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PESSIMISTIC AND OPTIMISTIC ROBUST SOLUTIONS WITH PREFERENCE STRUCTURES

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Abstract. In this paper, a pessimistic and optimistic robust approach is introduced to deal with uncertain optimization problems whose final space is equipped with a preference structure, which provides new insights concerning the so-called minmax and minmin solutions of uncertain vector and set optimization problems. In addition, a scalarization scheme is stated to address this robust approach which is applied to vector and set optimization problems with variable domination structures. Finally, pessimistic and optimistic robust solutions of uncertain set optimization problems with variable domination structures and a finite set of scenarios are characterized through scalarization.

Keywords. Robustness; Ordered space; Set optimization; Scalarization; Uncertainty; Vector optimization.

1. INTRODUCTION

In recent years, practitioners and researchers have considered uncertainty as a key point in optimization modelling, where the input data are not known exactly due to empirical measurements of parameters of devices or processes, errors in many entries such as forecasts of future demands or returns, etc. Therefore, optimization problems involving uncertainty are nowadays an active research field.

One of the more popular ways to deal with uncertain optimization problems is the robust approach. It is based on considering as a solution of the problem any point that is feasible in each scenario and optimal in the worst one (see, e.g., [3, 4, 33, 40]). Namely, a deterministic counterpart of the problem is built, which is named robust counterpart, whose feasible set is the intersection of all possible feasible sets and the objective function considers at each feasible point its worst value taking into account all scenarios. The solutions of this "pessimistic" reformulation are named minmax/worst case/strict robust solutions or simply robust solutions of the problem. The "optimistic" version was introduced in [2] to study duality in robust optimization problems. In this case, the value of objective function at each feasible point is the best one between all possible scenarios.

There are lots of results addressing pessimistic and optimistic robust solutions when the final space in the uncertain optimization problem is $\mathbb{R}$, i.e. in scalar optimization (see [3]). In this

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setting, the worst and best cases at each feasible point are easily defined by the supremum and the infimum of the objective values over all scenarios, respectively. However, this definition cannot be considered in optimization problems whose final space is not totally ordered since the notions of infimum and supremum of a bounded family of elements could not exist (see [26, 27]). This difficulty happens, for instance, in vector and set optimization problems and it has motivated several contributions over the last decade (see [5, 10, 25, 26, 27, 28, 34, 39]). All of them refer to problems equipped with constant ordering structures. In [15, 16, 32, 38], robust solutions of uncertain vector optimization problems equipped with variable domination structures have been studied.

On the one hand, the first notion of minmax robust solution of an uncertain vector optimization problem was introduced by Kuroiwa and Lee [34] in the setting of a Pareto problem, i.e. the final space is finite dimensional and it is ordered by components. As robust counterpart of the problem, they considered a new Pareto problem whose components are the supremum over all scenarios of each objective in the uncertain problem (objective-wise worst approach). Hence, the minmax robust solutions of the uncertain Pareto problem are defined as the efficient solutions of its robust counterpart. In the same setting and by a different approach, a new concept of minmax robust solution was introduced in [10]. In this case, the robust counterpart is a set optimization problem whose objective value at each feasible point is the set of all its images along the scenarios. Concerning the minmax robust solutions, they are nondominated solutions of the robust counterpart when the upper set less order relation is considered (set-based approach). For some motivations of this definition see [25, pg. 105] and [26, pg. 8]. In [25] this approach was extended to the optimistic point of view, and in [5, 26] to problems where the final space is a topological linear space partially ordered by a pointed convex cone, i.e. to uncertain vector optimization problems.

On the other hand, papers addressing uncertain set optimization problems are very recent. Namely, as far as we know, set-based pessimistic and optimistic robustness concepts were introduced for the first time in [39, Definiton 5]. However, they are not completely satisfactory since they are based on a robust counterpart whose final space coincides with the final space of the uncertain problem, which modifies the structure of the set-based approach for this setting. Both notions were characterized by scalarization in [6, 7] via image space analysis and versions of the so-called Gerstewitz function and oriented distance for set optimization problems. Other robustness notions were defined in [9], which are motivated in binary relations on the unions of families of sets.

This work has three main contributions and it is structured as follows. First, in Section 3, two general binary relations are defined, which provide a framework to introduce pessimistic and optimistic robust solution concepts in uncertain optimization problems. This framework and their corresponding pessimistic and optimistic robustness notions can be applied whenever the final space of the problem is equipped with a variable domination structure. Moreover, it makes sense of the set-based approach in defining robust solution concepts. Second, in Section 4, both introduced binary relations are characterized by scalarization. The obtained results allow us also to characterize set-relations based on variable domination structures (see [1, 14, 24, 29, 31, 37, 42]). Finally, in Section 5, these characterizations are applied to formulate pessimistic robust solutions of an uncertain set optimization problem whose domination structure is variable and the uncertainty set is finite. The paper is completed with introduction and preliminaries

sections, where the state of the art concerning pessimistic and optimistic robustness notions in multiobjective, vector and set optimization problems is summarized and the basic definitions and mathematical tools needed in the sequel are collected, respectively.

2. PRELIMINARIES

Throughout this paper, we denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and $\mathrm{cl}A$, $\mathrm{int}A$ the closure and topological interior of a set A in a topological space, respectively. Also, $\mathrm{cone}A$ is the generated cone by a set A in a linear space. Moreover, $\mathbb{R}^p_+$ is the nonnegative orthant of $\mathbb{R}^p$ and $\mathbb{R}_+ := \mathbb{R}^1_+$. The unit closed ball of a normed space is denoted by $\mathbb{B}$, the power set of a nonempty set H is written $\mathscr{P}(H)$ and $\mathscr{P}_0(H) := \mathscr{P}(H)\backslash\{\emptyset\}$.

Roughly speaking, to scalarize pessimistic and optimistic robust solutions of an uncertain optimization problem whose final space is equipped with a preference structure means to formulate them as solutions of associated scalar optimization problems. A key point in this technique is to have functions pointwise representing and preserving the preferences. Namely, let $\mathscr{E}$ be a nonempty set equipped with a binary relation $\preceq$ that models the preferences in $\mathscr{E}$. A function $\psi : \mathscr{E} \to \overline{\mathbb{R}}$ is said to be upper (resp. lower) $\preceq$-monotone at a point $v_0 \in \mathscr{E}$ in a nonempty set $A \subset \mathscr{E}$, if for each $a \in A$ such that $v_0 \preceq a$ (resp. $a \preceq v_0$) we have $\psi(v_0) \leq \psi(a)$ (resp. $\psi(a) \leq \psi(v_0)$); ψ is called upper (resp. lower) $\preceq$-monotone at v_0 if it is upper (resp. lower) $\preceq$-monotone at v_0 in $\mathscr{E}$.

Also, we say that ψ is upper (resp. lower) $\preceq$-representing at $v_0 \in \mathscr{E}$ in A, denoted $v_0 \in A^u_\psi$ (resp. $v_0 \in A^l_\psi$) if $\psi(v_0) \leq \psi(a)$ (resp. $\psi(a) \leq \psi(v_0)$) and $a \in A$, then $v_0 \preceq a$ (resp. $a \preceq v_0$); Finally, ψ is called upper (resp. lower) $\preceq$-representing at v_0 if $v_0 \in \mathscr{E}^u_\psi$ (resp. $v_0 \in \mathscr{E}^l_\psi$) .

Next, we recall several important examples of these functions in the linear setting (see [17, 19, 36, 41, 43]). Let Y be a real Hausdorff topological linear space, $K : Y \rightrightarrows Y$ be a closed convex cone valued mapping, $q : Y \to Y$ be such that $q(y) \in K(y)\backslash(-K(y))$, for all $y \in Y$, and the functions $\lambda, \mu : Y \times Y \to \overline{\mathbb{R}}$,

$$\lambda(y,z) := \inf\{t \in \mathbb{R} : y \in z + tq(y) - K(y)\},$$
$$\mu(y,z) := \sup\{t \in \mathbb{R} : y \in z + tq(y) + K(y)\}.$$

By the set-valued mapping K, we define two variable domination structures (see [11, 12]):

$$y \leq^1_K z : \iff z \in y + K(y),$$
$$y \leq^2_K z : \iff z \in y + K(z).$$

Clearly, $y \leq^1_K z$ iff $z \leq^2_{-K} y$ and $y \leq^2_K z$ iff $z \leq^1_{-K} y$. Function λ was recently introduced in [36] to address the above variable domination structures. Since the set $K(y)$ is a cone, for all $y \in Y$, throughout we suppose without loss of generality that the image set $q(Y)$ is bounded.

Several basic properties of functions λ, μ in connection with the binary relations $\leq^i_K$, $i = 1, 2$, are recalled in the following proposition (see [17, 36, 41]). Given $y \in Y$ and $i \in \{1, 2\}$, we denote

$$[y \leq^i_K \cdot] := \{z \in Y : y \leq^i_K z\},$$
$$[\cdot \leq^i_K y] := \{z \in Y : z \leq^i_K y\}.$$

Proposition 2.1. *We have that:*

(i) λ, μ are finite-valued whenever $q(y) \in \mathrm{int}K(y)$, for all $y \in Y$;

(ii) $\mu(0,z) = -\lambda(0,-z)$, for all $z \in Y$;

(iii) $\{(y,z)\in Y\times Y:\lambda(y,z)\leq 0\}=\bigcup_{y\in Y}(\{y\}\times[y\leq_K^1\cdot])$ *and* $\lambda(y,y)=0$*, for all* $y\in Y$*. Also,* $\{(y,z)\in Y\times Y:\mu(y,z)\geq 0\}=\bigcup_{y\in Y}(\{y\}\times[\cdot\leq_K^2 y])$ *and* $\mu(y,y)=0$*, for all* $y\in Y$*;*
(iv) $y\leq_K^1 z$ *iff* $\lambda(y,z)\leq\lambda(z,z)$ *and* $z\leq_K^2 y$ *iff* $\mu(y,z)\geq\mu(z,z)$*.*
(v) $\lambda(y,z-cq(y))=\lambda(y,z)+c$ *and* $\mu(y,z-cq(y))=\mu(y,z)+c$*, for all* $y,z\in Y$ *and* $c\in\mathbb{R}$*.*

Assume that Y is a normed space and $\emptyset\neq M\subsetneq Y$. The function $\delta_M:Y\to\mathbb{R}$ defined by $\delta_M(y)=d(y,M)-d(y,Y\backslash M)$ is called oriented (or signed) distance to the set M, where $d(\cdot,M):Y\to\mathbb{R}$, $d(y,M):=\inf_{z\in M}\|z-y\|$, for all $y\in Y$ (see [43] and the references therein). We denote $\Delta,\Theta:Y\times Y\to\mathbb{R}$, $\Delta(y,z)=\delta_{-K(y)}(y-z)$, $\Theta(y,z)=-\delta_{-K(y)}(z-y)$, for all $y,z\in Y$. The following properties are easily to prove.

Proposition 2.2. *We have that:*
(i) $\Theta(0,z)=-\Delta(0,-z)$*, for all* $z\in Y$*;*
(ii) $\{(y,z)\in Y\times Y:\Delta(y,z)\leq 0\}=\bigcup_{y\in Y}(\{y\}\times[y\leq_K^1\cdot])$ *and* $\Delta(y,y)=0$*, for all* $y\in Y$*. Also,* $\{(y,z)\in Y\times Y:\Theta(y,z)\geq 0\}=\bigcup_{y\in Y}(\{y\}\times[\cdot\leq_K^2 y])$ *and* $\Theta(y,y)=0$*, for all* $y\in Y$*;*
(iii) $y\leq_K^1 z$ *iff* $\Delta(y,z)\leq\Delta(z,z)$ *and* $z\leq_K^2 y$ *iff* $\Theta(y,z)\geq\Theta(z,z)$*.*

Remark 2.1. (i) Notice that $\lambda(\cdot,z):Y\to\overline{\mathbb{R}}$ and $\Delta(\cdot,z):Y\to\mathbb{R}$ are lower $\leq_K^1$-monotone and lower $\leq_K^1$-representing at z, for all $z\in Y$. Analogously, $\mu(\cdot,z):Y\to\overline{\mathbb{R}}$ and $\Theta(\cdot,z):Y\to\mathbb{R}$ are upper $\leq_K^2$-monotone and upper $\leq_K^2$-representing at z, for all $z\in Y$.

