 \leq \varepsilon_0 \rightarrow 0. \end{aligned}$$

Case 2. If $\frac{\varepsilon_0}{2} \leq \vartheta_1 < \vartheta_2 \leq b$, where ε_0 is small enough, then

$$\begin{aligned}
& \|y(\vartheta_2) - y(\vartheta_1)\|_{E_1} \\
& \leq \|S_q(\vartheta_2) - S_q(\vartheta_1)\| \|\theta_0\|_{E_1} + \left\| \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - s)^{q-1} T_q(\vartheta_2 - s) [E(s, \theta(s), (Q\theta)(s)) + g(s)] ds \right\|_{E_1} \\
& \quad + \left\| \int_0^{\vartheta_1} [(\vartheta_2 - s)^{q-1} T_q(\vartheta_2 - s) - (\vartheta_1 - s)^{q-1} T_q(\vartheta_1 - s)] [E(s, \theta(s), (Q\theta)(s)) + g(s)] ds \right\|_{E_2} \\
& \leq \|S_q(\vartheta_2) - S_q(\vartheta_1)\| \|\theta_0\|_{E_1} \\
& \quad + \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - s)^{q-1} \|T_q(\vartheta_2 - s)\| \|E(s, \theta(s), (Q\theta)(s)) + g(s)\|_{E_1} ds \\
& \quad + \int_0^{\vartheta_1} |(\vartheta_2 - s)^{q-1} - (\vartheta_1 - s)^{q-1}| \|T_q(\vartheta_2 - s)\| \|E(s, \theta(s), (Q\theta)(s)) + g(s)\|_{E_1} ds \\
& \quad + \int_0^{\vartheta_1} (\vartheta_1 - s)^{q-1} \|T_q(\vartheta_2 - s) - T_q(\vartheta_1 - s)\| \|E(s, \theta(s), (Q\theta)(s)) + g(s)\|_{E_1} ds \\
& := I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

As $T(\vartheta)$ is compact for $\vartheta > 0$, according to Lemma 2.1 (3), $S_q(\vartheta)$ is also compact. Thus,

$$I_1 = \|S_q(\vartheta_2) - S_q(\vartheta_1)\| \|\theta_0\|_{E_1} \rightarrow 0, \text{ as } |\vartheta_2 - \vartheta_1| \rightarrow 0. \quad (4.1)$$

From $H((E))$ (3), $H((G))$ (4), and the Hölder inequality, we have

$$\begin{aligned}
I_2 &= \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - s)^{q-1} \|T_q(\vartheta_2 - s)\| \|E(s, \theta(s), (Q\theta)(s)) + g(s)\|_{E_1} ds \\
&\leq \frac{a_M}{\Gamma(q)} \int_{\vartheta_1}^{\vartheta_2} (\vartheta_2 - s)^{q-1} (a_E(s) + a_\eta(s)) ds \\
&\leq \frac{a_M}{\Gamma(q)\sqrt{2q-1}} (\|a_E\|_{L^2(I; \mathbb{R}^+)} + \|a_\eta\|_{L^2(I; \mathbb{R}^+)}) (\vartheta_2 - \vartheta_1)^{q-\frac{1}{2}} \\
&\rightarrow 0, \text{ as } |\vartheta_2 - \vartheta_1| \rightarrow 0,
\end{aligned} \quad (4.2)$$

and

$$\begin{aligned}
I_3 &= \int_0^{\vartheta_1} |(\vartheta_2 - s)^{q-1} - (\vartheta_1 - s)^{q-1}| \|T_q(\vartheta_2 - s)\| \|E(s, \theta(s), (Q\theta)(s)) + g(s)\|_{E_1} ds \\
&\leq \frac{a_M}{\Gamma(q)} \int_0^{\vartheta_1} ((\vartheta_1 - s)^{q-1} - (\vartheta_2 - s)^{q-1}) (a_E(s) + a_\eta(s)) ds \\
&\leq \frac{a_M}{\Gamma(q)\sqrt{2q-1}} (\|a_E\|_{L^2(I; \mathbb{R}^+)} + \|a_\eta\|_{L^2(I; \mathbb{R}^+)}) (\vartheta_2 - \vartheta_1)^{q-\frac{1}{2}} \\
&\rightarrow 0, \text{ as } |\vartheta_2 - \vartheta_1| \rightarrow 0.
\end{aligned} \quad (4.3)$$

For $\varepsilon_0 > 0$ small enough, we obtain

$$\begin{aligned}
I_4 &= \int_0^{\vartheta_1} (\vartheta_1 - s)^{q-1} \|T_q(\vartheta_2 - s) - T_q(\vartheta_1 - s)\| \|E(s, \theta(s), (Q\theta)(s)) + g(s)\|_{E_1} ds \\
&\leq \int_{\vartheta_1 - \varepsilon_0}^{\vartheta_1} (\vartheta_1 - s)^{q-1} \|T_q(\vartheta_2 - s) - T_q(\vartheta_1 - s)\| \|E(s, \theta(s), (Q\theta)(s)) + g(s)\|_{E_1} ds \\
&\quad + \int_0^{\vartheta_1 - \varepsilon_0} (\vartheta_1 - s)^{q-1} \|T_q(\vartheta_2 - s) - T_q(\vartheta_1 - s)\| \|E(s, \theta(s), (Q\theta)(s)) + g(s)\|_{E_1} ds \\
&\leq \int_{\vartheta_1 - \varepsilon_0}^{\vartheta_1} (\vartheta_1 - s)^{q-1} \|T_q(\vartheta_2 - s) - T_q(\vartheta_1 - s)\| (a_E(s) + a_\eta(s)) ds \\
&\quad + \sup_{s \in [0, \vartheta_1 - \varepsilon_0]} \|T_q(\vartheta_2 - s) - T_q(\vartheta_1 - s)\| \int_0^{\vartheta_1 - \varepsilon_0} (\vartheta_1 - s)^{q-1} (a_E(s) + a_\eta(s)) ds \\
&\leq \frac{2a_M \varepsilon_0^{q-\frac{1}{2}}}{\Gamma(q) \sqrt{2q-1}} + \frac{b^{q-\frac{1}{2}} (\|a_E\|_{L^2(I; \mathbb{R}^+)} + \|a_\eta\|_{L^2(I; \mathbb{R}^+)})}{\sqrt{2q-1}} \sup_{s \in [0, \vartheta_1 - \varepsilon_0]} \|T_q(\vartheta_2 - s) - T_q(\vartheta_1 - s)\| \\
&\rightarrow 0, \text{ as } |\vartheta_2 - \vartheta_1| \rightarrow 0, \varepsilon_0 \rightarrow 0.
\end{aligned} \tag{4.4}$$

From (4.1)-(4.4), it follows that

$$\|y(\vartheta_2) - y(\vartheta_1)\|_{E_1} \rightarrow 0, \text{ as } |\vartheta_2 - \vartheta_1| \rightarrow 0.$$

Therefore, for each $\theta \in C(I; E_1)$, it can be deduced that $\Phi(\theta)$ is equicontinuous. [36, Arzela-Ascoli theorem] implies that $\Phi(\theta)$ is relatively compact for each $\theta \in C(I; E_1)$. To complete the proof of this step, it suffices to confirm that $\Phi(\theta)$ is a closed set in $C(I; E_1)$ for every $\theta \in C(I; E_1)$. Assume that the sequence $y_n \in \Phi(\theta)$ and $y_n \rightarrow y$ in $C(I; E_1)$ as $n \rightarrow \infty$. So there exists a sequence $g_n \in P_{\tilde{G}}(\theta)$ satisfying

$$y_n(\vartheta) = S_q(\vartheta)\theta_0 + \int_0^\vartheta (\vartheta - s)^{q-1} T_q(\vartheta - s) [E(s, \theta(s), (Q\theta)(s)) + g_n(s)] ds, \vartheta \in I.$$

By $H(G)$ (4), we conclude that $\{g_n\} \subset L^2(I; E_1)$ is weakly relatively compact. We may assume that $g_n \rightharpoonup g \in P_{\tilde{G}}(\theta)$ in $L^2(I; E_1)$, where $g \in P_{\tilde{G}}(\theta)$ is obtained by Lemma 4.1. In addition, the compactness of $T_q(\vartheta)$ for $\vartheta > 0$ implies

$$y_n(\vartheta) \rightarrow y(\vartheta) := S_q(\vartheta)\theta_0 + \int_0^\vartheta (\vartheta - s)^{q-1} T_q(\vartheta - s) [E(s, \theta(s), (Q\theta)(s)) + g(s)] ds, \vartheta \in I,$$

which means $y \in \Phi(\theta)$, i.e., $\Phi(\theta) \in C_{cc}(C(I; E_1))$.

Step 2. Φ is a closed set-valued mapping.

