J. Nonlinear Var. Anal. 11 (2027), No. 1, pp. 133-153
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.11.2027.1.07

ON DC MAPPINGS BETWEEN NORMED SPACES AND APPLICATIONS IN DC VECTOR OPTIMIZATION

L. HUERGA*, B. JIMÉNEZ, V. NOVO

Department of Applied Mathematics, E.T.S.I. Industriales, Universidad Nacional de Educación a Distancia (UNED), Madrid, Spain

Abstract. In this paper, we introduce several variants of the most known notions of vector DC mappings defined on normed spaces, and we study their main properties and the relationships between them. These new variants are weaker than the usual concepts of vector DC mappings and involve DC decompositions interesting from a practical point of view. Then, we provide a characterization of approximate and exact proper efficient solutions, respectively, of a constrained vector DC optimization problem through linear scalarization and in terms of the classical subdifferential of Convex Analysis. The characterizations are also particularized to the finite dimensional setting, where the feasible set of the multiobjective DC optimization problem is given by inequality constraints. These results are generalizations to the vector case of the well-known characterizations of approximate and exact global minima of an unconstrained scalar DC optimization problem due to Hiriart-Urruty [J.B. Hiriart-Urruty, Generalized differentiability, duality and optimization for problems dealing with differences of convex functions, In: J. Ponstein, (eds) Convexity and Duality in Optimization, Lecture Notes in Econom. and Math. Systems, vol. 256, pp. 37-70, Springer, Berlin, Heidelberg, 1985].

Keywords. Approximate proper efficiency; DC mapping; Scalarization techniques; Vector DC optimization.

1. INTRODUCTION

Many of the optimization problems that appear in practice do not fall within the framework of Convex Analysis. Even in the scalar optimization case, the difficulty of solving such problems increases considerably when they are defined by means of nonconvex functions or conditions. A special class within the family of nonconvex scalar optimization problems consists of those in which the objective function is defined as the difference of two convex functions. These functions are called DC (*Difference of Convex* functions), i.e., a scalar function $h : X \to \mathbb{R}$ is DC if there exist two convex functions $h_1, h_2 : X \to \mathbb{R}$ such that

$$h(x) = h_1(x) - h_2(x), \tag{1.1}$$

being X a real linear space.

*Corresponding author.
E-mail address: lhuerga@ind.uned.es (L. Huerga).
Received 31 December 2025; Accepted 30 September 2026; Published online 10 November 2026.
©2027 Journal of Nonlinear and Variational Analysis

Although a DC function is not convex in general, its representation as a difference of two convex functions allows the use of the set of tools from Convex Analysis, which is very useful to derive optimality conditions for an optimization problem defined by means of DC functions.

Then, over the last three decades, the branch of DC Optimization has attracted great interest in the field of scalar optimization, both at the theoretical level and in algorithm design (the so called DCA, *Difference of Convex Algorithms*). To acquire an overview of the properties of DC functions and the many applications in DC optimization, see, for instance, [2–5].

From the very definition, it is clear that the class of DC functions is given by the linear space generated by the class of convex functions. It should be emphasized that the set of scalar functions that can be represented as the difference of two convex functions is quite broad. In fact, it contains the set of lower $\mathscr{C}^2$ functions, introduced and studied by Rockafellar in [6], which in particular includes the class of $\mathscr{C}^2$ functions, and the class of differentiable functions whose gradient is locally Lipschitz (the class $\mathscr{C}^{1,1}$) (see [6, 7]).

Therefore, the literature on scalar DC Optimization is very extensive. The following additional references [8–16] constitute a small sample of recent studies on these problems.

In [1], Hiriart-Urruty derived a characterization of exact and approximate global minima of an unconstrained scalar DC optimization problem through the classical subdifferential of Convex Analysis. Later, other authors provided different characterizations for the solutions of this problem. For instance, Toland [17] and Singer [18], independently, derived the same characterization by means of conjugate functions, Flores-Bazán [19] established one in terms of a generalized subdifferential, and another results in this line were also given by Horst, Pardalos and Thoai [20], Horst and Thoai [2], and Martínez-Legaz and Seeger [21].

Regarding vector DC optimization problems and, in particular, multiobjective ones, the literature is not as extensive compared to the scalar case. At this point, the first question that arises is how to extend the notion of DC function from the scalar to the vector case. In the finite dimensional setting, the most natural extension is the one given by components, i.e., a function is DC if its components are all DC. In [22], Hartman considered this notion in one of his results about composition of DC functions.

With respect to mappings defined between infinite dimensional spaces, the most important notions of DC mapping known from the literature can be found in the works by Veselý and Zajíček (see [23–25]).

Herein, we will introduce several new variants of the notions of DC mappings from [23–25], and we study their properties, as well as the connections between them. These variants are interesting from a practical point of view, since it is easier to derive a DC decomposition for them.

Note that Gadhi and Metrane [26] derived sufficient optimality conditions for local weak efficient solutions of a DC vector optimization problem defined in Banach spaces. These conditions were obtained via a linear scalarization approach, and in terms of approximate strong vector subdifferentials.

Later, Guo and Li [27] derived necessary and sufficient optimality conditions for local approximate weak and proper (in the sense of Henig [28]) solutions of a DC vector optimization problem, defined in more general vector spaces, following the line of [26]. However, in [29], Shafie and Bozorgnia provided several counterexamples to refute the main results by Guo and

Li, and they corrected the mistakes. In [30], Shafie obtained alternative necessary and sufficient optimality conditions for local approximate weak efficient solutions of the problem.

Another interesting reference is the one by Taa [31], in which the author provided a necessary optimality condition for local weak efficient solutions of a DC multiobjective optimization problem by means of a scalarization approach, in terms of the oriented distance function due to Hiriart-Urruty (see, for instance, [32]). In [33], Gadhi, Laghdir and Metrane also studied optimality conditions for weak efficient solutions a DC vector optimization problem defined in Banach spaces through this scalarizing function.

In this paper, we study a DC constrained optimization problem defined in normed spaces, by considering a new variant of DC notion, and we characterize the approximate and exact proper solutions in the sense of Henig of this problem through linear scalarization, in terms of the classical subdifferential. The results presented here extend to the vector case the classical characterization of global minimality for an unconstrained scalar DC optimization problem due to Hiriart-Urruty [1].

The paper is structured as follows. In Section 2, we present the main notations, previous notions and results needed along the paper. In Section 3, we introduce several variants of DC notion, and we study their main properties as well as the connections between them. In Section 4, we focus on a constrained DC vector optimization problem defined in Banach spaces, whose objective mapping satisfies a specific DC condition. For this problem, we characterize the approximate and exact proper solutions through linear scalarization. In particular, we study the case of a DC multiobjective optimization problem with inequality constraints. In Section 5, we state the main conclusions and, finally, we indicate the references.

2. PRELIMINARIES

Let X be a real normed linear space and Y be a Banach space. We denote by Y^* the topological dual space of Y. The norm in Y will be denoted by $\|\cdot\|$ and the dual norm in Y^* by $\|\cdot\|_*$. The topological closure and interior with respect to the norm topology of a nonempty set $A \subset Y$ will be denoted by $\mathrm{cl}A$ and $\mathrm{int}A$, respectively. We say that A is solid if $\mathrm{int}A \neq \emptyset$.

On the other hand, we define the cone generated by A as $\mathrm{cone}A := \bigcup_{\alpha \geq 0} \alpha A$.

The closed unit ball in Y will be denoted by $\mathbb{B}_Y$ and the nonnegative orthant of $\mathbb{R}^m$ by $\mathbb{R}^m_+$.

We consider a preference relation in Y, defined by means of a proper cone E ($\{0\} \neq E \subset Y$) in the usual way: $y_1 \leq_E y_2 \Longleftrightarrow y_2 - y_1 \in E$, $\forall y_1, y_2 \in Y$. In what follows, we suppose that E is closed, convex and pointed ($E \cap (-E) = \{0\}$).

