J. Nonlinear Var. Anal. 11 (2027), No. 1, pp. 77-114
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.11.2027.1.05

UNIFICATION OF SEMICONTINUITIES AND APPLICATIONS IN ANALYZING STABILITY FOR SET OPTIMIZATION PROBLEMS

LAM QUOC ANH[1], LAM VAN DAY[2,3,4], LAM THI VAN KHANH[5,6], XIAOPENG ZHAO[7,*]

[1]Faculty of Mathematics and Informatics, Teacher College, Can Tho University, Can Tho City, Vietnam
[2]Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam
[3]Vietnam National University, Ho Chi Minh City, Vietnam
[4]Department of Mathematics, Faculty of Basic Sciences, Nam Can Tho University, Can Tho City, Vietnam
[5]Department of Mathematics and Statistics, Jerome J. Lohr College of Engineering,
South Dakota State University, Brookings, SD 57007, USA
[6]Department of Mathematics and Statistics, Kennedy College of Sciences, University of Massachusetts Lowell,
Lowell, MA 01854, USA
[7]School of Mathematical Sciences, Tiangong University, Tianjin 300387, China

Abstract. In this paper, we unify the lower and upper semicontinuity of a set-valued map and apply this to analyze the qualitative properties of weakly efficient and efficient solutions in set optimization problems. Our approach involves a proposed unification of the set less relation based on the Minkowski difference, including existence and stability conditions. First, we unify the upper and lower semicontinuity of a set-valued map and discuss its properties and characterizations. Next, building on these ideas, we propose a unification of the set less relations via the Minkowski difference and examine its properties. Then, we observe that the unification of semicontinuity is highly compatible with the unified form of these set relations in studying the qualitative properties of set optimization problems. Furthermore, in analyzing these solution properties, we introduce new approaches that allow us to establish stability conditions for set optimization problems without relying on coercive conditions, as required in previous works. To demonstrate the practical relevance of the results obtained, we further employ them to analyze the convergence behavior of interval optimal control problems and the irrigation management model under uncertainties in rainfall and cost.

Keywords. Minkowski difference; Qualitative properties; Set optimization problems; Stability conditions; Weakly efficient solutions.

1. INTRODUCTION

The lower and upper semicontinuities of set-valued maps play a crucial role in studying the solution properties of optimization models. As is well known, the emergence of these concepts originates from the notion of continuity of a single-valued map. That is, while the values of a single-valued map belong to a certain set, in the case of a set-valued map, two different perspectives are considered: first, the value set of the map is a subset of this set, or second, their

*Corresponding author.
E-mail address: zhaoxiaopeng.2007@163.com (X. Zhao)
Received 1 July 2025; Accepted 5 October 2025; Published online 10 November 2026.
©2027 Journal of Nonlinear and Variational Analysis

intersection is nonempty. From this, two distinct developmental directions arise, namely upper semicontinuity and lower semicontinuity, which, as we know, are applied to examine various properties of mathematical models. Drawing from observations of natural laws, the processes of separation and unification or, in other words, integration and dispersion, of objects, entities, and phenomena are often reflected through certain patterns. This leads us to wonder: given this perspective, since these two concepts originate from the same object, is there a way to unify them in some sense naturally at a higher level than their original definition? Moreover, what significance would such unification hold as a tool for studying mathematical models. More specifically, in our field of expertise, this new tool can be used to examine the solution properties of certain optimization models.

Building on this research direction, we find set optimization problems involving set less relations to be a highly suitable model. As is well known, studying set optimization problems with set less order relations is a compelling topic that has attracted considerable interest, particularly due to its ability to address the limitations of the vector approach, as highlighted in [23, 33]. Although set optimization problems have only recently attracted research interest, a large number of studies on these problems are published annually and widely distributed across the world. These studies cover various topics, such as optimality conditions [19, 32, 35, 44, 45], existence of solutions [3], stability analysis [4, 5, 13, 46], algorithms [10, 12, 14, 31], and their practical applications [20, 44]. This demonstrates the timeliness and strong appeal of this research area to mathematicians.

For the class of these problems, set less relations commonly used to evaluate the efficiency of models include the upper set less order relation (denoted by $\preceq^u$), the lower set less order relation (denoted by $\preceq^l$), and the set less order relation (denoted by $\preceq^s$) [24, 34, 43]. Naturally, for these relations to be used in studying the solution structures of these models, they must satisfy certain prescribed criteria of an order relation. However, as we have seen, one of the major drawbacks of these relations is that they focus only on comparing the maximal or minimal elements of the corresponding sets while disregarding their cardinality. This leads to less accurate comparative information between objects in nature, which often does not reflect reality. For example, when comparing the level of economic development between two countries, if one considers only the average annual income of individuals without taking into account the population size, the evaluation based on these comparison relations will fail to fully capture the actual economic potential of the countries. Perhaps stemming from similar real-world scenarios, Karaman [21] proposed set comparison relations based on the Minkowski difference, referred to as the $\preceq^{m_1}$ and $\preceq^{m_2}$ relations. The key distinction of these relations compared to those mentioned above is the necessary condition: for a set A to be smaller than a set B under these relations, the cardinality of A cannot exceed the cardinality of B. However, this is not only a mathematical principle but also a natural law: the more accurately a model reflects natural phenomena and aligns with real-life situations, the greater the complexity involved in analyzing and evaluating its solution properties and structure. Consequently, the number of research works on set optimization problems concerning set less relations based on the Minkowski difference remains very limited, both in quantity and in the range of topics covered. To date, almost all studies on this class of problems have focused on topics such as optimality conditions [21, 32, 35, 44], and stability analysis [8, 15, 46]. This indicates that dedicating time to investigating and advancing research in this area is highly meaningful and worthwhile. When applying mathematical models

to real-world situations, most of the problem data are derived from statistical methods, surveys, and similar approaches. These methods inherently involve approximations or errors when compared to the original data. Consequently, the topic of convergence conditions for mathematical models has garnered significant attention in recent years [3, 4, 5, 8, 15, 16, 17, 22, 25, 36]. The primary objective of this topic is to study the behavior of solutions to the original problem by examining the solutions of sequences of approximating problems whose data converge to the data of the original problem. In recent years, the convergence conditions for solutions to set optimization problems involving set less order relations have garnered considerable interest from researchers. However, most studies focus on the relations $\preceq^u$, $\preceq^l$, and $\preceq^s$ (see, e.g., [3, 4, 5, 16, 17, 22, 25, 36] and the references therein). In contrast, research on set optimization problems involving order relations defined by Minkowski differences, such as $\preceq^{m_1}$ and $\preceq^{m_2}$, remains limited. Furthermore, as previously mentioned, due to the differences between these relations and $\preceq^u$, $\preceq^l$, and $\preceq^s$, the techniques used for the latter are no longer applicable, which may lead to potential errors. These issues will be discussed in detail through comments and remarks provided in the subsequent sections of this study.

Returning to the objective mentioned at the beginning, the purpose of this study is to unify the lower and upper semicontinuities and use this unified concept to investigate the solution properties of mathematical models. More specifically, we first introduce a unification of the lower and upper semicontinuities of set-valued maps and discuss their properties, along with their relationships to the classical forms. Next, we construct a unified form of the order relations on sets introduced by Karaman [21] based on the Minkowski difference and utilize this new relation to examine the concept of efficient solutions in set optimization problems. As an example of the role of the unification concept of the lower and upper semicontinuities, we apply it to establish properties of weakly efficient and efficient solutions for the reference problems. In pursuit of this objective, we have explored new approaches to studying the convergence of these problems. These novel approaches allow us to avoid certain technical conditions that researchers often refer to as coercive conditions, such as inverse condition [5], property condition [41], or induction phenomena [1]. To highlight the usefulness of our theoretical developments, we extend their application to investigate the behavior of solutions in interval optimal control models, particularly in the context of irrigation planning under uncertain rainfall and fluctuating costs.

2. Preliminaries

Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and $\emptyset \neq X \subset \mathbb{X}$, $\emptyset \neq K \subset \mathbb{Y}$. For a nonempty subset A of a Banach space, by $\mathrm{cl}A$, $\mathrm{bd}A$, and $\mathrm{int}A$, we mean the closure, boundary, and topological interior of A, respectively. The open unit ball and closed unit ball of $\mathbb{Y}$ are defined by $\mathbb{B}$ and $\bar{\mathbb{B}}$, respectively. Let $\mathscr{P}(\mathbb{Y})$ denote the collection of all nonempty subsets of $\mathbb{Y}$, and let $\mathscr{P}_{\mathrm{cc}}(\mathbb{Y})$ denote the collection of all nonempty compact convex subsets of $\mathbb{Y}$.

First, we recall the concepts of convergence of sequences of sets in different senses, which will be used in the subsequent sections of this study.

Definition 2.1. [23, p. 83] Let $\{A_n\} \subset \mathscr{P}(\mathbb{Y})$. Then $\{A_n\}$ is said to converge to A of $\mathscr{P}(\mathbb{Y})$ in the

(a) *upper Painlevé-Kuratowski* convergence sense, denoted by $A_n \xrightarrow{K} A$, if $\mathrm{Ls}(A_n) \subseteq A$;

(b) *lower Painlevé-Kuratowski* convergence sense, denoted by $A_n \xrightarrow{K} A$, if $A \subseteq \mathrm{Li}(A_n)$;
(c) *Painlevé-Kuratowski* convergence sense, denoted by $A_n \xrightarrow{K} A$, if $\mathrm{Ls}(A_n) \subseteq A \subseteq \mathrm{Li}(A_n)$;
where the *limit superior* and *limit inferior* are defined by

$$\mathrm{Ls}(A_n) := \{y \in \mathbb{Y} : y = \lim_{k\to+\infty} y_k, y_k \in A_{n_k}, \{n_k\} \text{ is a subsequence of } \{n\}\},$$
$$\mathrm{Li}(A_n) := \{y \in \mathbb{Y} : y = \lim_{n\to+\infty} y_n, y_n \in A_n \text{ for sufficiently large } n\}, \text{ respectively.}$$

For $z \in \mathbb{Y}$, and $A, B \in \mathscr{P}(\mathbb{Y})$, we recall

$$d(z,A) := \inf_{a\in A} d(z,a),\ e(A,B) := \sup_{a\in A} d(a,B) = \inf\{\lambda \geq 0 : A \subseteq B + \lambda\bar{\mathbb{B}}\},$$

and the concepts of Hausdorff convergence.

Definition 2.2. [16] Let $\{A_n\}$ be a sequence of $\mathscr{P}(\mathbb{Y})$. The sequence is said to converge to a nonempty subset A of $\mathbb{Y}$ in the

(a) *upper Hausdorff* convergence sense, denoted by $A_n \xrightarrow{H} A$ if $e(A_n, A) \to 0$;
(b) *lower Hausdorff* convergence sense, denoted by $A_n \xrightarrow{H} A$ if $e(A, A_n) \to 0$;
(c) *Hausdorff* convergence sense, denoted by $A_n \xrightarrow{H} A$ if

$$H(A, A_n) := \max\{e(A_n, A), e(A, A_n)\} \to 0.$$

In the study of convergence conditions for optimization models, the conversion between the types of convergence presented above sometimes helps simplify the presentation and facilitates the transition from sequence convergence to set excess. The result below provide the necessary properties to ensure the attainment of our desired objective.

Lemma 2.1. [5] *Let $A \in \mathscr{P}(\mathbb{Y})$ and $\{A_n\} \subset \mathscr{P}(\mathbb{Y})$. If $A_n \xrightarrow{H} A$ and A is a compact set, then, for any $\{a_n\}$ with $a_n \in A_n$, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} \to a \in A$.*

Passing to the convergence of sequence of maps, based on the convergence of a sequence of sets and motivated by [1, 17, 36], we propose the convergence of a sequence of maps as follows.

Definition 2.3. Let $F, F_n : X \rightrightarrows \mathbb{Y}$, and $\emptyset \neq D = \cap_{n\in\mathbb{N}}(\mathrm{dom}\, F_n \cap \mathrm{dom}\, F)$. We say that $\{F_n\}$ converges to F in the

(a) *upper Hausdorff* sense on D, denoted by $F_n \xrightarrow{H} F$, if and only if, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$F_n(x) \subseteq F(x) + \varepsilon\mathbb{B} \text{ for all } n \geq n_0 \text{ and for all } x \in D.$$

(b) *upper Hausdorff* sense on D, denoted by $F_n \xrightarrow{H} F$, if and only if, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$F(x) \subseteq F_n(x) + \varepsilon\mathbb{B} \quad \text{for all } n \geq n_0 \text{ and for all } x \in D.$$

(c) *Hausdorff* sense on D, denoted by $F_n \xrightarrow{H} F$, if $F_n \xrightarrow{H} F$ and $F_n \xrightarrow{H} F$ on D.

We conclude this section by recalling an important result, often known as the Rådström cancellation lemma. Its main content provides the necessary conditions to ensure validity when simplifying both sides of an inclusion by a set.

Lemma 2.2. [37, Rådström cancelation lemma] *Let A, B and D be in $\mathscr{P}(\mathbb{Y})$. Assume that B is closed and convex, D is bounded and $A+D \subseteq B+D$. Then, $A \subseteq B$.*

3. Unification of Lower and Upper Semicontinuities and Set Less Relations

In this section, we focus on finding the way to unify the concept of lower and upper semicontinuities of set-valued maps. Let $\mathbb{X}$, $\mathbb{Y}, K$ and X be defined as in Section 2. We first recall these classical mentioned concepts.

Definition 3.1. [23, pp. 81-82] Let $F : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued map and $\bar{x} \in X$. Then, the set-valued map F is said to be

(a) *Hausdorff upper semicontinuous* (Husc) at $\bar{x}$ if, for any neighborhood B of the original $0_{\mathbb{Y}}$ in $\mathbb{Y}$, there exists a neighbourhood N of $\bar{x}$ satisfying
$$F(x) \subseteq F(\bar{x}) + B \text{ for all } x \in N \cap \operatorname{dom} F;$$
(b) *Hausdorff lower semicontinuous* (Hlsc) at $\bar{x}$ if, for any neighborhood B of the original $0_{\mathbb{Y}}$ in $\mathbb{Y}$, there exists a neighbourhood N of $\bar{x}$ satisfying
$$F(\bar{x}) \subseteq F(x) + B \text{ for all } x \in N \cap \operatorname{dom} F;$$
(c) *Hausdorff continuous* at $\bar{x}$ if F is both Husc and Hlsc at $\bar{x}$.

Now we suggest a unification of the above concepts characterized by a scalar element $\alpha \in \mathbb{R}$.

Definition 3.2. Let $\alpha \in \mathbb{R}$, and $F : X \rightrightarrows \mathbb{Y}$. Then, F is said to be *α-continuous* at $\bar{x} \in X$ if and only if, for any neighborhood B of the original $0_{\mathbb{Y}}$ in $\mathbb{Y}$, there exists a neighborhood N of $\bar{x}$ such that
$$-\alpha F(x) + (1-\alpha)F(\bar{x}) \subseteq -\alpha F(\bar{x}) + (1-\alpha)F(x) + B \text{ for all } x \in N. \tag{3.1}$$

Remark 3.1. If F is a single-valued map, then F is continuous at $\bar{x}$ if and only if F is α-continuous at $\bar{x}$ for any $\alpha \in \mathbb{R}$.

Next, we study the properties of the recently proposed continuous form in (3.1), as well as its relationships with classical concepts.

Proposition 3.1. *Let $F, G : X \rightrightarrows \mathbb{Y}$. The following statements hold true*

(a) *F is 0-continuous at $\bar{x}$ if and only if F is H-lsc at $\bar{x}$.*
(b) *F is 1-continuous at $\bar{x}$ if and only if F is H-usc at $\bar{x}$.*
(c) *F is 0-continuous at $\bar{x}$ if and only if F is α-continuous at $\bar{x}$ for all $\alpha \in]-\infty, 0]$, whenever F is convex valued on* $\operatorname{dom} F$.
(d) *F is 1-continuous at $\bar{x}$ if and only if F is α-continuous at $\bar{x}$ for all $\alpha \in [1, +\infty[$, whenever F is convex valued on* $\operatorname{dom} F$.
(e) *F is 0-continuous and 1-continuous at $\bar{x}$ if and only if F is α-continuous at $\bar{x}$ for all $\alpha \in [0, 1]$.*
(f) *If F and G are α-continuous at $\bar{x} \in X$, then $F+G$ is also α-continuous at $\bar{x}$.*
(g) *If F is α-continuous at $\bar{x} \in X$, then λF is also α-continuous at $\bar{x}$ for all $\lambda \in \mathbb{R}$.*

Proof. The conclusions (a), (b), and (g) can be readily derived from the definitions under standard mathematical transformations, and we therefore leave their proofs to the interested reader. In what follows, we provide detailed proofs for statements (c)-(f).

(c) By the 0-continuity of F at $\bar{x}$, for any $\varepsilon > 0$ there exists $U_{\bar{x}}$ such that

$$F(\bar{x}) \subseteq F(x) + \varepsilon\mathbb{B} \text{ for all } x \in U_{\bar{x}}.$$

Hence, for all $x \in U_{\bar{x}}$ we have

$$-\alpha F(x) - \alpha F(\bar{x}) + F(\bar{x}) \subseteq -\alpha F(x) - \alpha F(\bar{x}) + F(x) + \varepsilon\mathbb{B}.$$

From $\alpha \in (-\infty, 0)$ and the convex-valuedness of F on E, we obtain

$$-\alpha F(x) + (1-\alpha)F(\bar{x}) \subseteq (1-\alpha)F(x) - \alpha F(\bar{x}) + \varepsilon\mathbb{B} \text{ for all } x \in U_{\bar{x}}.$$

Therefore, we have verified the conclusion (c).

(d) By a similar argument as in (c), we also obtain conclusion (d).

(e) We need only to show that if F is 0 and 1-continuous then it is α-continuous for all $\alpha \in]0,1[$. From F is 0-continuity and 1-continuity of F at $\bar{x}$, for any $\varepsilon > 0$ we can find a neighborhood $U_{\bar{x}}$ of $\bar{x}$ for all $x \in U_{\bar{x}}$, we have

$$F(\bar{x}) \subseteq F(x) + \varepsilon\mathbb{B}, \text{ and } F(x) \subseteq F(\bar{x}) + \varepsilon\mathbb{B}.$$

Consequently, for every $\alpha \in]0,1[$ and for every $x \in U_{\bar{x}}$, we obtain

$$(1-\alpha)F(\bar{x}) \subseteq (1-\alpha)F(x) + (1-\alpha)\varepsilon\mathbb{B}, \text{ and } -\alpha F(x) \subseteq -\alpha F(\bar{x}) - \alpha\varepsilon\mathbb{B}.$$

Together with the balance and convexity of $\varepsilon\mathbb{B}$, these lead to

$$-\alpha F(x) + (1-\alpha)F(\bar{x}) \subseteq -\alpha F(\bar{x}) + (1-\alpha)F(x) + \varepsilon\mathbb{B} \text{ for all } x \in U_{\bar{x}}.$$

Hence, assertion (e) is claimed.

(f) By the definition, for each $\varepsilon > 0$, there exists a neighborhood $U_{\bar{x}}$ of $\bar{x}$ such that for every $x \in U_{\bar{x}}$, we have

$$-\alpha F(x) + (1-\alpha)F(\bar{x}) \subseteq -\alpha F(\bar{x}) + (1-\alpha)F(x) + 2^{-1}\varepsilon\mathbb{B}, \text{ and}$$

$$-\alpha G(x) + (1-\alpha)G(\bar{x}) \subseteq -\alpha G(\bar{x}) + (1-\alpha)G(x) + 2^{-1}\varepsilon\mathbb{B} \text{ for all } x \in U_{\bar{x}}^2.$$

Thus, for every $x \in U_{\bar{x}}$, we obtain

$$-\alpha(F+G)(x) + (1-\alpha)(F+G)(\bar{x}) \subseteq -\alpha(F+G)(\bar{x}) + (1-\alpha)(F+G)(x) + \varepsilon\mathbb{B}.$$

Equivalently, the map $F+G$ is α-continuous at $\bar{x}$. □

Motivated by the above result, in the sequel of this study, we only consider the α-continuity for the case of $\alpha \in [0,1]$.