Let $\theta_n \rightarrow \theta$ and $y_n \rightarrow y$ in $C(I; E_1)$ with $y_n \in \Phi(\theta_n)$. Indeed, we need to demonstrate that $y \in \Phi(\theta)$. According to the definition of Φ , we can select $g_n \in P_{\tilde{G}}(\theta_n)$ such that

$$y_n(\vartheta) = S_q(\vartheta)\theta_0 + \int_0^\vartheta (\vartheta - s)^{q-1} T_q(\vartheta - s) [E(s, \theta_n(s), (Q\theta_n)(s)) + g_n(s)] ds, \vartheta \in I. \tag{4.5}$$

From $H(G)$ (4), it is straightforward to show that $\{g_n\}$ is bounded in $L^2(I; E_1)$. Therefore, we can assume that $g_n \rightharpoonup g \in L^2(I; E_1)$. By Lemma 4.1, one gets $g \in P_{\tilde{G}}(\theta)$. From (4.5), $H(E)$ (2),

$H(G)$, $H(Q)$ and compactness of $T_q(\vartheta)$ for $\vartheta > 0$, we have, for all $\vartheta \in I$,

$$y_n(\vartheta) \rightarrow y(\vartheta) := S_q(\vartheta)\theta_0 + \int_0^\vartheta (\vartheta - s)^{q-1} T_q(\vartheta - s)[E(s, \theta(s), (Q\theta)(s)) + g(s)]ds.$$

Thus, $y \in \Phi(\theta)$.

Step 3. Φ is χ_T -condensing.

Suppose that $X \in C_b(C(I; E_1))$ is not relatively compact. We need to demonstrate that $\chi_T(X) \not\leq \chi_T(\Phi(X))$. Since $X \subset C(I; E_1)$ is bounded, it is evident that $\Phi(X)$ is relatively compact by employing the same reasoning as in Step 1. Thus, $\chi_T(\Phi(X)) = 0$. By $\chi_T(X) \leq \chi_T(\Phi(X)) = 0$, one has $\chi_T(X) = 0$. According to the regularity of χ_T , we have that X is relatively compact. This confirms that Φ is χ_T -condensing.

Step 4. There exists a constant $r_0 > 0$ such that $\Phi(B_{r_0}) \subset B_{r_0} := \{\theta \in C(I; E_1) : \|\theta\|_C \leq r_0\}$. To reach the conclusion of this step, we opt for

$$a_M \|\theta_0\|_{E_1} + \frac{a_M b^{q-\frac{1}{2}}}{\Gamma(q)\sqrt{2q-1}} (\|a_E\|_{L^2(I; \mathbb{R}^+)} + \|a_\eta\|_{L^2(I; \mathbb{R}^+)}) \leq r_0.$$

Let $\theta \in B_{r_0}$. For any $y \in \Phi(\theta)$, from $H(E)$ (3), $H(G)$ (4), Lemma 2.1 (1) and Hölder inequality, we have

$$\begin{aligned} \|y(\vartheta)\|_{E_1} &\leq \|S_q(\vartheta)\| \|\theta_0\|_{E_1} + \int_0^\vartheta (\vartheta - s)^{q-1} \|T_q(\vartheta - s)\| (\|E(s, \theta(s), (Q\theta)(s))\|_{E_1} + \|g(s)\|_{E_1}) ds \\ &\leq \|S_q(\vartheta)\| \|\theta_0\|_{E_1} + \frac{a_M}{\Gamma(q)} \int_0^\vartheta (\vartheta - s)^{q-1} (a_E(s) + a_\eta(s)) ds \\ &\leq a_M \|\theta_0\|_{E_1} + \frac{a_M b^{q-\frac{1}{2}}}{\Gamma(q)\sqrt{2q-1}} (\|a_E\|_{L^2(I; \mathbb{R}^+)} + \|a_\eta\|_{L^2(I; \mathbb{R}^+)}) \\ &\leq r_0. \end{aligned}$$

Hence, all the conditions outlined in Lemma 2.5 are met. Consequently, we confirm that $\text{Fix}(\Phi) \neq \emptyset$. Thus, FFDIQVHVI (1.1) possesses at least one mild solution (θ, u) .

Step 5. We show that the set of all mild trajectories for FFDIQVHVI (1.1) is compact.

To utilize Lemma 2.6, the remaining task is to confirm that $\text{Fix}(\Phi)$ is bounded in $C(I, E_1)$. For any $\theta \in \Phi(\theta)$, leveraging Lemma 2.1 (1) alongside the Hölder inequality, we deduce that

$$\begin{aligned} \|\theta(\vartheta)\|_{E_1} &= \|S_q(\vartheta)\theta_0 + \int_0^\vartheta (\vartheta - s)^{q-1} T_q(\vartheta - s)[E(s, \theta(s), (Q\theta)(s)) + g(s)]ds\|_{E_1} \\ &\leq \|S_q(\vartheta)\| \|\theta_0\|_{E_1} \\ &\quad + \int_0^\vartheta (\vartheta - s)^{q-1} \|T_q(\vartheta - s)\| [\|E(s, \theta(s), (Q\theta)(s))\|_{E_1} + \|g(s)\|_{E_1}] ds \\ &\leq a_M \|\theta_0\|_{E_1} + \frac{a_M}{\Gamma(q)} \int_0^\vartheta (\vartheta - s)^{q-1} (a_E(s) + a_\eta(s)) ds \\ &\leq a_M \|\theta_0\|_{E_1} + \frac{a_M b^{q-\frac{1}{2}}}{\Gamma(q)\sqrt{2q-1}} (\|a_E\|_{L^2(I; \mathbb{R}^+)} + \|a_\eta\|_{L^2(I; \mathbb{R}^+)}) , \text{ for all } \vartheta \in I. \end{aligned}$$

Thus, we have

$$\|\theta\|_{C(I; E_1)} \leq a_M \|\theta_0\|_{E_1} + \frac{a_M b^{q-\frac{1}{2}}}{\Gamma(q)\sqrt{2q-1}} (\|a_E\|_{L^2(I; \mathbb{R}^+)} + \|a_\eta\|_{L^2(I; \mathbb{R}^+)}).$$

Hence, all the conditions stipulated in Lemma 2.6 are met. Consequently, we deduce that the set of all mild trajectories for FFDIVHVI (1.1) is compact in $C(I; E_1)$. This concludes the proof. $\square$

5. APPLICATIONS TO THE THERMOELASTIC FRICTIONAL CONTACT PROBLEM WITH IMPLICIT OBSTACLES

In this section, we use a quasistatic thermalelastic frictional contact problem featuring implicit obstacles to illustrate our abstract results in Section 4. We consider a thermoelastic body which subjected to a heat source, occupies a bounded domain $\Xi \subset \mathbb{R}^d$ ($d = 2, 3$) and characterized by a Lipschitz boundary $\Gamma = \partial\Xi$. The boundary Γ comprise three disjoint measurable sections Γ_1, Γ_2 , and Γ_3 , with $\text{meas}(\Gamma_1) > 0$. The body is clamped on Γ_1 , the surface tractions of density $\mathbf{f}_N$ act on Γ_2 , and the volume force of density $\mathbf{f}_0$ act in Ξ . Moreover, there is contact between the body and Γ_3 . Throughout this section, boldface notation is employed for tensors and vectors. $\mathbf{v}$ represents the unit outward normal vector on Γ . The tangential and normal components of a displacement field $\mathbf{u}$ on Γ are expressed as $\mathbf{u}_\tau = \mathbf{u} - u_\nu \mathbf{v}$ and $u_\nu = \mathbf{u} \cdot \mathbf{v}$, respectively. The symbols $\boldsymbol{\sigma}_\nu$ and $\boldsymbol{\sigma}_\tau$ denote the tangential and normal components of the stress field $\boldsymbol{\sigma}$ on Γ , defined as $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma} \mathbf{v} - \sigma_\nu \mathbf{v}$ and $\sigma_\nu = (\boldsymbol{\sigma} \mathbf{v}) \cdot \mathbf{v}$. Moreover, $\mathbb{S}^d$ represents the space of real symmetric $d \times d$ matrices. On $\mathbb{R}^d$ and $\mathbb{S}^d$ we employ the standard notation for inner products and norms determined by

$$\boldsymbol{\eta} \cdot \boldsymbol{\xi} = \eta_i \xi_i, \|\boldsymbol{\eta}\| = (\boldsymbol{\eta} \cdot \boldsymbol{\eta})^{\frac{1}{2}}, \text{ for } \boldsymbol{\eta} = (\eta_i), \boldsymbol{\xi} = (\xi_i) \in \mathbb{R}^d,$$

$$\mathbf{e} : \mathbf{m} = e_{ij} m_{ij}, \|\mathbf{e}\| = (\mathbf{e} : \mathbf{e})^{\frac{1}{2}}, \text{ for } \mathbf{e} = (e_{ij}), \mathbf{m} = (m_{ij}) \in \mathbb{S}^d,$$

where $i, j \in \{1, 2, \dots, d\}$. Let $\bar{\Xi} = \Xi \times I$, $\bar{\Gamma}_1 = \Gamma_1 \times I$, $\bar{\Gamma}_2 = \Gamma_2 \times I$ and $\bar{\Gamma}_3 = \Gamma_3 \times I$. For simplicity's sake, we typically omit the variable $\mathbf{y}$ in functions.