The positive and the strict positive polar cone of E, are defined, respectively, in the following way:

$$E^+ = \{\xi \in Y^* : \xi(e) \geq 0,\ \forall e \in E\},$$
$$E^{s+} = \{\xi \in Y^* : \xi(e) > 0,\ \forall e \in E \backslash \{0\}\}.$$

Remark 2.1. (i) It is clear that E^+ and $E^{s+} \cup \{0\}$ are convex cones. Also, it is known that E^+ is closed with respect to the weak* topology, i.e., $E^+ = \mathrm{cl}_{w^*} E^+$, where cl_{w*} denotes the closure with respect to this topology (see, for instance, [34, Chapter 1]).

In particular, if Y is reflexive, then the weak* topology coincides with the weak topology on Y^*, and by a classical Mazur result we have that E^+ is closed with respect to the strong topology as well, i.e., $E^+ = \mathrm{cl}(E^+)$.

(ii) On the other hand, if Y is reflexive, then (see [35, Lemma 3.21(d)])

$$\operatorname{int}(E^+) \neq \emptyset \Rightarrow \operatorname{int}(E^+) = E^{s+}.$$

Definition 2.1. (a) It is said that a nonempty convex set $B \subset E$ is a base of E if $\operatorname{cone} B = E$ and $0 \notin \operatorname{cl} B$. If E has a base, we say that E is based.

(b) We say that E is well-based if it has a bounded base B.

(c) It is said that E is normal (with respect to the norm topology) if there exists $\delta > 0$ such that $\delta\|x\| \leq \|y\|$, whenever $0 \leq_E x \leq_E y$.

The following implications hold (see, for instance, [35, 36]):

$$\operatorname{int}(E^+) \neq \emptyset \Leftrightarrow E \text{ well-based} \Rightarrow E \text{ normal} \Leftrightarrow E^+ - E^+ = Y^* \Rightarrow E \text{ pointed}, \tag{2.1}$$

$$E^{s+} \neq \emptyset \Leftrightarrow E \text{ based} \Rightarrow E \text{ pointed}. \tag{2.2}$$

The strict right implications in (2.1) and (2.2) become equivalences if Y is a finite-dimensional space.

We also remind that a mapping $H : X \to Y$ is E-convex on a nonempty convex set $W \subset X$ if

$$H(\alpha x_1 + (1-\alpha)x_2) \leq_E \alpha H(x_1) + (1-\alpha)H(x_2),\ \forall x_1, x_2 \in W,\ \forall \alpha \in [0,1].$$

Along the work, we consider that a scalar mapping $h : X \to \mathbb{R}$ is DC if the mappings h_1, h_2 of the DC decomposition (see (1.1)) are continuous in addition to being convex, and we want to work with extensions of this scalar DC notion to the vector case. So, from now on, we consider a continuous mapping $F : X \to Y$, which is DC in some sense. The most known concepts of DC mapping were introduced by Veselý and Zajíček in [23–25]. We recall them in the following definition.

Definition 2.2. Let $F : X \to Y$. It is said that

(a) F is weakly DC if $\xi \circ F$ is a DC function for each $\xi \in Y^*$.

(b) F is delta convex if there exists a continuous convex function $\varphi : X \to \mathbb{R}$ (called the control function) such that $\xi \circ F + \varphi$ is continuous and convex for each $\xi \in \mathbb{B}_{Y^*}$.

(c) F is order DC if it can be represented as the difference of two continuous E-convex mappings, $F_1, F_2 : X \to Y$, i.e., $F = F_1 - F_2$.

Remark 2.2. The three notions given in Definition 2.2 are equivalent when $Y = \mathbb{R}^m$ and $E = \mathbb{R}^m_+$, and coincide with the concept of DC function by components (see, for instance, [24, 25]).

Indeed, let $Y = \mathbb{R}^m$, $E = \mathbb{R}^m_+$, and $F := f = (f_1, f_2, \ldots, f_m)$, $f_i : X \to \mathbb{R}$, $i \in \{1, 2, \ldots, m\}$. Suppose first that f is delta convex. Then, there exists a continuous and convex function $\varphi : X \to \mathbb{R}$ such that $\xi \circ f + \varphi$ is convex, for all $\xi \in \mathbb{B}_{\mathbb{R}^m}$ (we consider the Euclidean closed unit ball in $\mathbb{R}^m$). Thus, in particular, we have that $e_i \circ f + \varphi = f_i + \varphi$ is convex, for all $i \in \{1, 2, \ldots, m\}$, where e_i denotes the i-th vector of the canonical basis, so f_i is DC, for all $i \in \{1, 2, \ldots, m\}$ ($f_i = (f_i + \varphi) - \varphi$).

Reciprocally, suppose that f_i is DC, for all $i \in \{1, 2, \ldots, m\}$. Then, for each i, there exists a continuous and convex function $\varphi_i : X \to \mathbb{R}$ such that $f_i + \varphi_i$ is convex. Let

$$\varphi(x) := \sum_{i=1}^{m} (f_i + 2\varphi_i).$$

It is clear that φ is continuous and convex, since it is given by a sum of continuous convex functions. Let $\xi \in \mathbb{B}_{\mathbb{R}^m}$. We have that

$$\begin{aligned}\xi \circ f + \varphi &= \sum_{i=1}^{m} \xi_i f_i + \sum_{i=1}^{m} f_i + 2\sum_{i=1}^{m} \varphi_i \\ &= \sum_{i=1}^{m} (1+\xi_i) f_i + \sum_{i=1}^{m} \varphi_i + \sum_{i=1}^{m} \xi_i \varphi_i - \sum_{i=1}^{m} \xi_i \varphi_i + \sum_{i=1}^{m} \varphi_i \\ &= \sum_{i=1}^{m} (1+\xi_i)(f_i+\varphi_i) + \sum_{i=1}^{m} (1-\xi_i)\varphi_i.\end{aligned}$$

Since $|\xi_i| \leq 1$, for all $i \in \{1,2,\ldots,m\}$, we have that $1+\xi_i, 1-\xi_i \geq 0$, for all i, so $(1+\xi_i)(f_i+\varphi_i)$ and $(1-\xi_i)\varphi_i$ are convex functions for all i and, hence, $\xi \circ f + \varphi$ is a continuous convex function. Since ξ is a general vector of $\mathbb{B}_{\mathbb{R}^m}$, we deduce that f is delta convex, as we wanted to see.

The other two equivalences, i.e., f weakly DC if and only if f_i is DC, for all i, and if and only if f is $\mathbb{R}^m_+$-order DC, follow straightforward (for the last equivalence, take into account Remark 2.3(i)).

On the other hand, it is clear that the first two concepts of vector DC mapping given in Definition 2.2 are based on scalarization, in terms of compositions of F with continuous linear functionals, while the third one is stated in terms of vector mappings. Nevertheless, we can also characterize the order DC mappings through scalarization, taking into account the following proposition, whose proof is easy to check.

Proposition 2.1. *Let $W \subset X$ be a nonempty convex set and $H : X \to Y$. The following statements are equivalent:*

(*a*) *H is E-convex on W.*
(*b*) *$\xi \circ H$ is convex on W, for all $\xi \in E^+$.*

Remark 2.3. (i) Note that if $Y = \mathbb{R}^m$ and $E = \mathbb{R}^m_+$, then H is $\mathbb{R}^m_+$- convex on W if and only if each component of H is convex on W.

(ii) Observe that, taking into account Proposition 2.1, we have that F is order DC if and only if it can be represented as the difference of two continuous mappings, $F = F_1 - F_2$, with $F_1, F_2 : X \to Y$, such that $\xi \circ F_1$ and $\xi \circ F_2$ are convex for all $\xi \in E^+$.

In [25], the authors studied the relationships between the three concepts given in Definition 2.2, that we collect in the next result, for the convenience of the reader.

Proposition 2.2. *Let $F : X \to Y$.*

(*a*) *If F is delta convex, then it is weakly DC.*
(*b*) *Suppose that E is solid. If F is delta convex, then it is order DC.*
(*c*) *Suppose that E is well-based. If F is order DC, then it is delta convex.*
(*d*) *Suppose that E is normal. If F is order DC, then it is weakly DC.*

We finish this section by recalling a special family of dilating cones for E that we are going to use along the whole paper.