(ii) The functions λ,Δ (resp. μ,Θ) characterize the binary relation $\leq_K^1$ (resp. $\leq_K^2$) by scalarization since $y\leq_K^1 z$ (resp. $z\leq_K^2 y$) if and only if $\lambda(y,z)\leq\lambda(z,z)$ or $\Delta(y,z)\leq\Delta(z,z)$ (resp. $\mu(z,z)\leq\mu(y,z)$ or $\Theta(z,z)\leq\Theta(y,z)$) and these last inequalities involve the total order in $\overline{\mathbb{R}}$.

3. Best/Worst Scenario

The general formulation of an optimization problem is as follows:

$$\mathrm{Min}_{\preceq}\{f(x):x\in S\},\tag{OP}$$

where X is the decision space, $f:X\to\mathscr{E}$ is the objective function, $\mathscr{E}$ is the objective space, which is equipped with a binary relation $\preceq$ that models the preferences of the decision maker, and $\emptyset\neq S\subset X$ is the feasible set. Two popular examples of problem (OP) are vector optimization problems (see [12, 17]), where $(\mathscr{E},\preceq)=(Y,\leq_K^i)$, and set optimization problems (see [13, 28]), where $(\mathscr{E},\preceq)=(\mathscr{P}_0(Y),\precsim_K^{i,j})$, $i\in\{1,2\}$, $j\in\{l,u\}$, and

$$\begin{aligned}M_1,M_2\in\mathscr{P}_0(Y),\quad M_1\precsim_K^{i,l}M_2&:\Longleftrightarrow[\cdot\leq_K^i y_2]\cap M_1\neq\emptyset,\text{ for all }y_2\in M_2,\\ M_1\precsim_K^{i,u}M_2&:\Longleftrightarrow[y_1\leq_K^i\cdot]\cap M_2\neq\emptyset,\text{ for all }y_1\in M_1.\end{aligned}$$

Clearly,

$$M_1\precsim_K^{1,l}M_2\iff M_2\subset\bigcup_{y_1\in M_1}(y_1+K(y_1)),\tag{3.1}$$

$$M_1\precsim_K^{1,u}M_2\iff(y_1+K(y_1))\cap M_2\neq\emptyset,\text{ for all }y_1\in M_1,\tag{3.2}$$

$$M_1\precsim_K^{2,l}M_2\iff(y_2-K(y_2))\cap M_1\neq\emptyset,\text{ for all }y_2\in M_2,\tag{3.3}$$

$$M_1\precsim_K^{2,u}M_2\iff M_1\subset\bigcup_{y_2\in M_2}(y_2-K(y_2)).\tag{3.4}$$

The following definition collects the basic solution concepts of problem (OP).

Definition 3.1. (i) A point $x_0 \in S$ is said to be an efficient (nondominated) solution to problem (OP), denoted $x_0 \in \mathrm{E}(f,S,\preceq)$, if

$$x \in S,\ f(x) \preceq f(x_0) \Rightarrow f(x) = f(x_0).$$

(ii) A point $x_0 \in S$ is said to be a minimal solution to problem (OP), denoted $x_0 \in \mathrm{M}(f,S,\preceq)$, if

$$x \in S,\ f(x) \preceq f(x_0) \Rightarrow f(x_0) \preceq f(x).$$

Clearly, $\mathrm{E}(f,S,\preceq) \subset \mathrm{M}(f,S,\preceq)$ and $\mathrm{E}(f,S,\preceq) = \mathrm{M}(f,S,\preceq)$ as long as $\preceq$ fulfills the antisymmetric property.

In the sequel, $\leqslant$ denotes a partial order in $\mathscr{E}$, i.e. $\leqslant$ refers to a reflexive, transitive and antisymmetric binary relation in $\mathscr{E}$. A point $b \in \mathscr{E}$ is a lower (resp. upper) bound of a nonempty set $A \subset \mathscr{E}$ if $b \leqslant a$ (resp. $a \leqslant b$), for all $a \in A$. Moreover, b is said to be the infimum (resp. supremum) of A if it is a lower (resp. upper) bound of A and for each lower (resp. upper) bound $b' \in \mathscr{E}$ we have $b' \leqslant b$ (resp. $b \leqslant b'$). They are denoted $\inf A$ and $\sup A$, respectively. If $\inf A \in A$ (resp. $\sup A \in A$), then it is named minimum (resp. maximum) of A and it is referred as $\min A$ (resp. $\max A$). Notice that $\inf A$ and $\sup A$ are unique. Thus, $\inf A = \min A$ (resp. $\sup A = \max A$) whenever $\min A$ (resp. $\max A$) exists. A partially ordered space is said to be a complete lattice if the infimum and supremum exist for every nonempty set in the space (see [35]).

Consider the uncertain version of problem (OP):

$$\mathrm{Min}_{\preceq}\{f(x,u) : x \in S(u)\}, \tag{OP$_u$}$$

where $f : X \times \mathscr{U} \to \mathscr{E}$, $S : \mathscr{U} \rightrightarrows X$, $\mathscr{U}$ is the uncertainty set and $u \in \mathscr{U}$. Define $F_{\mathscr{U}} : X \rightrightarrows \mathscr{E}$, $F_{\mathscr{U}}(x) = \{f(x,u) : u \in \mathscr{U}\}$, for all $x \in X$. If the binary relation $\preceq$ is a partial order $\leq$ such that there exists $\inf F_{\mathscr{U}}(x)$ (resp. $\sup F_{\mathscr{U}}(x)$), for all $x \in S$ –for instance, whenever $\leq$ is a complete lattice, then one can introduce the optimistic (resp. pessimistic) robust counterpart of problem (OP$_u$), which considers the feasible set $S := \bigcap_{u\in\mathscr{U}} S(u)$ and chooses the best (resp. worst) case at every point:

$$\mathrm{Min}_{\leqslant}\{\inf F_{\mathscr{U}}(x) : x \in S\} \tag{3.5}$$

$$(\text{resp. } \mathrm{Min}_{\leqslant}\{\sup F_{\mathscr{U}}(x) : x \in S\}). \tag{3.6}$$

The simplest example of this setting is a total order $\leq$, i.e. for each $e_1, e_2 \in \mathscr{E}$, we have $e_1 \leq e_2$ or $e_2 \leq e_1$, and $\mathscr{U} = \{u_1, u_2, \ldots, u_p\}$, i.e. U is finite. In this case,

$$\inf F_{\mathscr{U}}(x) = \min\{f(x,u_1), f(x,u_2), \ldots, f(x,u_p)\},$$
$$\sup F_{\mathscr{U}}(x) = \max\{f(x,u_1), f(x,u_2), \ldots, f(x,u_p)\}, \text{ for all } x \in X.$$

For instance, the solutions of problem (3.6) in the framework $(\mathscr{E},\leqslant) = (\mathbb{R},\leq)$ and $\mathscr{U}$ finite are the popular minmax robust solutions.

Next, we introduce two binary relations $\preceq^o$ and $\preceq^p$ in $\mathscr{P}_0(\mathscr{E})$, which allow us to deal with the uncertain optimization problem (OP$_u$) through the optimistic and pessimistic robust point of view in the general setting where the infimum and supremum of a set in $(\mathscr{E},\preceq)$ do not exist. It is motivated by [9, Definition 2.3].

Definition 3.2. Given $A_1, A_2 \in \mathscr{P}_0(\mathscr{E})$, we write $A_1 \preceq^o A_2$ if, for each $a_2 \in A_2$, there exists $a_1 \in A_1$ satisfying $a_1 \preceq a_2$. Analogously, we denote $A_1 \preceq^p A_2$ if, for each $a_1 \in A_1$, there exists $a_2 \in A_2$ such that $a_1 \preceq a_2$.

Remark 3.1. By applying the binary relation $\trianglelefteq^o$ (resp. $\trianglelefteq^p$) to $(\mathscr{E},\preceq)=(\mathscr{P}_0(Y),\precsim_K^{1,u})$ and a constant set-valued mapping K we obtain that given $\mathscr{A}^1=\{A_i^1\}_{i\in I_1},\mathscr{A}^2=\{A_i^2\}_{i\in I_2}\subset\mathscr{P}_0(Y)$, $\mathscr{A}^1\trianglelefteq^o\mathscr{A}^2$ (resp. $\mathscr{A}^1\trianglelefteq^p\mathscr{A}^2$) if and only if for all $i_2\in I_2$ (resp. $i_1\in I_1$) there exists $i_1\in I_1$ (resp. $i_2\in I_2$) such that $A_{i_1}^1\subset A_{i_2}^2-K$. Analogously, if $\preceq=\precsim_K^{1,l}$, then $\mathscr{A}^1\trianglelefteq^o\mathscr{A}^2$ (resp. $\mathscr{A}^1\trianglelefteq^p\mathscr{A}^2$) if and only if for all $i_2\in I_2$ (resp. $i_1\in I_1$) there exists $i_1\in I_1$ (resp. $i_2\in I_2$) such that $A_{i_2}^2\subset A_{i_1}^1+K$.

These versions of the binary relations $\trianglelefteq^o$ and $\trianglelefteq^p$ were introduced in [9]. Specifically, the authors consider the union sets $A_1=\bigcup_{i_1\in I_1}A_{i_1}^1$ and $A_2=\bigcup_{i_2\in I_2}A_{i_2}^2$ and define $A_1\leq_K^L A_2$ if $\mathscr{A}^1\trianglelefteq^o\mathscr{A}^2$ with respect to $\preceq_K^{1,l}$ and $A_1\leq_K^U A_2$ if $\mathscr{A}^1\trianglelefteq^p\mathscr{A}^2$ with respect to $\preceq_K^{1,u}$. Notice that this definition is not consistent since depends on the representation of the sets A_1 and A_2 as a union of the elements in the families $\mathscr{A}_1$ and $\mathscr{A}_2$. This difficulty is illustrated in the next example.

Example 3.1. Consider $Y=\mathbb{R}^2$, $K(y)=\mathbb{R}^2_+$, for all $y\in\mathbb{R}^2$, and the sets $A_1=A_2=A+\mathbb{R}^2_+$, where

$$A=\{(y_1,y_2)\in\mathbb{R}^2_+ : y_1y_2=1\}.$$

Clearly, A_1 and A_2 are the union of the elements in the singleton $\mathscr{A}=\{A+\mathbb{R}^2\}$ and so $A_1\leq_K^L A_2$. However, they are also the union of the elements in the families $\mathscr{A}_1=\{y+\mathbb{R}^2_+ : y\in A\}$ and $\mathscr{A}_2=\{A+\mathbb{R}^2\}$ and $A+\mathbb{R}^2_+\not\subset y+\mathbb{R}^2_+$, for all $y\in A$, i.e. $A_1\not\leq_K^L A_2$.

Notice that addressing the minimization of elements in $(\mathscr{E},\preceq)$, comparisons $A_1\trianglelefteq^o A_2$ and $A_1\trianglelefteq^p A_2$ between two nonempty sets $A_1,A_2\subset\mathscr{E}$ mean that the best element in A_1 is lower than the best element in A_2 and the worst element in A_1 is less than the worst element in A_2, respectively. Hence, the following optimization problems formulate optimistic and pessimistic robust counterparts of the uncertain optimization problem (OP_u), respectively:

$$\mathrm{Min}_{\trianglelefteq^o}\{F_{\mathscr{U}}(x) : x\in S\}, \tag{OOP_u}$$

$$\mathrm{Min}_{\trianglelefteq^p}\{F_{\mathscr{U}}(x) : x\in S\}. \tag{POP_u}$$

Namely, in the next proposition, it is shown that the solutions of problems (3.5) and (OOP_u) (resp. (3.6) and (POP_u)) coincide. Thus, problems (OOP_u) and (POP_u) can be viewed as the optimistic and pessimistic robust counterparts of problem (OP_u) when the infimum and supremum of a family of elements in the final space of the problem do not exist.

Lemma 3.1. *Let $(\mathscr{E},\leqslant)$ be a complete lattice and consider two sets $A_1,A_2\in\mathscr{P}_0(\mathscr{E})$. The following assertions are true:*

(i) If $A_1\trianglelefteq^o A_2$, then $\inf A_1\leqslant\inf A_2$. Reciprocally, if $\min A_1\leqslant\inf A_2$, then $A_1\trianglelefteq^o A_2$.

(ii) If $A_1\trianglelefteq^p A_2$, then $\sup A_1\leqslant\sup A_2$. Reciprocally, if $\sup A_1\leqslant\max A_2$, then $A_1\trianglelefteq^p A_2$.

Proof. We only prove the first part as the second one is similar.

Assume that $A_1\trianglelefteq^o A_2$. Then, for each $a_2\in A_2$, there exists $a_1\in A_1$ satisfying $a_1\leqslant a_2$. As $\inf A_1$ is a lower bound of A_1, it follows that $\inf A_1\leqslant a_1$, for all $a_1\in A_1$. Therefore, $\inf A_1\leqslant a_2$, for all $a_2\in A_2$ and $\inf A_1$ is a lower bound of A_2. Hence, $\inf A_1\leqslant\inf A_2$.