We consider the following contact issues.

Problem 5.1. Find a temperature $\theta : \bar{\Xi} \rightarrow \mathbb{R}$, a stress field $\boldsymbol{\sigma} : \bar{\Xi} \rightarrow \mathbb{S}^d$ and a displacement field $\mathbf{u} : \bar{\Xi} \rightarrow \mathbb{R}^d$ satisfying $\mathbf{u}(\vartheta) \in K(\mathbf{u}(\vartheta))$ and

$$\boldsymbol{\sigma}(\vartheta) = \mathcal{A} \boldsymbol{\varepsilon}(\mathbf{u}(\vartheta)) + \mathcal{B} \theta(\vartheta) \quad \text{in } \bar{\Xi}, \quad (5.1)$$

$$-\text{Div } \boldsymbol{\sigma}(\vartheta) = \mathbf{f}_0 \quad \text{in } \bar{\Xi}, \quad (5.2)$$

$$\mathbf{u}(\vartheta) = \mathbf{0} \quad \text{on } \bar{\Gamma}_1, \quad (5.3)$$

$$\boldsymbol{\sigma}(\vartheta) \mathbf{v} = \mathbf{f}_N(\vartheta) \quad \text{on } \bar{\Gamma}_2, \quad (5.4)$$

$$u_\nu(\vartheta) \leq g_0, \sigma_\nu(\vartheta) + \eta_\nu \leq 0, (u_\nu(\vartheta) - g_0)(\sigma_\nu(\vartheta) + \eta_\nu) = 0 \quad \text{on } \bar{\Gamma}_3, \quad (5.5)$$

$$\eta_\nu \in \partial j_\nu(u_\nu(\vartheta)) \quad \text{on } \bar{\Gamma}_3, \quad (5.6)$$

$$-\boldsymbol{\sigma}_\tau(\vartheta) \in \mathcal{F}_b(u_\nu(\vartheta)) \partial \|\mathbf{u}_\tau(\vartheta)\| \quad \text{on } \bar{\Gamma}_3, \quad (5.7)$$

$${}^C D_\vartheta^q \theta(\vartheta) \in \Delta \theta(\vartheta) + \mathcal{E}(\vartheta, \theta(\vartheta), \mathbf{u}(\vartheta)) + [\mathcal{G}_{(\vartheta, \theta(\vartheta))}]_\alpha \quad \text{in } \bar{\Xi}, \quad (5.8)$$

$$\theta(0) = \theta_0 \quad \text{in } \bar{\Xi}, \quad (5.9)$$

where $K : C \rightarrow 2^C$ is $K(\boldsymbol{\eta}) = \{\boldsymbol{\xi} \in C : r_1(\boldsymbol{\xi}) \leq m_1(\boldsymbol{\eta})\}$, for $\boldsymbol{\eta} \in C$, where $C \subset V$ is specified by

$$C = \{\boldsymbol{\xi} \in V : v_\nu \leq g_0 \text{ on } \Gamma_3\}, \quad (5.10)$$

and the space V is defined in (5.11) below.

Now, let us briefly introduce the equations and conditions in Problem 5.1. Conditions (5.1) and (5.2) represent the elastic constitutive law and the equilibrium equation, respectively. In these equations, $\mathcal{A}$ denotes the elasticity operator, $\mathcal{B}$ represents the thermal expansion operator, $\text{Div } \boldsymbol{\sigma} = (\sigma_{ij,j})$ denotes the divergence operator, and $\boldsymbol{\varepsilon}(\mathbf{u})$ indicates the linearized strain tensor, i.e. $\boldsymbol{\varepsilon}(\mathbf{u}) = \varepsilon_{ij}(\mathbf{u})$, $\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$, for $i, j \in \{1, 2, \dots, d\}$. Conditions (5.3) and (5.4) represent the displacement-traction boundary conditions. The frictional Signorini-type unilateral contact conditions (5.5) and (5.6) model contact with a foundation consisting of a rigid body covered by a layer of deformable material with thickness g_0 , for further details, refer to [27, Sections 8.4, 8.5]. Condition (5.7) depicts a frictional condition represented by the Coulomb law of dry friction in its subgradient form, where $\mathcal{F}_b$ denotes the friction bound dependent on the normal penetration, refer to [20] for further details. Considering the inherent errors and uncertainties associated with measuring and estimating the density of volume heat sources in real world scenarios, coupled with potential time delay effects on heat conduction, we establish the evolution of the temperature field using equation (5.8). Here, $\mathcal{E}$ signifies the heat generated by the work of internal forces, while $\mathcal{G}$ represents the density of volume heat sources with a time delay. θ_0 represents the initial temperature.

For the variational analysis of Problem 5.1, we examine the subsequent function spaces

$$V = \{\boldsymbol{\xi} \in H^1(\Xi, \mathbb{R}^d) | \boldsymbol{\xi} = \mathbf{0} \text{ a.e. on } \Gamma_1\}, H_1 = L^2(\Xi, \mathbb{R}^d), H_2 = L^2(\Xi, \mathbb{S}^d). \quad (5.11)$$

Given that $\text{meas}(\Gamma_1) > 0$, it follows that the space V constitutes a Hilbert space with the inner product defined as $\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle_V = \langle \boldsymbol{\varepsilon}(\boldsymbol{\eta}), \boldsymbol{\varepsilon}(\boldsymbol{\xi}) \rangle_{H_2}$ for all $\boldsymbol{\eta}, \boldsymbol{\xi} \in V$, and the corresponding norm represented as $\|\cdot\|_V$. We designate V^* as the dual space of V , and $\langle \cdot, \cdot \rangle$ signifies the duality pairing between V^* and V . Simultaneously, it is evident that the space H_2 also constitutes a Hilbert space with the inner product denoted as

$$\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle_{H_2} = \int_{\Xi} \eta_{ij}(\mathbf{y}) \xi_{ij}(\mathbf{y}) d\mathbf{y}, \text{ for } \boldsymbol{\eta}, \boldsymbol{\xi} \in H_2,$$

and the corresponding norm indicated as $\|\cdot\|_{H_2}$. Following that, the trace of an element $\boldsymbol{\xi} \in V$ is likewise represented by $\boldsymbol{\xi}$. Remember that $V \subset L^2(\Xi; \mathbb{R}^d)$ entails a compact embedding, hence there exists a positive constant D_0 satisfying $\|\boldsymbol{\xi}\|_{L^2(\Xi; \mathbb{R}^d)} \leq D_0 \|\boldsymbol{\xi}\|_V$. By virtue of the Sobolev trace theorem, we can determine the minimal constant $r_0 > 0$ (dependent solely on Ξ and Γ_3) satisfying $\|\boldsymbol{\xi}\|_{L^2(\Gamma_3; \mathbb{R}^d)} \leq r_0 \|\boldsymbol{\xi}\|_V$ for all $\boldsymbol{\xi} \in V$. Furthermore, the trace operator $\tilde{\gamma}: V \rightarrow L^2(\Gamma_3; \mathbb{R}^d)$ is a continuous, compact, and linear operator, with $\|\tilde{\gamma}\|$ denoting its norm.

To establish our result, we will rely on some assumptions.

$H(\mathcal{A})$: $\mathcal{A}: \Xi \times \mathbb{S}^d \rightarrow \mathbb{S}^d$ is an operator such that

- (1) for each $\boldsymbol{\eta} \in \mathbb{S}^d$, $\mathcal{A}(\cdot, \boldsymbol{\eta})$ is measurable on Ξ ;
- (2) for a.e. $\mathbf{y} \in \Xi$, $\mathcal{A}(\mathbf{y}, \cdot)$ is continuous on $\mathbb{S}^d$;
- (3) there exist $a_0 \in L^2(\Xi; \mathbb{R}^+)$ and $a_1 > 0$ such that $\|\mathcal{A}(\mathbf{y}, \boldsymbol{\eta})\|_{\mathbb{S}^d} \leq a_0(\mathbf{y}) + a_1 \|\boldsymbol{\eta}\|_{\mathbb{S}^d}$, for all $\boldsymbol{\eta} \in \mathbb{S}^d$, a.e. $\mathbf{y} \in \Xi$;
- (4) there exists $a_{\mathcal{A}} > 0$ such that $(\mathcal{A}(\mathbf{y}, \mathbf{e}_1) - \mathcal{A}(\mathbf{y}, \mathbf{e}_2)) : (\mathbf{e}_1 - \mathbf{e}_2) \geq a_{\mathcal{A}} \|\mathbf{e}_1 - \mathbf{e}_2\|_{\mathbb{S}^d}^2$, for all $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{S}^d$, a.e. $\mathbf{y} \in \Xi$;
- (5) $\mathcal{A}(\mathbf{y}, \mathbf{0}) = \mathbf{0}$, for a.e. $\mathbf{y} \in \Xi$.