Suppose that E has a closed base B, and let $\delta := \inf\{\|b\| : b \in B\}$. Then, we define

$$E_\rho := \mathrm{cl}(\mathrm{cone}(B + \rho \mathbb{B}_Y)), \tag{2.3}$$

for $0 \leq \rho < \delta$. These cones are known as Henig dilating cones, and satisfy the properties collected in the following proposition (see, for instance, [37, Theorem 1.1], [36, Lemma 3.2.51] and [38, Remark 2.2]).

Proposition 2.3. (a) *E_ρ is a solid closed convex pointed cone, for all $0 < \rho < \delta$.*
(b) *$E\backslash\{0\} \subset E_{\rho_1}\backslash\{0\} \subset \operatorname{int} E_{\rho_2}$, for all $0 < \rho_1 < \rho_2 < \delta$.*
(c) *$E = \bigcap_{\rho\in(0,\delta)} E_\rho = \bigcap_{n\in\mathbb{N}} E_{\rho_n}$, where $\{\rho_n\} \subset (0,\delta)$ converges to 0.*

In the next lemma, we study an additional property of the family of cones $\{E_\rho\}_{0<\rho\leq\delta}$. We have not found its proof explicitly, so we add it for the convenience of the reader.

Lemma 2.1. *Suppose that Y is reflexive and B is bounded. Then,*

$$E^{s+} = \bigcup_{\rho\in(0,\delta)} E_\rho^{s+} = \bigcup_{n\in\mathbb{N}} E_{\rho_n}^{s+}, \tag{2.4}$$

where $\{\rho_n\} \subset (0,\delta)$ converges to 0.

The equalities also hold by replacing E_ρ^{s+} by $E_\rho^{+}\backslash\{0\}$.

Proof. Since Y is a Banach space and E is a closed convex cone with a closed base B, by [37, Theorem 1.1(1)], for all $\rho \in (0,\delta)$, we have that

$$E_\rho = \operatorname{cone}(\operatorname{cl}(B + \rho\mathbb{B}_Y)).$$

The set $B + \rho\mathbb{B}_Y$ is closed, since $\mathbb{B}_Y$ is weakly compact by the reflexivity of Y, and B is closed, hence

$$E_\rho = \operatorname{cone}(B + \rho\mathbb{B}_Y) \tag{2.5}$$

and $B + \rho\mathbb{B}_Y$ generates E_ρ. It is a base, since $0 \notin B + \rho\mathbb{B}_Y$. By (2.2), $E_\rho^{s+} \neq \emptyset$, for all $\rho \in (0,\delta)$, and taking into account that $E \subset E_\rho$, we have $E_\rho^{s+} \subset E^{s+}$, for all $\rho \in (0,\delta)$, so inclusions "$\supset$" in (2.4) are proved.

Now, consider $\xi \in E^{s+}$ and suppose by reasoning to the contrary that $\xi \notin E_{\rho_n}^{s+}$, for all n. Then, by (2.5), for each $n \in \mathbb{N}$, there exists $b_n \in B$ and $y_n \in \mathbb{B}_Y$ such that $\xi(b_n + \rho_n y_n) \leq 0$.

Since B is a bounded closed set in the reflexive space Y, it is weakly compact and without loss of generality we can assume that $\{b_n\}$ converges weakly to $\bar{b} \in B$. Also $\{y_n\} \subset \mathbb{B}_Y$ converges weakly to $\bar{y} \in \mathbb{B}_Y$. Hence, $\{b_n + \rho_n y_n\}$ converges weakly to $\bar{b}$ and $\xi(\bar{b}) \leq 0$, but $\xi \in E^{s+}$, so $\xi(\bar{b}) > 0$, and we obtain a contradiction. Hence, $\bigcup_{n\in\mathbb{N}} E_{\rho_n}^{s+} \subset E^{s+}$ and both equalities in (2.4) follow straightforward. □

3. New Variants of DC Mapping

In this section, we introduce several variants of the notions of DC mapping given in Definition 2.2 and study the connections between all of them.

If in the notions of Definition 2.2 we only consider functionals in E^+, we obtain new concepts of vector DC mapping, which are collected in the following definition.

In what follows, we suppose that Y is reflexive, and that E is based (which is equivalent to say that $E^{s+} \neq \emptyset$ by (2.2)) and has a closed base B.

Definition 3.1. It is said that

(a) F is E^+-weakly DC if $\xi \circ F$ is a DC function for each $\xi \in E^+$.
(b) F is E^+-delta convex if there exists a continuous convex function $\varphi : X \to \mathbb{R}$ such that $\xi \circ F + \varphi$ is convex for each $\xi \in E^+ \cap \mathbb{B}_{Y^*}$.

On the other hand, in view of Remark 2.3(ii), to highlight that the definition of order DC mapping only involves functionals of E^+, we will use the terminology "E^+-order DC" instead of just "order DC".

Remark 3.1. From Definitions 2.2 and 3.1, it follows immediately that

(i) F is weakly DC $\Rightarrow$ F is E^+-weakly DC,
(ii) F is delta convex $\Rightarrow$ F is E^+-delta convex,
(iii) F is E^+-delta convex $\Rightarrow$ F is E^+-weakly DC,
(iv) F is E^+-order DC$\Rightarrow$ F is E^+-weakly DC.

Next, in relation to the above concepts, we define two notions of DC mapping that depend on a vector $(q \in E\backslash\{0\}) \cap \mathbb{B}_Y$. Their study is interesting in order to find more specific DC decompositions of a mapping.

Definition 3.2. Let $q \in (E\backslash\{0\}) \cap \mathbb{B}_Y$. We say that

(a) F is (q, E^+)-delta convex if there exists a continuous convex function $\varphi : X \to \mathbb{R}$ (called q-control function) such that $\xi \circ F + \xi(q)\varphi$ is convex for all $\xi \in E^+ \cap \mathbb{B}_{Y^*}$.
(b) F is (q, E^+)-order DC if F admits a DC decomposition of the form

$$F_1(x) = F(x) + \varphi(x)q, \quad F_2(x) = \varphi(x)q, \quad \forall x \in X, \tag{3.1}$$

where $\xi \circ F_1$ is convex on X, for all $\xi \in E^+$, and $\varphi : X \to \mathbb{R}$ is a continuous convex function.

Remark 3.2. Let us observe that, according to Proposition 2.1:

(i) If F is (q, E^+)-order DC, then it is E^+-order DC, for all $q \in (E\backslash\{0\}) \cap \mathbb{B}_Y$.
(ii) F_2 in (3.1) verifies that $\xi \circ F_2$ is convex on X, for all $\xi \in E^+$.

In the next proposition, we study more connections between the notions given in Definitions 2.2, 3.1 and 3.2.

Proposition 3.1. *Let $q \in (E\backslash\{0\}) \cap \mathbb{B}_Y$.*

(a) F is (q, E^+)-delta convex if and only if it is (q, E^+)-order DC.
(b) If F is (q, E^+)-delta convex, then it is E^+-delta convex.
(c) Suppose that E is solid and $q \in \operatorname{int} E$. If F is delta convex, then it is (q, E^+)-order DC. If additionally we suppose that $\operatorname{int}(E^+) \neq \emptyset$, then the reciprocal implication also holds.
(d) Suppose that E is normal. If F is E^+-weakly DC, then it is weakly DC.
(e) Suppose that $\operatorname{int}(E^+) \neq \emptyset$. If F is E^+-delta convex, then it is delta convex.

Proof. (a) From the definition and Proposition 2.1, it follows that F is (q, E^+)-order DC if and only if there exists a continuous convex function $\varphi : X \to \mathbb{R}$ such that $\xi \circ (F + \varphi q) : X \to \mathbb{R}$, defined as $\xi \circ (F + \varphi q)(x) := \xi(F(x) + \varphi(x)q)$, is convex, for all $\xi \in E^+$, if and only if $\xi \circ (F + \varphi q)$ is convex, for all $\xi \in E^+ \cap \mathbb{B}_{Y^*}$, if and only if F is (q, E^+)-delta convex, with q-control function φ.