Reciprocally, suppose that $\min A_1\leqslant\inf A_2$. Then the point $a_1:=\min A_1\in A_1$ satisfies

$$a_1\leqslant\inf A_2\leqslant a_2,\ \text{ for all } a_2\in A_2$$

and clearly $A_1\trianglelefteq^o A_2$, which finishes the proof. □

Theorem 3.1. *Let $(\mathscr{E},\leqslant)$ be a complete lattice and suppose that* $\min A$ *and* $\max A$ *exist, for all nonempty set $A\subset\mathscr{E}$. Concerning problems* (3.5) *and* (OOP_u) *(resp.* (3.6) *and* (POP_u)*), we have*

$$\mathrm{E}(\min F_{\mathscr{U}},S,\leqslant)=\mathrm{M}(F_{\mathscr{U}},S,\trianglelefteq^o) \tag{3.7}$$

$$(\textit{resp. } \mathrm{E}(\max F_{\mathscr{U}},S,\leqslant)=\mathrm{M}(F_{\mathscr{U}},S,\trianglelefteq^p)). \tag{3.8}$$

Proof. We only state statement (3.7) since statement (3.8) can be obtained similarly.

Consider $x_0\in\mathrm{E}(\min F_{\mathscr{U}},S,\leqslant)$ and suppose that $x\in S$ satisfies $F_{\mathscr{U}}(x)\trianglelefteq^o F_{\mathscr{U}}(x_0)$. By the necessary condition in Lemma 3.1(i), we deduce that $\min F_{\mathscr{U}}(x)\leqslant\min F_{\mathscr{U}}(x_0)$. As x_0 is an efficient solution to problem (3.5), we have $\min F_{\mathscr{U}}(x)=\min F_{\mathscr{U}}(x_0)$. Thus, $\min F_{\mathscr{U}}(x_0)\leqslant\inf F_{\mathscr{U}}(x)$. From the sufficient condition in Lemma 3.1(i), we obtain that $F_{\mathscr{U}}(x_0)\trianglelefteq^o F_{\mathscr{U}}(x)$. Hence, $x_0\in\mathrm{M}(F_{\mathscr{U}},S,\trianglelefteq^o)$ and the inclusion $\mathrm{E}(\min F_{\mathscr{U}},S,\leqslant)\subset\mathrm{M}(F_{\mathscr{U}},S,\trianglelefteq^o)$ is proved.

Conversely, assume that $x_0\in\mathrm{M}(F_{\mathscr{U}},S,\trianglelefteq^o)$ and consider a point $x\in S$ satisfying $\min F_{\mathscr{U}}(x)\leqslant\min F_{\mathscr{U}}(x_0)$. As $\min F_{\mathscr{U}}(x_0)=\inf F_{\mathscr{U}}(x_0)$, by the the sufficient condition in Lemma 3.1(i), we state that $F_{\mathscr{U}}(x)\trianglelefteq^o F_{\mathscr{U}}(x_0)$. Since x_0 is a minimal solution to problem (OOP_u), we deduce that $F_{\mathscr{U}}(x_0)\trianglelefteq^o F_{\mathscr{U}}(x)$. Therefore, by the necessary condition in Lemma 3.1(i), we state that

$$\min F_{\mathscr{U}}(x_0)=\inf F_{\mathscr{U}}(x_0)\leqslant\inf F_{\mathscr{U}}(x)=\min F_{\mathscr{U}}(x)$$

and we obtain $\min F_{\mathscr{U}}(x)=\min F_{\mathscr{U}}(x_0)$ because of the antisymmetric property of the binary relation $\leqslant$. Thus, the inclusion $\mathrm{M}(F_{\mathscr{U}},S,\trianglelefteq^o)\subset\mathrm{E}(\min F_{\mathscr{U}},S,\leqslant)$ holds true and the proof finishes. □

Theorem 3.1 shows that the set-based approach to deal with pessimistic robust solutions of uncertain Pareto optimization problems introduced in [10] is suitable. However, minimal solutions of the robust counterpart should be considered instead of the nondominated (efficient) ones. Indeed, equalities (3.7) and (3.8) cannot be extended to the sets $\mathrm{E}(F_{\mathscr{U}},S,\trianglelefteq^o)$ and $\mathrm{E}(F_{\mathscr{U}},S,\trianglelefteq^p)$, respectively, as the next example illustrates for statement (3.8).

Example 3.2. Let the data of problem (OP_u) be $X=S=\{x_1,x_2\}$, $Y=\mathbb{R}^2$, $K(y)=\mathbb{R}^2_+$, for all $y\in\mathbb{R}^2$, $\preceq=\preceq^1_K$, $\mathscr{U}=\{u_1,u_2\}$, $f(x_1,u_1)=(1,1)$, $f(x_1,u_2)=(1,0)$, $f(x_2,u_1)=(0,1)$ and $f(x_2,u_2)=(1,1)$. Clearly, $\max F_{\mathscr{U}}(x_1)=\max F_{\mathscr{U}}(x_2)=(1,1)$ and function $\max F_{\mathscr{U}}$ is constant. Therefore, $\mathrm{E}(\max F_{\mathscr{U}},S,\preceq)=S$. Moreover, $F_{\mathscr{U}}(x_1)-\mathbb{R}^2_+=F_{\mathscr{U}}(x_2)-\mathbb{R}^2_+$. Hence, $F_{\mathscr{U}}(x_1)\trianglelefteq^p F_{\mathscr{U}}(x_1)$, $F_{\mathscr{U}}(x_2)\trianglelefteq^p F_{\mathscr{U}}(x_1)$ and $\mathrm{E}(F_{\mathscr{U}},S,\trianglelefteq^p)=\emptyset$ since $F_{\mathscr{U}}(x_1)\neq F_{\mathscr{U}}(x_2)$. Thus,

$$\mathrm{E}(F_{\mathscr{U}},S,\trianglelefteq^p)\subsetneq\mathrm{M}(F_{\mathscr{U}},S,\trianglelefteq^p)=\mathrm{E}(\max F_{\mathscr{U}},S,\preceq)$$

and equality (3.8) cannot be extended to the set $\mathrm{E}(F_{\mathscr{U}},S,\trianglelefteq^p)$.

According to Theorem 3.1, [5, Section 2], [39, Remark 1] and the above example, we introduce the following concepts of optimistic and pessimistic robust solution of the uncertain optimization problem (OP_u).

Definition 3.3. A point $x_0\in S$ is said to be an optimistic (resp. pessimistic) robust solution to problem (OP_u) if $x_0\in\mathrm{M}(F_{\mathscr{U}},S,\trianglelefteq^o)$ (resp. $x_0\in\mathrm{M}(F_{\mathscr{U}},S,\trianglelefteq^p)$).

As the robustness concepts introduced in [10, 26, 25], the above ones are set-based. However, they are different since in [10, 26, 25] nondominated solutions of the set-valued robust counterpart are considered, whereas in Definition 3.3, the solutions of the set-valued robust counterpart are minimal. Notice that statement (3.8) states the equality between minmax and minimal solutions instead of minmax and efficient (nondominated) solutions.

Example 3.3. Consider problem (OP$_u$) with the data $X = S = \{x_1, x_2, x_3\}$, $Y = \mathbb{R}^2$, $K(y) = \mathbb{R}^2_+$, for all $y \in \mathbb{R}^2$, $\preceq = \leq^1_K$, $\mathscr{U} = \{u_1, u_2\}$, $f(x_1,u_1) = f(x_2,u_1) = f(x_3,u_1) = (1,1)$, $f(x_1,u_2) = f(x_3,u_2) = (1,1/2)$ and $f(x_2,u_2) = (2,1/2)$. The solutions of this problem should be x_1, x_3 since f is constant at scenario u_1, $f(x_1,u_2) = f(x_3,u_2) \neq f(x_2,u_2)$ and $f(x_1,u_2) \leq^1_K f(x_2,u_2)$, i.e. at scenario u_2, points x_1 and x_3 obtain better values than x_2.

It is easy to check that $\mathrm{M}(F_{\mathscr{U}}, S, \trianglelefteq^p) = \{x_1, x_3\}$. Thus, x_1 and x_3 are the pessimistic robust solutions of the problem. However, addressing the robustness concepts in [10, 26], all feasible points are robust weakly efficient solutions and robust efficient solutions of the problem and there is not any robust strictly efficient solution.

The following proposition provides the binary relations $\trianglelefteq^o$ and $\trianglelefteq^p$ in the spaces $(Y, \leq^i_K)$ and $(\mathscr{P}_0(Y), \precsim^{i,j}_K)$, $i \in \{1,2\}$, $j \in \{l,u\}$. It follows directly from the corresponding definitions and statements (3.1)-(3.4). Notice that for each element $\mathscr{A} \in \mathscr{P}_0(\mathscr{P}_0(Y))$ there exists a nonempty family $\{A_i\}_{i\in I} \subset \mathscr{P}_0(Y)$ such that $\mathscr{A} = \{A_i\}_{i\in I}$.

Proposition 3.1. *The following assertions are true:*

(i) If $(\mathscr{E}, \preceq) = (Y, \leq^i_K)$, *then* $\trianglelefteq^p = \precsim^{i,u}_K$ *and* $\trianglelefteq^o = \precsim^{i,l}_K$.

(ii) Given $\mathscr{A}^1 = \{A^1_i\}_{i\in I_1}, \mathscr{A}^2 = \{A^2_i\}_{i\in I_2} \subset \mathscr{P}_0(Y)$, *we have:*

If $(\mathscr{E}, \preceq) = (\mathscr{P}_0(Y), \precsim^{1,l}_K)$, *then* $\mathscr{A}^1 \trianglelefteq^o \mathscr{A}^2$ *(resp.* $\mathscr{A}^1 \trianglelefteq^p \mathscr{A}^2$*) if for all* $i_2 \in I_2$ *there exists* $i_1 \in I_1$ *(resp. for all* $i_1 \in I_1$ *there exists* $i_2 \in I_2$*) such that* $A^2_{i_2} \subset \bigcup_{y\in A^1_{i_1}} (y + K(y))$;

If $(\mathscr{E}, \preceq) = (\mathscr{P}_0(Y), \precsim^{1,u}_K)$, *then* $\mathscr{A}^1 \trianglelefteq^o \mathscr{A}^2$ *(resp.* $\mathscr{A}^1 \trianglelefteq^p \mathscr{A}^2$ *) if for all* $i_2 \in I_2$ *there exists* $i_1 \in I_1$ *(resp. for all* $i_1 \in I_1$ *there exists* $i_2 \in I_2$*) such that* $(y + K(y)) \cap A^2_{i_2} \neq \emptyset$ *for all* $y \in A^1_{i_1}$;

If $(\mathscr{E}, \preceq) = (\mathscr{P}_0(Y), \precsim^{2,l}_K)$, *then* $\mathscr{A}^1 \trianglelefteq^o \mathscr{A}^2$ *(resp.* $\mathscr{A}^1 \trianglelefteq^p \mathscr{A}^2$ *) if for all* $i_2 \in I_2$ *there exists* $i_1 \in I_1$ *(resp. for all* $i_1 \in I_1$ *there exists* $i_2 \in I_2$*) such that* $(y - K(y)) \cap A^1_{i_1} \neq \emptyset$ *for all* $y \in A^2_{i_2}$;

If $(\mathscr{E}, \preceq) = (\mathscr{P}_0(Y), \precsim^{2,u}_K)$, *then* $\mathscr{A}^1 \trianglelefteq^o \mathscr{A}^2$ *(resp.* $\mathscr{A}^1 \trianglelefteq^p \mathscr{A}^2$ *) if for all* $i_2 \in I_2$ *there exists* $i_1 \in I_1$ *(resp. for all* $i_1 \in I_1$ *there exists* $i_2 \in I_2$*) such that* $A^1_{i_1} \subset \bigcup_{y\in A^2_{i_2}} (y - K(y))$.