$H(\mathcal{B})$: $\mathcal{B}: \Xi \times \mathbb{R} \rightarrow \mathbb{S}^d$ is an operator such that

- (1) for all $\tau \in \mathbb{R}$, $\mathcal{B}(\cdot, \tau)$ is measurable on Ξ ;
- (2) for a.e. $\mathbf{y} \in \Xi$, $\mathcal{B}(\mathbf{y}, \cdot)$ is continuous on $\mathbb{R}$;

(3) there exists $a_B \in L^2(\Xi; \mathbb{R}^+)$ such that $\|\mathcal{B}(\mathbf{y}, \xi)\| \leq a_B(\mathbf{y})$, for all $\xi \in \mathbb{R}$, a.e. $\mathbf{y} \in \Xi$.

$H(j_V)$: $j_V : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}$ is a function such that

(1) for every $\xi \in \mathbb{R}$, $j_V(\cdot, \xi)$ is measurable on Γ_3 and there is $\eta \in L^2(\Gamma_3)$ satisfying $j_V(\cdot, \eta(\cdot)) \in L^1(\Gamma_3)$;

(2) for a.e. $\mathbf{y} \in \Gamma_3$, $j_V(\mathbf{y}, \cdot)$ is locally Lipschitz on $\mathbb{R}$;

(3) there exist $\bar{c}_0 \geq 0$ and $\bar{c}_1 > 0$ such that $\|\partial j_V(\mathbf{y}, \xi)\| \leq \bar{c}_0 + \bar{c}_1|\xi|$, for all $\xi \in \mathbb{R}$, a.e. $\mathbf{y} \in \Gamma_3$;

(4) there exists $a_{j_V} \geq 0$ satisfying

$$j_V^0(\mathbf{y}, e_1; e_2 - e_1) + j_V^0(\mathbf{y}, e_2; e_1 - e_2) \leq a_{j_V}|e_1 - e_2|^2, \text{ for all } e_1, e_2 \in \mathbb{R}, \text{ a.e. } \mathbf{y} \in \Gamma_3.$$

$H(\mathcal{F})$: $\mathcal{F}_b : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}^+$ is a function such that

(1) for each $e \in \mathbb{R}$, $\mathcal{F}_b(\cdot, e)$ is measurable on Γ_3 ;

(2) for a.e. $\mathbf{y} \in \Gamma_3$, $\mathcal{F}_b(\mathbf{y}, \cdot)$ is continuous on $\mathbb{R}$;

(3) $|\mathcal{F}_b(\mathbf{y}, e_1) - \mathcal{F}_b(\mathbf{y}, e_2)| \leq a_{\mathcal{F}}|e_1 - e_2|$, for all $e_1, e_2 \in \mathbb{R}$, a.e. $\mathbf{y} \in \Gamma_3$ with $a_{\mathcal{F}} > 0$;

(4) $\mathcal{F}_b(\mathbf{y}, r) \leq \widetilde{a}_{\mathcal{F}}$, for all $r \in \mathbb{R}$, a.e. $\mathbf{y} \in \Gamma_3$ with $\widetilde{a}_{\mathcal{F}} > 0$.

$H(\mathcal{E})$: $\mathcal{E} : \Xi \times I \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a function such that

(1) for all $(r, \mathbf{e}) \in \mathbb{R} \times \mathbb{R}^d$, $\mathcal{E}(\cdot, \cdot, r, \mathbf{e})$ is measurable on $\Xi \times I$;

(2) for a.e. $\mathbf{y} \in \Xi$, $\mathcal{E}(\mathbf{y}, \cdot, \cdot, \cdot)$ is continuous on $I \times \mathbb{R} \times \mathbb{R}^d$;

(3) there exists $\varphi_{\mathcal{E}} \in L^2(I, \mathbb{R}^+)$ satisfying

$$|\mathcal{E}(\mathbf{y}, \vartheta, e, \xi)| \leq \varphi_{\mathcal{E}}(\vartheta), \text{ for all } (\vartheta, e, \xi) \in I \times \mathbb{R} \times \mathbb{R}^d, \text{ a.e. } \mathbf{y} \in \Xi.$$

$H(\mathcal{G})$: $\mathcal{G} : \Xi \times I \times \mathbb{R} \rightarrow F(\mathbb{R})$ is a fuzzy mapping such that

(1) for all $e \in \mathbb{R}$, $\mathcal{G}(\cdot, \cdot, e)$ is measurable on $\Xi \times I$;

(2) for a.e. $\mathbf{y} \in \Xi$, $\mathcal{G}(\mathbf{y}, \cdot, \cdot)$ is closed, convex, and continuous on $I \times \mathbb{R}$;

(3) $\widetilde{\mathcal{G}} : I \times \mathbb{R} \rightarrow F(\mathbb{R})$ defined by $\widetilde{\mathcal{G}}(\vartheta, r) = [\mathcal{G}_{(\vartheta, r)}]_{\alpha}$ has nonempty convex compact values for each $(\vartheta, r) \in I \times \mathbb{R}$;

(4) for every $\eta > 0$, there exists $a_{\eta} \in L^2(I, \mathbb{R}^+)$ satisfying

$$\|\mathcal{G}_{(\vartheta, s)}\| \leq a_{\eta}(\vartheta), \text{ for all } s \in \mathbb{R}, \text{ a.e. } \vartheta \in I,$$

with $|s| \leq \eta$.

$H(r_1, m_1)$: $r_1 : V \rightarrow \mathbb{R}$ and $m_1 : H_1 \rightarrow \mathbb{R}$ are two functions such that

(1) r_1 is lower semicontinuous, subadditive and positively homogeneous and $|r_1(\xi)| \leq \|\xi\|$, for all $\xi \in V$;

(2) m_1 is continuous, $m_0 := \inf_{\mathbf{e} \in H_1} m_1(\mathbf{e}) > 0$ and $r_1(\mathbf{0}) \leq m_0$.

$H(\mathbf{f}_0, \mathbf{f}_N, g_0)$: $\mathbf{f}_0 \in L^\infty(I, L^2(\Xi; \mathbb{R}^d))$, $\mathbf{f}_N \in L^\infty(I, L^2(\Gamma_2; \mathbb{R}^d))$, $g_0 \in L^2(\Gamma_3)$, $g_0 > 0$.

We posit that $(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\theta})$ constitutes a trio of smooth functions that satisfies (5.1)-(5.9). For $\mathbf{e} \in K(\mathbf{u}(\vartheta))$ fixed, we proceed to multiply the equilibrium equation (5.2) by $\mathbf{e} - \mathbf{u}(\vartheta)$ and subsequently apply the Green formula, as outlined in, for instance, [20, Theorem 2.25], yielding

$$\langle \boldsymbol{\sigma}(\vartheta), \boldsymbol{\varepsilon}(\mathbf{e}) - \boldsymbol{\varepsilon}(\mathbf{u}(\vartheta)) \rangle_{H_2} = \langle \mathbf{f}_0(\vartheta), \mathbf{e} - \mathbf{u}(\vartheta) \rangle_{H_1} + \int_{\Gamma} \boldsymbol{\sigma}(\vartheta) \mathbf{v} \cdot (\mathbf{e} - \mathbf{u}(\vartheta)) d\Gamma.$$

Let us revisit

$$\begin{aligned} \int_{\Gamma} \boldsymbol{\sigma}(\vartheta) \mathbf{v} \cdot (\mathbf{e} - \mathbf{u}(\vartheta)) d\Gamma &= \int_{\Gamma_1} \boldsymbol{\sigma}(\vartheta) \mathbf{v} \cdot (\mathbf{e} - \mathbf{u}(\vartheta)) d\Gamma + \int_{\Gamma_2} \boldsymbol{\sigma}(\vartheta) \mathbf{v} \cdot (\mathbf{e} - \mathbf{u}(\vartheta)) d\Gamma \\ &\quad + \int_{\Gamma_3} \boldsymbol{\sigma}(\vartheta) \mathbf{v} \cdot (\mathbf{e} - \mathbf{u}(\vartheta)) d\Gamma. \end{aligned}$$