(b) Suppose that F is (q, E^+)-delta convex. Then, by definition, there exists a q-control function φ such that $\xi \circ F + \xi(q)\varphi$ is convex for each $\xi \in E^+ \cap \mathbb{B}_{Y^*}$. Thus, since $|\xi(q)| =$

$\xi(q) \leq \|\xi\|_*\|q\| \leq 1$, it follows that $\xi \circ F + \varphi = (\xi \circ F + \xi(q)\varphi) + (1-\xi(q))\varphi$ is convex, for each $\xi \in E^+ \cap \mathbb{B}_{Y^*}$, so F is E^+-delta convex.

(c) Suppose that F is delta convex and let φ be an associated control function. Since $q \in \operatorname{int} E$, there exists $\gamma > 0$ such that $q + \gamma\mathbb{B}_Y \subset E$, i.e., $q - \gamma\dfrac{y}{\|y\|} \in E$, for all $y \in Y\backslash\{0\}$, which is equivalent to say that

$$y \leq_E \frac{\|y\|}{\gamma} q, \ \forall y \in Y. \tag{3.2}$$

Let $\hat{\varphi} := \frac{\varphi}{\gamma}$ and $F_2(x) := \hat{\varphi}(x)q$. It is clear that F_2 is E-convex. By statement (3.2) we have, for every $x_1, x_2 \in X$ and $\lambda \in (0,1)$,

$$\begin{aligned}
&F(\lambda x_1 + (1-\lambda)x_2) - \lambda F(x_1) - (1-\lambda)F(x_2) \\
&\leq_E \|F(\lambda x_1 + (1-\lambda)x_2) - \lambda F(x_1) - (1-\lambda)F(x_2)\|\frac{q}{\gamma} \\
&= \|\lambda F(x_1) + (1-\lambda)F(x_2) - F(\lambda x_1 + (1-\lambda)x_2)\|\frac{q}{\gamma} \\
&\leq_E (\lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) - \varphi(\lambda x_1 + (1-\lambda)x_2)\frac{q}{\gamma} \qquad (3.3) \\
&= \lambda F_2(x_1) + (1-\lambda)F_2(x_2) - F_2(\lambda x_1 + (1-\lambda)x_2),
\end{aligned}$$

where inequality (3.3) is due to [23, Proposition 1.13].

Therefore, by the inequalities above, we deduce that

$$(F+F_2)(\lambda x_1 + (1-\lambda)x_2) \leq_E \lambda(F+F_2)(x_1) + (1-\lambda)(F+F_2)(x_2),$$

for every $x_1, x_2 \in X$ and $\lambda \in (0,1)$, i.e., $F_1 := F + F_2 = F + \hat{\varphi}q$ is E-convex, so by Proposition 2.1 we have that $\xi \circ F_1$ is convex on X, for all $\xi \in E^+$, i.e., F is (q, E^+)-order DC.

Suppose additionally that $\operatorname{int}(E^+) \neq \emptyset$, i.e., E is well-based (see (2.1)), and that F is (q, E^+)-order DC. Then, by Remark 3.2(i) we have that F is E^+-order DC and by Proposition 2.2(c) we deduce that F is delta convex.

(d) Suppose that F is E^+-weakly DC. By statement (2.1), we have that $Y^* = E^+ - E^+$. Let $\xi \in Y^*$. Then, there exist $\xi_1, \xi_2 \in E^+$ such that $\xi = \xi_1 - \xi_2$, so $\xi \circ F = \xi_1 \circ F - \xi_2 \circ F$. Since $\xi_1 \circ F$ and $\xi_2 \circ F$ are DC by hypothesis, we conclude that $\xi \circ F$ is DC as well, so F is weakly DC.

(e) Suppose that F is E^+-delta convex and let φ be an associated control function. Then, it follows that $\xi \circ F + \varphi$ is convex for all $\xi \in E^+ \cap \mathbb{B}_{Y^*}$. Let $\hat{\xi} \in \operatorname{int}(E^+)$, with $\|\hat{\xi}\|_* \leq 1$. Then, there exists $\gamma > 0$ small enough such that $\hat{\xi} + \gamma\mathbb{B}_{Y^*} \subset E^+$. Therefore, since F is E^+-delta convex, we have that $\frac{1}{1+\gamma}(\hat{\xi} + \gamma y^*) \circ F + \varphi$ is convex, for all $y^* \in \mathbb{B}_{Y^*}$, i.e., $\frac{\gamma}{1+\gamma} y^* \circ F + \frac{1}{1+\gamma}\hat{\xi} \circ F + \varphi$ is convex for all $y^* \in \mathbb{B}_{Y^*}$, which is equivalent to say that $y^* \circ F + \hat{\varphi}$ is convex, for all $y^* \in \mathbb{B}_{Y^*}$, where $\hat{\varphi} := \frac{1}{\gamma}(\hat{\xi} \circ F + (1+\gamma)\varphi)$. It is clear that $\hat{\varphi}$ is convex, so F is delta convex, and the proof is complete. □

Remark 3.3. We can extract more conclusions from the results above, by considering the family of cones $\{E_\rho\}_{0<\rho<\delta}$. Let $q \in (E\backslash\{0\}) \cap \mathbb{B}_Y$. Then, in particular, it follows, according to Proposition 2.3(b), that $q \in \operatorname{int} E_\rho$ for all $\rho \in (0, \delta)$. Thus, by Proposition 3.1(c) we deduce that if F is delta convex, then it is (q, E_ρ^+)-order DC, for all $\rho \in (0, \delta)$.

The reciprocal implication holds if we suppose that $\operatorname{int}(E^+) \neq \emptyset$. Indeed, note that, by statement (2.1), we can assume without loss of generality that base B of E is bounded, additionally to being closed. In particular, $B+\rho\mathbb{B}_Y$ is a bounded base of E_ρ, for all $\rho \in (0,\delta)$, i.e., E_ρ is well-based too, so $\operatorname{int}(E_\rho^+) \neq \emptyset$. Hence, by Proposition 3.1(c) we deduce that if F is (q,E_ρ^+)-order DC, then it is delta convex, for all $\rho \in (0,\delta)$.

Taking into account Propositions 2.2 and 3.1, Remarks 3.1 and 3.3 and statement (2.1), we deduce the following corollary. In Figure 1 we collect all the connections between the different notions of DC mapping.

Corollary 3.1. *Suppose that* $\operatorname{int}(E^+) \neq \emptyset$ *and let* $q \in (E\backslash\{0\}) \cap \mathbb{B}_Y$. *It follows that*

(a) *F is weakly DC if and only if it is E^+-weakly DC.*
(b) *F is delta convex if and only if it is E^+-delta convex.*
(c) *Let $\rho \in (0,\delta)$. It follows that F is delta convex if and only if F is (q,E_ρ^+)-order DC, if and only if F is E_ρ^+-order DC. If, in addition, E is solid, then the equivalences also hold for $\rho = 0$ and $q \in \operatorname{int} E$.*
(d) *Let $\rho \in (0,\delta)$. It follows that F is delta convex if and only if it is (q,E_ρ^+)-delta convex.*

Proof. (a) Weakly DC $\overset{\text{Rem. 3.1(i)}}{\Rightarrow}$ E^+-weakly DC $\overset{\text{Prop. 3.1}(d), (2.1)}{\Rightarrow}$ weakly DC.

(b) Delta convex $\overset{\text{Rem. 3.1(ii)}}{\Rightarrow}$ E^+-delta convex $\overset{\text{Prop. 3.1}(e)}{\Rightarrow}$ delta convex.

(c) Delta convex $\overset{\text{Rem. 3.3}}{\Leftrightarrow}$ (q,E_ρ^+)-order DC $\overset{\text{Rem. 3.2(i)}}{\Rightarrow}$ E_ρ^+-order DC $\overset{\text{Prop. 2.2}(c),(2.1)}{\Rightarrow}$ delta convex.