4. Scalarization Results

The first aim of this section is to characterize the binary relations $\trianglelefteq^o$ and $\trianglelefteq^p$ corresponding to an arbitrary binary relation $\preceq$ in a set $\mathscr{E}$. In the sequel, we denote

$$A^{\preceq} := \bigcup_{a\in A} \{v \in \mathscr{E} : a \preceq v\},$$
$$A^{\succeq} := \bigcup_{a\in A} \{v \in \mathscr{E} : v \preceq a\}.$$

Theorem 4.1. *Consider* $\psi : \mathscr{E} \to \overline{\mathbb{R}}$, $\Psi : \mathscr{E} \times \mathscr{E} \to \overline{\mathbb{R}}$ *and* $A, B \in \mathscr{P}_0(\mathscr{E})$. *The following conditions concerning the relationships* $A \trianglelefteq^p B$ *and* $A \trianglelefteq^o B$ *are true.*

(i) Let ψ *be upper (resp. lower)* $\preceq$*-monotone at every point* $a \in A$ *in* B *(resp. at every point* $b \in B$ *in* A*). If* $A \trianglelefteq^p B$ *(resp.* $A \trianglelefteq^o B$*), then*

$$\sup_{a\in A} \psi(a) \leq \sup_{b\in B} \psi(b) \quad (\textit{resp.} \ \inf_{a\in A} \psi(a) \leq \inf_{b\in B} \psi(b)).$$

(ii) We have that

$$\bigcup_{v\in\mathscr{E}}\bigcup_{a\in A}(\{y\in\mathscr{E}:\Psi(a,v)\leq\Psi(y,v)\}\cap A^{l}_{\Psi(\cdot,v)})\subset A^{\preceq}, \tag{4.1}$$

$$\bigcup_{v\in\mathscr{E}}\bigcup_{b\in B}(\{y\in\mathscr{E}:\Psi(v,y)\leq\Psi(v,b)\}\cap B^{u}_{\Psi(v,\cdot)})\subset B^{\succeq}. \tag{4.2}$$

(iii) Suppose that $a\in B^{u}_{\Psi(a,\cdot)}$ *(resp.* $b\in A^{l}_{\Psi(\cdot,b)}$*), for all* $a\in A$ *(resp.* $b\in B$*). If* $\sup_{b\in B}\Psi(a,b)$ *(resp.* $\inf_{a\in A}\Psi(a,b)$*) is attained, for all* $a\in A$ *(resp.* $b\in B$*), and*

$$\inf_{a\in A}\sup_{b\in B}\Psi(a,b)\geq\sup_{a\in A}\Psi(a,a)\quad(\text{resp. }\sup_{b\in B}\inf_{a\in A}\Psi(a,b)\leq\inf_{b\in B}\Psi(b,b)),$$

then $A\trianglelefteq^{p}B$ *(resp.* $A\trianglelefteq^{o}B$*).*

(iv) Consider $\Psi(a,\cdot):\mathscr{E}\to\overline{\mathbb{R}}$ *(resp.* $\Psi(\cdot,b):\mathscr{E}\to\overline{\mathbb{R}}$*) is upper (resp. lower)* $\preceq$*-monotone at* a *in* B *(resp. at* b *in* A*), for all* $a\in A$ *(resp.* $b\in B$*). If* $A\trianglelefteq^{p}B$ *(resp.* $A\trianglelefteq^{o}B$*), then*

$$\inf_{a\in A}\sup_{b\in B}\Psi(a,b)\geq\inf_{a\in A}\Psi(a,a)\quad(\text{resp. }\sup_{b\in B}\inf_{a\in A}\Psi(a,b)\leq\sup_{b\in B}\Psi(b,b)).$$

Proof. (i) Assume that $A\trianglelefteq^{p}B$ and consider an arbitrary element a in A. By the definition of the binary relation $\trianglelefteq^{p}$, it follows that there exists $b_a\in B$ such that $a\preceq b_a$. Since ψ is upper $\preceq$-monotone at a in B, we obtain that $\psi(a)\leq\psi(b_a)$. Hence, $\sup_{a\in A}\psi(a)\leq\sup_{b\in B}\psi(b)$ and the result addressing the case $A\trianglelefteq^{p}B$ is obtained. The proof for the case $A\trianglelefteq^{o}B$ is similar, and it is omitted.

(ii) Let $v,y\in\mathscr{E}$ and $a\in A$ satisfy $\Psi(a,v)\leq\Psi(y,v)$ and $y\in A^{l}_{\Psi(\cdot,v)}$. By the definition of the set $A^{l}_{\Psi(\cdot,v)}$, we deduce that $a\preceq y$. Thus, $y\in A^{\preceq}$ and inclusion (4.1) is proved. The proof for the second part (4.2) is omitted since it is the same as (4.1).

(iii) We only state the first part concerning $A\trianglelefteq^{p}B$ as the other one is proved analogously. Consider a point $a\in A$. By the assumptions, we obtain that there exists $b_a\in B$ such that $\Psi(a,b_a)\geq\Psi(a,a)$. Hence,

$$a\in\{y\in\mathscr{E}:\Psi(a,y)\leq\Psi(a,b_a)\}\cap B^{u}_{\Psi(a,\cdot)}\subset\bigcup_{v\in\mathscr{E}}\bigcup_{b\in B}(\{y\in\mathscr{E}:\Psi(v,y)\leq\Psi(v,b)\}\cap B^{u}_{\Psi(v,\cdot)})$$

and the result follows by applying inclusion (4.2).

(iv) We prove the first part since the proof of the second one is similar. Take $\bar{a}\in A$. Since $A\trianglelefteq^{p}B$, we see that there exists $b_{\bar{a}}\in B$ satisfying $\bar{a}\preceq b_{\bar{a}}$. Thus, $\Psi(\bar{a},\bar{a})\leq\Psi(\bar{a},b_{\bar{a}})$ as $\Psi(\bar{a},\cdot)$ is upper $\preceq$-monotone at $\bar{a}$ in B, and it follows that

$$\inf_{a\in A}\Psi(a,a)\leq\Psi(\bar{a},\bar{a})\leq\Psi(\bar{a},b_{\bar{a}})\leq\sup_{b\in B}\Psi(\bar{a},b).$$

Since $\bar{a}\in A$ was arbitrarily chosen, we have $\inf_{a\in A}\Psi(a,a)\leq\inf_{a\in A}\sup_{b\in B}\Psi(a,b)$. This completes the proof. □

Theorem 4.1 allows us to characterize by scalarization the binary relations $\trianglelefteq^{o}$ and $\trianglelefteq^{p}$ in several settings. Next we obtain these characterizations in the spaces $(Y,\leq^{i}_{K})$, $i\in\{1,2\}$.

Theorem 4.2. *Consider* $(\mathscr{E},\preceq)=(Y,\leq^{1}_{K})$*,* $A,B\in\mathscr{P}_0(Y)$ *and* $\rho\in\{\lambda,\Delta\}$*.*

(i) If $A\trianglelefteq^{p}B$*, then*

$$\sup_{a\in A}\inf_{b\in B}\rho(a,b)\leq 0. \tag{4.3}$$

(ii) Assume that $\inf_{b\in B}\rho(a,b)$ *is attained, for all* $a\in A$. *If inequality* (4.3) *is fulfilled, then* $A\trianglelefteq^p B$.

(iii) If $y\in Y$ *satisfies* $\inf_{b\in B}\rho(y,b)\leq 0$, *then* $0\in \mathrm{cl}\,(B-[y\leq_K^1 \cdot])$.

(iv) Suppose that the set $B-[a\leq_K^1 \cdot]$ *is closed, for all* $a\in A$. *If inequality* (4.3) *is true, then* $A\trianglelefteq^p B$.

Proof. (i) Define $\Psi: Y\times Y\to\overline{\mathbb{R}}$ as $\Psi(y,z):=-\rho(y,z)$, for all $y,z\in Y$. Given $a\in A$, $b\in B$ such that $a\leq_K^1 b$, by parts (iii) and (iv) in Proposition 2.1 and parts (ii) and (iii) in Proposition 2.2, it is obtained

$$\Psi(a,a)=-\rho(a,a)=-\rho(b,b)\leq-\rho(a,b)=\Psi(a,b).$$

Therefore, $\Psi(a,\cdot):Y\to\overline{\mathbb{R}}$ is upper $\leq_K^1$-monotone at a in B, for all $a\in A$. By Theorem 4.1(iv), Proposition 2.1(iii) and Proposition 2.2(ii), it follows that

$$-\sup_{a\in A}\inf_{b\in B}\rho(a,b)=\inf_{a\in A}\sup_{b\in B}-\rho(a,b)=\inf_{a\in A}\sup_{b\in B}\Psi(a,b)\geq\inf_{a\in A}\Psi(a,a)=0$$

and inequality (4.3) is proved.

(ii) We claim that $a\in B^u_{\Psi(a,\cdot)}$, for all $a\in A$. Indeed, if $a\in A$, $b\in B$ satisfies $\Psi(a,a)\leq\Psi(a,b)$, then $\rho(a,b)\leq\rho(a,a)$ and by parts (iii) and (iv) in Proposition 2.1 and parts (ii) and (iii) in Proposition 2.2, we see that $a\leq_K^1 b$, which proves the assertion. Hence, by Theorem 4.1(iii), we have then $A\trianglelefteq^p B$.

(iii) Assume that $\inf_{b\in B}\rho(y,b)\leq 0$. Then, for each $\varepsilon>0$, there exists $b_\varepsilon\in B$ fulfilling $\rho(y,b_\varepsilon)\leq\varepsilon$. If $\rho=\lambda$, by Proposition 2.1(iii)-(v), we deduce that $y\leq_K^1 b_\varepsilon+\varepsilon q(y)$. Therefore,

$$-\varepsilon q(y)\in b_\varepsilon-(y+K(y))\subset B-[y\leq_K^1\cdot].$$

Considering the limit $\varepsilon\downarrow 0$ we see that $0\in\mathrm{cl}\,(B-[y\leq_K^1\cdot])$. In the case $\rho=\Delta$, we have

$$\delta_{-K(y)}(y-b_\varepsilon)=d(y-b_\varepsilon,-K(y))-d(y-b_\varepsilon,Y\backslash(-K(y)))\leq\varepsilon.$$

We claim that $d(y-b_\varepsilon,-K(y))\leq\varepsilon$. Indeed, if $y-b_\varepsilon\notin-K(y)$, then $d(y-b_\varepsilon,Y\backslash(-K(y)))=0$ and $d(y-b_\varepsilon,-K(y))\leq\varepsilon$. Otherwise, $y-b_\varepsilon\in-K(y)$ and $d(y-b_\varepsilon,-K(y))=0<\varepsilon$. Therefore,

$$d(b_\varepsilon,[y\leq_K^1\cdot])=d(b_\varepsilon,y+K(y))=d(y-b_\varepsilon,-K(y))\leq\varepsilon.$$

Hence, $0\in\mathrm{cl}\,(B-[y\leq_K^1\cdot])$ and part (iii) is proved.

(iv) Suppose that the set $B-[a\leq_K^1\cdot]$ is closed, for all $a\in A$. If inequality (4.3) holds true, we have $\inf_{b\in B}\rho(a,b)\leq 0$, for all $a\in A$, and by part (iii), we obtain

$$0\in\bigcap_{a\in A}\mathrm{cl}\,(B-[a\leq_K^1\cdot])=\bigcap_{a\in A}(B-[a\leq_K^1\cdot]).$$

Thus, $B\cap[a\leq_K^1\cdot]\neq\emptyset$, for all $a\in A$, and it follows that $A\trianglelefteq^p B$, which finishes the proof. □

Remark 4.1. If the set-valued mapping K is constant, i.e., $K(y)=K$, then

$$\mathrm{cl}(B-[y\leq_K^1\cdot])=\mathrm{cl}(B-(y+K))=\mathrm{cl}(B-K)-y$$

and then $0\in\mathrm{cl}(B-[y\leq_K^1\cdot])$ if and only if $y\in\mathrm{cl}(B-K)$. In addition, $B-[y\leq_K^1\cdot]$ is closed whenever $B-K$ is closed.

Next result can be stated by the function $\Psi=\rho$ and the same arguments as in the proof of the previous theorem.

Theorem 4.3. *Consider* $(\mathscr{E},\preceq) = (Y,\leq_K^1)$, $A,B \in \mathscr{P}_0(Y)$ *and* $\rho \in \{\lambda,\Delta\}$.

(i) If $A \trianglelefteq^o B$, *then*

$$\sup_{b\in B}\inf_{a\in A}\rho(a,b) \leq 0. \tag{4.4}$$

(ii) Assume that $\inf_{a\in A}\rho(a,b)$ *is attained, for all* $b \in B$. *If inequality* (4.4) *is fulfilled, then* $A \trianglelefteq^o B$.

(iii) If $y \in Y$ *satisfies* $\inf_{a\in A}\rho(a,y) \leq 0$, *then* $y \in \operatorname{cl}\bigcup_{a\in A}[a \leq_K^1 \cdot]$.

(iv) Suppose that the set $\bigcup_{a\in A}[a \leq_K^1 \cdot]$ *is closed. If inequality* (4.4) *is true, then* $A \trianglelefteq^o B$.