Given $\mathbf{e} - \mathbf{u}(\vartheta) = 0$ on Γ_1 , we can infer

$$\begin{aligned} \langle \boldsymbol{\sigma}(\vartheta), \boldsymbol{\varepsilon}(\mathbf{e}) - \boldsymbol{\varepsilon}(\mathbf{u}(\vartheta)) \rangle_{H_2} &= \langle \mathbf{f}_0(\vartheta), \mathbf{e} - \mathbf{u}(\vartheta) \rangle_{H_1} + \int_{\Gamma_2} \boldsymbol{\sigma}(\vartheta) \mathbf{v} \cdot (\mathbf{e} - \mathbf{u}(\vartheta)) d\Gamma \\ &\quad + \int_{\Gamma_3} \boldsymbol{\sigma}(\vartheta) \mathbf{v} \cdot (\mathbf{e} - \mathbf{u}(\vartheta)) d\Gamma. \end{aligned}$$

This equality, in conjunction with boundary condition (5.4), implies

$$\langle \boldsymbol{\sigma}(\vartheta), \boldsymbol{\varepsilon}(\mathbf{e}) - \boldsymbol{\varepsilon}(\mathbf{u}(\vartheta)) \rangle_{H_2} = \langle \mathbf{f}(\vartheta), \mathbf{e} - \mathbf{u}(\vartheta) \rangle_{V^* \times V} + \int_{\Gamma_3} \boldsymbol{\sigma}(\vartheta) \mathbf{v} \cdot (\mathbf{e} - \mathbf{u}(\vartheta)) d\Gamma,$$

where $\mathbf{f} \in C(I; V^*)$ is specified by

$$\langle \mathbf{f}(\vartheta), \mathbf{e} \rangle_{V^* \times V} = \langle \mathbf{f}_0(\vartheta), \mathbf{e} \rangle_{H_1} + \langle \mathbf{f}_N(\vartheta), \mathbf{e} \rangle_{L^2(\Gamma_2; \mathbb{R}^d)}, \text{ for all } \mathbf{e} \in V \text{ and } \vartheta \in I.$$

Moreover, we utilize the decomposition formula, as referenced in (6.33) of [20],

$$\boldsymbol{\sigma}(\vartheta) \mathbf{v} \cdot (\mathbf{e} - \mathbf{u}(\vartheta)) = \sigma_v(\vartheta)(e_v - u_v(\vartheta)) + \boldsymbol{\sigma}_\tau(\vartheta) \cdot (\mathbf{e}_\tau - \mathbf{u}_\tau(\vartheta)), \text{ a.e. on } \Gamma_3,$$

to get

$$\begin{aligned} \langle \boldsymbol{\sigma}(\vartheta), \boldsymbol{\varepsilon}(\mathbf{e}) - \boldsymbol{\varepsilon}(\mathbf{u}(\vartheta)) \rangle_{H_2} &= \langle \mathbf{f}(\vartheta), \mathbf{e} - \mathbf{u}(\vartheta) \rangle_{V^* \times V} \\ &\quad + \int_{\Gamma_3} (\sigma_v(\vartheta)(e_v - u_v(\vartheta)) + \boldsymbol{\sigma}_\tau(\vartheta) \cdot (\mathbf{e}_\tau - \mathbf{u}_\tau(\vartheta))) d\Gamma. \end{aligned} \quad (5.12)$$

Next, we analyze the boundary term on Γ_3 in (5.12). From (5.5) and (5.6), we get

$$\begin{aligned} \sigma_v(\vartheta)(e_v - u_v(\vartheta)) &= (\sigma_v(\vartheta) + \eta_v)(e_v - g_0) + (\sigma_v(\vartheta) + \eta_v)(g_0 - u_v(\vartheta)) \\ &\quad - \eta_v(e_v - u_v(\vartheta)) \\ &= (\sigma_v(\vartheta) + \eta_v)(e_v - g_0) - \eta_v(e_v - u_v(\vartheta)) \\ &\geq -\eta_v(e_v - u_v(\vartheta)) \\ &\geq -j_v^0(u_v(\vartheta); e_v - u_v(\vartheta)), \text{ on } \Gamma_3. \end{aligned} \quad (5.13)$$

By (5.7) and the definition of generalized subgradient, one has

$$-\boldsymbol{\sigma}_\tau(\vartheta) \cdot (\mathbf{e}_\tau - \mathbf{u}_\tau(\vartheta)) \leq \mathcal{F}_b(u_v(\vartheta))(\|\mathbf{e}_\tau\| - \|\mathbf{u}_\tau(\vartheta)\|), \text{ on } \Gamma_3. \quad (5.14)$$

We insert (5.13) and (5.14) into (5.12) to obtain

$$\begin{aligned} \langle \mathbf{f}(\vartheta), \mathbf{e} - \mathbf{u}(\vartheta) \rangle_{V^* \times V} &\leq \langle \boldsymbol{\sigma}(\vartheta), \boldsymbol{\varepsilon}(\mathbf{e}) - \boldsymbol{\varepsilon}(\mathbf{u}(\vartheta)) \rangle_{H_2} + \int_{\Gamma_3} j_v^0(u_v(\vartheta); e_v - u_v(\vartheta)) d\Gamma \\ &\quad + \int_{\Gamma_3} \mathcal{F}_b(u_v(\vartheta))(\|\mathbf{e}_\tau\| - \|\mathbf{u}_\tau(\vartheta)\|) d\Gamma. \end{aligned} \quad (5.15)$$

Considering the constitutive laws (5.1), (5.8), (5.9), and (5.15), we derive the following variational formulation for Problem 5.1.

Problem 5.2. Find a temperature $\theta : I \rightarrow L^2(\Xi; \mathbb{R})$ and a displacement field $\mathbf{u} : I \rightarrow C$ such that

$$\left\{ \begin{array}{l} {}^C D_{\vartheta}^q \theta(\vartheta) \in \Delta \theta(\vartheta) + \mathcal{E}(\vartheta, \theta(\vartheta), \mathbf{u}(\vartheta)) + [\mathcal{G}_{(\vartheta, \theta(\vartheta))}]_{\alpha}, \text{ a.e. } \vartheta \in I, \\ \langle \mathbf{f}(\vartheta), \mathbf{e} - \mathbf{u}(\vartheta) \rangle_{V^* \times V} \leq \langle \mathcal{A} \boldsymbol{\varepsilon}(\mathbf{u}(\vartheta)) + \mathcal{B} \theta(\vartheta), \boldsymbol{\varepsilon}(\mathbf{e}) - \boldsymbol{\varepsilon}(\mathbf{u}(\vartheta)) \rangle_{H_2} \\ \quad + \int_{\Gamma_3} j_v^0(u_v(\vartheta); e_v - u_v(\vartheta)) d\Gamma \\ \quad + \int_{\Gamma_3} \mathcal{F}_b(u_v(\vartheta))(\|\mathbf{e}_\tau\| - \|\mathbf{u}_\tau(\vartheta)\|) d\Gamma, \text{ for all } \mathbf{e} \in K(\mathbf{u}(\vartheta)), \\ \theta(0) = \theta_0. \end{array} \right.$$

We now provide the existence of mild solutions for Problem 5.2.

Theorem 5.1. Assume that hypotheses $H(\mathcal{A})$, $H(\mathcal{B})$, $H(j_v)$, $H(\mathcal{F})$, $H(\mathcal{E})$, $H(\mathcal{G})$, $H(\mathbf{f}_0, \mathbf{f}_N, g_0)$ and $H(r_1, m_1)$ hold. If

$$a_{\mathcal{A}} > \max\{a_{j_v}, \sqrt{2c_1}\}\|\gamma\|^2 + a_{\mathcal{F}}\|r_0\|^2, \quad (5.16)$$

and

$$\begin{aligned} m_0 \geq & \frac{\|a_{\mathcal{B}}\|_{L^2(\Xi; \mathbb{R}^+)} + \bar{c}_0 \sqrt{2\text{meas}(\Gamma_3)}\|\gamma\| + \widetilde{a_{\mathcal{F}}}\|r_0\|\sqrt{\text{meas}(\Gamma_3)}}{a_{\mathcal{A}} - a_{\mathcal{F}}\|r_0\|^2 - a_{j_v}\|\gamma\|^2} \\ & + \frac{\max\{r_0, D_0\}(\|\mathbf{f}_0\|_{L^\infty(I, L^2(\Xi; \mathbb{R}^d))} + \|\mathbf{f}_N\|_{L^\infty(I, L^2(\Gamma_2; \mathbb{R}^d))})}{a_{\mathcal{A}} - a_{\mathcal{F}}\|r_0\|^2 - a_{j_v}\|\gamma\|^2}, \end{aligned} \quad (5.17)$$

then Problem 5.2 has at least one mild solution $(\theta, \mathbf{u})$.