Analogous reasoning if E is solid and $q \in \operatorname{int} E$, taking into account Proposition 3.1(c).

(d) It follows by part (c) and Proposition 3.1(a). □

Remark 3.4. (i) Suppose that $\operatorname{int}(E^+) \neq \emptyset$ and let $q \in (E\backslash\{0\}) \cap \mathbb{B}_Y$. Then, observe that Corollary 3.1(c) means that if F is E_ρ^+-order DC, for any $\rho \in (0,\delta)$, then we can ensure the existence of E_ρ-convex mappings F_1 and F_2 satisfying (3.1), and that F is delta convex if and only if such a DC decomposition exists. This is interesting since mapping F_2 in (3.1) is easy to handle. Moreover, by Corollary 3.1(b) it is enough to consider functionals in $E^+ \cap \mathbb{B}_{Y^*}$ when dealing with delta convex mappings, which is also interesting from a practical point of view.

(ii) If F is delta convex with control function φ, and we consider $q \in B \cap \mathbb{B}_Y$, then for each $\rho \in (0,\delta)$ we have that $q+\rho\mathbb{B}_Y \subset E_\rho$, so following an analogous reasoning as in the proof of Proposition 3.1(c), we deduce that F is (q,E_ρ^+)-order DC, for $F_2 := \frac{\varphi}{\rho} q$.

(iii) If X is a Hilbert space and F is of the class $\mathscr{C}^{1,1}(X)$ (the class of functions $T : X \to Y$ such that the Fréchet derivative $T'(x)$ exists for all x and the mapping T' is Lipschitz on X), then F is delta convex with control function $\varphi(x) := L\|x\|^2$, where L denotes the Lipschitz constant of F' (see, for instance, [23, Proposition 1.11]). Thus, in this case, given $q \in B \cap \mathbb{B}_Y$ and $\rho \in (0,\delta)$ and taking into account part (b), we have the following DC decomposition for F in terms of E_ρ-convex mappings:

$$F_2(x) := \frac{L}{\rho}\|x\|^2 q, \quad F_1(x) := F(x) + F_2(x),\ \forall x \in X.$$

(iv) Suppose that E_1 and E_2 are two pointed closed convex cones such that $\operatorname{int}(E_1^+) \neq \emptyset$ and $\operatorname{int}(E_2^+) \neq \emptyset$. Then, by Corollary 3.1(a)-(b) we have that

$$F \text{ is } E_1^+\text{-delta convex} \Leftrightarrow F \text{ is delta convex} \Leftrightarrow F \text{ is } E_2^+\text{-delta convex},$$
$$F \text{ is } E_1^+\text{-weakly DC} \Leftrightarrow F \text{ is weakly DC} \Leftrightarrow F \text{ is } E_2^+\text{-weakly DC}.$$

(v) Suppose that $Y = \mathbb{R}^m$. Then, by (2.1) and the comment below, we have that $\operatorname{int}(E^+) \neq \emptyset$ and E is normal. Thus, taking into account Proposition 3.1(d) and Remark 2.2, we have that

$$\begin{aligned} &F \text{ is } E^+\text{-weakly DC} \Leftrightarrow F \text{ is weakly DC} \Leftrightarrow F \text{ is } \mathbb{R}^m_+\text{-weakly DC} \\ \Leftrightarrow &F \text{ is } E^+\text{-delta convex} \Leftrightarrow F \text{ is delta convex} \Leftrightarrow F \text{ is } \mathbb{R}^m_+\text{-delta convex} \\ \Leftrightarrow &F \text{ is } \mathbb{R}^m_+\text{-order DC} \Leftarrow F \text{ is } E^+\text{-order DC}, \end{aligned}$$

where the single left implication holds by Proposition 2.2(c). Moreover, if E is additionally solid, then this implication becomes an equivalence, by Proposition 2.2(b).

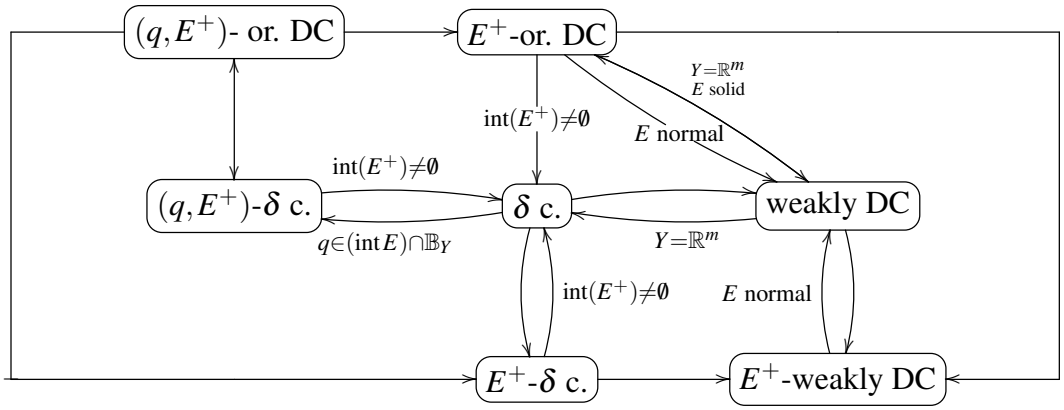


FIGURE 1. Connections between the different generalized DC notions. Notations: "-δ c." refers to "delta convex", and "or. DC" refers to "order DC".

4. Optimality Conditions for Approximate Proper Efficient Solutions of a DC Vector Optimization Problem

In this section we are going to focus on a type of approximate solutions of the following vector optimization problem

$$\operatorname{Min}_E\{F(x) : x \in S\}, \qquad (\mathscr{VOP}_{DC})$$

where $F : X \to Y$ is the objective mapping, which is going to satisfy a type of DC condition, and $S \neq \emptyset$ is the feasible set, that we suppose to be closed and convex.

As in the previous section, we suppose that Y is reflexive and F is continuous. Also, to deal with solutions of problem ($\mathscr{VOP}_{DC}$), we consider again that E has a closed base B. By (2.2) we also have that $E^{s+} \neq \emptyset$.

Specifically, we focus on a notion of approximate proper efficiency, which is an approximate version of the concept of proper efficiency due to Henig in [28].

Given a nonempty set $C \subset Y\backslash\{0\}$, we define the set-valued mapping with the same name $C : \mathbb{R}_+ \to 2^Y$ as follows:

$$C(\varepsilon) = \begin{cases} \varepsilon C & \text{if} \quad \varepsilon > 0 \\ (\operatorname{cone} C)\backslash\{0\} & \text{if} \quad \varepsilon = 0. \end{cases}$$

Definition 4.1. Let $\varepsilon \geq 0$, $\emptyset \neq C \subset Y\backslash\{0\}$ and $x_0 \in S$.

(a) Suppose that $C \cap (-E) = \emptyset$. It is said that x_0 is a (C,ε)-efficient solution of ($\mathscr{VOP}_{DC}$) if

$$(F(S) - F(x_0)) \cap (-C(\varepsilon)) = \emptyset.$$

The set of (C,ε)-efficient solutions is denoted by $\mathrm{AE}(F,S,C,\varepsilon)$.

(b) Suppose that $\operatorname{cl}(\operatorname{cone} C) \cap (-E\backslash\{0\}) = \emptyset$. It is said that x_0 is a (C,ε)-proper efficient solution of ($\mathscr{VOP}_{DC}$) in the sense of Henig if there exists a solid pointed convex cone $E' \subset Y$ such that $E\backslash\{0\} \subset \operatorname{int} E'$, $C \cap (-\operatorname{int} E') = \emptyset$ and

$$x_0 \in \mathrm{AE}(F,S,C + \operatorname{int} E', \varepsilon).$$

The set of (C,ε)-proper efficient solutions of ($\mathscr{VOP}_{DC}$) is denoted by $\mathrm{He}(F,S,C,\varepsilon)$.

Remark 4.1. (i) The concept of (C,ε)-efficiency was introduced in [39]. As it can be seen in this reference, this notion extends and unifies the most important concepts of approximate efficiency known from the literature.