Proof. (i) Define $\Psi : Y \times Y \to \overline{\mathbb{R}}$ as $\Psi(y,z) := \rho(y,z)$, for all $y,z \in Y$. Given $a \in A$, $b \in B$ such that $a \leq_K^1 b$, by parts (iii) and (iv) in Proposition 2.1 and parts (ii) and (iii) in Proposition 2.2, it is obtained

$$\Psi(a,b) = \rho(a,b) \leq \rho(b,b) = \Psi(b,b).$$

Therefore, $\Psi(\cdot,b) : Y \to \overline{\mathbb{R}}$ is lower $\leq_K^1$-monotone at b in A, for all $b \in B$. By Theorem 4.1(iv), Proposition 2.1(iii) and Proposition 2.2(ii) it follows that

$$\sup_{b\in B}\inf_{a\in A}\rho(a,b) = \sup_{b\in B}\inf_{a\in A}\Psi(a,b) \leq \sup_{b\in B}\Psi(b,b) = 0$$

and inequality (4.4) is proved.

(ii) We claim that $b \in A^l_{\Psi(\cdot,b)}$, for all $b \in B$. Indeed, if $a \in A$, $b \in B$ satisfy $\Psi(a,b) \leq \Psi(b,b)$, then $\rho(a,b) \leq \rho(b,b)$ and by parts (iii) and (iv) in Proposition 2.1 and parts (ii) and (iii) in Proposition 2.2 we see that $a \leq_K^1 b$, which proves the assertion. Hence, by Theorem 4.1(iii) we have then $A \trianglelefteq^o B$.

(iii) Assume that $\inf_{a\in A}\rho(a,y) \leq 0$. Then, for each $\varepsilon > 0$, there exists $a_\varepsilon \in A$ fulfilling $\rho(a_\varepsilon,y) \leq \varepsilon$. Suppose that $\rho = \lambda$. By Proposition 2.1(iv),(v) we deduce that $a_\varepsilon \leq_K^1 y + \varepsilon q(a_\varepsilon)$. Therefore,

$$-\varepsilon q(a_\varepsilon) \in y - (a_\varepsilon + K(a_\varepsilon)) \subset y - \bigcup_{a\in A}[a \leq_K^1 \cdot].$$

Since the set $q(Y)$ is bounded, by considering the limit $\varepsilon \downarrow 0$ we obtain that

$$0 \in \operatorname{cl}\left(y - \bigcup_{a\in A}[a \leq_K^1 \cdot]\right).$$

Thus $y \in \operatorname{cl}\bigcup_{a\in A}[a \leq_K^1 \cdot]$. If $\rho = \Delta$, then

$$\delta_{-K(a_\varepsilon)}(a_\varepsilon - y) = d(a_\varepsilon - y, -K(a_\varepsilon)) - d(a_\varepsilon - y, Y\backslash(-K(a_\varepsilon))) \leq \varepsilon.$$

We claim that $d(a_\varepsilon - y, -K(a_\varepsilon)) \leq \varepsilon$. Indeed, if $a_\varepsilon - y \notin -K(a_\varepsilon)$, then $d(a_\varepsilon - y, Y\backslash(-K(a_\varepsilon))) = 0$ and $d(a_\varepsilon - y, -K(a_\varepsilon)) \leq \varepsilon$. Otherwise, $a_\varepsilon - y \in -K(a_\varepsilon)$ and $d(a_\varepsilon - y, -K(a_\varepsilon)) = 0 < \varepsilon$. Therefore,

$$d\left(y, \bigcup_{a\in A}[a \leq_K^1 \cdot]\right) \leq d(y,[a_\varepsilon \leq_K^1 \cdot]) = d(y, a_\varepsilon + K(a_\varepsilon)) = d(a_\varepsilon - y, -K(a_\varepsilon)) < \varepsilon.$$

Hence, $y \in \operatorname{cl}\bigcup_{a\in A}[a \leq_K^1 \cdot]$ and part (iii) is proved.

(iv) Suppose that the set $\bigcup_{a\in A}[a \leq_K^1 \cdot]$ is closed. If inequality (4.4) holds true, we have $\inf_{a\in A}\rho(a,b) \leq 0$, for all $b \in B$, and by part (iii), we obtain $B \subset \bigcup_{a\in A}[a \leq_K^1 \cdot]$. Thus, $A \trianglelefteq^o B$ and the proof finishes. □

Similarly to Theorems 4.2 and 4.3, one can obtain scalarization results for the relationships $A \trianglelefteq^p B$ and $A \trianglelefteq^o B$ in the setting $(Y, \leq_K^2)$ via the functions μ and Θ. They are formulated in the sequel.

Theorem 4.4. *Consider* $(\mathscr{E}, \preceq) = (Y, \leq_K^2)$, $A, B \in \mathscr{P}_0(Y)$ *and* $\rho \in \{\mu, \Theta\}$.

(i) If $A \trianglelefteq^p B$, *then*

$$\inf_{a\in A}\sup_{b\in B}\rho(b,a) \geq 0. \tag{4.5}$$

(ii) Assume that $\sup_{b\in B}\rho(b,a)$ *is attained, for all* $a \in A$. *If inequality* (4.5) *is fulfilled, then* $A \trianglelefteq^p B$.

(iii) If $y \in Y$ *satisfies* $\sup_{b\in B}\rho(b,y) \geq 0$, *then* $y \in \operatorname{cl}\bigcup_{b\in B}[\cdot \leq_K^2 b]$.

(iv) Suppose that the set $\bigcup_{b\in B}[\cdot \leq_K^2 b]$ *is closed. If inequality* (4.5) *is true, then* $A \trianglelefteq^p B$.

Proof. Define $\Psi(y,z) := \rho(z,y)$, for all $y, z \in Y$, and repeat the same arguments as in the proofs of Theorems 4.2 and 4.3. □

Theorem 4.5. *Consider* $(\mathscr{E}, \preceq) = (Y, \leq_K^2)$, $A, B \in \mathscr{P}_0(Y)$ *and* $\rho \in \{\mu, \Theta\}$.

(i) If $A \trianglelefteq^o B$, *then*

$$\inf_{b\in B}\sup_{a\in A}\rho(b,a) \geq 0. \tag{4.6}$$

(ii) Assume that $\sup_{a\in A}\rho(b,a)$ *is attained, for all* $b \in B$. *If inequality* (4.6) *is fulfilled, then* $A \trianglelefteq^o B$.

(iii) If $y \in Y$ *satisfies* $\sup_{a\in A}\rho(y,a) \geq 0$, *then* $0 \in \operatorname{cl}(A - [\cdot \leq_K^2 y])$.

(iv) Suppose that the set $A - [\cdot \leq_K^2 b]$ *is closed, for all* $b \in B$. *If inequality* (4.6) *is true, then* $A \trianglelefteq^o B$.

Proof. Define $\Psi(y,z) := -\rho(z,y)$, for all $y, z \in Y$, and repeat the same arguments as in the proofs of Theorems 4.2 and 4.3. □

Remark 4.2. (i) It was stated in Proposition 3.1(i) that binary relation $\trianglelefteq^p$ (resp. $\trianglelefteq^o$) correspond to the domination structure $\precsim_K^{i,u}$ (resp. $\precsim_K^{i,l}$), provided that $(\mathscr{E}, \preceq) = (Y, \leq_K^i)$. Therefore, Theorems 4.2 and 4.3 (resp. Theorems 4.4 and 4.5) characterize the domination structures $\precsim_K^{1,u}$ and $\precsim_K^{1,l}$ (resp. $\precsim_K^{2,u}$ and $\precsim_K^{2,l}$) and extend some results in the literature. For instance, Theorem 4.2 encompasses [30, Theorem 3.3], Theorem 4.3 encompasses [1, Theorem 3.5], [6, Lemma 3(iii)] and [30, Theorem 3.8], [31, Theorem 5] and [42, Proposition 3.5], and Theorem 4.4 encompasses [1, Theorem 3.9], [6, Lemma 3(iv)], and [31, Theorem 8].

(ii) In order to state the relationships $A \trianglelefteq^o B$ and $A \trianglelefteq^p B$, Theorems 4.2 and 4.3-4.5 consider "attainment" or "closedness" assumptions, which are independent. For instance, in [9, Example 4] two sets A, B fulfill $A \trianglelefteq^p B$ but the "attainment" hypothesis is not satisfied. Specifically, let $Y = \mathbb{R}^2$, $K(y) = K := \operatorname{cone}\{(1,0)\}$ for all $y \in \mathbb{R}^2$, $A = \operatorname{cone}\{(-1,0)\}$ and $B = K$. Clearly, $A \subset B - K$ and $A \trianglelefteq^p B$. Notice that $B - [a \leq_K^1 \cdot] = \mathbb{R} \times \{0\}$, for all $a \in A$, and it is a closed set. However, for any bounded function $q = (q_1, q_2) : \mathbb{R}^2 \to K\backslash\{(0,0)\}$, $a = (a_1, a_2) \in A$ and

$b=(b_1,b_2)\in B$, we have $\lambda(a,b)=\dfrac{a_1-b_1}{q_1(a)}$ and so, for each $a\in A$, $\inf\limits_{b\in B}\lambda(a,b)=-\infty$. Hence, part (iv) of Theorem 4.2 can be applied to obtain the relationship $A\trianglelefteq^p B$, whereas part (ii) cannot be applied.

In the same setting as above, consider $K=\operatorname{cone}\{(-1,0)\}$, $q(y)=(-1,0)$ for all $y\in\mathbb{R}^2$, $B=\{(t,e^t):t\leq 0\}$ and $A=\{0\}\times(0,1]$. It follows that

$$(0,t)=(\ln t,t)-(\ln t,0)\in B-K, \text{ for all } t\in(0,1].$$

Hence, $A\subset B-K$ and $A\trianglelefteq^p B$. Notice that

$$\lambda(a,b)=\begin{cases} b_1-a_1 & \text{if} \quad a_2=b_2\\ +\infty & \text{if} \quad a_2\neq b_2\end{cases}$$

and it follows that

$$\inf_{b\in B}\lambda(a,b)=\lambda(a,(\ln a_2,a_2))=\ln a_2-a_1=\ln a_2, \text{ for all } a\in A.$$

However, $B-[a\leq_K^1\ \cdot]=-a+B-K$, and this set is not closed, for all $a\in A$. Thus, part (ii) of Theorem 4.3 can be applied to obtain the relationship $A\trianglelefteq^p B$, whereas part (iv) cannot be applied.

(iii) In [7, Remark 4.1], [20, Lemma 3.2], [21, Proposition 4.2], [22, Proposition 4.4], [24, Propostition 3] and [30, Remark 1], several conditions have been stated to check the attainment and closedness assumptions of Theorems 4.2 and 4.3-4.5 in different settings.

In the sequel, characterizations by scalarization of the binary relations $\trianglelefteq^o$ and $\trianglelefteq^p$ in the spaces $(\mathscr{P}_0(Y),\precsim_K^{i,j})$, $i\in\{1,2\}$, $j\in\{l,u\}$ are stated.

Theorem 4.6. *Consider* $(\mathscr{E},\preceq)=(\mathscr{P}_0(Y),\precsim_K^{1,l})$, $\mathscr{A},\mathscr{B}\in\mathscr{P}_0(\mathscr{P}_0(Y))$ *and* $\rho\in\{\lambda,\Delta\}$. *Suppose that* $\sup\limits_{a_2\in A}\inf\limits_{a_1\in A}\rho(a_1,a_2)=0$, *for all* $A\in\mathscr{A}$.

(i) If $\mathscr{A}\trianglelefteq^p\mathscr{B}$, *then*

$$\sup_{A\in\mathscr{A}}\inf_{B\in\mathscr{B}}\sup_{b\in B}\inf_{a\in A}\rho(a,b)\leq 0. \tag{4.7}$$

(ii) Assume that $\inf\limits_{B\in\mathscr{B}}\sup\limits_{b\in B}\inf\limits_{a\in A}\rho(a,b)$ *is attained in* $\mathscr{B}$, *for all* $A\in\mathscr{A}$, *and* $\inf\limits_{a\in A}\rho(a,b)$ *is attained or* $\bigcup\limits_{a\in A}[a\leq_K^1\cdot]$ *is closed, for all* $b\in\bigcup\limits_{B\in\mathscr{B}}B$ *and* $A\in\mathscr{A}$. *If inequality (4.7) is fulfilled, then* $\mathscr{A}\trianglelefteq^p\mathscr{B}$.