Proof. Let $E_1 = L^2(\Xi; \mathbb{R})$, $E_3 = L^2(\Gamma_3; \mathbb{R})$, $E_2 = V$ and $\gamma: V \rightarrow L^2(\Gamma_3; \mathbb{R})$ be the normal trace operator defined respectively by

$$\tilde{\gamma}(\mathbf{u}) = \gamma(\mathbf{u}) = u_v. \quad (5.18)$$

We consider the functions $A: E_2 \rightarrow E_2^*$, $B: E_1 \rightarrow E_2^*$, $J: E_3 \rightarrow \mathbb{R}$, $\varphi: E_2 \times E_2 \rightarrow \mathbb{R}$, $E: I \times E_1 \times E_2 \rightarrow E_1$, $G: I \times E_1 \rightarrow 2^{E_1}$, $f: I \rightarrow E_2^*$ determined by

$$\begin{aligned} \langle A\mathbf{u}, \mathbf{e} \rangle &= \int_{\Xi} \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{e}) d\mathbf{y}, \quad \langle B\theta, \mathbf{e} \rangle = \langle \mathcal{B}\theta, \boldsymbol{\varepsilon}(\mathbf{e}) \rangle_{H_2}, \\ J(z) &= \int_{\Gamma_3} j_v(z_v) d\Gamma, \quad \varphi(\mathbf{u}, \mathbf{e}) = \int_{\Gamma_3} \mathcal{F}_b(u_v) \|\mathbf{e}_\tau\| d\Gamma, \\ E(\vartheta, \theta, \mathbf{u}) &= \mathcal{E}(\vartheta, \theta, \mathbf{u}), \quad f(\vartheta) = \mathbf{f}(\vartheta), \quad G(\vartheta, \theta) = \mathcal{G}(\vartheta, \theta). \end{aligned}$$

for all $\mathbf{u}, \mathbf{e} \in E_2$, $\vartheta \in I$, $\theta \in E_1$, $z \in E_3$. Finally, for (5.8), we define the operator $M: D(M) \subset E_1 \rightarrow E_1$ as follows

$$M = \Delta, \quad D(M) = H^2(\Xi) \cap H_0^1(\Xi). \quad (5.19)$$

According to [8], it can be concluded that the semigroup $T(\vartheta)$ generated by M is compact. Subsequently, Problem 5.2 can be reformulated as follows:

$$\begin{cases} {}^C D_{\vartheta}^q \theta(\vartheta) \in M\theta(\vartheta) + E(\vartheta, \theta(\vartheta), \mathbf{u}(\vartheta)) + [G(\vartheta, \theta(\vartheta))]_{\alpha}, & \text{a.e. } \vartheta \in I, \\ \langle \mathbf{f}(\vartheta), \mathbf{e} - \mathbf{u}(\vartheta) \rangle_{E_2^* \times E_2} \leq \langle A\mathbf{u}(\vartheta) + B\theta(\vartheta), \mathbf{e} - \mathbf{u}(\vartheta) \rangle_{E_2} + J^0(\gamma\mathbf{u}(\vartheta); \gamma(\mathbf{e} - \mathbf{u}(\vartheta))) \\ \quad + \varphi(\mathbf{u}(\vartheta), \mathbf{e}) - \varphi(\mathbf{u}(\vartheta), \mathbf{u}(\vartheta)), & \text{for all } \mathbf{e} \in K(\mathbf{u}(\vartheta)), \\ \theta(0) = \theta_0. \end{cases} \quad (5.20)$$

It is evident that if $(\theta, \mathbf{u})$ constitutes a solution to (5.20), then it also serves as a solution to Problem 5.2. Leveraging this observation, we employ Theorem 4.1 to demonstrate the solvability of Problem 5.2.

To complete the proof of $H(A)$, we employ reasoning akin to that employed in [20, Theorem 7.3], and establish that $A: E_2 \rightarrow E_2^*$ is pseudomonotone and strongly monotone, i.e.,

$$\begin{aligned} \langle A(\mathbf{e}) - A(\mathbf{u}), \mathbf{e} - \mathbf{u} \rangle &= \langle \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{e}) - \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{e}) - \boldsymbol{\varepsilon}(\mathbf{u}) \rangle_{H_2} \\ &\geq a_{\mathcal{A}} \|\boldsymbol{\varepsilon}(\mathbf{e}) - \boldsymbol{\varepsilon}(\mathbf{u})\|_{\mathbb{S}^d}^2 \\ &= a_{\mathcal{A}} \|\mathbf{e} - \mathbf{u}\|_{E_2}^2, \end{aligned}$$

for all $\mathbf{e}, \mathbf{u} \in E_2$, while $H(A)$ is valid.

Next, we confirm the validity of $H(B)$. Applying $H(\mathcal{B})$ (3) and the Hölder inequality, we obtain

$$\begin{aligned}\|\langle B\theta, \mathbf{e} \rangle\| &\leq \int_{\Xi} \|\mathcal{B}(\theta) : \boldsymbol{\varepsilon}(\mathbf{e})\| dy \\ &\leq \int_{\Xi} \|\mathcal{B}(\theta)\| \|\boldsymbol{\varepsilon}(\mathbf{e})\| dy \\ &\leq \left(\int_{\Xi} \|\mathcal{B}\theta\|^2 dy \right)^{\frac{1}{2}} \left(\int_{\Xi} \|\boldsymbol{\varepsilon}(\mathbf{e})\|^2 dy \right)^{\frac{1}{2}} \\ &\leq \|a_{\mathcal{B}}\|_{L^2(\Xi; \mathbb{R}^+)} \|\mathbf{e}\|, \text{ for } \theta \in E_1, \mathbf{e} \in E_2.\end{aligned}$$

Thus $\|\langle B\theta, \mathbf{e} \rangle\| \leq \|a_{\mathcal{B}}\|_{L^2(\Xi; \mathbb{R}^+)}$, which means that $H(B)$ is satisfied.

We now turn to condition $H(J)$. By [20, Theorem 3.47] and $H(j_v)$, we deduce that $J : E_2 \rightarrow \mathbb{R}$ is locally Lipschitz continuous and fulfills $J^0(z_1; z_2) \leq \int_{\Gamma_3} j_v^0(z_1; z_2) d\Gamma$, and $\partial J(z) \subset \int_{\Gamma_3} \partial j_v(z) d\Gamma$, for all $z_1, z_2 \in E_3$. Hence, $H(J)$ (1) holds. From $H(j_v)$ (4), for any $z_1, z_2 \in E_3$, we have

$$\begin{aligned}J^0(z_1; z_2 - z_1) + J^0(z_2; z_1 - z_2) &\leq \int_{\Gamma_3} j_v^0(z_1; z_2 - z_1) d\Gamma + \int_{\Gamma_3} j_v^0(z_2; z_1 - z_2) d\Gamma \\ &\leq a_{j_v} \int_{\Gamma_3} \|z_1 - z_2\|^2 d\Gamma \\ &\leq a_{j_v} \|z_1 - z_2\|_{E_3}^2.\end{aligned}$$

So, $H(J)$ (3) is satisfied by $a_J = a_{j_v}$. Furthermore, from [20, Corollary 4.15(v)], it can be inferred that

$$\|\partial J(z)\|_{E_3^*} \leq \sqrt{2\text{meas}(\Gamma_3)\overline{c_0}} + \sqrt{2\overline{c_1}} \|z\|_{E_3}, \text{ for } z \in E_3.$$

Hence $H(J)$ (2) is satisfied with $d_0 = \sqrt{2\text{meas}(\Gamma_3)\overline{c_0}}$ and $d_1 = \sqrt{2\overline{c_1}}$. We next verify $H(\varphi)$. From hypothesis $H(\mathcal{F})$ and the convexity of norms, it is evident that, for each $\boldsymbol{\eta} \in E_2$, $\varphi(\boldsymbol{\eta}, \cdot)$ is a convex function. Consequently, it is bounded from below by an affine function. By employing a standard argument and Fatou's lemma, it is demonstrated that $\varphi(\boldsymbol{\eta}, \cdot)$ is lower semicontinuous on E_2 . Therefore, $H(\varphi)$ (1) is satisfied. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{e}_1, \mathbf{e}_2 \in E_2$. According to the continuity of the trace operator, $H(\mathcal{F})$ (2), and the Hölder inequality, one has