On the other hand, the notion of (C,ε)-proper efficiency in the sense of Henig was given in [40]. It is clear that

$$\mathrm{He}(F,S,C,\varepsilon) \subset \mathrm{AE}(F,S,C + E\backslash\{0\}, \varepsilon).$$

Conditions $C \cap (-E) = \emptyset$ and $C \cap (-\operatorname{int} E') = \emptyset$ in the definitions of (C,ε)-efficiency and (C,ε)-proper efficiency, respectively, are given to avoid superfluous notions (see, for instance, [40]).

(ii) It is easy to see that if we consider $C = E\backslash\{0\}$ or $\varepsilon = 0$ and $C \subset Y\backslash\{0\}$ such that $\operatorname{cone} C = E$, then the notions given in Definition 4.1 reduce to their respective exact versions of efficiency and proper efficiency in the sense of Henig. Indeed, we remind that $x_0 \in S$ is an exact efficient solution of ($\mathscr{VOP}_{DC}$) if

$$(F(S) - F(x_0)) \cap (-E\backslash\{0\}) = \emptyset.$$

On the other hand, it is said that x_0 is an exact proper efficient solution of ($\mathscr{VOP}_{DC}$) in the sense of Henig if there exists a solid pointed convex cone $E' \subset Y$ such that $E\backslash\{0\} \subset \operatorname{int} E'$ and

$$(F(S) - F(x_0)) \cap (-\operatorname{int} E') = \emptyset.$$

In what follows, we denote by $\mathrm{He}(F,S,E)$ the set of exact proper efficient solutions.

Also, let us note that when $Y = \mathbb{R}^m$ and $E = \mathbb{R}^m_+$, the concept of proper efficiency in the sense of Henig is equivalent to the notion of proper efficiency in the sense of Geoffrion [41], probably better known in the setting of finite dimensional spaces (see, for instance, [42, 43]).

(iii) Let

$$\mathscr{H}_E(C) := \{E' \subset Y : E' \text{ solid convex pointed cone}, E\backslash\{0\} \subset \operatorname{int}(E'), C \cap (-\operatorname{int} E') = \emptyset\}.$$

From the definition, it is clear that

$$\mathrm{He}(F,S,C,\varepsilon) = \bigcup_{E' \in \mathscr{H}_E(C)} \mathrm{AE}(F,S,C + \operatorname{int} E', \varepsilon).$$

Suppose that base B of E is compact. Then, by [44, Proposition 2.4.6(iii)] and Proposition 2.3, it follows that

$$\mathrm{He}(F,S,C,\varepsilon)=\bigcup_{0<\rho<\delta}\mathrm{AE}(F,S,C+\operatorname{int}E_\rho,\varepsilon)=\bigcup_{0<\rho<\delta}\mathrm{AE}(F,S,C+E_\rho\backslash\{0\},\varepsilon), \tag{4.1}$$

where $\{E_\rho\}_{0<\rho<\delta}$ is the family of dilating cones defined in (2.3). The equalities given in (4.1) are interesting from a practical point of view, since they imply that the set of proper efficient solutions can be expressed only by means of the specific family $\{E_\rho\}_{0<\rho<\delta}$, so it is not necessary to deal with the abstract family $\mathscr{H}_E(C)$.

(iv) Sets $C_{\rho,q}:=q+E_\rho$, where $q\in E\backslash\{0\}$ and $\rho\in(0,\delta)$, are especially interesting to obtain approximate proper solutions. The reason is twofold: On the one hand, they are easy to construct and, on the other hand, the set of $(C_{\rho,q},\varepsilon)$-proper efficient solutions approximates suitably the set of exact proper solutions for $\varepsilon>0$ small enough. Indeed, if $x_0\in S$ and there exist sequences $\{x_n\}\subset X$ and $\{\varepsilon_n\}\subset\mathbb{R}_+\backslash\{0\}$ such that $x_n\in\mathrm{He}(F,S,C_{\rho,q},\varepsilon_n)$ for all n, and $\varepsilon_n\downarrow 0$, then $x_0\in\mathrm{He}(F,S,E)$ (see [45, Theorem 4.6(a)]).

The latter property is not as evident as it seems. Although it seems to be natural that the set of approximate solutions represents the set of exact solutions with a small error, there exist simple examples for which, for instance, the set of exact efficient solutions is a singleton and the set of approximate solutions is unbounded, with any precision error (see, for instance, [45, Example 4.1]).

(v) With respect again to sets $C_{\rho,q}$, let us note that

$$x_0\in\mathrm{He}(F,S,C_{\rho,q},\varepsilon)\Leftrightarrow(F(S)-F(x_0)+\varepsilon q)\cap(-\operatorname{int}E_\rho)=\emptyset.$$

Indeed, suppose that $x_0\in\mathrm{He}(F,S,C_{\rho,q},\varepsilon)$. Then, there exists $E'\in\mathscr{H}_E(C_{\rho,q})$ such that

$$(F(S)-F(x_0))\cap(-C_{\rho,q}(\varepsilon)-\operatorname{int}E')=\emptyset. \tag{4.2}$$

By definition, one has $C_{\rho,q}(\varepsilon)=\varepsilon q+E_\rho$, if $\varepsilon>0$, and $C_{\rho,q}(0)=\operatorname{int}E_\rho$. Indeed, inclusion "$\subset$" of the last equality is clear. For the reciprocal inclusion, let $e\in\operatorname{int}E_\rho$. Then, there exists $\eta>0$ such that $e+\eta\mathbb{B}_Y\subset E_\rho$, and there exists $\alpha>0$ small enough such that $-\alpha q\in\eta\mathbb{B}_Y$. Hence,

$$e=e-\alpha q+\alpha q\in e+\eta\mathbb{B}_Y+\alpha q=\alpha\left(q+\frac{1}{\alpha}(e+\eta\mathbb{B}_Y)\right)\subset\alpha(q+E_\rho)\subset C_{\rho,q}(0).$$

Thus, it is not difficult to check that

$$\varepsilon q+\operatorname{int}E_\rho=\varepsilon q+\operatorname{int}E_\rho+E\backslash\{0\}\subset C_{\rho,q}(\varepsilon)+\operatorname{int}E',$$

so statement (4.2) implies that $(F(S)-F(x_0)+\varepsilon q)\cap(-\operatorname{int}E_\rho)=\emptyset$. In fact, it is an equivalence, since the left implication follows directly from the definition of approximate proper efficiency and the properties of the family of cones $\{E_\rho\}_{0<\rho<\delta}$. Indeed, it is enough to choose $E'=E_\rho$ to see that $E'\in\mathscr{H}_E(C_{\rho,q})$ and $x_0\in\mathrm{AE}(F,S,C_{\rho,q}+\operatorname{int}E',\varepsilon)$.

In Theorem 4.1, we provide a characterization through scalarization for a feasible point x_0 to be an approximate proper efficient solution of problem ($\mathscr{VOP}_{DC}$). To motivate this result, we need to introduce some preliminaries concerning an unconstrained scalar DC optimization problem, defined in the following way:

$$\mathrm{Min}\{f(x):x\in X\}, \tag{$\mathscr{SOP}_{DC}$}$$

where $f := f_1 - f_2$, and $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ are proper convex and lower semicontinuous functions. By convention, we suppose that $+\infty - (+\infty) = +\infty$. We also suppose that $\operatorname{dom} f_1 \subset \operatorname{dom} f_2$, so in this way function f does not take the value $-\infty$ (we remind that $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$).