Remark 3.2. Middle-continuity, namely α-continuity at $\alpha = \frac{1}{2}$, captures a symmetric form of stability for set-valued mappings. In contrast to Hausdorff upper or lower semicontinuity, where the control is essentially one-sided, middle-continuity requires that perturbations in the input lead to proportionally bounded and balanced deviations in the output. This bidirectional control mechanism is particularly relevant in real-world systems where stability is not biased toward increase or decrease but must be preserved symmetrically.

Several concrete instances demonstrate the natural emergence of middle-continuity:

- *Ecological Systems:* In population dynamics models, such as simplified predator-prey systems, the ecosystem often exhibits resilience near equilibrium states. A small environmental disturbance in prey availability leads to a controlled and approximately symmetric adjustment in both prey and predator populations, preventing disproportionate extinction or overgrowth. The set-valued mapping from environmental parameters to feasible population pairs satisfies middle-continuity near equilibria.
- *Health Sciences:* In pharmacokinetic models, patient responses (e.g., blood pressure, glucose levels) to small changes in medication dosage are often required to vary symmetrically within safe ranges, avoiding overshooting or undershooting therapeutic targets. This safety constraint naturally leads to mappings that are middle-continuous.
- *Economic Equilibria:* In Walrasian or Edgeworth-type general equilibrium models, small perturbations in the economic environment (e.g., slight changes in initial endowments or preferences) induce adjustments in supply-demand correspondences. Middle-continuity models a setting where neither excess demand nor excess supply is disproportionately sensitive to perturbations, reflecting a stable and fair adjustment mechanism.
- *Engineering Control Systems:* In feedback control of mechanical or electrical systems, certain controllers are designed to ensure symmetric error tolerances around setpoints. Small disturbances must lead to proportionally bounded deviations on both sides to maintain system performance. The relation from disturbance inputs to output responses can be modeled as middle-continuous.
- *Machine Learning and Data Science:* In robust optimization and adversarial training, small perturbations of input data are expected to result in small, symmetric changes in prediction sets. Regularization techniques enforcing Lipschitz continuity or stability in set-valued outputs implicitly promote forms of middle-continuity.

Thus, middle-continuity serves as a mathematical abstraction of balanced stability, with applications across diverse fields such as ecology, economics, healthcare, engineering, and data science.

The following is an example to illustrate a map that is middle-continuous but neither Hausdorff upper semicontinuous nor Hausdorff lower semicontinuous.

Example 3.1. Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, and $K = X = \mathbb{R}_+$. Let F from X into $\mathbb{Y}$ be defined by

$$F(x) := \begin{cases}]-\infty, 0[& \text{if } x = 1, \\ ]0, +\infty[& \text{if } x \neq 1. \end{cases}$$

It is clear that F is neither lower nor upper Hausdorff continuous at $x = 1$. However, F is middle-continuous at $x = 1$.

For the set valued mappings F and F_n defined as in Section 2, we examine the following key property for our analysis.

Lemma 3.1. *Assume that $F_n \xrightarrow{H} F$ on D. Let $\{x_n\}$ be a sequence in D that converges to $\bar{x} \in D$ and $\alpha \in [0,1]$. If F is α-continuous at $\bar{x}$, then, for any $\varepsilon > 0$,*

$$-\alpha F_n(x_n) + (1-\alpha)F(\bar{x}) \subseteq -\alpha F(\bar{x}) + (1-\alpha)F_n(x_n) + \varepsilon\mathbb{B} \text{ for } n \text{ sufficiently large.}$$

Proof. $\diamond$ Case 1: $\alpha = 0$. By the 0-continuity of F at $\bar{x}$, we imply that, for any $\varepsilon > 0$,

$$F(\bar{x}) \subseteq F(x_n) + 2^{-1}\varepsilon\mathbb{B} \text{ for } n \text{ sufficiently large.} \tag{3.2}$$

Moreover, due to $F_n \xrightarrow{H}_{\tau} F$ on D, we have

$$F(x_n) \subseteq F_n(x_n) + 2^{-1}\varepsilon\mathbb{B} \text{ for } n \text{ sufficiently large.} \tag{3.3}$$

Combining (3.2) with (3.3), we conclude that, for n sufficiently large,

$$F(\bar{x}) \subseteq F(x_n) + 2^{-1}\varepsilon\mathbb{B} \subseteq F_n(x_n) + 2^{-1}\varepsilon\mathbb{B} + 2^{-1}\varepsilon\mathbb{B} = F_n(x_n) + \varepsilon\mathbb{B}.$$

Thus the conclusion holds true for $\alpha = 0$.

◇ Case 2: $\alpha = 1$. By using the same arguments as above, we also obtain the needed inclusion.

◇ Case 3: $\alpha \in]0,1[$. From $F_n \xrightarrow{H} F$ on D, for any $\varepsilon > 0$, we have

$$F_n(x_n) \subseteq F(x_n) + 2^{-1}\varepsilon\mathbb{B} \text{ for } n \text{ sufficiently large;}$$

$$F(x_n) \subseteq F_n(x_n) + 2^{-1}\varepsilon\mathbb{B} \text{ for } n \text{ sufficiently large.}$$

Multiplying both sides of the above inclusions by α and $(1-\alpha)$ respectively, we obtain

$$-\alpha F_n(x_n) \subseteq -\alpha F(x_n) - \alpha 2^{-1}\varepsilon\mathbb{B} \text{ for } n \text{ sufficiently large;} \tag{3.4}$$

$$(1-\alpha)F(x_n) \subseteq (1-\alpha)F_n(x_n) + (1-\alpha)2^{-1}\varepsilon\mathbb{B} \text{ for } n \text{ sufficiently large.} \tag{3.5}$$

On the other hand, because of the α-continuity of F at $\bar{x}$ and $\{x_n\} \to \bar{x}$, we have

$$-\alpha F(x_n) + (1-\alpha)F(\bar{x}) \subseteq -\alpha F(\bar{x}) + (1-\alpha)F(x_n) + 2^{-1}\varepsilon\mathbb{B} \text{ for } n \text{ sufficiently large.} \tag{3.6}$$

Combining (3.4) and (3.5) with (3.6), we obtains

$$\begin{aligned} -\alpha F_n(x_n) + (1-\alpha)F(\bar{x}) &\subseteq -\alpha F(x_n) - \alpha 2^{-1}\varepsilon\mathbb{B} + (1-\alpha)F(\bar{x}) \\ &\subseteq -\alpha F(\bar{x}) + (1-\alpha)F_n(x_n) + \varepsilon\mathbb{B} \text{ for } n \text{ sufficiently large.} \end{aligned}$$

Therefore, the proof is completed. □

Now, we move on to studying the construction of a unification of set less relations which is compatible with the form of α-continuity we discussed above. For A, B belonging to $\mathscr{P}(\mathbb{Y})$ and $t \in \mathbb{R}$, we denote the algebraic sum and the Minkowski (Pontryagin) difference of A and B

$$A + B := \{a + b : a \in A, b \in B\}, \quad tA := \{ta : a \in A\}, \quad \text{and}$$

$$B \dot{-} A := \{z \in \mathbb{Y} : z + A \subseteq B\}, \text{ respectively.}$$

We recall the binary relations on $\mathscr{P}(\mathbb{Y})$ with respect to a nonempty subset K of $\mathbb{Y}$ as in [21]:

$$A \preceq^{m_1} B \Leftrightarrow (B \dot{-} A) \cap K \neq \emptyset \Leftrightarrow \exists k \in K, k + A \subseteq B,$$

$$A \preceq^{m_2} B \Leftrightarrow (A \dot{-} B) \cap (-K) \neq \emptyset \Leftrightarrow \exists k \in K, -k + B \subseteq A.$$

For more details on the relationships between the above relations and the upper and lower set less relations, as well as their properties and practical significance, we refer the reader to some representative works [8, 15]. Motivated by [2], in which the authors considered a unified form of the lower and upper set less relations and investigated the well-posedness of set optimization problems, we propose a unified framework for these two relations as follows: For each $\alpha \in \mathbb{R}$ and $\Theta \subset \mathbb{R}$, we define

$$A \preceq^{\alpha} B \Leftrightarrow \text{ there exists } k \in K \text{ such that } k + (1-\alpha)A - \alpha B \subseteq (1-\alpha)B - \alpha A,$$

$$A \preceq^{\Theta} B \Leftrightarrow A \preceq^{\alpha} B \text{ for all } \alpha \in \Theta.$$

We next discuss some properties of the new order relation. Interestingly, the relation $\preceq^\alpha$ also shares properties similar to those of α-continuity that we examined in Proposition 3.1. To keep the paper concise and succinct, we choose to omit the details of the proof.

Proposition 3.2. *For all $A,B \in \mathscr{P}(\mathbb{Y})$, the following assertions are true.*
(a) $A \preceq^0 B$ *if and only if* $A \preceq^{m_1} B$.
(b) $A \preceq^1 B$ *if and only if* $A \preceq^{m_2} B$.
(c) $A \preceq^0 B$ *if and only if* $A \preceq^\alpha B$ *for all* $\alpha \in]-\infty, 0]$, *whenever* A,B *are convex.*
(d) $A \preceq^1 B$ *if and only if* $A \preceq^\alpha B$ *for all* $\alpha \in [1, +\infty[$, *whenever* A,B *are convex.*
(e) $A \preceq^0 B$ *and* $A \preceq^1 B$ *if and only if* $A \preceq^\alpha B$ *for all* $\alpha \in [0,1]$.

In order to consider set optimization problems involving the suggested order relation, we continue to study further fundamental properties of this set less relation.

Lemma 3.2. *Let $A,B,C,D,K \in \mathscr{P}(\mathbb{Y})$ and $\alpha \in \mathbb{R}$. The following statements are true*
(a) $\preceq^\alpha$ *is compatible with addition, i.e, if* $A \preceq^\alpha B$ *then* $A+C \preceq^\alpha B+C$.
(b) $\preceq^\alpha$ *is preserved under the same inequalities, i.e, if* $A \preceq^\alpha B$ *and* $C \preceq^\alpha D$, *then* $A+C \preceq^\alpha B+D$, *whenever* K *is a convex cone.*
(c) $\preceq^\alpha$ *is compatible with multiplying by a positive number if and only if* K *is a cone, i.e, if* $A \preceq^\alpha B$ *then* $\lambda A \preceq^\alpha \lambda B$, *for all* $\lambda \in [0, +\infty[$.
(d) $\preceq^\alpha$ *is reflexive if and only if* $0_{\mathbb{Y}} \in K$.
(e) $\preceq^\alpha$ *is transitive on* $\mathscr{P}_{cc}(\mathbb{Y})$ *if* K *is a convex cone.*

Proof. Because the conclusions in (a)-(d) are easy to verify from the definition of $\preceq^\alpha$, we only present the proof of statement (e). For each $A,B,D \in \mathscr{P}_{cc}(\mathbb{Y})$, assume that

$$A \preceq^\alpha B, \text{ and } B \preceq^\alpha D. \tag{3.7}$$

We prove that $A \preceq^\alpha D$. From (3.7), there exist $k_1, k_2 \in K$ such that

$$k_1 + (1-\alpha)A - \alpha B \subseteq -\alpha A + (1-\alpha)B, \text{ and}$$
$$k_2 + (1-\alpha)B - \alpha D \subseteq -\alpha B + (1-\alpha)D.$$

By adding the corresponding sides of the above inclusions, we obtain

$$k_1 + (1-\alpha)A - \alpha B + k_2 + (1-\alpha)B - \alpha D \subseteq -\alpha A + (1-\alpha)B - \alpha B + (1-\alpha)D.$$

Because A,B,D belong to $\mathscr{P}_{cc}(\mathbb{Y})$, $-\alpha B + (1-\alpha)B$ is bounded and $(1-\alpha)A - \alpha D + K$ is closed and convex. Using Lemma 2.2, we obtain

$$k_1 + k_2 + (1-\alpha)A - \alpha D \subseteq -\alpha A + (1-\alpha)D.$$

This together with the convexity of the cone K implies that $A \preceq^\alpha D$. □

However, in general, the relation $\preceq^\alpha$ does not satisfy the antisymmetric property. For instance, in $\mathbb{R}^2$, consider $K = \mathbb{R}^2_+$, $A = \{(z_1,z_2) : z_1^2 + z_2^2 \leq 1\}$, $B = \{(z_1,z_2) : z_1^2 + z_2^2 \leq 4\}$ and $\alpha = \frac{1}{2}$. Through some basic computations, we obtain

$$(1-\alpha)A - \alpha B = (1-\alpha)B - \alpha A = \left\{(z_1,z_2) : z_1^2 + z_2^2 \leq \frac{9}{4}\right\}.$$

Therefore, by taking $k_1 = k_2 = (0,0)$, we conclude that

$$k_1 + (1-\alpha)A - \alpha B \subseteq (1-\alpha)B - \alpha A, \quad k_2 + (1-\alpha)B - \alpha A \subseteq (1-\alpha)A - \alpha B.$$

Equivalently, $A \preceq^{\alpha} B$ and $B \preceq^{\alpha} A$, although A is different from B. Thanks to Lemma 2.2, we derive the following characterization of A and B when they are equivalent, namely $A \preceq^{\alpha} B$ and $B \preceq^{\alpha} A$.

Proposition 3.3. *For every nonempty bounded subsets A, B of $\mathbb{Y}$, and $\alpha \in \mathbb{R}$,*

$$[A \preceq^{\alpha} B \text{ and } B \preceq^{\alpha} A] \Leftrightarrow [-\alpha A + (1-\alpha)B = -\alpha B + (1-\alpha)A].$$

In this section, we assume the cone K is convex, closed, pointed, and solid. Now, we consider set optimizations involving the relation $\preceq^{\alpha}$. Let $\Phi : X \rightrightarrows \mathbb{Y}$ be a set valued mapping and Ω be a nonempty subset of $X \cap \operatorname{dom}\Phi$. We consider the following problem:

$$\operatorname{Min}\Phi(x) \quad \text{subject to} \quad x \in \Omega. \tag{SOP$_\alpha(\Phi,\Omega)$}$$

We define *α-efficient solution* (denoted by $\alpha - \operatorname{Eff}(\Phi,\Omega)$) and *$\alpha$-weakly efficient solution* (denoted by $\alpha - \operatorname{WEff}(\Phi,\Omega)$) of (SOP$_\alpha(\Phi,\Omega)$) as follows:

$$\alpha - \operatorname{Eff}(\Phi,\Omega) = \{\bar{x} \in \Omega : [\forall x \in \Omega, \Phi(x) \preceq^{\alpha} \Phi(\bar{x})] \Rightarrow [\Phi(\bar{x}) \preceq^{\alpha} \Phi(x)]\},$$
$$\alpha - \operatorname{WEff}(\Phi,\Omega) = \{\bar{x} \in \Omega : [\forall x \in \Omega, \Phi(x) \prec^{\alpha} \Phi(\bar{x})] \Rightarrow [\Phi(\bar{x}) \prec^{\alpha} \Phi(x)]\},$$

where $M \prec^{\alpha} N :\Leftrightarrow \exists k \in \operatorname{int}K : k + (1-\alpha)M - \alpha N \subseteq (1-\alpha)N - \alpha M$ for any $M, N \in \mathscr{P}(\mathbb{Y})$.

Next, we discuss some properties and characterizations of the α-efficient and α-weakly efficient solutions of (SOP$_\alpha(\Phi,\Omega)$).

Lemma 3.3. *Assume that Φ is bounded valued on Ω. Then, the following statements hold.*
(a) $\alpha - \operatorname{WEff}(\Phi,\Omega) = \{\bar{x} \in \Omega : \forall z \in \Omega : \Phi(z) \not\prec^{\alpha} \Phi(\bar{x})\}$.
(b) $\alpha - \operatorname{Eff}(\Phi,\Omega) \subseteq \alpha - \operatorname{WEff}(\Phi,\Omega)$.

Proof. (a) If $\Phi(z) \not\prec^{\alpha} \Phi(\bar{x})$ for all $z \in \Omega$, then $\bar{x} \in \alpha - \operatorname{WEff}(\Phi,\Omega)$. Conversely, suppose that $\bar{x} \in \alpha - \operatorname{WEff}(\Phi,\Omega)$ and there exists $z \in E$ such that $\Phi(z) \prec^{\alpha} \Phi(\bar{x})$. Then, we find an element $k_1 \in \operatorname{int}K$ such that

$$k_1 + (1-\alpha)\Phi(z) - \alpha\Phi(\bar{x}) \subseteq (1-\alpha)\Phi(\bar{x}) - \alpha\Phi(z). \tag{3.8}$$

By the definition of α-weakly efficient solution and $\Phi(z) \prec^{\alpha} \Phi(\bar{x})$, we imply that $\Phi(\bar{x}) \prec^{\alpha} \Phi(z)$. Therefore, we can identify $k_2 \in \operatorname{int}K$ satisfying

$$k_2 + (1-\alpha)\Phi(\bar{x}) - \alpha\Phi(z) \subseteq (1-\alpha)\Phi(z) - \alpha\Phi(\bar{x}). \tag{3.9}$$

Combining (3.8) with (3.9), we obtain

$$\begin{aligned} k_1 + k_2 + (1-\alpha)\Phi(z) - \alpha\Phi(\bar{x}) &\subseteq k_2 + (1-\alpha)\Phi(\bar{x}) - \alpha\Phi(z) \\ &\subseteq (1-\alpha)\Phi(z) - \alpha\Phi(\bar{x}). \end{aligned}$$

Using the bounded valuedness of Φ and Lemma 2.2, we obtain $k_1 + k_2 = 0_{\mathbb{Y}}$. This is impossible. Thus statement (a) is true.
(b) By the similar arguments as in the proof of (a), we also verify conclusion (b). □

For $\varepsilon \in \mathbb{R}_+$ and $A, B \in \mathscr{P}(\mathbb{Y})$, we consider some additional set less relations that are used to discuss convergence conditions for (SOP$_\alpha(\Phi,\Omega)$) in the following sections.

$$A \preceq^{\alpha}_{\varepsilon} B \Leftrightarrow \exists k_0 \in K : k_0 + (1-\alpha)A - \alpha B \subseteq (1-\alpha)B - \alpha A + \varepsilon\bar{\mathbb{B}},$$
$$A \prec^{\alpha}_{\varepsilon} B \Leftrightarrow \exists k_0 \in \operatorname{int}K : k_0 + (1-\alpha)A - \alpha B \subseteq (1-\alpha)B - \alpha A + \varepsilon\bar{\mathbb{B}}.$$

It is easy to check that if $A \preceq^{\alpha} B$ ($A \prec^{\alpha} B$), then $A \preceq^{\alpha}_{\varepsilon} B$ ($A \prec^{\alpha}_{\varepsilon} B$, respectively).

Based on the above set less relations, we suggest concepts of approximate and associated solutions of (SOP), say *associated α-weakly efficient solutions*, *associated α-efficient solutions* and *approximately α-efficient solutions* and (denoted by $\alpha-\mathrm{WEff}^{\varepsilon}(\Phi,\Omega)$, $\alpha-\mathrm{Eff}^{\varepsilon}(\Phi,\Omega)$ and $\alpha-\mathrm{Eff}_{\varepsilon}(\Phi,\Omega)$, respectively) as follows

$$\begin{aligned}
\alpha-\mathrm{WEff}^{\varepsilon}(\Phi,\Omega) &= \{\bar{x}\in\Omega : \forall x\in\Omega : \Phi(x) \prec^{\alpha}_{\varepsilon} \Phi(\bar{x}) \Rightarrow \Phi(\bar{x}) \prec^{\alpha}_{\varepsilon} \Phi(x)\},\\
\alpha-\mathrm{Eff}^{\varepsilon}(\Phi,\Omega) &= \{\bar{x}\in\Omega : \forall x\in\Omega : \Phi(x) \preceq^{\alpha}_{\varepsilon} \Phi(\bar{x}) \Rightarrow \Phi(\bar{x}) \preceq^{\alpha}_{\varepsilon} \Phi(x)\},\\
\alpha-\mathrm{Eff}_{\varepsilon}(\Phi,\Omega) &= \{\bar{x}\in\Omega : \forall x\in\Omega : \Phi(x) \preceq^{\alpha} \Phi(\bar{x}) \Rightarrow \Phi(\bar{x}) \preceq^{\alpha}_{\varepsilon} \Phi(x)\}.
\end{aligned}$$

Now, we consider an example to illustrate the computation of the solution sets introduced above, demonstrating their existence and feasibility. For simplicity, to highlight the distinction between different types of solutions, we examine the case of the problem in a finite-dimensional real space with a single-valued objective function.