$$\begin{aligned}\varphi(\mathbf{u}_1, \mathbf{e}_2) - \varphi(\mathbf{u}_1, \mathbf{e}_1) + \varphi(\mathbf{u}_2, \mathbf{e}_1) - \varphi(\mathbf{u}_2, \mathbf{e}_2) &= \int_{\Gamma_3} \mathcal{F}_b(u_{1v}) \|\mathbf{e}_{2\tau}\| d\Gamma - \int_{\Gamma_3} \mathcal{F}_b(u_{1v}) \|\mathbf{e}_{1\tau}\| d\Gamma \\ &\quad + \int_{\Gamma_3} \mathcal{F}_b(u_{2v}) \|\mathbf{e}_{1\tau}\| d\Gamma - \int_{\Gamma_3} \mathcal{F}_b(u_{2v}) \|\mathbf{e}_{2\tau}\| d\Gamma \\ &= \int_{\Gamma_3} (\mathcal{F}_b(u_{1v}) - \mathcal{F}_b(u_{2v})) (\|\mathbf{e}_{1\tau}\| - \|\mathbf{e}_{2\tau}\|) d\Gamma \\ &\leq a_{\mathcal{F}} \|u_{1v} - u_{2v}\|_{L^2(\Gamma_3; \mathbb{R})} \|\mathbf{e}_{1\tau} - \mathbf{e}_{2\tau}\|_{L^2(\Gamma_3; \mathbb{R}^d)} \\ &\leq a_{\mathcal{F}} \|r_0\|^2 \|\mathbf{u}_1 - \mathbf{u}_2\|_{E_2} \|\mathbf{e}_1 - \mathbf{e}_2\|_{E_2},\end{aligned}$$

which implies that $H(\varphi)$ (2) is satisfied by $a_\varphi = a_{\mathcal{F}} \|r_0\|^2$. Further, for $\mathbf{u} \in E_2$, we have

$$\begin{aligned} \varphi(\mathbf{u}, \mathbf{e}_1) - \varphi(\mathbf{u}, \mathbf{e}_2) &= \int_{\Gamma_3} \mathcal{F}_b(u_v)(\|\mathbf{e}_{1\tau}\| - \|\mathbf{e}_{2\tau}\|) d\Gamma \\ &\leq \widetilde{a_{\mathcal{F}}} \int_{\Gamma_3} \|\mathbf{e}_{1\tau} - \mathbf{e}_{2\tau}\| d\Gamma \\ &\leq \widetilde{a_{\mathcal{F}}} \sqrt{\text{meas}(\Gamma_3)} \|r_0\| \|\mathbf{e}_1 - \mathbf{e}_2\|_{E_2}. \end{aligned}$$

Hence, $H(\varphi)$ (3) holds. Consider sequences $\{\mathbf{u}_n\}$, $\{\mathbf{e}_n\}$, $\{\mathbf{z}_n\}$ such that $\mathbf{u}_n \rightharpoonup \mathbf{u}$, $\mathbf{e}_n \rightarrow \mathbf{e}$, $\mathbf{z}_n \rightharpoonup \mathbf{z}$ in E_2 . Given the compactness of the trace operator, it follows that $u_{nv} \rightarrow u_v$ in $L^2(\Gamma; \mathbb{R})$, $\mathbf{e}_{n\tau} \rightarrow \mathbf{e}_\tau$, $\mathbf{z}_{n\tau} \rightarrow \mathbf{z}_\tau$ in $L^2(\Gamma; \mathbb{R}^d)$ and

$$\begin{aligned} \varphi(\mathbf{u}_n, \mathbf{e}_n) - \varphi(\mathbf{u}_n, \mathbf{z}_n) &= \int_{\Gamma_3} \mathcal{F}_b(u_{nv})(\|\mathbf{e}_{n\tau}\| - \|\mathbf{z}_{n\tau}\|) d\Gamma \\ &\rightarrow \int_{\Gamma_3} \mathcal{F}_b(u_v)(\|\mathbf{e}_\tau\| - \|\mathbf{z}_\tau\|) d\Gamma \\ &= \varphi(\mathbf{u}, \mathbf{e}) - \varphi(\mathbf{u}, \mathbf{z}). \end{aligned}$$

Consequently, $H(\varphi)$ (4) is met, thus establishing the fulfillment of $H(\varphi)$.

We next verify $H(K)$. It is clear from (5.10) that C is closed, convex, and nonempty subset of E_2 . From the hypothesis $H(r_1, m_1)$ (2), we have $\mathbf{0} \in K(\mathbf{e})$ for all $\mathbf{e} \in C$. Consider $\mathbf{e} \in C$ and $\{\mathbf{u}_n\} \subset K(\mathbf{e})$ satisfying $\mathbf{u}_n \rightarrow \mathbf{u}$ as $n \rightarrow \infty$ for some $\mathbf{u} \in C$. Due to the lower semicontinuity of r_1 , one has $r_1(\mathbf{u}) \leq \liminf_{n \rightarrow +\infty} r_1(\mathbf{u}_n) \leq m_1(\mathbf{e})$. Hence, $K(\mathbf{e})$ is closed for all $\mathbf{e} \in C$. For each $\mathbf{e} \in C$, $\lambda \in (0, 1)$, Let $\mathbf{u}, \mathbf{w} \in K(\mathbf{e})$. $H(r_1, m_1)$ (1) gives

$$r_1(\lambda \mathbf{u} + (1 - \lambda) \mathbf{w}) \leq \lambda r_1(\mathbf{u}) + (1 - \lambda) r_1(\mathbf{w}) \leq \lambda m_1(\mathbf{e}) + (1 - \lambda) m_1(\mathbf{e}) = m_1(\mathbf{e}),$$

and $\lambda \mathbf{u} + (1 - \lambda) \mathbf{w} \in K(\mathbf{e})$. Therefore, $K(\mathbf{e})$ is a convex set. Hence, we infer that $H(K)$ (1) is satisfied.

Next, we consider $\{\mathbf{e}_n\} \subset C$ such that $\mathbf{e}_n \rightharpoonup \mathbf{e}$ in C as $n \rightarrow \infty$. To demonstrate that $K(\mathbf{e}_n) \xrightarrow{M} K(\mathbf{e})$, we will confirm conditions (1) and (2) of Definition 2.2. To establish (1), consider any arbitrary $\mathbf{u} \in K(\mathbf{e})$, and set $\mathbf{u}_n = \frac{m_1(\mathbf{e}_n)}{m_1(\mathbf{e})} \mathbf{u}$. Then, by employing the positive homogeneity of r_1 and the condition $m_0 > 0$, we deduce that

$$r_1(\mathbf{u}_n) = \frac{m_1(\mathbf{e}_n)}{m_1(\mathbf{e})} r_1(\mathbf{u}) \leq m_1(\mathbf{e}_n),$$

which implies that $\mathbf{u}_n \in K(\mathbf{e}_n)$ for each $n \geq 1$. By continuity of m_1 and the compactness of the embedding of V into H_2 , we conclude that

$$\lim_{n \rightarrow +\infty} \|\mathbf{u}_n - \mathbf{u}\| = \lim_{n \rightarrow +\infty} \left\| \frac{m_1(\mathbf{e}_n)}{m_1(\mathbf{e})} \mathbf{u} - \mathbf{u} \right\| = \lim_{n \rightarrow +\infty} \frac{|m_1(\mathbf{e}_n) - m_1(\mathbf{e})|}{m_1(\mathbf{e})} \|\mathbf{u}\| = 0,$$

which implies $\mathbf{u}_n \rightarrow \mathbf{u}$. Thus, condition (1) holds. To prove condition (2), let $\{\mathbf{u}_n\} \subset C$ be such that $\mathbf{u}_n \in K(\mathbf{e}_n)$ and $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in C as $n \rightarrow \infty$. We use the lower semicontinuity of r_1 , the continuity of m_1 , and the compactness of the embedding of V into H_2 again, to infer

$$r_1(\mathbf{u}) \leq \liminf_{n \rightarrow +\infty} r_1(\mathbf{u}_n) \leq \liminf_{n \rightarrow +\infty} m_1(\mathbf{e}_n) = m_1(\mathbf{e}),$$

which implies $\mathbf{u} \in K(\mathbf{e})$. Thus $H(K)$ (2) holds. We only need to confirm condition (3.2) and $H(K)$ (3). For this purpose, setting $\mathbf{e}_0 = \mathbf{0}$ and $r = m_0$, from $H(r_1, m_1)$, we get $B(\mathbf{e}_0, r) \subset$