Besides, a well-known result from Hiriart-Urruty (see [46, Theorem 4.4]) establishes that given $x_0 \in \operatorname{dom} f$ with $\partial f_2(x_0) \neq \emptyset$, it follows that x_0 is a δ-global minimum of ($\mathscr{SOP}_{DC}$) (i.e., $f(x_0) \leq f(x) + \delta$, for all $x \in X$) if and only if

$$\partial_\varepsilon f_2(x_0) \subset \partial_{\varepsilon+\delta} f_1(x_0), \ \forall \varepsilon \geq 0,$$

where $\partial_\varepsilon h$ is the classical ε-subdifferential of Convex Analysis of a convex function $h : X \to \mathbb{R} \cup \{+\infty\}$, given for any $\bar{x} \in \operatorname{dom} h$ by

$$\partial_\varepsilon h(\bar{x}) := \{x^* \in X^* : h(x) \geq h(\bar{x}) - \varepsilon + x^*(x - \bar{x}), \forall x \in \operatorname{dom} h\},$$

and $\partial h(\bar{x}) := \partial_0 h(\bar{x})$. It follows that $\partial_\varepsilon h(\bar{x})$ is not empty for all $\varepsilon > 0$ if and only if h is lower semicontinuous at $\bar{x}$, and that $\partial h(\bar{x})$ is nonempty if h is continuous at $\bar{x}$ (see, for instance, [34, Theorems 2.4.4 and 2.4.9]).

In what follows, we fix $q \in (E \backslash \{0\}) \cap \mathbb{B}_Y$, and we focus on approximate proper efficient solutions with respect to the approximation sets $C_{\rho,q}$, due to the reasons exposed in Remark 4.1(iv).

We recall that the indicator function $I_A : X \to \mathbb{R} \cup \{+\infty\}$ of a nonempty set $A \subset X$ is given by $I_A(x) = 0$ if $x \in A$, and $I_A(x) = +\infty$, otherwise.

Theorem 4.1. *Let $\varepsilon \geq 0$, $\rho \in (0, \delta)$ and $x_0 \in S$. Suppose that F is (q, E_ρ^+)-order DC, with q-control function φ. Then, $x_0 \in \operatorname{He}(F, S, C_{\rho,q}, \varepsilon)$ if and only if*

$$\partial_\gamma \varphi(x_0) \subset \bigcup_{\substack{\mu \in E_\rho^+ \backslash \{0\} \\ \mu(q)=1}} \partial_{\gamma+\varepsilon}(\mu \circ F + \varphi + I_S)(x_0), \quad \forall \gamma \geq 0.$$

Proof. Suppose that $x_0 \in \operatorname{He}(F, S, C_{\rho,q}, \varepsilon)$. Fix $\gamma \geq 0$ and let $x^* \in \partial_\gamma \varphi(x_0)$. Also, let

$$H(x) := F(x) + \varphi(x)q - F(x_0) - \varphi(x_0)q - x^*(x - x_0)q + (\varepsilon + \gamma)q.$$

Note that H is E_ρ-convex because $H_1(x) := F(x) + \varphi(x)q$ is E_ρ-convex, $H_2(x) := -x^*(x - x_0)q$ is E_ρ-convex and H is of the form $H = H_1 + H_2 + y_0$, with $y_0 \in Y$.

It follows that $H(x) \in -\operatorname{int} E_\rho$ has no solution in S. Indeed, suppose that there exists $\bar{x} \in S$ such that

$$H(\bar{x}) \in -\operatorname{int} E_\rho. \tag{4.3}$$

Then, since $x^* \in \partial_\gamma \varphi(x_0)$ we have in particular that

$$[\varphi(\bar{x}) - \varphi(x_0) + \gamma - x^*(\bar{x} - x_0)]q \in E. \tag{4.4}$$

Thus, as

$$H(\bar{x}) = F(\bar{x}) - F(x_0) + [\varphi(\bar{x}) - \varphi(x_0) + \gamma - x^*(\bar{x} - x_0)]q + \varepsilon q,$$

by (4.3) and (4.4) we deduce that

$$F(\bar{x}) - F(x_0) + \varepsilon q = H(\bar{x}) - [\varphi(\bar{x}) - \varphi(x_0) + \gamma - x^*(\bar{x} - x_0)]q \in -\operatorname{int} E_\rho - E = -\operatorname{int} E_\rho,$$

which contradicts that $x_0 \in \mathrm{He}(F,S,C_{\rho,q},\varepsilon)$ (see Remark 4.1(v)). Therefore, we have that

$$(H(S)+E_\rho)\cap(-\operatorname{int}E_\rho)=\emptyset.$$

Since H is E_ρ-convex and S is a convex set, we have $H(S)+E_\rho$ is a convex set. Thus, by applying a standard separation result (see, for instance, [35, Theorem 3.16]), there exist $\bar\mu \in Y^*\backslash\{0\}$ and $\alpha\in\mathbb{R}$ such that

$$\bar\mu(-e_1)\le\alpha\le(\bar\mu\circ H)(x)+\bar\mu(e_2),\quad \forall x\in S,\forall e_1,e_2\in E_\rho, \tag{4.5}$$

$$\bar\mu(-e_1)<\alpha,\quad \forall e_1\in\operatorname{int}E_\rho. \tag{4.6}$$

By (4.6) we deduce that $\bar\mu\in E_\rho^+\backslash\{0\}$. On the other hand, by (4.5), with $e_1=e_2=0$, we derive

$$(\bar\mu\circ F)(x)+\varphi(x)\bar\mu(q)-(\bar\mu\circ F)(x_0)-\varphi(x_0)\bar\mu(q)$$
$$-x^*(x-x_0)\bar\mu(q)+(\varepsilon+\gamma)\bar\mu(q)\ge 0,\quad \forall x\in S,$$

i.e., $\bar\mu(q)x^*\in\partial_{\bar\mu(q)(\gamma+\varepsilon)}(\bar\mu\circ F+\bar\mu(q)\varphi+I_S)(x_0)$ or, equivalently,

$$x^*\in\partial_{\gamma+\varepsilon}(\mu\circ F+\varphi+I_S)(x_0),$$

with $\mu:=\frac{1}{\bar\mu(q)}\bar\mu$ (observe that $q\in E\backslash\{0\}\subset\operatorname{int}E_\rho$, so $\bar\mu(q)>0$), and the right implication is proved.

For the reciprocal implication, suppose that $x_0\notin\mathrm{He}(F,S,C_{\rho,q},\varepsilon)$. Then, by Remark 4.1(v), there exists $\bar x\in S$ such that

$$F(\bar x)-F(x_0)+\varepsilon q\in-\operatorname{int}E_\rho,$$

which implies that

$$(\mu\circ F)(\bar x)+I_S(\bar x)<(\mu\circ F)(x_0)+I_S(x_0)-\varepsilon\mu(q), \tag{4.7}$$

for all $\mu\in E_\rho^+\backslash\{0\}$.

Let $x^*\in\partial\varphi(\bar x)$ (this subdifferential is nonempty, since φ is convex and continuous on X). Then, in particular,

$$\varphi(x_0)\ge\varphi(\bar x)+x^*(x_0-\bar x).$$

Let

$$\gamma:=\varphi(x_0)-\varphi(\bar x)-x^*(x_0-\bar x)\ge 0. \tag{4.8}$$

It follows that

$$x^*\in\partial_\gamma\varphi(x_0).$$

On the other hand, by (4.9) we deduce that

$$(\mu\circ F)(\bar x)+I_S(\bar x)+\mu(q)\varphi(\bar x)-\mu(q)\varphi(\bar x)$$
$$<(\mu\circ F)(x_0)+I_S(x_0)+\mu(q)\varphi(x_0)-\mu(q)\varphi(x_0)-\varepsilon\mu(q)+x^*(x_0-\bar x)\mu(q)+x^*(\bar x-x_0)\mu(q), \tag{4.9}$$

and taking into account (4.8) we derive that

$$(\mu\circ F)(\bar x)+I_S(\bar x)+\mu(q)\varphi(\bar x)$$
$$<(\mu\circ F)(x_0)+I_S(x_0)+\mu(q)\varphi(x_0)-\varepsilon\mu(q)-\gamma\mu(q)+x^*(\bar x-x_0)\mu(q),$$

for all $\mu \in E_\rho^+\backslash\{0\}$, i.e.,

$$\mu(q)x^* \notin \partial_{\mu(q)(\gamma+\varepsilon)}(\mu\circ F+\mu(q)\varphi+I_S)(x_0), \quad \forall \mu \in E_\rho^+\backslash\{0\}.$$

Then, $x^* \notin \bigcup_{\substack{\mu\in E_\rho^+\backslash\{0\}\\ \mu(q)=1}} \partial_{\gamma+\varepsilon}(\mu\circ F+\varphi+I_S)(x_0)$, and the proof is complete. □

In the next result we characterize the exact proper efficient solutions of problem ($\mathscr{VOP}_{DC}$). For this aim, we need the following lemma. We denote

$$\mathscr{H} := \mathscr{H}_E(E\backslash\{0\}) = \{E' \subset Y \text{ solid convex pointed cone}, E\backslash\{0\} \subset E'\}.$$

Lemma 4.1. *It follows that*

$$E^{s+} = \bigcup_{E'\in\mathscr{H}} (E'^+\backslash\{0\}).$$

Proof. Inclusion "$\supset$" is clear, since $E\backslash\{0\} \subset \operatorname{int}E'$, for all $E' \in \mathscr{H}$ and so, if $y^* \in E'^+ \setminus \{0\}$ one has $y^*(e) > 0$ for all $e \in \operatorname{int}E'$, so $y^*(e) > 0$ for all $e \in E \setminus \{0\}$ and $y^* \in E^{s+}$.