Example 3.2. Let $\mathbb{X}=\mathbb{R}$, $\mathbb{Y}=\mathbb{R}^2$ with the norm defined by $\|(x_1,x_2)\| := |x_1|+|x_2|$. Let $K=\mathbb{R}^2_+$, $X=\mathbb{X}$, $\Omega_1=X$, $\Omega_2=[-2,2]$, $K=\mathbb{R}^2_+$, $\Omega_3=[-1,1]$, and $\varepsilon\in]0,1]$. Let $\varphi,\psi : X\to\mathbb{Y}$ be defined by

$$\varphi(x) := \begin{cases} (x,1) & \text{if } x\in]-\infty,-1[,\\ (x,-x) & \text{if } x\in[-1,1],\\ (x,-1) & \text{if } x\in]1,+\infty[,\end{cases}$$

and

$$\psi(x) := \begin{cases} (x,-1) & \text{if } x\in]-\infty,-1[,\\ (x,x) & \text{if } x\in[-1,1],\\ (x,1) & \text{if } x\in]1,+\infty[.\end{cases}$$

By direct computations, we obtain

$$0-\mathrm{Eff}(\varphi,\Omega_1)=]-1,1],\ 0-\mathrm{Eff}^{\varepsilon}(\varphi,\Omega_1)=]-1+\varepsilon,1],\ 0-\mathrm{Eff}_{\varepsilon}(\varphi,\Omega_1)=]-1,1+\varepsilon],$$

$$0-\mathrm{Eff}(\psi,\Omega_2)=[-2,-2+\varepsilon[,\ 0-\mathrm{Eff}^{\varepsilon}(\psi,\Omega_2)=[-2,-2+\varepsilon],\ 0-\mathrm{Eff}_{\varepsilon}(\psi,\Omega_2)=[-2,-2+\varepsilon],$$

$$0-\mathrm{WEff}(\varphi,\Omega_1)=\Omega_1,\ 0-\mathrm{WEff}^{\varepsilon}(\varphi,\Omega_1)=[-1+\varepsilon,1],$$

$$0-\mathrm{WEff}(\psi,\Omega_2)=[-2,-1],\ 0-\mathrm{WEff}(\psi,\Omega_3)=\{-1\},\ 0-\mathrm{WEff}^{\varepsilon}(\psi,\Omega_2)=\{-2\},$$

$$0-\mathrm{WEff}^{\varepsilon}(\psi,\Omega_3)=[-1,-1+2^{-1}\varepsilon[.$$

Now, we continue to examine the relationships among the solution sets considered in this study to facilitate the investigation of the convergence for the problem ($\mathrm{SOP}_{\alpha}(\Phi,\Omega)$).

Lemma 3.4. *The following statements hold*

(a) $\alpha-\mathrm{Eff}^{0}(\Phi,\Omega)=\alpha-\mathrm{Eff}_{0}(\Phi,\Omega)=\alpha-\mathrm{Eff}(\Phi,\Omega)$.

(b) $\alpha-\mathrm{Eff}(\Phi,\Omega)\subseteq\alpha-\mathrm{Eff}_{\varepsilon}(\Phi,\Omega)$.

(c) $\alpha-\mathrm{Eff}_{\varepsilon_1}(\Phi,\Omega)\subseteq\alpha-\mathrm{Eff}_{\varepsilon_2}(\Phi,\Omega)$, *whenever* $\varepsilon_2\geq\varepsilon_1$.

(d) *If Φ is compact valued on Ω, then*

$$\alpha-\mathrm{Eff}(\Phi,\Omega)=\bigcap_{\varepsilon>0}[\alpha-\mathrm{Eff}_{\varepsilon}(\Phi,\Omega)],\ \textit{and}$$

$$\bigcap_{\varepsilon>0}[\alpha-\mathrm{Eff}^{\varepsilon}(\Phi,\Omega)]\subseteq\alpha-\mathrm{Eff}(\Phi,\Omega).$$

Proof. Since statements (a)-(c) are straightforward to verify, we leave their proofs to the reader. Here, we provide a detailed proof for case (d) only.

Let $\bar{x}$ be an arbitrary point in $\bigcap_{\varepsilon>0}[\alpha-\mathrm{Eff}_\varepsilon(\Phi,\Omega)]$. Then, for any sequence $\{\varepsilon_n\}$ with $\varepsilon_n\downarrow 0$ we always have $\bar{x}\in\alpha-\mathrm{Eff}_{\varepsilon_n}(\Phi,\Omega)$ for all $n\in\mathbb{N}$. We assume that there exists $z\in\Omega$ such that $\Phi(z)\preceq^{\alpha}\Phi(\bar{x})$. Hence, we achieve $\Phi(\bar{x})\preceq_{\varepsilon_n}\Phi(z)$, and hence there exists $k_n\in K$ such that

$$k_n+(1-\alpha)\Phi(\bar{x})-\alpha\Phi(z)\subseteq(1-\alpha)\Phi(z)-\alpha\Phi(\bar{x})+\varepsilon_n\bar{\mathbb{B}}. \tag{3.10}$$

For each $v_0\in(1-\alpha)\Phi(\bar{x})-\alpha\Phi(z)$, there exist $v_n\in(1-\alpha)\Phi(z)-\alpha\Phi(\bar{x})$ and $b_n\in\varepsilon_n\bar{\mathbb{B}}$ such that

$$k_n+v_0=v_n+b_n \text{ for all } n\in\mathbb{N}. \tag{3.11}$$

By the compactness of $(1-\alpha)\Phi(z)-\alpha\Phi(\bar{x})$, we get a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ converging to $\bar{v}_0\in(1-\alpha)\Phi(z)-\alpha\Phi(\bar{x})$. Additionally, from $\varepsilon_n\downarrow 0$, we get $b_n\to 0$, which together with (3.11) gives us $k_{n_k}=v_{n_k}-v_0+b_{n_k}\to\bar{v}_0-v_0:=k_0\in K$, as K is closed. We next show that

$$k_0+(1-\alpha)\Phi(\bar{x})-\alpha\Phi(z)\subseteq(1-\alpha)\Phi(z)-\alpha\Phi(\bar{x}).$$

Suppose that we can find $v_1\in(1-\alpha)\Phi(\bar{x})-\alpha\Phi(z)$ such that

$$k_0+v_1\notin(1-\alpha)\Phi(z)-\alpha\Phi(\bar{x}). \tag{3.12}$$

By (3.10), we have $k_n+v_1\in(1-\alpha)\Phi(z)-\alpha\Phi(\bar{x})+\varepsilon_n\bar{\mathbb{B}}$. By the same arguments as above, we also have $k_0+v_1\in(1-\alpha)\Phi(z)-\alpha\Phi(\bar{x})$. This contradicts (3.12). Thus, for any v_0, we derive $k_0+v_0\in(1-\alpha)\Phi(z)-\alpha\Phi(\bar{x})$. Equivalently, we have $\Phi(\bar{x})\preceq^{\alpha}\Phi(z)$. and hence $\bar{x}\in\alpha-\mathrm{Eff}(\Phi,\Omega)$. Namely,

$$\bigcap_{\varepsilon>0}[\alpha-\mathrm{Eff}_\varepsilon(\Phi,\Omega)]\subseteq\alpha-\mathrm{Eff}(\Phi,\Omega). \tag{3.13}$$

Combining Lemma 3.4(b) and (3.13), we obtain

$$\bigcap_{\varepsilon>0}[\alpha-\mathrm{Eff}_\varepsilon(\Phi,\Omega)]=\alpha-\mathrm{Eff}(\Phi,\Omega).$$

By the same arguments, we also obtain $\bigcap_{\varepsilon>0}[\alpha-\mathrm{Eff}^{\varepsilon}(\Phi,\Omega)]\subseteq\alpha-\mathrm{Eff}(\Phi,\Omega)$. □

The following example is provided to demonstrate that the converse inclusions of the above conclusions (b), (c) are not true in general.

Example 3.3. Let $\mathbb{X}=\mathbb{R}$, $\mathbb{Y}=\mathbb{R}^2$, $X=\mathbb{R}_+$, $\Omega=[0,1]$, and $K=\mathbb{R}^2_+$. Let $\Phi:X\rightrightarrows\mathbb{Y}$ be defined by

$$\Phi(x):=\{(z_1,z_2)\in\mathbb{R}^2:(z_1-x)^2+z_2^2\leq 1\}.$$

By direct computations, we have

$$\alpha-\mathrm{Eff}(\Phi,\Omega)=\{0\};\ \alpha-\mathrm{Eff}_\varepsilon(\Phi,\Omega)=[0,\varepsilon].$$

It is clear that the reverse inclusions in conclusions (b) and (c) mentioned above do not always hold.

The main contributions of this section are twofold. First, we propose the unification of various types of semicontinuities and set-based relations based on the Minkowski difference, with the corresponding terms being α-continuity and $\preceq^{\alpha}$. Second, we demonstrate that the concept of α-continuity and the set-based relation $\preceq^{\alpha}$ share many similar properties, and simultaneously,

each satisfies the necessary criteria to meet standard requirements for corresponding objects of the same class.

4. Existence Conditions for (SOP)

In this section, we aim to unify the lower and upper semicontinuity of set-valued maps to discuss the existence conditions for set optimization problems involving the set less relation $\preceq^{\alpha}$ when the feasible solution sets are not required to be compact. We begin our discussion with a generalized α-continuity of a set-valued mapping, which resembles a weakened version of the unification forms provided in the previous section. Let $\mathbb{X}, \mathbb{Y}, K, \Omega$ and the set valued mapping Φ be defined as in Section 3.

Definition 4.1. Let $\mathscr{R}$ be a binary relation on $\mathscr{P}(\mathbb{Y})$, denoted by $(\mathscr{P}(\mathbb{Y}), \mathscr{R})$. The map $\Phi : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be

(a) *$\mathscr{R}$-pseudo demi-lower continuous* at $\hat{x} \in \operatorname{dom}\Phi$ if, for each sequence $\{x_n\} \subset \operatorname{dom}\Phi$ converging to $\hat{x}$ and $\Phi(x_{n+1})\mathscr{R}\Phi(x_n)$ for all $n \in \mathbb{N}$, $\Phi(\hat{x})\mathscr{R}\Phi(x_1)$.

(b) *$\mathscr{R}$-pseudo demi-upper continuous* at $\hat{x} \in \operatorname{dom}\Phi$ if, for each sequence $\{x_n\}$ with $x_n \to \hat{x}$ and $\Phi(x_n)\mathscr{R}\Phi(x_{n+1})$ for all $n \in \mathbb{N}$, $\Phi(x_1)\mathscr{R}\Phi(\hat{x})$.

To illustrate the motivation behind the above concepts, we discuss some meaningful special cases of the binary relation $\mathscr{R}$ provided in the following remark.

Remark 4.1. (a) Case $\mathscr{R} = \preceq^{l}$. Then,
$\diamond$ if $\Phi(\hat{x}) + K$ is closed and Φ is K-Hausdorff upper semicontinuous at $\hat{x}$, then Φ is $\preceq^{l}$-pseudo demi-lower continuous at $\hat{x}$;
$\diamond$ if $\Phi(x_1) + K$ is closed and Φ is K-Hausdorff lower semicontinuous at $\hat{x}$, then Φ is $\preceq^{l}$-pseudo demi-upper continuous at $\hat{x}$.

(b) Case $\mathscr{R} = \preceq^{u}$. Then,
$\diamond$ if Φ is $(-K)$-closed valued at x_1 and Φ is $(-K)$-Hausdorff lower semicontinuous at $\hat{x}$, then Φ is $\preceq^{u}$-pseudo demi-lower continuous at $\hat{x}$;
$\diamond$ if Φ is $(-K)$-closed valued at $\hat{x}$, and Φ is $(-K)$-Hausdorff upper semicontinuous at $\hat{x}$, then Φ is $\preceq^{u}$-pseudo demi-upper continuous at $\hat{x}$.

(c) Case $\mathscr{R} = \preceq^{m_1}$. Then,
$\diamond$ if Φ is compact valued at x_1, and Φ is Hausdorff lower semicontinuous at $\hat{x}$, then Φ is $\preceq^{m_1}$-pseudo demi-lower continuous at $\hat{x}$.
$\diamond$ if Φ is Hausdorff upper semicontinuous and compact valued at $\hat{x}$, then Φ is $\preceq^{m_1}$-pseudo demi-upper continuous at $\hat{x}$.

(d) Case $\mathscr{R} = \preceq^{m_2}$. Then,
$\diamond$ if Φ is compact valued and Hausdorff upper semicontinuous at $\hat{x}$, then Φ is $\preceq^{m_2}$-pseudo demi-lower continuous at $\hat{x}$.
$\diamond$ if Φ is compact valued at x_1 and Φ Hausdorff lower continuous at $\hat{x}$, then Φ is $\preceq^{m_2}$-pseudo demi-upper continuous at $\hat{x}$.

In the general case of $\mathscr{R} = \preceq^{\alpha}$, we also obtain relationships between the α-continuity property and the pseudo demi-continuity property shown as the following result.

Proposition 4.1. *Let Φ be a set valued mapping from X into $\mathbb{Y}$ and let $\hat{x}$ be an element of* $\operatorname{dom}\Phi$. *Then, the following statements are true:*

(a) *If Φ is compact and convex valued on* $\operatorname{dom}\Phi$ *and Φ is α-continuous at $\hat{x}$, then Φ is $\preceq^{\alpha}$-pseudo demi-lower continuous at $\hat{x}$.*
(b) *If Φ is compact and convex valued on* $\operatorname{dom}\Phi$ *and Φ is $(1-\alpha)$-continuous at $\hat{x}$, then Φ is $\preceq^{\alpha}$-pseudo demi-upper continuous at $\hat{x}$.*

Proof. It suffices to prove that (a) holds, as the validity of (b) can be established in a similar way. Assume that we have $x_n \to \hat{x}$ and $\Phi(x_{n+1}) \preceq^{\alpha} \Phi(x_n)$ for all $n \in \mathbb{N}$. By the transitive property of $\preceq^{\alpha}$, we have $\Phi(x_n) \preceq^{\alpha} \Phi(x_1)$ for all $n \in \mathbb{N}$. Hence, there exist $k_n \in K$ such that

$$k_n + (1-\alpha)\Phi(x_n) - \alpha\Phi(x_1) \subseteq (1-\alpha)\Phi(x_1) - \alpha\Phi(x_n) \text{ for all } n \in \mathbb{N}. \tag{4.1}$$

Because Φ is α-continuous at $\hat{x}$, for any $\varepsilon > 0$, we derive that, for n large enough,

$$-\alpha\Phi(x_n) + (1-\alpha)\Phi(\hat{x}) \subseteq (1-\alpha)\Phi(x_n) - \alpha\Phi(\hat{x}) + \varepsilon\bar{\mathbb{B}}. \tag{4.2}$$

Combining (4.1) with (4.2), for n large enough, we obtain

$$\begin{aligned} k_n - \alpha\Phi(x_n) + (1-\alpha)\Phi(\hat{x}) - \alpha\Phi(x_1) &\subseteq k_n + (1-\alpha)\Phi(x_n) - \alpha\Phi(\hat{x}) + \varepsilon\bar{\mathbb{B}} - \alpha\Phi(x_1) \\ &\subseteq (1-\alpha)\Phi(x_1) - \alpha\Phi(x_n) - \alpha\Phi(\hat{x}) + \varepsilon\bar{\mathbb{B}}. \end{aligned}$$

Because of the convex and compact valuedness of Φ on $\operatorname{dom}\Phi$, Lemma 2.2 implies that, for n large enough,

$$k_n + (1-\alpha)\Phi(\hat{x}) - \alpha\Phi(x_1) \subseteq (1-\alpha)\Phi(x_1) - \alpha\Phi(\hat{x}) + \varepsilon\bar{\mathbb{B}}. \tag{4.3}$$

Therefore, for any $\hat{z}_0 \in \Phi(\hat{x})$ and $z_0 \in \Phi(x_1)$ we can find $\hat{z}_n \in \Phi(\hat{x})$, $z_n \in \Phi(x_1)$ and $b_n \in \varepsilon\bar{\mathbb{B}}$ such that, for n large enough,

$$k_n + (1-\alpha)\hat{z}_0 - \alpha z_0 = (1-\alpha)z_n - \alpha\hat{z}_n + b_n. \tag{4.4}$$

By the compact valuedness of Φ and the arbitrariness of ε, there exist subsequences $\{z_{n_k}\}$, $\{\hat{z}_{n_k}\}$ of $\{z_{n_k}\}$, $\{\hat{z}_{n_k}\}$ converge to $v_0 \in \Phi(x_1)$, $\hat{v}_0 \in \Phi(\hat{x})$, respectively, and $b_n \to 0$. This along with (4.4) gives us

$$\begin{aligned} k_{n_k} &= (1-\alpha)z_{n_k} - \alpha\hat{z}_{n_k} + b_{n_k} - (1-\alpha)\hat{z}_0 + \alpha z_0 \\ &\overset{k\to+\infty}{\longrightarrow} (1-\alpha)v_0 - \alpha\hat{v}_0 - (1-\alpha)\hat{z}_0 + \alpha z_0 := k_0 \in K. \end{aligned}$$

Suppose that there exist $\hat{z}_1 \in \Phi(\hat{x})$ and $z_1 \in \Phi(x_1)$ such that

$$k_0 + (1-\alpha)\hat{z}_1 - \alpha z_1 \notin (1-\alpha)\Phi(x_1) - \alpha\Phi(\hat{x}). \tag{4.5}$$

By the same arguments as above, we see a contradiction to (4.3). So, $\Phi(\hat{x}) \preceq^{\alpha} \Phi(x_1)$, which means that Φ is pseudo lower-demi continuous at $\hat{x}$. □

In the rest of this section, we aim to discuss the solvability of the problem (SOP) without assuming the compactness of the feasible solution set. Instead, we assume that Φ is convex and compact valued on Ω. Motivated by [1, 3], we first consider the level set defined by

$$\mathscr{L}_{\preceq}(\Phi, \Omega, x_0) = \{x \in \Omega : \Phi(x) \preceq^{\alpha} \Phi(x_0)\}.$$

Then, the following statements are not hard to verify.

Lemma 4.1. *Let $x_0 \in \Omega$, and $\alpha \in [0,1]$. The following statements hold*
(a) *If $\bar{x} \in \alpha - \operatorname{Eff}(\Phi, \mathscr{L}_{\preceq^{\alpha}}(\Phi, \Omega, x_0))$, then $\bar{x} \in \alpha - \operatorname{Eff}(\Phi, \Omega)$.*
(b) *If $\bar{x} \in \alpha - \operatorname{WEff}(\Phi, \mathscr{L}_{\preceq^{\alpha}}(\Phi, \Omega, x_0))$, then $\bar{x} \in \alpha - \operatorname{WEff}(\Phi, \Omega)$.*

The above result puts us in a position to present the main result of this section.

Theorem 4.1. *Let $\alpha \in [0,1]$. Assume that there exists $x_0 \in \Omega$ such that $\mathscr{L}_{\preceq^\alpha}(\Phi,\Omega,x_0)$ is compact, and Φ is $\preceq^\alpha$-pseudo demi-lower continuous on $\mathscr{L}_{\preceq^\alpha}(\Phi,\Omega,x_0)$. Then, $\alpha-\mathrm{Eff}(\Phi,\Omega)$ is nonempty.*

Proof. Let an arbitrary chain $\{\Phi(x_n)\}$ with $\Phi(x_{n+1}) \preceq^\alpha \Phi(x_n)$, $x_n \in \mathscr{L}_{\preceq^\alpha}(\Phi,\Omega,x_0)$ for all $n \in \mathbb{N}$. By the compactness of $\mathscr{L}_{\preceq^\alpha}(\Phi,\Omega,x_0)$, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to $\hat{x} \in \mathscr{L}_{\preceq^\alpha}(\Phi,\Omega,x_0)$. We next prove that $\Phi(\hat{x})$ is a lower bounded of the chain $\{\Phi(x_n)\}$. On the contrary, suppose that there exists $\hat{n} \in \mathbb{N}$ such that $\Phi(\hat{x}) \not\preceq^\alpha \Phi(x_{\hat{n}})$. By the decreasing property of the chain $\{\Phi(x_n)\}$, it follows that

$$\Phi(x_{\hat{n}+k}) \preceq^\alpha \cdots \preceq^\alpha \Phi(x_{\hat{n}+1}) \preceq^\alpha \Phi(x_{\hat{n}}) \text{ for all } k \in \mathbb{N}.$$

Setting $\hat{x}_{k+1} := x_{\hat{n}+k}$, we obtain $\Phi(\hat{x}_{k+1}) \preceq^\alpha \Phi(\hat{x}_k)$ for all $k \in \mathbb{N}$ and $\hat{x}_k \to \hat{x}$. By Definition 4.1(a), it follows that $\Phi(\hat{x}) \preceq^\alpha \Phi(\hat{x}_1) = \Phi(x_{\hat{n}})$, which contradicts our assumption. Therefore, $\Phi(\hat{x})$ is a lower bound of the chain $\{\Phi(x_n)\}$. By Zorn's Lemma, $\alpha-\mathrm{Eff}(\Phi,\mathscr{L}_{\preceq^\alpha}(\Phi,\Omega,x_0))$ is a nonempty set. This together with Lemma 4.1(a) give us the conclusion as in Theorem 4.1. □

To illustrate the applicability of Theorem 4.1, we consider the following example.