(iii) Consider $M\in\mathscr{P}_0(Y)$. *If*

$$\inf_{B\in\mathscr{B}}\sup_{b\in B}\inf_{y\in M}\lambda(y,b)\leq 0 \tag{4.8}$$

and the function q is constant in M (for instance, if M is a singleton), then

$$0\in\operatorname{cl}\left(\bigcup_{B\in\mathscr{B}}\bigcap_{b\in B}\left(b-\bigcup_{y\in M}[y\leq_K^1\cdot]\right)\right). \tag{4.9}$$

(iv) Suppose that the set $\bigcup\limits_{B\in\mathscr{B}}\bigcap\limits_{b\in B}\left(b-\bigcup\limits_{a\in A}[a\leq_K^1\cdot]\right)$ *is closed and the function q is constant in A, for all* $A\in\mathscr{A}$. *If inequality (4.7) is true with* $\rho=\lambda$, *then* $\mathscr{A}\trianglelefteq^p\mathscr{B}$.

(v) Consider $M \in \mathscr{P}_0(Y)$. If $\inf_{B\in\mathscr{B}} \sup_{b\in B} \inf_{y\in M} \Delta(y,b) \leq 0$, then, for each $\varepsilon > 0$, there exists $B \in \mathscr{B}$ such that $B \subset \bigcup_{y\in M}[y \leq_K^1 \cdot] + \varepsilon\mathbb{B}$.

(vi) Suppose that $\bigcup_{a\in A}[a \leq_K^1 \cdot]$ is closed, for all $A \in \mathscr{A}$, and the family $\mathscr{B}$ is finite. If inequality (4.7) is true with $\rho = \Delta$, then $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

Proof. (i) Define $\Psi : \mathscr{P}_0(Y) \times \mathscr{P}_0(Y) \to \overline{\mathbb{R}}$ as follows:

$$\Psi(A,B) := -\sup_{b\in B} \inf_{a\in A} \rho(a,b), \text{ for all } A,B \in \mathscr{P}_0(Y).$$

We claim that $\Psi(A,\cdot) : \mathscr{P}_0(Y) \to \overline{\mathbb{R}}$ is upper $\precsim_K^{1,l}$-monotone at A, for all $A \in \mathscr{P}_0(Y)$. Indeed, suppose that $A,B \in \mathscr{P}_0(Y)$ satisfy $A \precsim_K^{1,l} B$. By Proposition 3.1(i), we see that $A \trianglelefteq_1^o B$, where $\trianglelefteq_1^o$ denotes the optimistic binary relation $\trianglelefteq^o$ corresponding to $(\mathscr{E},\preceq) = (Y,\leq_K^1)$. Hence, by inequality (4.4) and the assumption, we have that

$$-\sup_{b\in B} \inf_{a\in A} \rho(a,b) \geq 0 = -\sup_{a_2\in A} \inf_{a_1\in A} \rho(a_1,a_2).$$

Thus, the assertion is proved and the result follows by Theorem 4.1(iv).

(ii) We claim that $A \in \mathscr{B}^u_{\Psi(A,\cdot)}$, for all $A \in \mathscr{A}$. Indeed, let $A \in \mathscr{A}$ and $B \in \mathscr{B}$ satisfy $\Psi(A,A) \leq \Psi(A,B)$. By the assumptions and the definition of Ψ, we obtain $\sup_{b\in B} \inf_{a\in A} \rho(a,b) \leq 0$. In view of the the parts (ii) and (iv) of Theorem 4.3 and Proposition 3.1(i), we deduce that $A \precsim_K^{1,l} B$ and the assertion $A \in \mathscr{B}^u_{\Psi(A,\cdot)}$ is stated. Hence, part (ii) is a consequence of Theorem 4.1(iii).

(iii) Suppose that inequality (4.8) is satisfied. For each $\varepsilon > 0$, there exists $B_\varepsilon \in \mathscr{B}$ such that $\inf_{y\in M} \lambda(y,b) \leq \varepsilon/2$, for all $b \in B_\varepsilon$. Fix a point $b_\varepsilon \in B_\varepsilon$. There exists $y_{b_\varepsilon} \in M$ fulfilling $\lambda(y_{b_\varepsilon}, b_\varepsilon) \leq \varepsilon$. If $q(y) = q$, for all $y \in M$, by parts (iii) and (v) of Proposition 2.1, we obtain that $b_\varepsilon + \varepsilon q \in y_{b_\varepsilon} + K(y_{b_\varepsilon})$. Hence,

$$-\varepsilon q \in b_\varepsilon - (y_{b_\varepsilon} + K(y_{b_\varepsilon})) = b_\varepsilon - [y_{b_\varepsilon} \leq_K^1 \cdot] \subset b_\varepsilon - \bigcup_{y\in M} [y \leq_K^1 \cdot]$$

and $-\varepsilon q \in \bigcap_{b\in B_\varepsilon}(b - \bigcup_{y\in M}[y \leq_K^1 \cdot])$. Therefore, given a neighborhood V of zero in Y, there exists $\varepsilon > 0$ such that $-\varepsilon q \in V$ and $V \cap \left(\bigcap_{b\in B_\varepsilon}(b - \bigcup_{y\in M}[y \leq_K^1 \cdot])\right) \neq \emptyset$, which implies assertion (4.9).

(iv) By part (iii), we see that for each set $A \in \mathscr{A}$ there exists $B \in \mathscr{B}$ such that

$$0 \in \bigcap_{b\in B} \left(b - \bigcup_{a\in A} [a \leq_K^1 \cdot] \right).$$

Thus, for all $b \in B$ there exists $a \in A$ fulfilling $a \leq_K^1 b$, i.e., $A \precsim_K^{1,l} B$. Therefore, $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

(v) By the same arguments as in part (iii) but replacing ε with $\varepsilon/2$, we obtain that for each $\varepsilon > 0$ there exists $B_\varepsilon \in \mathscr{B}$ such that for all $b_\varepsilon \in B_\varepsilon$ there exists $y_{b_\varepsilon} \in M$ satisfying $\Delta(y_{b_\varepsilon}, b_\varepsilon) \leq \varepsilon/2$. If $b_\varepsilon \notin y_{b_\varepsilon} + K(y_{b_\varepsilon})$, then

$$d(y_{b_\varepsilon} - b_\varepsilon, -K(y_{b_\varepsilon})) = \delta_{-K(y_{b_\varepsilon})}(y_{b_\varepsilon} - b_\varepsilon) = \Delta(y_{b_\varepsilon}, b_\varepsilon) \leq \varepsilon/2.$$

Hence,

$$b_\varepsilon \in y_{b_\varepsilon} + K(y_{b_\varepsilon}) + \varepsilon\mathbb{B} \subset \bigcup_{y\in M} [y \leq_K^1 \cdot] + \varepsilon\mathbb{B}.$$

Clearly, the above statement is also true if $b_\varepsilon \in y_{b_\varepsilon} + K(y_{b_\varepsilon})$ and part (v) is stated.

(vi) Consider a set $A \in \mathscr{A}$. By applying part (v) to $M = A$ and $\varepsilon_n = 1/n$, we obtain that there exists $B_n \in \mathscr{B}$ such that $B_n \subset \bigcup_{a\in A}[a \leq_K^1 \cdot] + (1/n)\mathbb{B}$. Since the family $\mathscr{B}$ is finite, we deduce that there exists $B \in \mathscr{B}$ such that

$$B \subset \operatorname{cl}\left(\bigcup_{a\in A}[a \leq_K^1 \cdot]\right) = \bigcup_{a\in A}[a \leq_K^1 \cdot].$$

Therefore, there exists $B \in \mathscr{B}$ such that $A \precsim_K^{1,l} B$ and it is proved that $\mathscr{A} \trianglelefteq^p \mathscr{B}$. □

The proofs of the following theorems follow the same assertions as the previous one. Thus, they are omitted and we only provide the scalarization function that allows to obtain the corresponding results.

Theorem 4.7. *Consider* $(\mathscr{E}, \preceq) = (\mathscr{P}_0(Y), \precsim_K^{1,l})$, $\mathscr{A}, \mathscr{B} \in \mathscr{P}_0(\mathscr{P}_0(Y))$ *and* $\rho \in \{\lambda, \Delta\}$. *Suppose that* $\sup_{b_2\in B}\inf_{b_1\in B}\rho(b_1,b_2) = 0$, *for all* $B \in \mathscr{B}$.

(i) If $\mathscr{A} \trianglelefteq^o \mathscr{B}$, *then*

$$\sup_{B\in\mathscr{B}}\inf_{A\in\mathscr{A}}\sup_{b\in B}\inf_{a\in A}\rho(a,b) \leq 0. \tag{4.10}$$

(ii) Assume that $\inf_{A\in\mathscr{A}}\sup_{b\in B}\inf_{a\in A}\rho(a,b)$ *is attained in* $\mathscr{A}$ *for all* $B \in \mathscr{B}$ *and* $\inf_{a\in A}\rho(a,b)$ *is attained or* $\bigcup_{a\in A}[a \leq_K^1 \cdot]$ *is closed, for all* $A \in \mathscr{A}$ *and* $b \in \bigcup_{B\in\mathscr{B}} B$. *If inequality* (4.10) *is fulfilled, then* $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

(iii) Consider $M \in \mathscr{P}_0(Y)$. *If* $\inf_{A\in\mathscr{A}}\sup_{y\in M}\inf_{a\in A}\lambda(a,y) \leq 0$ *and the function* q *is constant in* A *(for instance, if* A *is a singleton), for all* $A \in \mathscr{A}$, *then*

$$0 \in \operatorname{cl}\left(\bigcup_{A\in\mathscr{A}}\bigcap_{y\in M}\left(y - \bigcup_{a\in A}[a \leq_K^1 \cdot]\right)\right).$$

(iv) Suppose that the set $\bigcup_{A\in\mathscr{A}}\bigcap_{b\in B}\left(b - \bigcup_{a\in A}[a \leq_K^1 \cdot]\right)$ *is closed, for all* $B \in \mathscr{B}$, *and the function* q *is constant in* A, *for all* $A \in \mathscr{A}$. *If inequality* (4.10) *is true with* $\rho = \lambda$, *then* $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

(v) Consider $M \in \mathscr{P}_0(Y)$. *If* $\inf_{A\in\mathscr{A}}\sup_{y\in M}\inf_{a\in A}\Delta(a,y) \leq 0$, *then for each* $\varepsilon > 0$ *there exists* $A \in \mathscr{A}$ *such that*

$$M \subset \bigcup_{a\in A}[a \leq_K^1 \cdot] + \varepsilon\mathbb{B}.$$

(vi) Suppose that $\bigcup_{a\in A}[a \leq_K^1 \cdot]$ *is closed, for all* $A \in \mathscr{A}$, *and the family* $\mathscr{A}$ *is finite. If inequality* (4.10) *is true with* $\rho = \Delta$, *then* $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

Proof. Define $\Psi : \mathscr{P}_0(Y) \times \mathscr{P}_0(Y) \to \overline{\mathbb{R}}$ as follows:

$$\Psi(A,B) := \sup_{b\in B}\inf_{a\in A}\rho(a,b), \text{ for all } A,B \in \mathscr{P}_0(Y),$$

and proceed as the proof of Theorem 4.6. □

Theorem 4.8. *Consider* $(\mathscr{E}, \preceq) = (\mathscr{P}_0(Y), \precsim_K^{2,l})$, $\mathscr{A}, \mathscr{B} \in \mathscr{P}_0(\mathscr{P}_0(Y))$ *and* $\rho \in \{\mu, \Theta\}$. *Suppose that* $\inf_{a_1\in A}\sup_{a_2\in A}\rho(a_1,a_2) = 0$, *for all* $A \in \mathscr{A}$.

(i) If $\mathscr{A} \trianglelefteq^p \mathscr{B}$, *then*

$$\inf_{A\in\mathscr{A}} \sup_{B\in\mathscr{B}} \inf_{b\in B} \sup_{a\in A} \rho(b,a) \geq 0. \tag{4.11}$$

(ii) Assume that $\sup_{B\in\mathscr{B}} \inf_{b\in B} \sup_{a\in A} \rho(b,a)$ *is attained in* $\mathscr{B}$ *for all* $A \in \mathscr{A}$ *and* $\sup_{a\in A} \rho(b,a)$ *is attained or* $A - [\cdot \leq_K^2 b]$ *is closed for all* $A \in \mathscr{A}$ *and* $b \in \bigcup_{B\in\mathscr{B}} B$. *If inequality* (4.11) *is fulfilled, then* $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

(iii) Consider $M \in \mathscr{P}_0(Y)$. *If* $\sup_{B\in\mathscr{B}} \inf_{b\in B} \sup_{y\in M} \mu(b,y) \geq 0$ *and the function* q *is constant on* B *(for instance, if* B *is a singleton), for all* $B \in \mathscr{B}$, *then*

$$0 \in \operatorname{cl}\left(\bigcup_{B\in\mathscr{B}} \bigcap_{b\in B} \left(M - [\cdot \leq_K^2 b]\right)\right).$$

(iv) Suppose that the set $\bigcup_{B\in\mathscr{B}} \bigcap_{b\in B} \left(A - [\cdot \leq_K^2 b]\right)$ *is closed, for all* $A \in \mathscr{A}$, *and the function* q *is constant in* B, *for all* $B \in \mathscr{B}$. *If inequality* (4.11) *is true with* $\rho = \mu$, *then* $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

(v) Consider $M \in \mathscr{P}_0(Y)$. *If* $\sup_{B\in\mathscr{B}} \inf_{b\in B} \sup_{y\in M} \Theta(b,y) \geq 0$, *then for each* $\varepsilon > 0$ *there exists* $B \in \mathscr{B}$ *such that* $(M - [\cdot \leq_K^2 b]) \cap \varepsilon\mathbb{B} \neq \emptyset$, *for all* $b \in B$.