$\cap_{\mathbf{e} \in C} K(\mathbf{e})$, as $B(\mathbf{0}, m_0) \subset K(\mathbf{e})$ for each $\mathbf{e} \in C$. Hence, $H(K)$ (3) holds. It follows from (5.17) that (3.2) are satisfied with the following data:

$$a_A = a_{\mathcal{A}}, a_B = \|a_{\mathcal{B}}\|_{L^2(\Xi; \mathbb{R}^+)}, d_0 = \sqrt{2\text{meas}(\Gamma_3)\overline{c_0}}, d_1 = \sqrt{2\overline{c_1}},$$

$$a_J = a_{j_v}, a_\varphi = a_{\mathcal{F}}\|r_0\|^2, c_\varphi(z) = \widetilde{a_{\mathcal{F}}}\sqrt{\text{meas}(\Gamma_3)}\|r_0\|,$$

where $z \in \mathbb{R}^+$. According to the trace theorem, as indicated in [20, Theorem 2.21], it is established that the operator γ determined by (5.18) is compact, linear, and bounded. Consequently, $H(\gamma)$ is satisfied. It follows from $H(\mathbf{f}_0, \mathbf{f}_N, g_0)$, (5.16) and (5.19) that $H(f)$, H_0 and $H(M)$ are satisfied. Further, it is evident that $H(E)$ and $H(G)$ are fulfilled by $H(\mathcal{E})$ and $H(\mathcal{G})$, respectively. Hence, we have verified all hypotheses of Theorem 4.1. So, Problem 5.2 has at least one mild solution $(\theta, \mathbf{u})$. $\square$

6. CONCLUSIONS

In this paper, we investigated a class of fuzzy fractional differential inclusion quasi-variational-hemivariational inequality (FFDIQVHVI) in Banach spaces. First, we obtained a uniqueness and existence theorem for quasi-variational-hemivariational inequality under certain mild conditions. Then, by utilizing the fixed point theorem of condensing set-valued operators, fuzzy set theory and theory of measure of noncompactness, we obtained the nonemptiness of the mild solutions set for FFDIQVHVI and the compactness of the set of all mild trajectories for FFDIQVHVI. Finally, we demonstrated the applicability of the abstract theory by addressing a nonlinear quasistatic thermoelastic frictional contact problem featuring implicit obstacles. We then presented existence results for this problem. After confirming the existence of solutions to FFDIQVHVI, a central focus frequently shifts towards assessing the stability of these solutions and devising algorithms for their approximation. Hence, it becomes both intriguing and pivotal to explore the stability of solutions to FFDIQVHVI and to devise algorithms for approximating their solution. Our forthcoming research will be built upon these objectives.

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REFERENCES

- [1] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal.* 72 (2010), 2859-2862.
- [2] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, Interscience, New York, 1983.
- [3] S. S. Chang, *Variational Inequality and Complementarity Problem Theory with Applications*, Shanghai Scientific and Technological Literature, Shanghai. 1991.
- [4] J. X. Cen, L. J. Li, S. Migórski, V. T. Nguyen, Convergence of a generalized penalty and regularization method for quasi-variational-hemivariational inequalities, *Commun. Nonlinear Sci. Numer. Simul.* 103 (2021), 105998.

- [5] S. S. Chang, Y. G. Zhu, On variational inequalities for fuzzy mappings, *Fuzzy Sets Syst.* 32 (1989), 359-367.
- [6] K. Deimling, *Multivalued Differential Equations*, Walter de Gruyter, Berlin, 1992.
- [7] Z. Denkowski, S. Migórski, N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications*, Kluwer Academic Publishers, Boston, Dordrecht, London, New York, 2003.
- [8] K. J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York, 2000.
- [9] J. Gwinner, On a class of mixed random variational inequalities, *Commun. Optim. Theory* 2024 (2024), 22.
- [10] E. Hüllermeier, An approach to modelling and simulation of uncertain dynamical systems, *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems.* 5 (1997), 117-137.
- [11] N. V. Hung, Generalized Levitin-Polyak well-posedness for controlled systems of FMQHI-fuzzy mixed quasi-hemivariational inequalities of Minty type, *J. Comput. Appl. Math.* 386 (2021), 113263.
- [12] T. J. Jiang, D. L. Cai, Y.B. Xiao, S. Migórski, Time-dependent elliptic quasi-variational-hemivariational inequalities: well-posedness and application, *J. Global Optim.* 88 (2024), 509-530.
- [13] M. Kamenski, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, Walter De Gruyter, Berlin, 2001.
- [14] S. Kesornprom, S. Suantai, P. Chalamjiak, A splitting algorithm with inertial and correction terms for solving split variational inclusion problems and its applications, *J. Nonlinear Funct. Anal.* 2025 (2025), 22.
- [15] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, 2006.
- [16] P. Lv, Y.J. Liu, X. Qin, A stochastic multiobjective optimization problem via a stochastic variational inequality, *J. Nonlinear Convex Anal.* 28 (2027), 1-12.
- [17] P. Lv, On fixed points of nonexpansive operators on Banach spaces, *J. Nonlinear Convex Anal.* 27 (2026), 1-10.
- [18] U. Mosco, Convergence of convex sets and of solutions of variational inequalities, *Adv. Math.* 3 (1969), 510-585.
- [19] S. Migórski, S. Dudek, A new class of elliptic quasi-variational-hemivariational inequalities for fluid flow with mixed boundary conditions. *Comput. Math. Appl.* 100 (2021), 51-61.
- [20] S. Migórski, A. Ochal, M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities, Models and Analysis of Contact Problems*, Adv. Appl. Math. Mech. Springer, New York, 2013.
- [21] Z. Naniewicz, P. D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.
- [22] P. D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.
- [23] P. D. Panagiotopoulos, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, 1993.
- [24] Z. J. Peng, Z. H. Liu, Inverse problems for nonlinear quasi-variational-hemivariational inequalities with application to obstacle problems of elliptic type, *Commun. Nonlinear Sci. Numer. Simul.* 101 (2021), 105889.
- [25] X. Qin, A weakly convergent method for splitting problems with nonexpansive mappings, *J. Nonlinear Convex Anal.* 24 (2023), 1033-1043.
- [26] A. U. Raghunathan, J. R. Pérez-Correa, E. Agosin, L.T. Biegler, Parameter estimation in metabolic flux balance models for batch fermentation formulation and solution using differential variational inequalities, *Ann. Oper. Res.* 148 (2006), 251-270.
- [27] M. Sofonea, S. Migórski, *Variational-Hemivariational Inequalities with Applications*, Chapman & Hall/CRC Press, Boca Raton-London, 2018.
- [28] H. Sun, R. Kwon, Robust contextual bandit method for optimal loan offering, *J. Appl. Numer. Optim.* 7 (2025), 333-358.
- [29] V. M. Tam, N. V. Hung, D. O'Regan, An existence result for a new class of fuzzy fractional differential inclusions with Clarke's subdifferential via resolvent operators in Banach spaces, *Fuzzy Sets Syst.* 443 (2022), 221-240.
- [30] R. Toledo-Hernandez, V. Rico-Ramirez, G. A. Iglesias-Silva, U. M. Diwekar, A fractional calculus approach to the dynamic optimization of biological reactive systems. Part I: fractional models for biological reactions, *Chem. Eng. Sci.* 117 (2014), 217-228.
- [31] D. F. Wu, M. H. Chen, Fuzzy fractional delay differential inclusions driven by hemivariational inequalities in Banach spaces, *Commun. Nonlinear Sci. Numer. Simul.* 134 (2024), 108009.

- [32] Z. B. Wu, C. Min, N. J. Huang, On a system of fuzzy fractional differential inclusions with projection operators, *Fuzzy Sets Syst.* 347 (2018), 70-88.
- [33] Z. B. Wu, X. Wang, N. J. Huang, T. Y. Wang, H. M. Wang, A new class of fuzzy fractional differential inclusions driven by variational inequalities, *Fuzzy Sets Syst.* 419 (2021), 99-121.
- [34] Z. B. Wu, X. Wang, N. J. Huang, Y. B. Xiao, G. H. Zhang, On a new system of fractional delay differential equations coupled with fuzzy variational inequalities, *Fuzzy Sets Syst.* 436 (2022), 55-81.
- [35] L. A. Zadeh, Fuzzy sets, *Inf. Control.* 8 (1965), 338-353.
- [36] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, vol.II/B: Nonlinear Monotone Operators, Springer, New York, 1990.
- [37] S. D. Zeng, Y. R. Bai, J. C. Yao, V. T. Nguyen, A new variant of fuzzy fractional dynamic system driven by time-dependent variational inequality, *Fractals.* 30 (2022), 1-13.
- [38] S. D. Zeng, J. X. Cen, V. T. Nguyen, Generalized fractional evolution equations driven by fuzzy variational inequalities, *Fuzzy Sets Syst.* 429 (2022), 60-73.
- [39] S. D. Zeng, S. Migórski, A. A. Khan, Nonlinear quasi-hemivariational inequalities: existence and optimal control, *SIAM J. Control Optim.* 59 (2021), 1246-1274.
- [40] S. D. Zeng, D. Motreanu, A. A. Khan, Evolutionary quasi-variational-hemivariational inequalities I: existence and optimal control, *J. Optim. Theory Appl.* 193 (2022), 950-970.
- [41] Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.* 59 (2010), 1063-1077.
- [42] Y. G. Zhu, L. Rao, Differential inclusions for fuzzy maps, *Fuzzy Sets Syst.* 112 (2000), 257-261.