Example 4.1. Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, and $X = K = \Omega = \mathbb{R}_+$. Let Φ be a set-valued mapping from X into $\mathbb{Y}$ defined by

$$\Phi(x) := [x-1, x+1] \text{ for all } x \in X.$$

By the basic computations, we get

$$\begin{aligned}(1-\alpha)\Phi(x) - \alpha\Phi(y) &= [(1-\alpha)(x-1) - \alpha(y+1), (1-\alpha)(x+1) - \alpha(y-1)] \\ &= (1-\alpha)x - \alpha y + \bar{\mathbb{B}} \text{ for all } x, y \in X.\end{aligned}$$

From this, we obtain the statement: $\Phi(x) \preceq^\alpha \Phi(y)$ if and only if $x \leq y$ for all $x, y \in X$. It follows that $\mathscr{L}_{\preceq^\alpha}(\Phi,\Omega,x_0) = [0,x_0]$ for all $x_0 \in \Omega$, which is a compact set. Moreover, it can be verified that Φ is $\preceq^\alpha$-pseudo demi-lower continuous on Ω for all $\alpha \in [0,1]$. Therefore, all the assumptions of Theorem 4.1 are satisfied, and hence the reference problem is solvable. In fact, by direct computations, we have $\alpha-\mathrm{Eff}(\Phi,\Omega) = \{0\}$.

Turning to the solvability of (SOP_α) for weakly efficient solutions, we obtain the following result by utilizing Lemma 3.3(b) and Theorem 4.1.

Corollary 4.1. *Assume that all assumptions of Theorem 4.1 are satisfied. Then, $\alpha-\mathrm{WEff}(\Phi,\Omega) \neq \emptyset$.*

Remark 4.2. The main contribution of this section lies in establishing existence conditions for set optimization problems by unifying the notions of lower and upper semicontinuities for set-valued maps. The results obtained here not only offer significant advances regarding the existence of solutions to reference problems with non-compact feasible sets but also highlight the internal consistency and applicability of the proposed unified framework. It is worth emphasizing that even in the two special cases where $\alpha \in \{0,1\}$, the results presented in this section represent new contributions to the solvability conditions of the reference problem.

5. Stability for Weakly Efficient Solution of (SOP)

Let $\mathbb{X}$, $\mathbb{Y}$, X and K, be defined as in Section 4. To present the research findings concisely and make the exposition of this paper more succinct, the remainder of this study introduces the notations and assumptions used in the subsequent sections. Consider set valued maps $F, F_n : X \rightrightarrows \mathbb{Y}$, and a sequence $\{E_n\}$ in $\mathscr{P}(X)$ such that

$$\bigcup_{n\in\mathbb{N}} E_n \subseteq D := \bigcap_{n\in\mathbb{N}} (\operatorname{dom} F_n \cap \operatorname{dom} F).$$

For a given set $E \subseteq D$, we consider the following set optimization problems:

$$\operatorname{Min} F(x) \quad \text{subject to} \quad x \in E, \tag{SOP$_\alpha(F,E)$}$$

and

$$\operatorname{Min} F_n(x) \quad \text{subject to} \quad x \in E_n. \tag{SOP$_\alpha(F_n,E_n)$}$$

For simplicity in presentation, we use the following assumptions regarding the data of the problems mentioned above.

$(\mathscr{A}_1)$: $E_n \xrightarrow{K} E$;
$(\mathscr{A}_2)$: $E_n \xrightarrow{H} E$, E is a compact set;
$(\mathscr{A}_3)$: $F_n \xrightarrow{H} F$ on D;
$(\mathscr{A}_4)$: F is α and $(1-\alpha)$-continuous on E;
$(\mathscr{A}_5)$: F is bounded valued on E;
$(\mathscr{A}_5^*)$: F is convex and compact valued on E;
$(\mathscr{A}_6)$: F_n is bounded valued on E_n for every $n \in \mathbb{N}$;
$(\mathscr{A}_6^*)$: F_n is convex and compact valued on E_n for every $n \in \mathbb{N}$.

We first prove the preservation through limits of the relation $\preceq^\alpha$ for all $\alpha \in [0,1]$ as follows.

Lemma 5.1. *For $\alpha \in]0,1[$, assume that $(\mathscr{A}_3)$, $(\mathscr{A}_4)$, $(\mathscr{A}_5^*)$ and $(\mathscr{A}_6)$ are satisfied. If for any sequence $\{(x_n,y_n)\}$ converging to $(x_0,y_0) \in E \times E$ with $x_n, y_n \in E_n$ and $F_n(x_n) \preceq^\alpha F_n(y_n)$ for all $n \in \mathbb{N}$, then $F(x_0) \preceq^\alpha F(y_0)$. In particular, when $\alpha \in \{0,1\}$, if assumptions $(\mathscr{A}_3)$, $(\mathscr{A}_4)$ are satisfied and F is compact valued on E, then, for any sequence $\{x_n,y_n\}$ converging to $(x_0,y_0) \in E \times E$ with $x_n, y_n \in E_n$ and $F_n(x_n) \preceq^\alpha F_n(y_n)$ for all $n \in \mathbb{N}$, $F(x_0) \preceq^\alpha F(y_0)$.*

Proof. Due to assumption $(\mathscr{A}_4)$ and Lemma 3.1, we imply that, for any $\varepsilon > 0$ and for n large enough,

$$-\alpha F_n(x_n) + (1-\alpha)F(x_0) \subseteq (1-\alpha)F_n(x_n) - \alpha F(x_0) + 2^{-1}\varepsilon\mathbb{B}, \text{ and} \tag{5.1}$$

$$(1-\alpha)F_n(y_n) - \alpha F(y_0) \subseteq -\alpha F_n(y_n) + (1-\alpha)F(y_0) + 2^{-1}\varepsilon\mathbb{B}. \tag{5.2}$$

Adding (5.1) and (5.2) side by side, we obtain that, for n large enough,

$$\begin{aligned}(1-\alpha)F(x_0) - \alpha F(y_0) + (1-\alpha)F_n(y_n) - \alpha F_n(x_n)\\ \subseteq -\alpha F_n(y_n) + (1-\alpha)F(y_0) + (1-\alpha)F_n(x_n) - \alpha F(x_0) + \varepsilon\mathbb{B}.\end{aligned}$$

Combining this with $F_n(x_n) \preceq^\alpha F_n(y_n)$, we conclude that, for n large enough, there exists $k_n \in K$ satisfying

$$\begin{aligned}k_n + (1-\alpha)F(x_0) - \alpha F(y_0) + (1-\alpha)F_n(y_n) - \alpha F_n(x_n)\\ \subseteq (1-\alpha)F(y_0) - \alpha F(x_0) - \alpha F_n(x_n) + (1-\alpha)F_n(y_n) + \varepsilon\bar{\mathbb{B}}.\end{aligned}$$

By assumptions $(\mathscr{A}_5^*)$ and $(\mathscr{A}_6)$, we imply that set $-\alpha F_n(x_n)+(1-\alpha)F_n(y_n)$ is bounded, and set $(1-\alpha)F(y_0)-\alpha F(x_0)+\varepsilon\bar{\mathbb{B}}$ is a closed convex set. Therefore, by Lemma 2.2, we imply that, for n large enough,

$$k_n+(1-\alpha)F(x_0)-\alpha F(y_0)\subseteq(1-\alpha)F(y_0)-\alpha F(x_0)+\varepsilon\bar{\mathbb{B}}. \tag{5.3}$$

Thus, for sufficiently large n, for any $z_0\in F(x_0)$, $v_0\in F(y_0)$, there exist $z_n\in F(x_0)$, $v_n\in F(y_0)$ and $b_n\in\varepsilon\bar{\mathbb{B}}$ satisfying

$$k_n+(1-\alpha)z_0-\alpha v_0=(1-\alpha)z_n-\alpha v_n+b_n. \tag{5.4}$$

Because of the compact valuedness of F at (x_0,y_0), there exists subsequences $\{(z_{n_j},v_{n_j})\}$ of $\{(z_n,v_n)\}$ converging to $(\bar{z}_0,\bar{v}_0)\in F(x_0)\times F(y_0)$. Hence, by (5.4) and the arbitrariness of ε, we obtain that $k_{n_j}:=[(1-\alpha)z_{n_j}-\alpha v_{n_j}+b_{n_j}-(1-\alpha)z_0+\alpha v_0]$ converges to $k_0:=[(1-\alpha)\bar{z}_0-\alpha\bar{v}_0-(1-\alpha)z_0+\alpha v_0]\in K$, as $j\to+\infty$. Suppose that there exist $z_1\in F(x_0)$, and $z_2\in F(y_0)$ such that

$$k_0+(1-\alpha)z_1-\alpha v_1\notin(1-\alpha)F(y_0)-\alpha F(x_0). \tag{5.5}$$

Then, by (5.3), we obtain some $\hat{b}_n\in\varepsilon\bar{\mathbb{B}}$ satisfying

$$k_n+(1-\alpha)z_1-\alpha v_1-\hat{b}_n\in(1-\alpha)F(y_0)-\alpha F(x_0).$$

Due to the compact valuedness of F and the arbitrariness of ε, when $j\to+\infty$, we imply that

$$k_{n_j}+(1-\alpha)z_1-\alpha v_1-\hat{b}_{n_j}\to k_0+(1-\alpha)z_1-\alpha v_1\in(1-\alpha)F(y_0)-\alpha F(x_0),$$

this contradicts (5.5). □

To see the convergence of the weak efficient solutions and efficient solutions of (SOP$_\alpha(F,E)$), inspired by [30], we introduce the concepts of the domination property, which is used to analyze the stability of the reference problem.

Definition 5.1. For $\alpha\in[0,1]$, the problem (SOP$_\alpha(\Phi,\Omega)$) is said to have a

(a) *weak domination property* if, for each $x\in\Omega$, then either $x\in\alpha-\mathrm{WEff}(\Phi,\Omega)$ or there exists $\bar{x}\in\alpha-\mathrm{WEff}(\Phi,\Omega)$ satisfying $\Phi(\bar{x})\prec^\alpha\Phi(x)$.

(b) *domination property* if, for each $x\in\Omega$, then either $x\in\alpha$-$\mathrm{Eff}(\Phi,\Omega)$ or there exists an element $\bar{x}\in\alpha$-$\mathrm{Eff}(\Phi,\Omega)$ satisfying $\Phi(\bar{x})\preceq^\alpha\Phi(\tilde{x})$.

Remark 5.1. Note that, in studying stability conditions for set optimization problems, the lower convergence of efficient solutions and the upper convergence of weakly efficient solutions are often addressed through corresponding conditions on the problem data (see [8, 17, 22] for details). However, the lower convergence of weakly efficient solutions and the upper convergence of efficient solutions remain highly challenging issues. Therefore, to establish the lower convergence of weakly efficient solutions, researchers commonly rely on appropriate conditions under which weakly efficient solutions coincide with efficient solutions. For example, in Theorem 4.4 of [22], the authors established the lower convergence of efficient solutions for set optimization problems involving the lower set less relation via a strict quasiconvexity condition on the objective mappings. Similarly, in Theorem 4.2 of both [8] and [17] , this condition is replaced by strict quasiconnectedness.

Unlike the above works, we propose a new approach to address the lower convergence of weakly efficient solutions for the reference problems.

Theorem 5.1. *Let $\alpha \in]0,1[$. Assume that assumptions $(\mathscr{A}_1)-(\mathscr{A}_4)$, $(\mathscr{A}_5^*)$ and $(\mathscr{A}_6)$ are satisfied, and assume further that, for each $n \in \mathbb{N}$, problem $(\mathrm{SOP}_\alpha(F_n, E_n))$ has the weak domination property. Then, for any $x_0 \in \alpha - \mathrm{WEff}(F,E)$, one of the following statements occurs:*

(a) *There exists a sequence $\{x_n\}$ with $x_n \in \alpha - \mathrm{WEff}(F_n, E_n)$ converging to x_0.*
(b) *There exists a sequence $\{\bar{x}_{n_k}\}$ with $\bar{x}_{n_k} \in \alpha - \mathrm{WEff}(F_{n_k}, E_{n_k})$ converging to some $\bar{x}$, where $\bar{x}$ is an improvement solution of x_0 in the sense that $\bar{x} \in \alpha - \mathrm{WEff}(F,E)$ and $F(\bar{x}) \preceq^\alpha F(x_0)$.*

Proof. For any $x_0 \in \alpha - \mathrm{WEff}(F,E)$, by assumption $(\mathscr{A}_1)$, we can find a sequence $\{x_n\}$ with $x_n \in E_n$, which converges to x_0. If $x_n \in \alpha - \mathrm{WEff}(F_n, E_n)$ for all n, then we obtain the conclusion (a). Otherwise, we can get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin \alpha - \mathrm{WEff}(F_{n_k}, E_{n_k})$ for all n_k. By the weak domination property of $((\mathrm{SOP})_\alpha(F_{n_k}, E_{n_k}))$, there exist $\bar{x}_{n_k} \in \alpha - \mathrm{WEff}(F_{n_k}, E_{n_k})$ such that

$$F_{n_k}(\bar{x}_{n_k}) \prec^\alpha F_{n_k}(x_{n_k}). \tag{5.6}$$

Due to assumption $(\mathscr{A}_2)$, we can assume that $\{\bar{x}_n\}$ converges to $\bar{x} \in E$. Therefore, by Lemma 5.1 and (5.6), we derive that

$$F(\bar{x}) \preceq^\alpha F(x_0). \tag{5.7}$$

Moreover, since the set less relation $\preceq^\alpha$ is transitive, we conclude that $\bar{x} \in \alpha - \mathrm{WEff}(F,E)$ as $x_0 \in \alpha - \mathrm{WEff}(F,E)$, and so the second assertion is derived. □

Note that, with the general relation $\preceq^\alpha$ and the specific relations $\preceq^{m_1}$ and $\preceq^{m_2}$, a sequence of inequalities $A_i \preceq^\alpha B_i$ typically generates a sequence $\{k_i\} \subset K$ that is compatible with the sequence of inequalities, rather than a single order cone K shared by all inequalities, as is the case for the relations $\preceq^u$ and $\preceq^l$. Consequently, the techniques and approaches for set optimization problems involving $\preceq^u$ and $\preceq^l$, particularly regarding convergence conditions, are not applicable to set optimization problems involving the relation $\preceq^\alpha$. This is the key point that leads to a minor logical flaw in the proof of Theorem 3.3 in [8], particularly concerning the induction step involving the limit with respect to the relation $\prec^{m_2}$. Specifically, under the condition $F(x) \prec^{m_2} F(y)$, the authors concluded that for any sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y, respectively, $F_n(x_n) \prec^{m_2} F_n(y_n)$ for sufficiently large n. Unfortunately, this result is later employed as a key step in analyzing the convergence conditions for weakly efficient solutions of set optimization problems involving the relation $\preceq^{m_2}$. As a consequence, some of the related conclusions in [8] may be influenced by this subtle issue. The following counterexample shows that the conditions imposed in Theorem 3.3 of [8] are insufficient to guarantee its claimed conclusion.

Example 5.1. Let $\mathbb{X} = \mathbb{R}$, $\mathbb{Y} = \mathbb{R}^2$ and $X = \mathbb{R}_+$, $K = \mathbb{R}^2_+$. Let $F, F_n : X \rightrightarrows \mathbb{Y}$ be defined respectively by

$$F(x) := \{(z_1,z_2) \in \mathbb{R}^2 : (z_1-x)^2 + (z_2-x)^2 \le 1\}, \text{ and}$$
$$F_n(x) := \left\{(z_1,z_2) \in \mathbb{R}^2 : (z_1-x)^2 + (z_2-x)^2 \le 1 - \frac{1}{(n+1)(x+1)}\right\}.$$

It is clear that F, F_n satisfy all assumptions in Theorem 3.3 of [8]. We choose $x_n = x_0 = 1$, and $y_n = y_0 = 0$. It is clear that $F(y_0) \prec^1 F(x_0)$ but $F_n(y_n) \not\prec^1 F_n(x_n)$ for all $n \in \mathbb{N}$.

Based on the above observations, finding new and appropriate approaches to handle the convergence for set optimization problems concerning these relations is crucial and serves as the primary motivation for our research. In line with this direction, we aim to propose an approach

to establish convergence conditions for set optimization problems involving the relation $\preceq^{\alpha}$ through a type of associated solutions for the reference problems.

Theorem 5.2. *Let $\alpha \in]0,1[$, $\varepsilon_0 > 0$, and $x_n \in \bigcap_{0<\varepsilon<\varepsilon_0} \alpha - \mathrm{WEff}^{\varepsilon}(F_n, E_n)$. If assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_3)$, $(\mathscr{A}_4)$, $(\mathscr{A}_5^*)$, and $(\mathscr{A}_6^*)$ are satisfied, then every cluster point of $\{x_n\}$ belongs to $\alpha - \mathrm{WEff}(F,E)$.*

Proof. Let x_0 be a cluster point of $\{x_n\}$, that is, there exists some subsequence $\{x_{n_i}\}$ such that x_{n_i} converges to $\bar{x}$. We show that $x_0 \in \alpha - \mathrm{WEff}(F,E)$. Suppose, to the contrary, that there exists $z \in E$ such that $F(z) \prec^{\alpha} F(x_0)$, or equivalently there exists an element $k \in \operatorname{int} K$ satisfying

$$k + (1-\alpha)F(z) - \alpha F(x_0) \subseteq (1-\alpha)F(x_0) - \alpha F(z). \tag{5.8}$$

Due to assumption $(\mathscr{A}_1)$, we can find a sequence $\{z_n\}$ converging to z with $z_n \in E_n$ for all n. Hence, by assumption $(\mathscr{A}_4)$, for i large enough, we get

$$-\alpha F_{n_i}(x_{n_i}) + (1-\alpha)F(x_0) \subseteq -\alpha F(x_0) + (1-\alpha)F_{n_i}(x_{n_i}) + 2^{-1}\varepsilon\bar{\mathbb{B}}, \text{ and} \tag{5.9}$$

$$(1-\alpha)F_{n_i}(z_{n_i}) - \alpha F(z) \subseteq (1-\alpha)F(z) - \alpha F_{n_i}(z_{n_i}) + 2^{-1}\varepsilon\bar{\mathbb{B}}. \tag{5.10}$$

Combining (5.8), (5.9) with (5.10), for i large enough, we obtain

$$\begin{aligned} k+&(1-\alpha)F_{n_i}(z_{n_i}) - \alpha F_{n_i}(x_{n_i}) - \alpha F(z) + (1-\alpha)F(x_0) \\ &\subseteq (1-\alpha)F_{n_i}(x_{n_i}) - \alpha F_{n_i}(z_{n_i}) + (1-\alpha)F(x_0) - \alpha F(z) + \varepsilon\bar{\mathbb{B}}. \end{aligned} \tag{5.11}$$

Due to $(\mathscr{A}_5)$ and $(\mathscr{A}_6^*)$, Lemma 2.2 provides that, for i large enough,

$$k + (1-\alpha)F_{n_i}(z_{n_i}) - \alpha F_{n_i}(x_{n_i}) \subseteq (1-\alpha)F_{n_i}(x_{n_i}) - \alpha F_{n_i}(z_{n_i}) + \varepsilon\bar{\mathbb{B}}.$$