(vi) Suppose that $A - [\cdot \leq_K^2 b]$ *is closed, for all* $A \in \mathscr{A}$ *and* $b \in \bigcup_{B\in\mathscr{B}} B$, *and the family* $\mathscr{B}$ *is finite. If inequality* (4.11) *is true with* $\rho = \Theta$, *then* $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

Proof. Define $\Psi : \mathscr{P}_0(Y) \times \mathscr{P}_0(Y) \to \overline{\mathbb{R}}$ as follows:

$$\Psi(A,B) := \inf_{b\in B} \sup_{a\in A} \rho(b,a), \text{ for all } A,B \in \mathscr{P}_0(Y),$$

and proceed as the proof of Theorem 4.6. □

Theorem 4.9. *Consider* $(\mathscr{E},\preceq) = (\mathscr{P}_0(Y), \precsim_K^{2,l})$, $\mathscr{A},\mathscr{B} \in \mathscr{P}_0(\mathscr{P}_0(Y))$ *and* $\rho \in \{\mu,\Theta\}$. *Suppose that* $\inf_{b_1\in B} \sup_{b_2\in B} \rho(b_1,b_2) = 0$, *for all* $B \in \mathscr{B}$.

(i) If $\mathscr{A} \trianglelefteq^o \mathscr{B}$, *then*

$$\inf_{B\in\mathscr{B}} \sup_{A\in\mathscr{A}} \inf_{b\in B} \sup_{a\in A} \rho(b,a) \geq 0. \tag{4.12}$$

(ii) Assume that $\sup_{A\in\mathscr{A}} \inf_{b\in B} \sup_{a\in A} \rho(b,a)$ *is attained in* $\mathscr{A}$ *for all* $B \in \mathscr{B}$ *and* $\sup_{a\in A} \rho(b,a)$ *is attained or* $A - [\cdot \leq_K^2 b]$ *is closed, for all* $A \in \mathscr{A}$ *and* $b \in \bigcup_{B\in\mathscr{B}} B$. *If inequality* (4.12) *is fulfilled, then* $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

(iii) Consider $M \in \mathscr{P}_0(Y)$. *If* $\sup_{A\in\mathscr{A}} \inf_{y\in M} \sup_{a\in A} \mu(y,a) \geq 0$ *and the function* q *is constant on* M *(for instance, if* M *is a singleton), then*

$$0 \in \operatorname{cl}\left(\bigcup_{A\in\mathscr{A}} \bigcap_{y\in M} \left(A - [\cdot \leq_K^2 y]\right)\right).$$

(iv) Suppose that the set $\bigcup_{A\in\mathscr{A}} \bigcap_{b\in B} \left(A - [\cdot \leq_K^2 b]\right)$ *is closed, for all* $B \in \mathscr{B}$, *and the function* q *is constant in* B, *for all* $B \in \mathscr{B}$. *If inequality* (4.12) *is true with* $\rho = \mu$, *then* $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

(v) Consider $M \in \mathscr{P}_0(Y)$. *If* $\sup_{A\in\mathscr{A}} \inf_{y\in M} \sup_{a\in A} \Theta(y,a) \geq 0$, *then for each* $\varepsilon > 0$ *there exists* $A \in \mathscr{A}$ *such that* $(A - [\cdot \leq_K^2 y]) \cap \varepsilon\mathbb{B} \neq \emptyset$, *for all* $y \in M$.

(vi) Suppose that $A-[\cdot \leq_K^2 b]$ *is closed, for all* $A \in \mathscr{A}$ *and* $b \in \bigcup_{B\in\mathscr{B}} B$, *and the family* $\mathscr{A}$ *is finite. If inequality* (4.12) *is true with* $\rho = \Theta$, *then* $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

Proof. Define $\Psi : \mathscr{P}_0(Y) \times \mathscr{P}_0(Y) \to \overline{\mathbb{R}}$ as follows:

$$\Psi(A,B) := -\inf_{b\in B}\sup_{a\in A}\rho(b,a), \text{ for all } A,B \in \mathscr{P}_0(Y),$$

and proceed as the proof of Theorem 4.6. □

Theorem 4.10. *Consider* $(\mathscr{E},\preceq) = (\mathscr{P}_0(Y), \precsim_K^{1,u})$, $\mathscr{A},\mathscr{B} \in \mathscr{P}_0(\mathscr{P}_0(Y))$ *and* $\rho \in \{\lambda,\Delta\}$. *Suppose that* $\sup_{a_1\in A}\inf_{a_2\in A}\rho(a_1,a_2) = 0$, *for all* $A \in \mathscr{A}$.

(i) If $\mathscr{A} \trianglelefteq^p \mathscr{B}$, *then*

$$\sup_{A\in\mathscr{A}}\inf_{B\in\mathscr{B}}\sup_{a\in A}\inf_{b\in B}\rho(a,b) \leq 0. \tag{4.13}$$

(ii) Assume that $\inf_{B\in\mathscr{B}}\sup_{a\in A}\inf_{b\in B}\rho(a,b)$ *is attained in* $\mathscr{B}$ *for all* $A \in \mathscr{A}$ *and* $\inf_{b\in B}\rho(a,b)$ *is attained or* $B-[a \leq_K^1 \cdot]$ *is closed, for all* $B \in \mathscr{B}$ *and* $a \in \bigcup_{A\in\mathscr{A}} A$. *If inequality* (4.13) *is fulfilled, then* $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

(iii) Consider $M \in \mathscr{P}_0(Y)$. *If* $\inf_{B\in\mathscr{B}}\sup_{y\in M}\inf_{b\in B}\lambda(y,b) \leq 0$ *and the function* q *is constant in* M *(for instance, if* M *is a singleton), then*

$$0 \in \operatorname{cl}\left(\bigcup_{B\in\mathscr{B}}\bigcap_{y\in M}\left(B-[y\leq_K^1\cdot]\right)\right).$$

(iv) Suppose that the set $\bigcup_{B\in\mathscr{B}}\bigcap_{a\in A}\left(B-[a\leq_K^1\cdot]\right)$ *is closed and the function* q *is constant in* A, *for all* $A \in \mathscr{A}$. *If inequality* (4.13) *is true with* $\rho = \lambda$, *then* $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

(v) Consider $M \in \mathscr{P}_0(Y)$. *If* $\inf_{B\in\mathscr{B}}\sup_{y\in M}\inf_{b\in B}\Delta(y,b) \leq 0$, *then for each* $\varepsilon > 0$ *there exists* $B \in \mathscr{B}$ *such that* $(B-[y\leq_K^1\cdot])\cap\varepsilon\mathbb{B} \neq \emptyset$, *for all* $y \in M$.

(vi) Suppose that $B-[a\leq_K^1\cdot]$ *is closed, for all* $B \in \mathscr{B}$ *and* $a \in \bigcup_{A\in\mathscr{A}} A$, *and the family* $\mathscr{B}$ *is finite. If inequality* (4.13) *is true with* $\rho = \Delta$, *then* $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

Proof. Define $\Psi : \mathscr{P}_0(Y) \times \mathscr{P}_0(Y) \to \overline{\mathbb{R}}$ as follows:

$$\Psi(A,B) := -\sup_{a\in A}\inf_{b\in B}\rho(a,b), \text{ for all } A,B \in \mathscr{P}_0(Y),$$

and proceed as the proof of Theorem 4.6. □

Theorem 4.11. *Consider* $(\mathscr{E},\preceq) = (\mathscr{P}_0(Y), \precsim_K^{1,u})$, $\mathscr{A},\mathscr{B} \in \mathscr{P}_0(\mathscr{P}_0(Y))$ *and* $\rho \in \{\lambda,\Delta\}$. *Suppose that* $\sup_{b_1\in B}\inf_{b_2\in B}\rho(b_1,b_2) = 0$, *for all* $B \in \mathscr{B}$.

(i) If $\mathscr{A} \trianglelefteq^o \mathscr{B}$, *then*

$$\sup_{B\in\mathscr{B}}\inf_{A\in\mathscr{A}}\sup_{a\in A}\inf_{b\in B}\rho(a,b) \leq 0. \tag{4.14}$$

(ii) Assume that $\inf_{A\in\mathscr{A}}\sup_{a\in A}\inf_{b\in B}\rho(a,b)$ *is attained in* $\mathscr{A}$ *for all* $B \in \mathscr{B}$ *and* $\inf_{b\in B}\rho(a,b)$ *is attained or* $B-[a\leq_K^1\cdot]$ *is closed, for all* $B \in \mathscr{B}$ *and* $a \in \bigcup_{A\in\mathscr{A}} A$. *If inequality* (4.14) *is fulfilled, then* $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

(iii) Consider $M \in \mathscr{P}_0(Y)$. If $\inf_{A\in\mathscr{A}} \sup_{a\in A} \inf_{y\in M} \lambda(a,y) \leq 0$ and the function q is constant in A (for instance, if A is a singleton), for all $A \in \mathscr{A}$, then

$$0 \in \mathrm{cl}\left(\bigcup_{A\in\mathscr{A}} \bigcap_{a\in A} \left(M - [a \leq_K^1 \cdot]\right)\right).$$

(iv) Suppose that the set $\bigcup_{A\in\mathscr{A}} \bigcap_{a\in A} \left(B - [a \leq_K^1 \cdot]\right)$ is closed, for all $B \in \mathscr{B}$, and the function q is constant in A, for all $A \in \mathscr{A}$. If inequality (4.14) is true with $\rho = \lambda$, then $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

(v) Consider $M \in \mathscr{P}_0(Y)$. If $\inf_{A\in\mathscr{A}} \sup_{a\in A} \inf_{y\in M} \Delta(a,y) \leq 0$, then for each $\varepsilon > 0$ there exists $A \in \mathscr{A}$ such that $(M - [a \leq_K^1 \cdot]) \cap \varepsilon\mathbb{B} \neq \emptyset$, for all $a \in A$.

(vi) Suppose that $B - [a \leq_K^1 \cdot]$ is closed, for all $B \in \mathscr{B}$ and $a \in \bigcup_{A\in\mathscr{A}} A$, and the family $\mathscr{A}$ is finite. If inequality (4.14) is true with $\rho = \Delta$, then $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

Proof. Define $\Psi : \mathscr{P}_0(Y) \times \mathscr{P}_0(Y) \to \overline{\mathbb{R}}$ as follows:

$$\Psi(A,B) := \sup_{a\in A} \inf_{b\in B} \rho(a,b), \text{ for all } A,B \in \mathscr{P}_0(Y),$$

and proceed as the proof of Theorem 4.6.