Equivalently, $F_{n_i}(z_{n_i}) \prec_{\varepsilon}^{\alpha} F_{n_i}(x_{n_i})$ for i sufficiently large. This along with $x_{n_i} \in \bigcap_{0<\varepsilon<\varepsilon_0} \alpha - \mathrm{WEff}^{\varepsilon}(F_{n_i}, E_{n_i}))$ gives us $F_{n_i}(x_{n_i}) \prec_{\varepsilon}^{\alpha} F_{n_i}(z_{n_i})$ for i sufficiently large. Hence, for i large enough, we obtain that $F_{n_i}(x_{n_i}) \preceq_{\varepsilon}^{\alpha} F_{n_i}(z_{n_i})$. Consequently, there exist a sequence $\{k_n\} \subset K$ and a subsequence $\{k_{n_i}\}$ of $\{k_n\}$ satisfying

$$k_{n_i} + (1-\alpha)F_{n_i}(x_{n_i}) - \alpha F_{n_i}(z_{n_i}) \subseteq (1-\alpha)F_{n_i}(z_{n_i}) - \alpha F_{n_i}(x_{n_i}) + \varepsilon\bar{\mathbb{B}}. \tag{5.12}$$

Using assumption $(\mathscr{A}_4)$, for i large enough, we conclude

$$-\alpha F_{n_i}(z_{n_i}) + (1-\alpha)F(z) \subseteq -\alpha F(z) + (1-\alpha)F_{n_i}(z_{n_i}) + 2^{-1}\varepsilon\bar{\mathbb{B}}, \text{ and} \tag{5.13}$$

$$(1-\alpha)F_{n_i}(x_{n_i}) - \alpha F(x_0) \subseteq (1-\alpha)F(x_0) - \alpha F_{n_i}(x_{n_i}) + 2^{-1}\varepsilon\bar{\mathbb{B}}. \tag{5.14}$$

It follows from (5.12), (5.13), and (5.14) that

$$k_{n_i} + (1-\alpha)F(x_0) - \alpha F(z) \subset (1-\alpha)F(z) - \alpha F(x_0) + 2\varepsilon\bar{\mathbb{B}} \text{ for } i \text{ sufficiently large.}$$

By a similar arguments as in the proof of Lemma 5.1, we obtain $F(x_0) \preceq^{\alpha} F(z)$, or equivalently, there exists $k_1 \in K$ satisfying

$$k_1 + (1-\alpha)F(x_0) - \alpha F(z) \subseteq (1-\alpha)F(z) - \alpha F(x_0). \tag{5.15}$$

By (5.8) and (5.15), we see that

$$\begin{aligned} k + k_1 + (1-\alpha)F(z) - \alpha F(x_0) &\subseteq k_1 + (1-\alpha)F(x_0) - \alpha F(z) \\ &\subseteq (1-\alpha)F(z) - \alpha F(x_0). \end{aligned}$$

By assumption $(\mathscr{A}_5)$ and Lemma 2.2, we obtain that $k + k_1 = 0_{\mathbb{Y}}$, which is impossible as $k_1 \in K$ and $k \in \operatorname{int} K$. Therefore, $x_0 \in \alpha - \mathrm{WEff}(F,E)$. □

One of the key points in the proof of Theorems 5.1 and 5.2 is the need to apply the Rådström cancellation lemma to simplify the complex inclusions, and thus we must employ the conditions of convex, closed, and bounded values for the objective mappings. In the special case where $\alpha \in \{0,1\}$, the use of the Rådström cancellation lemma is no longer necessary, and hence the conditions mentioned above can be relaxed. Since the proof techniques for the convergence results in this special case are closely related to the proofs of Theorems 5.1 and 5.2, and to avoid repetition and prevent unnecessary confusion, we leave the proof of the following results to the reader.

Theorem 5.3. *Let $\alpha \in \{0,1\}$ and $\varepsilon_0 > 0$. The following statements hold*

(i) *Assume that assumptions $(\mathscr{A}_1)-(\mathscr{A}_4)$ are satisfied, F is compact valued on E, and for each n the problem $(\mathrm{SOP}_\alpha)(F_n,E_n)$ has the weak domination property. Then, for any $x_0 \in \alpha-\mathrm{WEff}(F,E)$, one of the following statements occurs*
   (a) *There exists a sequence $\{x_n\}$ with $x_n \in \alpha-\mathrm{WEff}(F_n,E_n)$ converging to x_0.*
   (b) *There is a sequence $\{\bar{x}_{n_k}\}$ with $\bar{x}_{n_k} \in \alpha-\mathrm{WEff}(F_{n_k},E_{n_k})$ converging to some $\bar{x}$, where $\bar{x}$ is an improvement solution of x_0 in the sense that $\bar{x} \in \alpha-\mathrm{WEff}(F,E)$ and $F(\bar{x}) \preceq^{\alpha} F(x_0)$.*

(ii) *Assume that $(\mathscr{A}_1)$, $(\mathscr{A}_3)$ and $(\mathscr{A}_4)$ are satisfied and further that F is compact valued on E. Then, for $\varepsilon > 0$, for any sequence $\{x_n\}$ with $x_n \in \bigcap_{0<\varepsilon<\varepsilon_0} \alpha-\mathrm{WEff}^{\varepsilon}(F_n,E_n)$, its cluster points will belong to $\alpha-\mathrm{WEff}(F,E)$.*

To illustrate the applicability of Theorem 5.3, we provide the following examples.

Example 5.2. Let $\mathbb{X}=\mathbb{Y}=\mathbb{R}$, $X=K=\mathbb{R}_+$, $E=[0,1]$, and $E_n=\left[\sin\left(\frac{\pi}{2n}\right),\cos\left(\frac{\pi}{2n}\right)+\sin\left(\frac{\pi}{2n}\right)\right]$. Let $F,F_n : X \rightrightarrows \mathbb{Y}$ be defined by

$$F(x) := \left[\frac{x^3-x^2+x-2}{x^2+1}, \frac{x^3+x^2+x+2}{x^2+1}\right],$$

and

$$F_n(x) := \left[\frac{x^3-x^2+x-2}{x^2+1}-\frac{1}{n}, \frac{x^3+x^2+x+2}{x^2+1}+\frac{1}{n}\right].$$

Obviously, all the assumptions of Theorem 5.3(i) are satisfied, and so we can get the conclusion of Theorem 5.3(i). Furthermore, by direct computation, we have

$$0-\mathrm{WEff}(F,E)=[0,1]; 0-\mathrm{WEff}(F_n,E_n)=\left[\sin\left(\frac{\pi}{2n}\right),\cos\left(\frac{\pi}{2n}\right)+\sin\left(\frac{\pi}{2n}\right)\right].$$

Example 5.3. Let $\mathbb{X}=\mathbb{R}$, $\mathbb{Y}=\mathbb{R}^2$, $X=\mathbb{R}_+$, $K=\mathbb{R}^2_+$, $E=[0,1]$, $E_n=\left[0,1+\frac{1}{n}\right]$ and $\varepsilon_0=10^{-3}$. Let $F,F_n : X \rightrightarrows \mathbb{Y}$ be defined by

$$F(x) := (x,x)+\bar{\mathbb{B}}, \text{ and } F_n(x) := \left(x,x-\frac{1}{n}\right)+\bar{\mathbb{B}}.$$

It is easy to verify that all the assumptions in Theorem 5.3(ii) are satisfied, and hence the conclusion of Theorem 5.3(ii) is obtained. Moreover, by direct computation, we obtain

$$0-\mathrm{WEff}(F,E)=\{0\}; 0-\mathrm{WEff}^{\varepsilon}(F_n,E_n)=\left[0,\frac{\varepsilon}{\sqrt{2}}\right[\text{ for all } \varepsilon>0.$$

Remark 5.2. In [15], the authors obtained some results on the upper convergence for weakly efficient solutions of (SOP). Specifically, in Theorem 3.3 of [15], the authors derived the convergence condition for weak efficient solutions by using the condition of upper semicontinuity with open values of the objective mappings on the entire feasible set E. It is worthy noting here that the condition of upper semicontinuity with open values on E of the set-valued map F will leads to the fact that: at any point $\bar{x} \in E$, there always exists a neighborhood N of $\bar{x}$ such that $F(\bar{x})$ contains all values of F within this neighborhood. Thus, the arbitrariness of the point seems to imply that the mapping F behaves like a piecewise constant mapping, a very special type of mapping, as shown in Example 3.3 in [15]. This observation makes us not overly surprised by the remarks in [8], which state that a locally Lipschitz continuous mapping F does not guarantee upper semicontinuity. However, if F is upper semicontinuous with compact values, then it is covered by local Lipschitz continuity. Consequently, there is a minor flaw in the reasoning in Examples 2.4 and 2.5 of [8]. Unfortunately, these examples have been used in some subsequent study (see [42]). This observation could explain why upper semicontinuity is often associated with the condition of having compact or closed values.

Remark 5.3. The main contributions of this section are as follows. First, regarding the upper convergence for weakly efficient solutions, we establish sufficient conditions for this property for ($\mathrm{SOP}_\alpha(F,E)$), including the special cases where $\alpha \in \{0,1\}$. Furthermore, the results presented here are new and provide an approach that helps avoid mistakes in analyzing the stability of such problems. Second, with regard to the lower convergence for weakly efficient solutions, we propose a new approach to investigate this property in cases where the sets of weakly efficient solutions and efficient solutions are distinct. These results are new not only in the general case of ($\mathrm{SOP}_\alpha(F,E)$), but also in the special cases for $\alpha \in \{0,1\}$, specifically the reference problems concerning $\preceq^{m_1}$ and $\preceq^{m_2}$.

6. Stability for Efficient Solution of (SOP)

Of course, taking suitable additional conditions such as the strictly quasiconvex condition in [22] or the strictly quasiconnected condition in [8, 17] ensure that the sets of efficient solutions and weakly efficient solutions coincide, and then the convergence results established in the previous section for the weakly efficient solutions will lead to corresponding results for the efficient solutions. With the desire to contribute further to this topic concerning efficient solutions, we will examine their stability in cases where the sets of weakly efficient solutions and efficient solutions are not necessarily the same. We begin this endeavor by studying an important result below.

Theorem 6.1. *Let $\alpha \in]0,1[$. Assume that assumptions $(\mathscr{A}_1)-(\mathscr{A}_4)$, $(\mathscr{A}_5^*)$ and $(\mathscr{A}_6)$ are satisfied, and assume further that, for each n, problem ($\mathrm{SOP}_\alpha(F_n,E_n)$) has the domination property. Then, for any $x_0 \in \alpha\text{-}\mathrm{Eff}(F,E)$, one of the following statements occurs*

(a) *There exists a sequence $\{x_n\}$ with $x_n \in \alpha-\mathrm{Eff}(F_n,E_n)$ converging to x_0.*
(b) *There is a sequence $\{\bar{x}_{n_k}\}$ with $\bar{x}_{n_k} \in \alpha-\mathrm{Eff}(F_{n_k},E_{n_k})$ converging to some $\bar{x}$, where $F(\bar{x}) \preceq^\alpha F(x_0)$.*

In particular, when $\alpha \in \{0,1\}$, if assumptions $(\mathscr{A}_1)-(\mathscr{A}_4)$ are satisfied, F is compact valued on E, and for each n the problem ($\mathrm{SOP}_\alpha(F_n,E_n)$) has the domination property. Then, one of the following statements occurs

(a) *There exists a sequence* $\{x_n\}$ *with* $x_n \in \alpha - \mathrm{Eff}(F_n, E_n)$ *converging to* x_0.
(b) *There exists a sequence* $\{\bar{x}_{n_k}\}$ *with* $\bar{x}_{n_k} \in \alpha - \mathrm{Eff}(F_{n_k}, E_{n_k})$ *converging to some* $\bar{x}$, *where* $F(\bar{x}) \preceq^{\alpha} F(x_0)$.

Proof. Let $x_0 \in \alpha\text{-Eff}(F,E)$ be arbitrary. By $(\mathscr{A}_1)$ and $x_0 \in E$, we can choose a sequence $\{x_n\}$ with $x_n \in E_n$ such that $\{x_n\}$ converges to x_0. If $x_n \in \alpha\text{-Eff}(F_n, E_n)$ for all n, then we obtain the needed conclusion. Otherwise, we can get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin \alpha\text{-Eff}(F_{n_k}, E_{n_k})$ for all n_k. Due to the domination property of problem $(\mathrm{SOP})(F_{n_k}, E_{n_k})$, there exists $\bar{x}_{n_k} \in \mathrm{Eff}(F_{n_k}, E_{n_k})$ such that

$$F_{n_k}(\bar{x}_{n_k}) \preceq^{\alpha} F_{n_k}(x_{n_k}). \tag{6.1}$$

Due to assumption $(\mathscr{A}_2)$, we can obtain a subsequence of $\{\bar{x}_{n_k}\}$ (still denoted by $\{\bar{x}_{n_k}\}$) converging to some $\bar{x} \in E$. Combining (6.1) with Lemma 5.1, we obtain $F(\bar{x}) \preceq^{\alpha} F(x_0)$. Moreover, since x_0 is an efficient solution to $\gamma\text{-Eff}(F,E)$, we have $\bar{x} \in \gamma\text{-Eff}(F,E)$. By using similar arguments as above, we can also derive the remaining statements. □

Theorem 6.2. *For* $\alpha \in]0,1[$, *assume that assumptions* $(\mathscr{A}_1) - (\mathscr{A}_4)$, $(\mathscr{A}_5^*)$, *and* $(\mathscr{A}_6^*)$ *are satisfied. Then, for any* $x_0 \in \alpha - \mathrm{Eff}(F,E)$, *there exist a positive real sequence* $\{\varepsilon_n\}$ *with* $\varepsilon_n \to 0$ *and a sequence* $\{x_{n_k}\}$ *with* $x_{n_k} \in \alpha - \mathrm{Eff}_{\varepsilon_{n_k}}(F_{n_k}, E_{n_k})$ *converging to* x_0. *Furthermore, for* $\alpha \in \{0,1\}$, *assume that assumptions* $(\mathscr{A}_1) - (\mathscr{A}_4)$ *are satisfied and* F *is compact valued on* E. *Then, for any* $x_0 \in \alpha - \mathrm{Eff}(F,E)$ *there exists a positive real sequence* $\{\varepsilon_n\}$ *with* $\varepsilon_n \to 0$ *and a sequence* $\{x_{n_k}\}$ *with* $x_{n_k} \in \alpha - \mathrm{Eff}_{\varepsilon_{n_k}}(F_{n_k}, E_{n_k})$ *converging to* x_0.

Proof. Let x_0 be an arbitrary in $\alpha - \mathrm{Eff}(F,E)$. By assumption $(\mathscr{A}_1)$, we can find a sequence $\{x_n\}$ with $x_n \in E_n$ converging to x_0. Suppose that there exist $z_n \in E_n$ such that $F_n(z_n) \preceq^{\alpha} F_n(x_n)$ for all $n \in \mathbb{N}$, which means that, for each $n \in \mathbb{N}$, we can identify $k_n \in K$ satisfying

$$k_n + (1-\alpha)F_n(z_n) - \alpha F_n(x_n) \subseteq (1-\alpha)F_n(x_n) - \alpha F_n(z_n). \tag{6.2}$$

Due to $(\mathscr{A}_2)$, there exists a sequence $\{z_n\}$ which has a subsequence $\{z_{n_j}\}$ (still denoted by z_j) converging to $z \in E$. Then, assumption $(\mathscr{A}_4)$ implies that, for any $\varepsilon > 0$ and for j large enough,

$$(1-\alpha)F_j(x_j) - \alpha F(x_0) \subseteq (1-\alpha)F(x_0) - \alpha F_j(x_j) + 2^{-1}\varepsilon\bar{\mathbb{B}}, \text{ and} \tag{6.3}$$

$$-\alpha F_j(z_j) + (1-\alpha)F(z) \subseteq -\alpha F(z) + (1-\alpha)F_j(z_j) + 2^{-1}\varepsilon\bar{\mathbb{B}}. \tag{6.4}$$

Combining (6.2) with (6.3) and(6.4), we imply that, for j large enough,

$$\begin{aligned} k_j + (1-\alpha)F(z) - \alpha F(x_0) - \alpha F_j(z_j) + (1-\alpha)F_j(x_j) \\ \subseteq (1-\alpha)F_j(x_j) - \alpha F_j(z_j) - \alpha F(z) + (1-\alpha)F(x_0) + \varepsilon\bar{\mathbb{B}}. \end{aligned}$$

Thanks to assumptions $(\mathscr{A}_5^*)$ and $(\mathscr{A}_6)$, Lemma 2.2 provides us the following conclusion:

$$k_j + (1-\alpha)F(z) - \alpha F(x_0) \subseteq (1-\alpha)F(x_0) - \alpha F(z)) + \varepsilon\bar{\mathbb{B}}. \tag{6.5}$$

Utilizing an analogous argument to that used in the proof of Lemma 5.1, we get $F(z) \preceq^{\alpha} F(x_0)$. Consequently, $F(x_0) \preceq^{\alpha} F(z)$ as $x_0 \in \alpha - \mathrm{Eff}(F,E)$. Therefore, we can identify $k_0 \in K$ such that

$$k_0 + (1-\alpha)F(x_0) - \alpha F(z) \subseteq (1-\alpha)F(z) - \alpha F(x). \tag{6.6}$$

Combining (6.3), (6.4) and (6.6), we derive that, for j large enough,

$$k_0+(1-\alpha)F_j(x_j)-\alpha F_j(z_j)-\alpha F(x_0)+(1-\alpha)F(z)\\ \subseteq (1-\alpha)F(z)-\alpha F(x_0)-\alpha F_j(x_j)+(1-\alpha)F_j(z_j)+\varepsilon\bar{\mathbb{B}}.$$

Then, by assumptions $(\mathscr{A}_5)$ and $(\mathscr{A}_6^*)$, Lemma 2.2 yields that for j large enough,

$$k_0+(1-\alpha)F_j(x_j)-\alpha F_j(z_j)\subseteq -\alpha F_j(x_j)+(1-\alpha)F_j(z_j)+\varepsilon\bar{\mathbb{B}}.$$

Therefore, we conclude that

$$\forall\varepsilon>0,\exists j_\varepsilon,\forall j\geq j_\varepsilon : x_j\in\alpha-\mathrm{Eff}_\varepsilon(F_j,E_j). \tag{6.7}$$

For an arbitrary real positive numbers $\{\varepsilon_j\}$ converging to 0, employing (6.7), we obtain that

$$\forall i\in\mathbb{N},\exists k_i\in\mathbb{N},\forall j\geq k_i : x_j\in\alpha-\mathrm{Eff}_{\varepsilon_i}(F_j,E_j).$$

Without loss of generality, we can assume that the mapping $i\mapsto k_i$ is increasing. Thus, for each $j\geq k_1$, there exists a unique $i_j\in\mathbb{N}^*$ such that $k_{i_j}\leq n<k_{i_j+1}$. By setting $\bar{\varepsilon}_j:=\varepsilon_{i_j}$ for $j=k_1,k_1+1,\ldots$, we obtain the sequence $\{\bar{\varepsilon}_j\}$, which converges to 0 and satisfies $x_j\in\alpha-\mathrm{Eff}_{\bar{\varepsilon}_j}(F_j,E_j)$ for all j. □

We now present an example to demonstrate and validate the applicability of the preceding result.

Example 6.1. Let $\mathbb{X}=\mathbb{R},\mathbb{Y}=\mathbb{R}^2,X=\mathbb{R}_+,K=\mathbb{R}^2_+,E=[1,3],E_n=\left]1,3-\frac{1}{n}\right[$, and $F,F_n:X\rightrightarrows\mathbb{Y}$ be defined by

$$F(x):=\begin{cases}\{(z_1,z_2)\in\mathbb{R}^2:(z_1-x)^2+z_2^2\leq 1\} & \text{if } x\in[0,2],\\ \{(z_1,z_2)\in\mathbb{R}^2:(z_1-x)^2+(z_2+x-2)^2\leq 1\} & \text{if } x\in]2,+\infty[\,,\end{cases}$$

and

$$F_n(x):=\begin{cases}\{(z_1,z_2)\in\mathbb{R}^2:(z_1-x)^2+(z_2+\frac{1}{n})^2\leq 1\} & \text{if } x\in[0,2],\\ \{(z_1,z_2)\in\mathbb{R}^2:(z_1-x)^2+(z_2+x-2+\frac{1}{n})^2\leq 1\} & \text{if } x\in]2,+\infty[\,.\end{cases}$$

Then, all the assumptions of Theorem 6.2 are satisfied, and furthermore by directly computations, we obtain

$$0-\mathrm{Eff}(F,E)=\{1\};0-\mathrm{Eff}(F_n,E_n)=\left]2,\frac{3n-1}{n}\right[\text{ and}$$

$$0-\mathrm{Eff}_\varepsilon(F_n,E_n)=]1,1+\varepsilon]\bigcup\left]2,\frac{3n-1}{n}\right[.$$

It is evident that for $x_0=1\in 0-\mathrm{Eff}(F,E)$, there does not exist any sequences $\{x_n\}$ with $x_n\in 0-\mathrm{Eff}(F_n,E_n)$ that converges to x_0. Fortunately, for any positive real sequence numbers $\{\varepsilon_n\}$ converging to 0, we can choose a sequence $\{x_n\}$ with $x_n=1+\varepsilon_n\in 0-\mathrm{Eff}_{\varepsilon_n}(F_n,E_n)$ converging to x_0.

Turning to the upper convergence of efficient solutions, we introduce the following statement.

Theorem 6.3. *For $\alpha \in]0,1[$, $\varepsilon_0 > 0$, and $x_n \in \bigcap_{0<\varepsilon<\varepsilon_0} \alpha - \mathrm{Eff}^{\varepsilon}(F_n, E_n)$. Assume that $(\mathscr{A}_1)$, $(\mathscr{A}_3)$, $(\mathscr{A}_4)$, $(\mathscr{A}_5^*)$ and $(\mathscr{A}_6^*)$ are satisfied. Then, every cluster point of $\{x_n\}$ belongs to $\alpha - \mathrm{Eff}(F,E)$. Furthermore, for $\alpha \in \{0,1\}$, $\varepsilon_0 > 0$, and $x_n \in \bigcap_{0<\varepsilon<\varepsilon_0} \alpha - \mathrm{Eff}^{\varepsilon}(F_n, E_n)$. If assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_3)$ and $(\mathscr{A}_4)$ hold and F is compact valued on E, then every cluster points of $\{x_n\}$ belongs to $\alpha - \mathrm{Eff}(F,E)$.*

Proof. Let x_0 be a cluster point of $\{x_n\}$. We prove that $x_0 \in \alpha - \mathrm{Eff}(F,E)$. Based on the definition of the cluster point, there exists a subsequence of $\{x_n\}$ (still denoted by $\{x_n\}$) converging to x_0. Assume that we can find $z \in E$ satisfying

$$F(z) \preceq^{\alpha} F(x_0), \tag{6.8}$$

and then there exists $k_0 \in K$ such that

$$k_0 + (1-\alpha)F(z) - \alpha F(x_0) \subseteq (1-\alpha)F(x_0) - \alpha F(z). \tag{6.9}$$

In view of assumption $(\mathscr{A}_1)$, we can identify $z_n \in E_n$ such that $\{z_n\}$ converges to z. Thanks to $(\mathscr{A}_4)$, Lemma 3.1 provides that $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \geq n_\varepsilon$:

$$-\alpha F_n(x_n) + (1-\alpha)F(x_0) \subseteq -\alpha F(x_0) + (1-\alpha)F_n(x_n) + 2^{-1}\varepsilon\bar{\mathbb{B}}, \text{ and} \tag{6.10}$$

$$(1-\alpha)F_n(z_n) - \alpha F(z) \subseteq (1-\alpha)F(z) - \alpha F_n(z_n) + 2^{-1}\varepsilon\bar{\mathbb{B}}. \tag{6.11}$$

Combining (6.9) and (6.10) with (6.11), we obtain that for n large enough

$$\begin{aligned} k_0 + (1-\alpha)F_n(z_n) - \alpha F_n(x_n) + (1-\alpha)F(x_0) - \alpha F(z) \\ \subseteq (1-\alpha)F(x_0) - \alpha F(z) - \alpha F_n(z_n) + (1-\alpha)F_n(x_n) + \varepsilon\bar{\mathbb{B}}. \end{aligned}$$

In view of assumptions $(\mathscr{A}_5^*)$ and $(\mathscr{A}_6^*)$, Lemma 2.2 implies that for n large enough

$$k_0 + (1-\alpha)F_n(z_n) - \alpha F_n(x_n) \subseteq (1-\alpha)F_n(x_n) - \alpha F_n(z_n) + \varepsilon\bar{\mathbb{B}}.$$

Equivalently, $F_n(z_n) \preceq_{\varepsilon}^{\alpha} F_n(x_n)$, and hence we have $F_n(x_n) \preceq_{\varepsilon}^{\alpha} F_n(z_n)$ as

$$x_n \in \bigcap_{0<\varepsilon<\varepsilon_0} \alpha - \mathrm{Eff}^{\varepsilon}(F_n, E_n).$$

Applying a similar reasoning as in the proof of Lemma 5.1, we imply that $F(x_0) \preceq^{\alpha} F(z)$, and so inequality (6.8) leads to $x_0 \in \alpha - \mathrm{Eff}(F,E)$. Therefore, $x_0 \in \alpha - \mathrm{Eff}(F,E)$. □

Remark 6.1. With the aim of highlighting the novelty of our results and approach in studying the upper convergence for efficient solutions of the reference problems, we have attempted to search for related works on this topic. Naturally, if any exist, they would only cover the special cases where $\alpha \in \{0,1\}$. Unfortunately, we have not found any studies addressing this topic. Therefore, we tentatively conclude that the results obtained in this section constitute the first contributions to the upper convergence for set optimization problems concerning set relations based on the Minkowski difference. To conclude our study, we present a illustrative example that showcases the applicability of the obtained results.

Example 6.2. Let $\mathbb{X} = \mathbb{R}, \mathbb{Y} = \mathbb{R}^2, X = \mathbb{R}_+, K = \mathbb{R}_+^2, E = [0,1], E_n = \left[0, 1 + \frac{1}{n}\right]$, $\varepsilon_0 = 10^{-3}$ and let $F, F_n : X \rightrightarrows \mathbb{Y}$ be defined respectively by

$$F(x) := \begin{cases} (x,1) + \bar{\mathbb{B}} & \text{if } x \in [0,1], \\ (x, 2-x) + \bar{\mathbb{B}} & \text{if } x \in]1, +\infty[\,, \end{cases}$$

$$F_n(x) := \begin{cases} \left(x, \frac{1+n}{n}\right) + \bar{\mathbb{B}} & \text{if } x \in [0,1], \\ \left(x, \frac{1+2n-nx}{n}\right) + \bar{\mathbb{B}} & \text{if } x \in]1, +\infty[. \end{cases}$$

Then, all the assumptions of Theorem 6.3, and moreover, performing direct computation yields

$$0-\mathrm{Eff}(F,E) = \{0\}; 0-\mathrm{Eff}(F_n,E_n) = \{0\} \bigcup \left]1, 1+\frac{1}{n}\right],$$

$$0-\mathrm{Eff}^{\varepsilon}(F_n,E_n) = \begin{cases} [0,\varepsilon] \bigcup]1+\varepsilon, 1+\frac{1}{n}] & \text{if } \frac{1}{n} > \varepsilon, \\ [0,\varepsilon] & \text{if } \frac{1}{n} \leq \varepsilon; \end{cases}$$

and

$$0-\mathrm{Eff}_{\varepsilon}(F_n,E_n) = [0,\varepsilon] \bigcup \left]1, 1+\frac{1}{n}\right].$$

It is clear that

$$\mathrm{Ls}(0-\mathrm{Eff}(F_n,E_n)) \not\subseteq 0-\mathrm{Eff}(F,E), \text{ and } \mathrm{Ls}(0-\mathrm{Eff}_{\varepsilon}(F_n,E_n)) \not\subseteq 0-\mathrm{Eff}(F,E).$$

Fortunately, we have $\mathrm{Ls}\left(\bigcap_{0<\varepsilon<\varepsilon_0} 0-\mathrm{Eff}^{\varepsilon}(F_n,E_n)\right) \subseteq 0-\mathrm{Eff}(F,E)$.

7. Applications

7.1. **Interval optimal control problems.** In this subsection, we aim to apply the results and techniques obtained in previous sections to consider the existence and convergences for interval optimal control problems. We start by introducing the foundational elements of interval arithmetic that serve as a basis for the Hukuhara difference introduced by Stefanini [38]. This notion was subsequently applied by Stefanini and Bede [39] to develop a generalized concept of the Hukuhara derivative for interval-valued functions. Our discussion takes place within the setting of real interval spaces, denoted by

$$\mathscr{I}(\mathbb{R}) = \{[a_L, a_U] : a_L, a_U \in \mathbb{R},\ a_L \leq a_U\},$$

where a_L and a_U represent the lower and upper bounds, respectively, of the interval $A = [a_L, a_U]$. For simplicity, we shall refer to elements of $\mathscr{I}(\mathbb{R})$ as *intervals* throughout this section.

Given two intervals $A = [a_L, a_U]$ and $B = [b_L, b_U]$, and a real scalar $k \in \mathbb{R}$, the basic arithmetic operations are defined as follows:

$$A + B = [a_L + b_L,\ a_U + b_U],\ kA = \begin{cases} [ka_L,\ ka_U] & \text{if } k \geq 0, \\ [ka_U,\ ka_L] & \text{if } k < 0. \end{cases}$$

Then, the difference of the two intervals A and B is given by

$$A - B = A + (-1)B = [a_L - b_U,\ a_U - b_L].$$

Let $\mathbb{X}$ be a normed space and $\mathscr{A}$ be a nonempty subset of $\mathbb{X}$. Let $f : \mathscr{A} \to \mathscr{I}(\mathbb{R})$ be a interval-valued function. The interval optimization problem has the following form:

(IOP) $\min f(x)$ such that $x \in \mathscr{A}$.

We recall the solution concept introduced by Ishibuchi and Tanaka [18]. Consider a closed interval $A = [a_L, a_U]$ in $\mathscr{I}(\mathbb{R})$. Its center and half-width can be expressed as $a_C = \frac{1}{2}(a_L + a_U)$ and $a_W = \frac{1}{2}(a_U - a_L)$, respectively. With this representation, the interval A can be written in the form $\langle a_C, a_W \rangle$.

Let $A = \langle a_C, a_W \rangle$ and $B = \langle b_C, b_W \rangle$ denote two such intervals. The comparison between A and B is established by separating the analysis into the cases of maximization and minimization:

(a) In the context of maximization, we define the relation

$$A \preceq_{CW} B \quad \text{if and only if} \quad a_C \leq b_C \text{ and } a_W \geq b_W.$$

Here, the width reflects the degree of uncertainty. Hence, an interval with a larger center value and reduced uncertainty (smaller half-width) is more desirable.

(b) For minimization purposes, we define

$$A \preceq_{CW} B \quad \text{if and only if} \quad a_C \leq b_C \text{ and } a_W \leq b_W.$$

In this setting, preference is given to intervals that are both lower in center and exhibit less uncertainty.

We further denote

$$A \prec_{CW} B \text{ if and only if } A \preceq_{CW} B \text{ and } A \neq B.$$

This work focuses solely on the minimization framework, which is sufficient for general analysis. Motivated by [18, 40], we also consider the concept of efficient solution of (IOP) as follows.

Definition 7.1. A feasible point $\bar{x} \in \mathscr{A}$ is called a *center-width solution* of problem (IOP) if there is no $x \in \mathscr{A}$ such that $f(x) \prec_{CW} f(\bar{x})$.

Inspired by the above observations, we define the following ordering relation. For $A = [a_L, a_U] \in \mathscr{I}(\mathbb{R})$ and $\alpha \in [0,1]$, we define the α-representative point of A given by

$$\mathrm{Rep}_{[a_L,a_U]}(\alpha) = \alpha a_L + (1-\alpha) a_U.$$

The ordering between two intervals A and B can be characterized differently depending on whether the objective is to maximize or minimize:

(a) When dealing with maximization, the comparison rule is given by

$$A \preceq^{\alpha}_{\mathrm{RepW}} B \quad \text{if and only if} \quad \mathrm{Rep}_A(\alpha) \leq \mathrm{Rep}_B(\alpha) \text{ and } a_W \geq b_W.$$

In this case, a preferred interval is one that not only has a greater α-representative value but also reflects less uncertainty, as measured by its smaller half-width.

(b) In the case of minimization, we adopt the following criterion:

$$A \preceq^{\alpha}_{\mathrm{RepW}} B \quad \text{if and only if } \mathrm{Rep}_A(\alpha) \leq \mathrm{Rep}_B(\alpha) \text{ and } a_W \leq b_W.$$

Here, intervals with both smaller α-representative values and lower levels of uncertainty are considered superior.

Motivated by the center-width solution concept defined in Definition 7.1, we next introduce the notion of a α-representation-width solution for problem (IOP).

Definition 7.2. For $\alpha \in [0,1]$, a feasible point $\bar{x} \in \mathscr{A}$ is called a *α-representation-width solution* of problem (IOP), written as $\bar{x} \in \alpha\text{-Sol}_{\mathrm{RepW}}(\mathrm{IOP})(f, \mathscr{A})$, if there is no $x \in \mathscr{A}$ such that $f(x) \prec^{\alpha}_{\mathrm{RepW}} f(\bar{x})$, where $A \prec^{\alpha}_{\mathrm{RepW}} B :\Leftrightarrow [A \preceq^{\alpha}_{\mathrm{RepW}} B,\ A \neq B]$.

Now we investigate the relationship between the order relation $\preceq_\alpha$ in $\mathscr{P}(\mathbb{R})$ with the ordering cone $K=\mathbb{R}_+$, which we studied in Section 3, and the relation α-RepW that we just introduced in the space $\mathscr{I}(\mathbb{R})$ as follows.

Let $\alpha\in[0,1]$ and $A=[a_L,a_U],B=[b_L,b_U]\in\mathscr{I}(\mathbb{R})$. Then,

$$A\preceq^\alpha B\Leftrightarrow\exists k\in\mathbb{R}_+,\ k+(1-\alpha)A-\alpha B\subseteq(1-\alpha)B-\alpha A.$$

Consequently, there is $k\in\mathbb{R}_+$ such that

$$k+[(1-\alpha)a_L,(1-\alpha)]a_U]-[\alpha b_L,\alpha b_U]\subseteq[(1-\alpha)b_L,(1-\alpha)b_U]-[\alpha a_L,\alpha a_U].$$

Equivalently, there exists $k\in\mathbb{R}_+$ such that

$$\Leftrightarrow k+[(1-\alpha)a_L-\alpha b_U,(1-\alpha)]a_U-\alpha b_L]\subseteq[(1-\alpha)b_L-\alpha a_U,(1-\alpha)b_U-\alpha a_L].$$

This implies that $A\preceq_\alpha B$ if and only if

$$\begin{cases}(1-\alpha)a_U-\alpha b_L\leq(1-\alpha)b_U-\alpha a_L,\\ (1-\alpha)]a_U-\alpha b_L-((1-\alpha)a_L-\alpha b_U)\leq(1-\alpha)b_U-\alpha a_L-((1-\alpha)b_L-\alpha a_U).\end{cases}$$

$$\Leftrightarrow\begin{cases}\alpha a_L+(1-\alpha)a_U & \leq & \alpha b_L+(1-\alpha)b_U,\\ (1-2\alpha)(a_U-a_L) & \leq & (1-2\alpha)(b_U-b_L).\end{cases}$$

Thus, based on the $\preceq^\alpha$ and the $\preceq_{CW}$, we can modify the ordering relation in $\mathscr{I}(\mathbb{R})$ as follows

$$[a_L,a_U]\preccurlyeq^\alpha[b_L,b_U]:\Leftrightarrow\begin{cases}\mathrm{Rep}_{[a_L,a_U]}(\alpha)\leq\mathrm{Rep}_{[b_L,b_U]}(\alpha)\text{ and }a_W\leq b_W & \text{if } 0\leq\alpha\leq\frac{1}{2},\\ \mathrm{Rep}_{[a_L,a_U]}(\alpha)\leq\mathrm{Rep}_{[b_L,b_U]}(\alpha)\text{ and }a_W\geq b_W & \text{if } \frac{1}{2}<\alpha\leq 1.\end{cases}$$

By the definition we can verify that for any $\alpha\in[0,1]$, the following statement is true

$$[[a_L,a_U]\preccurlyeq^\alpha[b_L,b_U]\text{ and }[b_L,b_U]\preccurlyeq^\alpha[a_L,a_U]]\Leftrightarrow[a_L,a_U]=[b_L,b_U].$$

Therefore, the definition of α-representation-width solution of problem (IOP) defined in Definition 7.2 can restate under the following equivalent form.

Definition 7.3. A feasible point $\bar{x}\in\mathscr{A}$ is called a *α-representation-width solution* of problem (IOP) if $[\forall x\in\mathscr{A},F(x)\preccurlyeq^\alpha F(\bar{x})]\Rightarrow[F(\bar{x})\preccurlyeq^\alpha F(x)]$. Equivalently, there is no $x\in\mathscr{A}$ such that $F(x)\prec^\alpha F(\bar{x})$, where $A\prec^\alpha B:\Leftrightarrow A\preccurlyeq^\alpha B$ and $A\neq B$.

Motivated by above observation and the $\prec^\alpha$ defined in Section 3, we consider the following relation in $\mathscr{I}(\mathbb{R})$: For $\alpha\in[0,1]$ and $A=[a_L,a_U],B=[b_L,b_U]\in\mathscr{I}(\mathbb{R})$, we consider the following relations

$$[a_L,a_U]\prec^\alpha[b_L,b_U]:\Leftrightarrow\begin{cases}\mathrm{Rep}_{[a_L,a_U]}(\alpha)<\mathrm{Rep}_{[b_L,b_U]}(\alpha)\text{ and }a_W<b_W & \text{if } 0\leq\alpha\leq\frac{1}{2},\\ \mathrm{Rep}_{[a_L,a_U]}(\alpha)<\mathrm{Rep}_{[b_L,b_U]}(\alpha)\text{ and }a_W>b_W & \text{if } \frac{1}{2}<\alpha\leq 1.\end{cases}$$

Similar to Section 3, we define the sets of *α-weak representation-width solution* of (IOP) as follows

$$\alpha\text{-WSol}_{\mathrm{RepW}}(\mathrm{IOP})(f,\mathscr{A}):=\{x\in\mathscr{A}:[\forall z\in\mathscr{A},F(z)\prec^\alpha F(x)]\Rightarrow F(x)\prec^\alpha F(z)]\},$$

An interval-valued function $F:[a,b]\to\mathscr{I}(\mathbb{R})$ can be represented as $F(s)=[F_L(s),F_U(s)]$, where $F_L(s)\leq F_U(s)$ for all $s\in[a,b]$. The functions F_L and F_U are referred to as the *lower* and *upper* functions of F, respectively.

Next, we recall the basic concepts related to the Aumann integral for interval-valued functions. Let $F : [a,b] \to \mathscr{I}(\mathbb{R})$ be an interval-valued function. The set of integrable selectors of F, denoted by $\mathscr{S}(F)$, is defined as

$$\mathscr{S}(F) = \{\psi : [a,b] \to \mathbb{R} \mid \psi \text{ is integrable and } \psi(s) \in F(s) \text{ a.e. in } [a,b]\}.$$

The *Aumann integral* of F over the interval $[a,b]$ is given by

$$\int_a^b F(s)\,ds = \left\{ \int_a^b \psi(s)\,ds \;\middle|\; \psi \in \mathscr{S}(F) \right\}.$$

If $\mathscr{S}(F) \neq \emptyset$, then F is said to be *Aumann integrable*.

Lemma 7.1. [26] *Let $F : [a,b] \to \mathscr{I}(\mathbb{R})$ be a measurable and integrally bounded interval-valued function, represented as $F(s) = [F_L(s), F_U(s)]$. Then the following statements hold:*

(a) F is integrable, and its integral belongs to $\mathscr{I}(\mathbb{R})$:

$$\int_a^b F(s)\,ds \in \mathscr{I}(\mathbb{R}).$$

(b) The component functions F_L and F_U are continuous, and the integral of F satisfies

$$\int_a^b F(s)\,ds = \left[\int_a^b F_L(s)\,ds,\ \int_a^b F_U(s)\,ds \right].$$

Consider an interval-valued function F defined on $\mathbb{R}^n$ as

$$F(x) = F(x_1,\ldots,x_n) \in \mathscr{I}(\mathbb{R}) \quad \text{for all } x = (x_1,\ldots,x_n) \in \mathbb{R}^n,$$

where the functions $F_L(x)$ and $F_U(x)$ respectively define the lower and upper bounds of $F(x)$.