□

Theorem 4.12. *Consider $(\mathscr{E},\preceq) = (\mathscr{P}_0(Y), \precsim_K^{2,u})$, $\mathscr{A},\mathscr{B} \in \mathscr{P}_0(\mathscr{P}_0(Y))$ and $\rho \in \{\mu,\Theta\}$. Suppose that $\inf_{a_2\in A} \sup_{a_1\in A} \rho(a_1,a_2) = 0$, for all $A \in \mathscr{A}$.*

(i) If $\mathscr{A} \trianglelefteq^p \mathscr{B}$, then

$$\inf_{A\in\mathscr{A}} \sup_{B\in\mathscr{B}} \inf_{a\in A} \sup_{b\in B} \rho(b,a) \geq 0. \tag{4.15}$$

(ii) Assume that $\sup_{B\in\mathscr{B}} \inf_{a\in A} \sup_{b\in B} \rho(b,a)$ is attained in $\mathscr{B}$ for all $A \in \mathscr{A}$ and $\sup_{b\in B} \rho(b,a)$ is attained or $\bigcup_{b\in B} [\cdot \leq_K^2 b]$ is closed, for all $B \in \mathscr{B}$ and $a \in \bigcup_{A\in\mathscr{A}} A$. If inequality (4.15) is fulfilled, then $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

(iii) Consider $M \in \mathscr{P}_0(Y)$. If $\sup_{B\in\mathscr{B}} \inf_{y\in M} \sup_{b\in B} \mu(b,y) \geq 0$ and the function q is constant in B (for instance, if B is a singleton), for all $B \in \mathscr{B}$, then

$$0 \in \mathrm{cl}\left(\bigcup_{B\in\mathscr{B}} \bigcap_{y\in M} \left(y - \bigcup_{b\in B} [\cdot \leq_K^2 b]\right)\right).$$

(iv) Suppose that the set $\bigcup_{B\in\mathscr{B}} \bigcap_{a\in A} \left(a - \bigcup_{b\in B} [\cdot \leq_K^2 b]\right)$ is closed, for all $A \in \mathscr{A}$, and the function q is constant in B, for all $B \in \mathscr{B}$. If inequality (4.15) is true with $\rho = \mu$, then $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

(v) Consider $M \in \mathscr{P}_0(Y)$. If $\sup_{B\in\mathscr{B}} \inf_{y\in M} \sup_{b\in B} \Theta(b,y) \geq 0$, then for each $\varepsilon > 0$ there exists $B \in \mathscr{B}$ such that

$$M \subset \bigcup_{b\in B} [\cdot \leq_K^2 b] + \varepsilon\mathbb{B}.$$

(vi) Suppose that $\bigcup_{b\in B} [\cdot \leq_K^2 b]$ is closed, for all $B \in \mathscr{B}$, and the family $\mathscr{B}$ is finite. If inequality (4.15) is true with $\rho = \Theta$, then $\mathscr{A} \trianglelefteq^p \mathscr{B}$.

Proof. Define $\Psi : \mathscr{P}_0(Y) \times \mathscr{P}_0(Y) \to \overline{\mathbb{R}}$ as follows:

$$\Psi(A,B) := \inf_{a\in A} \sup_{b\in B} \rho(b,a), \text{ for all } A,B \in \mathscr{P}_0(Y),$$

and proceed as the proof of Theorem 4.6. □

Theorem 4.13. *Consider* $(\mathscr{E},\preceq) = (\mathscr{P}_0(Y), \precsim_K^{2,u})$, $\mathscr{A},\mathscr{B} \in \mathscr{P}_0(\mathscr{P}_0(Y))$ *and* $\rho \in \{\mu,\Theta\}$. *Suppose that* $\inf\limits_{b_2\in B} \sup\limits_{b_1\in B} \rho(b_1,b_2) = 0$, *for all* $B \in \mathscr{B}$.

(i) If $\mathscr{A} \trianglelefteq^o \mathscr{B}$, *then*

$$\inf_{B\in\mathscr{B}} \sup_{A\in\mathscr{A}} \inf_{a\in A} \sup_{b\in B} \rho(b,a) \geq 0. \tag{4.16}$$

(ii) Assume that $\sup\limits_{A\in\mathscr{A}} \inf\limits_{a\in A} \sup\limits_{b\in B} \rho(b,a)$ *is attained in* $\mathscr{A}$ *for all* $B \in \mathscr{B}$ *and* $\sup\limits_{b\in B} \rho(b,a)$ *is attained or* $\bigcup\limits_{b\in B} [\cdot \leq_K^2 b]$ *is closed, for all* $B \in \mathscr{B}$ *and* $a \in \bigcup\limits_{A\in\mathscr{A}} A$. *If inequality (4.16) is fulfilled, then* $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

(iii) Consider $M \in \mathscr{P}_0(Y)$. *If* $\sup_{A\in\mathscr{A}} \inf_{a\in A} \sup_{y\in M} \mu(y,a) \geq 0$ *and the function* q *is constant in* M *(for instance, if* M *is a singleton), then*

$$0 \in \operatorname{cl}\left(\bigcup_{A\in\mathscr{A}} \bigcap_{a\in A} \left(a - \bigcup_{y\in M} [\cdot \leq_K^2 y] \right) \right).$$

(iv) Suppose that the set $\bigcup\limits_{A\in\mathscr{A}} \bigcap\limits_{a\in A} \left(a - \bigcup\limits_{b\in B} [\cdot \leq_K^2 b] \right)$ *is closed and the function* q *is constant in* B, *for all* $B \in \mathscr{B}$. *If inequality (4.16) is true with* $\rho = \mu$, *then* $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

(v) Consider $M \in \mathscr{P}_0(Y)$. *If* $\sup_{A\in\mathscr{A}} \inf_{a\in A} \sup_{y\in M} \Theta(y,a) \geq 0$, *then for each* $\varepsilon > 0$ *there exists* $A \in \mathscr{A}$ *such that*

$$A \subset \left(\bigcup_{y\in M} [\cdot \leq_K^2 y] \right) + \varepsilon\mathbb{B}.$$

(vi) Suppose that $\bigcup\limits_{b\in B} [\cdot \leq_K^2 b]$ *is closed, for all* $B \in \mathscr{B}$, *and the family* $\mathscr{A}$ *is finite. If inequality (4.16) is true with* $\rho = \Theta$, *then* $\mathscr{A} \trianglelefteq^o \mathscr{B}$.

Proof. Define $\Psi : \mathscr{P}_0(Y) \times \mathscr{P}_0(Y) \to \overline{\mathbb{R}}$ as follows:

$$\Psi(A,B) := -\inf_{a\in A} \sup_{b\in B} \rho(b,a), \text{ for all } A,B \in \mathscr{P}_0(Y),$$

and proceed as the proof of Theorem 4.6. □

Remark 4.3. Notice that Theorems 4.6-4.13 consider one of the following assumptions: given $A \in \mathscr{P}_0(Y)$,

$$\sup_{a_1\in A} \inf_{a_2\in A} \rho(a_1,a_2) = 0, \rho \in \{\lambda,\Delta\},$$
$$\sup_{a_2\in A} \inf_{a_1\in A} \rho(a_1,a_2) = 0, \rho \in \{\lambda,\Delta\},$$
$$\inf_{a_1\in A} \sup_{a_2\in A} \rho(a_1,a_2) = 0, \rho \in \{\mu,\Theta\},$$
$$\inf_{a_2\in A} \sup_{a_1\in A} \rho(a_1,a_2) = 0, \rho \in \{\mu,\Theta\}.$$

There exist in the literature several results to check them in different settings, for instance, [7, Lemma 2.2(a)], [8, Proposition 2.5(v)], [18, Propositions 4.2(e), 4.5(b), 5.4(b) and 6.6(a)], [20, Lemma 3.4(ii)], [23, Theorem 3.10(i)] and [24, Remark 9(b)].

Papers concerning robust solutions of uncertain set optimization problems are very recent (see [6, 7, 9, 39]). They are based on a robust counterpart introduced in [39] whose image at each feasible point is the union of all objective values along all scenarios. This approach seems unsuitable since the final space of the robust counterpart does not allow addressing with the basic notions of pessimistic and optimistic choice of all objective values along all scenarios (see also Remark 3.1).

For instance, paper [7] addresses the characterization by scalarization of binary relations on union sets (see Remark 3.1). It has several mathematical issues since the elements in $\mathscr{P}(\mathscr{P}(Y))$ (i.e., families of subsets of Y) are compared through quasiorders in $\mathscr{P}(Y)$. All these issues are a consequence of the definition of the mapping $H_{\mathscr{U}}$ (see [7, equation (1)]), whose final space should be $\mathscr{P}(\mathscr{P}(Y))$ instead of $\mathscr{P}(Y)$. This mistake is spread in [7, Definition 2.5 and Propositions 3.1–3.4] since the unions of elements in $\mathscr{P}(Y)$ are considered as elements in Y instead of families of subsets of Y, i.e., elements in $\mathscr{P}(\mathscr{P}(Y))$. Similar drawbacks happen in [6, 9].

The results of this section not only correct the above mentioned mathematical difficulties, but also they allow to clarify the robust counterparts and the definitions of minmax and minmin robust solutions considered in [6, 7, 9, 39] to deal with uncertain set optimization problems.

5. Uncertain Set Optimization

In this last section, as an illustration of the results in Section 4, we obtain scalarization results for pessimistic solutions of an uncertain set optimization problem with finite scenarios. Although it is assumed that preferences are given by the binary relation $\precsim_K^{1,l}$, any other binary relation $\precsim_K^{i,j}$ could be addressed in a similar way, also if optimistic solutions are taken into account instead the pessimistic ones. Denote $[1:r] := \{1,2,\ldots,r\}$.

Consider the uncertain set optimization problem:

$$\operatorname{Min}_{\precsim_K^{1,l}} \{G(x,u) : x \in S(u)\}, \tag{SOP$_u$}$$

where $u \in \mathscr{U} := \{u_1,u_2,\ldots,u_r\}$, $G : X \times \mathscr{U} \rightrightarrows Y$, $G(x,u) \neq \emptyset$, for all $x \in X$ and $u \in \mathscr{U}$, $S : \mathscr{U} \rightrightarrows X$, and $K : Y \rightrightarrows Y$. Recall that a point x_0 is a pessimistic solution of problem (SOP$_u$) if $x_0 \in \mathrm{M}(G_U, S, \trianglelefteq^p)$ and $G_{\mathscr{U}} : X \rightrightarrows \mathscr{P}_0(Y)$, $G_{\mathscr{U}}(x) = \{G(x,u_i)\}_{i=1}^r$, for all $x \in X$ and $S = \bigcap_{i=1}^r S(u_i)$. In addition, it follows that $G_{\mathscr{U}}(x_1) \trianglelefteq^p G_{\mathscr{U}}(x_2)$ if for all $i \in [1:r]$ there exists $j \in [1:r]$ such that $G(x_1,u_i) \precsim_K^{1,l} G(x_2,u_j)$, i.e. $G(x_2,u_j) \subset \bigcup_{y \in G(x_1,u_i)} y + K(y)$. Next we denote $G_{\mathscr{U}}(S) := \bigcup_{\substack{x \in S \\ i \in [1:r]}} G(x,u_i)$.

Theorem 5.1. *Consider $\rho \in \{\lambda, \Delta\}$. Suppose that*

$$\sup_{y \in G(x,u_i)} \inf_{z \in G(x,u_i)} \rho(z,y) = 0, \text{ for all } x \in S, i \in [1:r], \tag{5.1}$$

and for each $x \in S$ and $i \in [1:r]$ assume that

$$\inf_{y \in G(x,u_i)} \rho(y,z) \text{ is attained, for all } z \in G_{\mathscr{U}}(S). \tag{5.2}$$

or

$$\bigcup_{y \in G(x,u_i)} y + K(y) \text{ is closed.} \tag{5.3}$$

Then $x_0 \in S$ is a pessimistic solution of problem (SOP_u) if and only if

$$x \in S, \sup_{i \in [1:r]} \inf_{j \in [1:r]} \sup_{z \in G(x_0,u_j)} \inf_{y \in G(x,u_i)} \rho(y,z) \leq 0 \tag{5.4}$$

$$\Longrightarrow \sup_{i \in [1:r]} \inf_{j \in [1:r]} \sup_{z \in G(x,u_j)} \inf_{y \in G(x_0,u_i)} \rho(y,z) \leq 0. \tag{5.5}$$

Proof. Consider a feasible point $x \in S$. We claim that $G_{\mathscr{U}}(x) \trianglelefteq^p G_{\mathscr{U}}(x_0)$ if and only if statement (5.4) is true. Indeed, assumptions (5.1), (5.2) and (5.3) allow us to apply parts (i) and (ii) of Theorem 4.6 to $\rho \in \{\lambda, \Delta\}$, $\mathscr{A} = G_{\mathscr{U}}(x)$ and $\mathscr{B} = G_{\mathscr{U}}(x_0)$. It follows that $\mathscr{A} \trianglelefteq^p \mathscr{B}$ if and only if statement (5.4) is satisfied, and the assertion is stated. Analogously, $G_{\mathscr{U}}(x_0) \trianglelefteq^p G_{\mathscr{U}}(x)$ is equivalent to statement (5.5). Therefore, by Definition 3.3, x_0 is a pessimistic solution of problem (SOP_u) if and only if

$$x \in S, G_{\mathscr{U}}(x) \trianglelefteq^p G_{\mathscr{U}}(x_0) \Rightarrow G_{\mathscr{U}}(x_0) \trianglelefteq^p G_{\mathscr{U}}(x)$$

which is equivalent to the implication in (5.4)-(5.5) and the proof finishes. □

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