A result by Aubin and Cellina [9] shows that the continuity of F is characterized by the continuity of its lower and upper functions.

Lemma 7.2. [9] *Let F be an interval-valued function defined on a set $X \subset \mathbb{R}^n$, and let $x_0 \in X$. Then, F is continuous at x_0 if and only if $F_L(x)$ and $F_U(x)$ are continuous at x_0.*

Now we turn to the problem model that is the main focus of this subsection, namely the interval optimal control problem involving the above ordering relation in $\mathscr{I}(\mathbb{R})$ along with the Hausdorff distance defined by

$$H(A,B) = \max\{|a_L - b_L|, |a_U - b_U|\} \text{ for all } A = [a_L, a_U],\ B = [b_L, b_U] \in \mathscr{I}(\mathbb{R}). \tag{7.1}$$

In what follows, we use $\mathbb{R}^n$ to denote the n-dimensional Euclidean space equipped with the norm $|\cdot|$, and we let $C([t_0,t_1],\mathbb{R}^n)$ denote the Banach space of continuous functions $x : [t_0,t_1] \to \mathbb{R}^n$, endowed with the supremum norm $|x| := \max_{t\in[t_0,t_1]} |x(t)|$. For each $1 \leq q \leq \infty$, the space $\mathscr{L}^q([t_0,t_1],\mathbb{R}^l)$ refers to the set of all $\mathbb{R}^l$-valued functions on $[t_0,t_1]$ that are q-integrable, equipped with the norm $|\cdot|_q$. The closed unit ball in the Euclidean space is denoted by $\overline{\mathbb{B}}$.

Let $f : [t_0,t_1] \times \mathbb{R}^n \times \mathbb{R}^l \to \mathscr{I}(\mathbb{R})$ and $\phi : [t_0,t_1] \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ be given mappings, and let $\Omega \subset \mathbb{R}^l$ be a closed set. We study the interval optimal control problem of determining a control function $u \in \mathscr{L}^q([t_0,t_1],\mathbb{R}^l)$, along with its associated trajectory $x \in C([t_0,t_1],\mathbb{R}^n)$, that together satisfy the following system:

$$\min \int_{t_0}^{t_1} f(t,x(t),u(t))dt,$$

subject to

$$x(t) = x_0 + \int_{t_0}^{t} \varphi(s, x(s), u(s))ds, \quad t \in [t_0, t_1], \tag{7.2}$$

and the control constraint

$$u(t) \in \Omega, \quad t \in [t_0, t_1], \tag{7.3}$$

where x_0 is a given vector in $\mathbb{R}^n$. A pair of functions (x,u) that satisfies the state equation (7.2) and the control constraint (7.3) is referred to as a feasible solution to the problem.

Motivated by [6, 7], we define Φ as the set of all functions φ that satisfy the following hypotheses:

$(\mathscr{H}1)$ φ is continuous on $[t_0,t_1] \times \mathbb{R}^n \times \Omega$.

$(\mathscr{H}2)$ There exists a function $\ell \in \mathscr{L}^1([t_0,t_1],\mathbb{R})$ such that

$$|\varphi(t,y_1,v) - \varphi(t,y_2,v)| \le \ell(t)|y_1 - y_2| \quad \text{a.e. } t \in [t_0,t_1], \forall (y_1,v),(y_2,v) \in \mathbb{R}^n \times \Omega.$$

$(\mathscr{H}3)$ there exists a constant ℓ_2 such that

$$|\langle y, \varphi(t,y,v)\rangle| \le \ell_2(1+|y|^2) \quad \forall (t,y,v) \in [t_0,t_1] \times \mathbb{R}^n \times \Omega.$$

A control $u \in \mathscr{L}^q([t_0,t_1],\mathbb{R}^l)$ is said to be admissible if $u(t) \in \Omega$ for all $t \in [t_0,t_1]$. The set of all admissible controls is denoted by $\widehat{U}$. Let U be a nonempty closed subset of $\widehat{U}$. Set

$$X := C([t_0,t_1],\mathbb{R}^n),\ Z := X \times U,$$

$$\mathbb{F} := \{f : [t_0,t_1] \times \mathbb{R}^n \times \Omega \to \mathscr{I}(\mathbb{R}) : f_L, f_U \text{ are continuous}\} \text{ and } \mathbb{P} := \mathbb{F} \times \Phi.$$

It is worth noting that, for every pair $(x,u) \in Z$, control constraint (7.3) is satisfied. For each $p := (f,\varphi) \in \mathbb{P}$ and $z = (x,u) \in Z$, we define a mapping $g : Z \to \mathscr{I}(\mathbb{R})$ by

$$g(z) := \int_{t_0}^{t_1} f(t,x(t),u(t))dt = \left[\int_{t_0}^{t_1} f_L(t,x(t),u(t))dt, \int_{t_0}^{t_1} f_U(t,x(t),u(t))dt\right]. \tag{7.4}$$

We also consider the feasible solution mapping $D : \Phi \rightrightarrows Z$, which is defined as follows:

$$D(\varphi) := \{z = (x,u) \in Z : (7.2) \text{ is satisfied}\}. \tag{7.5}$$

Then, for each $(f,\varphi) \in \mathbb{P}$, the interval optimal control problem under consideration can be reformulated as follows:

$$\text{(IOCP):} \quad \min g(z) \ \text{ subject to } \ z \in D(\varphi).$$

Let $f_k \in \mathbb{F}$ for each $k \in \mathbb{N}$. We define the sequence of mappings $g_k : Z \to \mathscr{I}(\mathbb{R})$ by

$$g_k(z) = \int_{t_0}^{t_1} f_k(t,x(t),u(t))dt. \tag{7.6}$$

Now, for each $(f_k,\varphi_k) \in \mathbb{P}$, we consider the interval optimal control problem (IOCP) under perturbations, where the perturbations are introduced through variations in both the objective mapping and the feasible solution set over the parameter space $\mathbb{P}$.

$$\text{(IOCP)}_k: \quad \min g_k(z) \ \text{ subject to } \ z \in D(\varphi_k).$$

For two vectors $p = (f,\varphi), q = (g,\psi) \in \mathbb{P}$, the distance between p and q is

$$d_{\mathbb{P}}(p,q) := d_{\mathbb{F}}(f,g) + d_{\Phi}(\varphi,\psi),$$

where

$$d_{\mathbb{F}}(f,g) := \max\left\{ \sup_{\substack{y\in\mathbb{R}^n\\ v\in\Omega}} \int_{t_0}^{t_1} |g_L(t,y,v) - f_L(t,y,v)|dt, \sup_{\substack{y\in\mathbb{R}^n\\ v\in\Omega}} \int_{t_0}^{t_1} |g_U(t,y,v) - f_U(t,y,v)|dt \right\},$$

$$d_{\Phi}(\varphi_1,\varphi_2) := \sup_{\substack{y\in\mathbb{R}^n\\ v\in\Omega}} \int_{t_0}^{t_1} |\varphi_1(t,y,v) - \varphi_2(t,y,v)|dt.$$

For each $p = (f,\varphi) \in \mathbb{P}$ and $\alpha \in [0,1]$, we denote the sets of α-efficient solutions and α-weakly efficient solutions of (IOCP) by α-Sol(IOCP)(f,φ) and α-WSol(IOCP)(f,φ), respectively, i.e.,

$$\alpha\text{-Sol(IOCP)}(f,\varphi) := \{\bar{z} \in D(\varphi) : [\forall z \in D(\varphi), g(z) \preccurlyeq^{\alpha}_{\cdot} g(\bar{z})] \Rightarrow [g(\bar{z}) \preccurlyeq^{\alpha}_{\cdot} g(z)]\},$$
$$\alpha\text{-WSol(IOCP)}(f,\varphi) := \{\bar{z} \in D(\varphi) : [\forall z \in D(\varphi), g(z) \prec^{\alpha}_{\cdot} g(\bar{z})] \Rightarrow [g(\bar{z}) \prec^{\alpha}_{\cdot} g(z)]\}.$$

Definition 7.4. [11] Let Ψ be the set of functions ψ defined on $[t_0,t_1]$. The set Ψ is said to be equimeasurable on $[t_0,t_1]$ if, for each $\psi \in \Psi$, there exists a sequence of step functions $\{\varphi_k\}$ such that

(a) For each k, φ_k has no more than k discontinuous points.
(b) For all $\varepsilon > 0$ and for all $\sigma > 0$, there exists a finite real number $k_0(\varepsilon,\sigma)$ such that

$$|\varphi_k(t) - \psi(t)| < \varepsilon \quad \forall k > k_0(\varepsilon,\sigma),$$

except on a set of total measure less than σ.

Lemma 7.3. [6] *If Ω is bounded and U is equimeasurable on $[t_0,t_1]$, then $D(\varphi)$ is nonempty and compact for all $\varphi \in \Phi$.*

Lemma 7.4. [7] *If $\{\varphi_k\} \subset \Phi$ converges to φ in Φ, then $D(\varphi_k) \xrightarrow{P.K} D(\varphi)$.*

By Lemma 7.3 and [30], we obtain the following result.

Lemma 7.5. *Assume that Ω is bounded and U is equimeasurable on $[t_0,t_1]$, and $(f,\varphi) \in \mathbb{F}\times\Phi$. Then,* (IOCP) *has the domination and weak domination properties.*

As an illustration of the applicability of the results obtained in Sections 5 and 6, we present the following result. Its proof is based on the structure of interval-valued functions together with Lemmas 7.3 - 7.5 and Theorems 5.1, 6.1.

Theorem 7.1. *Let $\alpha \in [0,1]$. Assume that Ω is bounded and U is equimeasurable on $[t_0,t_1]$, and $\{\varphi_k\} \subset \Phi$ converges to φ in Φ. Then, the following statements hold:*

(1) *For any $z_0 \in \alpha$-WSol(IOCP)(f,φ), one of the following statements occurs:*
  (a) *There exists a sequence $\{z_n\}$ with $z_n \in \alpha$-WSol(IOCP)(f_n,φ_n) converging to z_0.*
  (b) *There is a sequence $\{\bar{z}_{n_k}\}$ with $\bar{z}_{n_k} \in \alpha$-WSol(IOCP)(f_{n_k},φ_{n_k}) converging to some $\bar{z}$, where $\bar{z}$ is an improvement solution of z_0 in the sense that $\bar{z} \in \alpha$-WSol(IOCP)(f,φ) and $g(\bar{z}) \preccurlyeq^{\alpha}_{\cdot} g(z_0)$.*

(2) *For any $z_0 \in \alpha$-Sol(IOCP)(f,φ), one of the following statements occurs*
  (a) There exists a sequence $\{z_n\}$ with $z_n \in \alpha$-Sol(IOCP)(f_n,φ_n) converging to z_0.
  (b) There is a sequence $\{\bar{z}_{n_k}\}$ with $\bar{z}_{n_k} \in \alpha$-Sol(IOCP)(f_{n_k},φ_{n_k}) converging to some $\bar{z} \in \alpha$-Sol(IOCP)(f,φ) with $g(\bar{z}) = g(z_0)$.

7.2. **Interval optimal control model for irrigation management under uncertain rainfall and cost.** We consider the problem of optimizing irrigation planning over a time horizon $[0,T]$ using optimal control techniques. The state variable $x(t)$ represents the amount of water in the soil at time t, while the control variable $u(t)$ denotes the irrigation flow rate introduced into the soil [27]. The dynamics of the system are governed by the water balance equation:

$$\dot{x}(t) = u(t) + g(t) - \beta x(t), \quad x(0) = x_0,$$

where:

- $g(t)$ is the rainfall (assumed to be known and fixed),
- $\beta \in (0,1)$ is the percentage of water lost due to runoff and deep infiltration (see, [29]),
- x_0 is the initial water level in the soil.

The control is bounded:

$$u(t) \in [0,M] \quad \text{a.e. in } [0,T],$$

and the state must satisfy a minimum requirement:

$$x(t) \geq x_{\min} \quad \forall t \in [0,T].$$

The objective of the irrigation management problem (IMP) discussed in [28] is to minimize the total irrigation volume.

$$J(u) = \int_0^T u(t)\,dt.$$

This model implicitly assumes a unitary cost per unit of water, i.e., $c(t) \equiv 1$.

In practice, the actual cost of water usage may vary with time due to energy prices, local water tariffs, or irrigation scheduling policies. We thus introduce a time-dependent cost function $c(t)$, which leads to the generalized objective:

$$J(u) = \int_0^T c(t)u(t)\,dt,$$

where $c(t)$ is a known, positive, and bounded function over $[0,T]$. The rest of the model (dynamics and constraints) remains unchanged.

In realistic agricultural settings, the amount of rainfall is not exactly known in advance. Instead, we may only know that $g(t)$ belongs to a range:

$$g(t) \in [g_L(t), g_U(t)] \quad \text{for all } t \in [0,T],$$

where $g_L(t)$ and $g_U(t)$ are continuous functions defining the lower and upper bounds of rainfall at time t.

To guarantee a feasible irrigation strategy under all possible rainfall realizations, the control $u(t)$ must ensure that the state $x(t)$ satisfies:

$$x(t) \geq x_{\min} \quad \forall t \in [0,T], \quad \forall g(t) \in [g_L(t), g_U(t)].$$

In this context, the required irrigation effort becomes uncertain and depends on how favorable or unfavorable the rainfall actually is.

We now assume that not only is rainfall uncertain, but also the unit cost of water. That is, for each time t, the price per unit of irrigation water lies within an interval:

$$c(t) \in [\underline{c}(t), \overline{c}(t)],$$

where the bounds $\underline{c}(t)$ and $\overline{c}(t)$ are known, continuous functions.

Consequently, since both the control $u(t)$ and the cost coefficient $c(t)$ are uncertain or interval-valued, the total cost becomes an interval in $\mathscr{I}(\mathbb{R})$:

$$J(u) = \left[\int_0^T \underline{c}(t)u_{\min}(t)\,dt,\ \int_0^T \overline{c}(t)u_{\max}(t)\,dt\right],$$

where:

- $u_{\min}(t)$ is the irrigation required under the most favorable rainfall $g_U(t)$,
- $u_{\max}(t)$ is the irrigation required under the least rainfall $g_L(t)$.

This leads to the *interval optimal control problem*:

$$\begin{aligned}
\text{(IIMP)}\quad & \text{minimize} \quad J(x,u) = \left[\int_0^T \underline{c}(t)\,u_{\min}(t)\,dt,\ \int_0^T \overline{c}(t)\,u_{\max}(t)\,dt\right] \subset \mathbb{R},\\
\text{subject to:}\quad & \dot{x}(t) = u(t) + g(t) - \beta x(t), \quad x(0) = x_0,\\
& g(t) \in [g_L(t), g_U(t)] \quad \text{a.e. } t \in [0,T],\\
& x(t) \geq x_{\min} \quad \forall t \in [0,T],\\
& u(t) \in [0,M] \quad \text{a.e. } t \in [0,T],
\end{aligned} \tag{7.7}$$

where g_L, g_U and u belong to the space $C([0,T])$ of continuous functions on $[0,T]$ endowed with the uniform norm.

Lemma 7.6. *Let $\Phi : C([0,T]) \times C([0,T]) \to C([0,T])$ be the mapping defined by*

$$(u,g) \mapsto x(t) := x_0 e^{-\beta t} + \int_0^t e^{-\beta(t-s)}(u(s)+g(s))\,ds.$$

Then, Φ is Lipschitz continuous, and hence it is continuous.

Proof. Let $(u_1,g_1),(u_2,g_2) \in C([0,T]) \times C([0,T])$, and define $x_1 := \Phi(u_1,g_1)$, $x_2 := \Phi(u_2,g_2)$. Then, for every $t \in [0,T]$,

$$x_1(t) - x_2(t) = \int_0^t e^{-\beta(t-s)}\left[(u_1(s)-u_2(s)) + (g_1(s)-g_2(s))\right]ds.$$

Taking the absolute value and using the triangle inequality yields

$$|x_1(t) - x_2(t)| \leq \int_0^t e^{-\beta(t-s)}\left(|u_1(s)-u_2(s)| + |g_1(s)-g_2(s)|\right)ds.$$

Using the uniform norm $\|z\| = \sup_{s\in[0,T]} |z(s)|$, we have

$$|x_1(t) - x_2(t)| \leq (\|u_1-u_2\| + \|g_1-g_2\|)\int_0^t e^{-\beta(t-s)}ds.$$

Since $\int_0^t e^{-\beta(t-s)}ds = \frac{1-e^{-\beta t}}{\beta} \leq \frac{1}{\beta}$, we obtain

$$\|x_1 - x_2\| \leq \frac{1}{\beta}\left(\|u_1-u_2\| + \|g_1-g_2\|\right).$$

Thus, Φ is Lipschitz continuous. □

We next recall the classical concept of equicontinuity, which plays an important role in establishing the compactness of subsets of $C([0,T])$ via the well-known Arzelà–Ascoli theorem, as it will be crucial in the analysis that follows.

Definition 7.5. Let $\mathscr{P} \subset C([0,T])$ be a family of real-valued continuous functions defined on the closed interval $[0,T]$. We say that $\mathscr{P}$ is *equicontinuous* on $[0,T]$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $\varphi \in \mathscr{P}$ and for all $s,t \in [0,T]$ with $|t-s| < \delta$, $|\varphi(t) - \varphi(s)| < \varepsilon$.

Lemma 7.7 (Arzelà–Ascoli theorem)**.** *Let $\mathscr{P} \subset C([0,T])$. Then, $\mathscr{P}$ is relatively compact in $C([0,T])$ equipped with the uniform norm (i.e., the closure of $\mathscr{P}$ is compact) if and only if:*

(i) *$\mathscr{P}$ is uniformly bounded,i.e., there exists a constant $M > 0$ such that*

$$|\varphi(t)| \leq M \quad \text{for all } \varphi \in \mathscr{P} \text{ and all } t \in [0,T];$$

(ii) *$\mathscr{F}$ is equicontinuous on $[0,T]$.*

Let $a := \min_{t\in[0,T]} g_L(t)$ and $b := \max_{t\in[0,T]} g_U(t)$. We consider the following sets:

$$\mathscr{G} := \{g \in C([0,T]) : a \leq g(t) \leq b \text{ for all } t \in [0,T]\}, \tag{7.8}$$

$$\mathscr{U} := \{u \in C([0,T]) : 0 \leq u(t) \leq M \text{ for all } t \in [0,T]\}. \tag{7.9}$$

In what follows, we assume that both sets $\mathscr{G}$ and $\mathscr{U}$ are equicontinuous.

Remark 7.1. Let $\mathscr{P} := \{\varphi \in C([0,T]) : \alpha \leq \varphi(t) \leq \beta \text{ for all } t \in [0,T]\}$, where $\alpha < \beta$ and $T > 0$ are fixed constants. In general, the set Φ is not equicontinuous. However, the equicontinuity property is guaranteed if one of the following additional conditions holds:

(a) *Uniform Lipschitz continuity:* There exists a constant $L > 0$ such that

$$|\varphi(t) - \varphi(s)| \leq L|t-s| \quad \text{for all } \varphi \in \mathscr{P} \text{ and all } t,s \in [0,T].$$

(b) *Uniformly bounded derivatives:* Each function $\varphi \in \Phi$ is continuously differentiable on $[0,T]$, and there exists a constant $L > 0$ such that

$$\sup_{t\in[0,T]} |\varphi'(t)| \leq L \quad \text{for all } \varphi \in \mathscr{P}.$$

(c) *Uniform modulus of continuity:* There exists a continuous function $\omega : [0,T] \to \mathbb{R}_+$ with $\omega(0) = 0$, such that

$$|\varphi(t) - \varphi(s)| \leq \omega(|t-s|) \quad \text{for all } \varphi \in \mathscr{P} \text{ and all } t,s \in [0,T].$$

Lemma 7.8. *Let $\mathscr{G}$ and $\mathscr{U}$ be defined as in (7.8) and (7.9), respectively. Define the feasible set*

$$\mathscr{S} := \left\{ (x,u) \in C([0,T]) \times C([0,T]) \;\middle|\; \begin{array}{l} \exists g \in \mathscr{G} \\ \dot{x}(t) = u(t) + g(t) - \beta x(t),\ u \in \mathscr{U},\ x(0) = x_0 \\ x(t) \geq x_{\min} \ \ \forall t \in [0,T] \end{array} \right\}.$$

Then, $\mathscr{S}$ is compact in $C([0,T]) \times C([0,T])$ with the uniform (supremum) norm.

Proof. Define the sets:

- $\widehat{\mathscr{G}} := \{g \in C([0,T]) : g_L(t) \leq g(t) \leq g_U(t)\}$, which is closed, bounded, and equicontinuous, hence compact by the Arzelà–Ascoli theorem.
- $\mathscr{U} := \{u \in C([0,T]) : 0 \leq u(t) \leq M\}$, which is also compact by the Arzelà–Ascoli theorem.

Define the mapping $\Psi : C([0,T]) \times C([0,T]) \to C([0,T]) \times C([0,T])$ by $\Psi(u,g) = (u, x_{(u,g)})$, where $x_{(u,g)}$ solves $\dot{x}(t) = u(t) + g(t) - \beta x(t), \quad x(0) = x_0$. By Lemma 7.6, Ψ is continuous. Therefore, $\mathscr{S}_0 := \{(u, x_{(u,g)}) : (u,g) \in \mathscr{U} \times \mathscr{G}\}$ is the image of a compact set under a continuous map, and hence $\mathscr{S}_0$ is compact.

Finally, the set $\mathscr{S}$ is the subset of $\mathscr{S}_0$ consisting of all (x,u) such that

$$x(t) \geq x_{\min} \ \forall t \in [0,T],$$

which is a closed condition. Thus, $\mathscr{S} \subset \mathscr{S}_0$ is closed, and hence it is compact. □

Theorem 7.2. *Let $g_L^n, g_U^n, g_L, g_U, u \in C([0,T])$ satisfy*

$$g_L^n \to g_L, \quad g_U^n \to g_U \quad \textit{uniformly on } [0,T],$$

with $g_L^n(t) \leq g_U^n(t)$ for all $t \in [0,T]$ and $n \in \mathbb{N}$, and similarly for g_L, g_U. For each $n \in \mathbb{N}$, define the feasible set:

$$\mathscr{S}_n := \left\{ (x,u) \in C([0,T])^2 \;\middle|\; \begin{array}{l} \exists\, g \in C([0,T]),\ g_L^n(t) \leq g(t) \leq g_U^n(t) \\ \dot{x}(t) = u(t) + g(t) - \beta x(t), \quad x(0) = x_0 \\ x(t) \geq x_{\min},\ \forall t, \quad 0 \leq u(t) \leq M \end{array} \right\},$$

and let $\mathscr{S}$ be defined similarly with $g = (g_L, g_U)$. Then, the sequence $\mathscr{S}_n$ converges to $\mathscr{S}$ in both Hausdorff and Painlevé-Kuratowski senses.

Proof. We prove convergence in both Hausdorff and Painlevé–Kuratowski senses.

Let

$$\widehat{\mathscr{G}_n} := \{g \in C([0,T]) : g_L^n(t) \leq g(t) \leq g_U^n(t)\},$$

$$\widehat{\mathscr{G}} := \{g \in C([0,T]) : g_L(t) \leq g(t) \leq g_U(t)\}.$$

Since $g_L^n \to g_L$, $g_U^n \to g_U$ uniformly, it follows that $d_H(\widehat{\mathscr{G}_n}, \widehat{\mathscr{G}}) \to 0$ in the Hausdorff metric induced by the uniform norm $\|\cdot\|$ on $C([0,T])$. Let $\varepsilon > 0$. Since $\widehat{\mathscr{G}_n} \to \widehat{\mathscr{G}}$ in Hausdorff distance, for every $g \in \widehat{\mathscr{G}}$, there exists $g_n \in \widehat{\mathscr{G}_n}$ such that $\|g_n - g\| < \delta$, and vice versa. Fix any $(x,u) \in \mathscr{S}$. Then $x = \Phi(u,g)$ for some g. There exists $g_n \in \mathscr{G}_n$ with $\|g_n - g\| < \delta$. Define $x_n := \Phi(u, g_n)$. Then,

$$\|x_n - x\| \leq \int_0^T e^{-\beta(t-s)} \|g_n - g\|\, ds \leq \frac{1 - e^{-\beta T}}{\beta} \cdot \delta.$$

Choosing δ small enough gives $\|x_n - x\| < \varepsilon$. Since the constraints $x_n(t) \geq x_{\min}$ are preserved for large enough n, $(x_n, u) \in \mathscr{S}_n$. So, for any $(x,u) \in \mathscr{S}$, we can find a nearby $(x_n, u) \in \mathscr{S}_n$, proving that

$$\sup_{(x,u) \in \mathscr{S}} \inf_{(y,v) \in \mathscr{S}_n} \|(x,u) - (y,v)\| < \varepsilon.$$

The reverse inclusion $\mathscr{S}_n \subset \mathscr{S} + B_\varepsilon$ follows similarly. Hence, $\mathscr{S}_n$ converge to $\mathscr{S}$ in the Hausdorff sense.

Moreover, by the definitions we get that $\mathscr{S}_n$ and $\mathscr{S}$ are nonempty and closed for all $n \in \mathbb{N}$. Thus, by [1, Lemma 2.1], we also obtain that $\mathscr{S}_n$ converge to $\mathscr{S}$ in the Painlevé-Kuratowski sense. □

Remark 7.2. By applying Lemma 7.8 and Theorem 7.2, we see that, for each interval function $g = [g_L, g_U]$ with $g_L, g_U \in C([0,T])$, there exists a compact subset $\Omega \subset \mathbb{R}$ such that, for any sequence $\{g_n\}$, where $g_n = [g_L^n, g_U^n]$ with $g_L, g_U \in C([0,T])$, and $\{g_n\}$ converges uniformly to g, and for any $(x_n, u) \in \mathscr{S}_n$, it holds that $x_n(t) \in \Omega$ for all $t \in [0,T]$.

We now consider a sequence of perturbed problems, indexed by n:

$$\begin{aligned} (\mathrm{IIMP})_n \quad & \text{minimize} \quad J_n(x,u) = \left[\int_0^T \underline{c}(t)\, u_{\min}^n(t)\, dt,\ \int_0^T \overline{c}(t)\, u_{\max}^n(t)\, dt\right], \\ \text{subject to:} \quad & \dot{x}(t) = u(t) + g(t) - \beta x(t), \quad x(0) = x_0, \\ & g(t) \in [g_L^n(t), g_U^n(t)] \quad \text{a.e. } t \in [0,T], \\ & x(t) \geq x_{\min} \quad \forall t \in [0,T], \\ & u(t) \in [0,M] \quad \text{a.e. } t \in [0,T], \end{aligned} \tag{7.10}$$

where $u \in C([0,T])$ and $g_L^n, g_U^n \in C([0,T])$ converge to g_L, g_U, respectively, in $C([0,T])$ with respect to the uniform norm.

Lemma 7.9. *Let $g_L^n \to g_L$, and $g_U^n \to g_U$ uniformly in $C([0,T])$. Let $J_n(u)$ and $J(u)$ be the interval-valued objective functionals associated with problems (7.10) and (7.7), respectively. Then, for every admissible control $u \in C([0,T])$ with $0 \leq u(t) \leq M$, the sequence $J_n(x,u)$ converges to $J(x,u)$ in both the Hausdorff and Painlevé–Kuratowski senses in $\mathscr{I}(\mathbb{R})$.*

Proof. For each n, define the feasible control set:

$$\mathscr{U}_n := \left\{ u \in C([0,T]) \;\middle|\; \begin{array}{l} \dot{x}(t) = u(t) + g(t) - \beta x(t) \\ g(t) \in [g_L^n(t), g_U^n(t)], u(t) \in [0,M], x(t) \geq x_{\min}\ \forall t \in [0,T] \\ x(0) = x_0 \end{array} \right\}.$$

The corresponding objective interval is:

$$J_n(x,u) = \left[\int_0^T \underline{c}(t) u_{\min}^n(t) dt,\ \int_0^T \overline{c}(t) u_{\max}^n(t) dt\right].$$

As $g_L^n \to g_L$ and $g_U^n \to g_U$ uniformly, the corresponding sets of feasible controls $\mathscr{U}_n$ converge to $\mathscr{U}$ in both Hausdorff and Painlevé–Kuratowski senses (by continuity of the ODE solution map with respect to g). Thus, $u_{\min}^n$ and $u_{\max}^n$ converge to $u_{\min}$ and $u_{\max}$ in the uniform norm, respectively. Since the integral functionals $u \mapsto \int_0^T \underline{c}(t) u(t) dt$ and $u \mapsto \int_0^T \overline{c}(t) u(t) dt$ are continuous with respect to the uniform norm, then $J_n(x,u)$ converge to $J(x,u)$ in both Hausdorff and Painlevé–Kuratowski senses. □

Since the problem (IIMP) is a specific case of the problem (IOCP) considered in Subsection 7.1, for simplicity in presentation, we will reuse the system of notations and the ordering relations that were previously introduced in the study of the solution properties of (IOCP). Let

$$\mathscr{F} := \{f = [f_L, f_U] : f_L, f_U \in C([0,T])\}.$$

The following result on the convergence of the problem (IIMP) is obtained from the results of Lemmas 7.8, 7.9 and Theorems 7.1, 7.2.

Theorem 7.3. *Let $\alpha \in [0,1]$, and $\{g_n\} \subset \mathscr{F}$ be a sequence converging to $g \in \mathscr{G}$. Assume that $\mathscr{G}$ and $\mathscr{U}$ are equicontinuous. Then, the following statements hold:*

(1) *For any* $(x_0,u_0)\in \alpha$-WSol(IIMP)$(g,\mathscr{S})$, *one of the following statements occurs:*
   (a) *There exists a sequence* $\{(x_n,u_n)\}$ *with* $(x_n,u_n)\in \alpha$-WSol(IIMP)$(g_n,\mathscr{S}_n)$ *converging to* (x_0,u_0).
   (b) *There is a sequence* $\{(\bar{x}_{k_k},\bar{u}_{n_k})\}$ *with* $(\bar{x}_{k_k},\bar{u}_{n_k})\in \alpha$-WSol(IIMP)$(g_{n_k},\mathscr{S}_{n_k})$ *converging to some* $(\bar{x},\bar{u})$, *where* $(\bar{x},\bar{u})$ *is an improvement solution of* (x_0,u_0) *in the sense that* $(\bar{x},\bar{u})\in \alpha$-WSol(IIMP)$(g,\mathscr{S})$ *and* $J(\bar{x},\bar{u})\preccurlyeq^{\alpha} J(x_0,u_0)$.

(2) *For any* $(x_0,u_0)\in \alpha$-Sol(IIMP)$(g,\mathscr{S})$, *one of the following statements occurs*
   (a) *There exists a sequence* $\{(x_n,u_n)\}$ *with* $(x_n,u_n)\in \alpha$-Sol(IIMP)$(g_n,\mathscr{S}_n)$ *converging to* (x_0,u_0).
   (b) *There is a sequence* $\{(\bar{x}_{n_k},u_{n_k})\}$ *with* $(\bar{x}_{n_k},u_{n_k})\in \alpha$-Sol(IIMP)$(g_{n_k},\mathscr{S}_{n_k})$ *converging to some* $(\bar{x},\bar{u})\in \alpha$-Sol(IIMP)$(g,\mathscr{S})$ *with* $J(\bar{x},\bar{u})=J(x_0,u_0)$.

8. Conclusions

In this study, by introducing and utilizing a unification of lower and upper semicontinuities, we established the existence and convergence of weakly efficient and efficient solutions for set optimization problems concerning a unification of set less relations based on the Minkowski difference. The significant contributions of this research lie in the following key points. First, we unified the lower and upper semicontinuities of set-valued maps and provided a unification of set less relations based on the Minkowski difference. Various properties of these unification forms were also discussed. Second, we established the existence of two typical efficient solutions for set optimization models where the feasible solution sets are not required to satisfy compactness. Third, we suggested new approaches to studying the lower and upper convergence of the considered solutions. More importantly, through these new approaches, we avoided key coercive conditions that were used in previous studies. Finally, as an illustration of the applicability of the obtained results, we applied them to the study of the convergence for interval optimal control problems and irrigation management under uncertain rainfall and cost. The results obtained in this research not only provide significant contributions to the studied topic but also demonstrate the applicability of the introduced concepts and approaches, which have the potential to be applied to the analysis of solution properties in various other optimization models.

Acknowledgments
The authors are grateful to the reviewers for useful suggestions which improved the presentation of this paper. This study was funded by Vietnam National Program for the Development of Mathematics 2021–2030 under grant number B2025-CTT-03.

References

[1] L.Q. Anh, L.V. Day, T.Q. Duy, C. Tammer, J.C. Yao, External and internal stability for set optimization problems, J. Optim. Theory Appl. 43 (2026), 209.
[2] L.Q. Anh, T.Q. Duy, L.T.V. Khanh, T.N. Tam, T.T.T. Mai, On the pointwise and global well-posedness for nonconvex set optimization problems, Optimization 75 (2026), 2253–2273.
[3] L.Q. Anh, D.V. Hien, Convergence of solutions to set optimization problems with variable ordering structures, J. Glob. Optim. 91 (2024), 677–699.
[4] L.Q. Anh, T.Q. Duy, D.V. Hien, Stability of efficient solutions to set optimization problems, J. Glob. Optim. 78 (2020), 563–580.

[5] L.Q. Anh, T.Q. Duy, D.V. Hien, D. Kuroiwa, N. Petrot, Convergence of solutions to set optimization problems with the set less order relation, J. Optim. Theory Appl. 185 (2020), 416–432.
[6] L.Q. Anh, V.T. Tai, T.N. Tam, Convergence of solutions to nonlinear nonconvex optimal control problems, Optimization 66 (2023), 1–39.
[7] L.Q. Anh, V.T. Tai, T.N. Tam, Painlevé-Kuratowski convergences of solutions to nonlinear multiobjective optimal control problems, Evol. Equ. Control Theory 13 (2024), 1229–1249.
[8] Q.H. Ansari, N. Hussain, P.K. Sharma, Convergence of the solution sets for set optimization problems, J. Nonlinear Var. Anal. 6 (2022), 165–183.
[9] J.P. Aubin, A. Cellina, Differential Inclusions: Set-Valued Maps and Viability Theory, Springer, New York, 1984.
[10] G. Bouza, E. Quintana, C. Tammer, A steepest descent method for set optimization problems with set-valued mappings of finite cardinality, J. Optim. Theory Appl. 190 (2021), 711–743.
[11] S.S.L. Chang, An extension of Ascoli's theorem and its applications to the theory of optimal control, Trans. Amer. Math. Soc. 115 (1965), 445–470.
[12] G. Eichfelder, E. Quintana, Set-based robust optimization of uncertain multiobjective problems via epigraphical reformulations, Eur. J. Oper. Res. 313 (2024), 871–882.
[13] A. El Ayoubi, M. Ait Mansour, J. Lahrache, Sensitivity analysis of an optimal control problem under Lipschitzian perturbations, J. Appl. Numer. Optim. 6 (2024), 211-228.
[14] D. Ghosh, K. Kumar, J.C. Yao, X. Zhao, A projected gradient method for constrained set optimization problems with set-valued mappings of finite cardinality, Eng. Optim. 57 (2025), 3229–3258.
[15] M. Gupta, M. Srivastava, Approximate solutions and Levitin–Polyak well-posedness for set optimization using weak efficiency, J. Optim. Theory Appl. 186 (2020), 191–208.
[16] C. Gutiérrez, E. Miglierina, E. Molho, V. Novo, Convergence of solutions of a set optimization problem in the image space, J. Optim. Theory Appl. 170 (2016), 358–371.
[17] Y. Han, Painlevé–Kuratowski convergences of the solution sets for set optimization problems with cone-quasiconnectedness, Optimization 71 (2022), 2185–2208.
[18] H. Ishibuchi, H. Tanaka, Multiobjective programming in optimization of the interval objective function, Eur. J. Oper. Res. 48 (1990), 219–225.
[19] J. Jahn, Characterizations of the set less order relation in nonconvex set optimization, J. Optim. Theory Appl. 193 (2022), 523–544.
[20] J. Jahn, Order Analysis, Deep Learning, and Connections to Optimization, Springer, Berlin, 2024.
[21] E. Karaman, M. Soyertem, İ.A. Güvenç, D. Tozkan, M. Küçük, Y. Küçük, Partial order relations on family of sets and scalarizations for set optimization, Positivity 22 (2018), 783–802.
[22] Karuna, C. Lalitha, External and internal stability in set optimization, Optimization 68 (2019), 833–852.
[23] A.A. Khan, C. Tammer, C. Zălinescu, Set-Valued Optimization, Springer, Berlin, 2015.
[24] D. Kuroiwa, The natural criteria in set-valued optimization, RIMS Kokyuroku Kyoto Univ. 1031 (1998), 85–90.
[25] C. Lalitha, External and internal stability in set optimization using gamma convergence, Carpathian J. Math. 35 (2019), 393–406.
[26] U. Leal, W. Lodwick, G. Silva, G.G. Maqui-Huaman, Interval optimal control for uncertain problems, Fuzzy Sets Syst. 402 (2021), 142–154.
[27] A.P. Lemos-Paião, S.O. Lopes, M.D.R. de Pinho, (2025). A parametric optimal control problem applied to daily irrigation, Math. Mo. del. Nat. Phenom. 20 (2025), 33.
[28] S.O. Lopes, F.A. Fontes, R.M. Pereira, M. de Pinho, A.M. Gonçalves, Optimal control applied to an irrigation planning problem, Math. Probl. Eng. 2016 (2016), 5076879.
[29] S.O. Lopes, F.A. Fontes, R.M. Pereira, A.M. Gonçalves, G.J. Machado, Irrigation planning: an optimal control approach, AIP Conf. Proc. 1558 (2013), 622–625.
[30] D.T. Luc Theory of Vector Optimization. Springer, Berlin, 1989.
[31] A. Löhne, A solution method for arbitrary polyhedral convex set optimization problems, SIAM J. Optim. 35 (2025), 330–346.
[32] M. Mabrouk, M. Laghdir, A. Ed-dahdah, A. Rikouane, Sequential approximate weak optimality conditions for robust multiobjective fractional optimization problems, Appl. Set-Valued Anal. Optim. 7 (2025), 251-269.

[33] B.S. Mordukhovich, Variational Analysis and Applications. Springer, Cham, 2018.
[34] Z. Nishnianidze, Fixed points of monotonic multiple-valued operators, Bull. Georgian Natl. Acad. Sci. 114 (1984), 489–491.
[35] N.P. Osmolovskii, V. M. Veliov, Strong metric subregularity of the optimality mapping and second-order sufficient optimality conditions in extremal problems with constraints, Commun. Optim. Theory 2025 (2025), 31.
[36] Z.Y. Peng, X.J. Chen, Y.B. Zhao, X.B. Li, Painlevé-Kuratowski convergence of minimal solutions for set-valued optimization problems via improvement sets, J. Glob. Optim. 87 (2023), 759–781.
[37] H. Rådström, An embedding theorem for spaces of convex sets, Proc. Amer. Math. Soc. 3 (1952), 165–169.
[38] L. Stefanini, A generalization of Hukuhara difference, In: D. Dubois M.A. Lubiano, H. Prade, M.Á. Gil, P. Grzegorzewski, O. Hryniewicz, (Eds.) Soft Methods for Handling Variability and Imprecision, pp. 203–210, Springer, Berlin, 2008.
[39] L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Anal. 71 (2009), 1311–1328.
[40] H.C. Wu, The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function, Eur. J. Oper. Res. 176 (2007), 46-59.
[41] Y. Xu, S. Li, Continuity of the solution set mappings to a parametric set optimization problem, Optim. Lett. 8 (2014), 2315–2327.
[42] M. Yang, T. Li, G. Xu, Continuity of the solutions sets for parametric set optimization problems, J. Inequal. Appl. 2024 (2024), 1–12.
[43] R.C. Young, The algebra of many-valued quantities, Math. Ann. 104(1931), 260–290.
[44] Y. Zhai, G. Yu, T. Tang, W. Han, Optimal conditions in uncertain set-valued optimization problems via second-order subdifferential involving Minkowski difference with application to zero-sum matrix games, J. Comput. Appl. Math. 464 (2025), 116530.
[45] Z. Zhou, K. Liang, Q.H. Ansari, Optimality conditions for Benson proper efficiency of set-valued equilibrium problems, Math. Methods Oper. Res. 101 (2025), 111–134.
[46] Z. Zhou, K. Feng, Q.H. Ansari, Well-posedness of set optimization problems with set order defined by Minkowski difference, J. Optim. Theory Appl. 204 (2025), 31.