On the other hand, let $y^* \in E^{s+}$ and let

$$\hat{E} := \{y \in Y : y^*(y) > 0\} \cup \{0\}.$$

It is clear that $\hat{E} \in \mathscr{H}$ and $y^* \in \hat{E}^+\backslash\{0\}$, so the proof is complete. □

Theorem 4.2. *Let $x_0 \in S$. Suppose that F is (q,E^+)-order DC, with q-control function φ. Then, $x_0 \in \operatorname{He}(F,S,E)$ if and only if*

$$\partial_\gamma\varphi(x_0) \subset \bigcup_{\substack{\mu\in E^{s+}\\ \mu(q)=1}} \partial_\gamma(\mu\circ F+\varphi+I_S)(x_0), \quad \forall\gamma \geq 0. \tag{4.10}$$

Proof. Suppose that $x_0 \in \operatorname{He}(F,S,E)$. Then, by definition, there exists $E' \in \mathscr{H}$ such that

$$(F(S)-F(x_0)) \cap (-\operatorname{int}E') = \emptyset.$$

Thus, we reason in analogous way as in the proof of the right implication of Theorem 4.1, by considering $\varepsilon = 0$ and by replacing E_ρ by E' in (4.3), to deduce that

$$(H(S)+E) \cap (-\operatorname{int}E') = \emptyset.$$

If we continue following the proof of the right implication of Theorem 4.1, we conclude that (4.10) holds (note that $(E')^+\backslash\{0\} \subset E^{s+}$).

For the reciprocal implication, suppose that $x_0 \notin \operatorname{He}(F,S,E)$, and let $E' \in \mathscr{H}$. Then, there exists $\bar{x} \in S$ such that $F(\bar{x}) - F(x_0) \in -\operatorname{int}E'$. Thus, reasoning in analogous way as in the proof of the left implication of Theorem 4.1 (for $\varepsilon = 0$ and for E' instead of E_ρ), we deduce that

$$\partial_\gamma\varphi(x_0) \not\subset \bigcup_{\substack{\mu\in (E')^+\backslash\{0\}\\ \mu(q)=1}} \partial_\gamma(\mu\circ F+\varphi+I_S)(x_0), \quad \forall\gamma \geq 0.$$

Since the statement above holds for any cone $E' \in \mathscr{H}$, by Lemma 4.1 we deduce that

$$\partial_\gamma\varphi(x_0) \not\subset \bigcup_{\substack{\mu\in E^{s+}\\ \mu(q)=1}} \partial_\gamma(\mu\circ F+\varphi+I_S)(x_0), \quad \forall\gamma \geq 0,$$

and the proof is finished. □

Remark 4.2. (i) By Remark 3.3, we know that if F is delta convex, then it is (q,E_ρ^+)-order DC, for all $\rho\in(0,\delta)$, and it is (q,E^+)-order DC if E is solid and $q\in \operatorname{int}E$ by Corollary 3.1(c).

(ii) Since in Theorem 4.1 we have that $\mu\circ F+\varphi$ is convex and continuous for all $\mu\in E_\rho^+$, with $\mu(q)=1$, and I_S is convex, then we can apply the classical sum rule for approximate subdifferentials (see, for instance, [34, Theorem 2.8.7]) to obtain that

$$\partial_{\gamma+\varepsilon}(\mu\circ F+\varphi+I_S)(x_0)=\bigcup_{\substack{\gamma_1,\gamma_2\geq 0\\ \gamma_1+\gamma_2=\gamma+\varepsilon}}\left\{\partial_{\gamma_1}(\mu\circ F+\varphi)(x_0)+N_{\gamma_2}(S;x_0)\right\},$$

where $N_{\gamma_2}(S;x_0):=\partial_{\gamma_2}I_S(x_0)=\{x^*\in X^*: x^*(x-x_0)\leq\gamma_2,\ \forall x\in S\}$, is the γ_2-normal cone of S at x_0. The above also applies to Theorem 4.2.

To finish, we consider the particular case of a classical multiobjective optimization problem with inequality constraints, and characterize its approximate proper solutions. Indeed, we consider $Y=\mathbb{R}^m$ and $E=\mathbb{R}^m_+$, so we have the following problem:

$$\begin{aligned}&\operatorname{Min}\{f(x): x\in S\},\\ &S=\{x\in X: g(x)\in -\mathbb{R}^p_+\},\end{aligned}\qquad (\mathscr{MOP}_{DC})$$

where $f=(f_1,f_2,\dots,f_m):X\to\mathbb{R}^m$, and $g=(g_1,g_2,\dots,g_p):X\to\mathbb{R}^p$ is $\mathbb{R}^p_+$-convex on X. We also suppose that each g_i is continuous, for $i\in\{1,2,\dots,p\}$, and that the Slater constraint qualification holds, i.e., there exists $\bar{x}\in X$ such that $g(\bar{x})\in -\operatorname{int}\mathbb{R}^p_+$.

We consider the ∞-norm in $\mathbb{R}^m$, i.e.,

$$\|(y_1,y_2,\dots,y_m)\|_\infty:=\max_{i\in\{1,2,\dots,m\}}\{|y_i|\},\quad \forall (y_1,y_2,\dots,y_m)\in\mathbb{R}^m.$$

Let $q=(1,1,\dots,1)\in\mathbb{R}^m_+\cap\mathbb{B}_{\mathbb{R}^m}$, and denote $R_\rho:=(\mathbb{R}^m_+)_\rho$, for $\rho\in(0,\delta)$. By Proposition 3.1(c), it follows that f is delta convex (i.e., by Remark 2.2, each component f_i is a DC function, for all $i=1,2,\dots,m$) if and only if it is (q,R_ρ^+)-order DC, for any $\rho\in(0,\delta)$.

Thus, for the following result, we suppose that f is delta convex and that φ is a control function of f.

Corollary 4.1. *Let $x_0\in S$. Suppose that f is delta convex with control function φ.*

(a) Let $\varepsilon\geq 0$ and $\rho\in(0,\delta)$. It follows that $x_0\in \operatorname{He}(f,S,C_{\rho,q},\varepsilon)$ if and only if

$$\partial_\gamma\varphi(x_0)\subset\bigcup_{\substack{\mu\in R_\rho^+\setminus\{0\}\\ \sum_{i=1}^m\mu_i=1}}\ \bigcup_{\substack{\gamma_i\geq 0\\ \sum_{i=1}^{m+1}\gamma_i=\gamma+\varepsilon}}\left\{\sum_{i=1}^m\mu_i\partial_{\frac{\gamma_i}{\mu_i}}(f_i+\varphi)(x_0)+\bigcup_{\substack{\alpha_j,\beta_j\geq 0\\ \sum_{j=1}^p\eta_j=\gamma_{m+1}}}\partial_{\beta_j}(\alpha_j g_j)(x_0)\right\},$$

for all $\gamma\geq 0$, where $\eta_j=\beta_j-\alpha_j g_j(x_0)$, $j=1,2,\dots,p$.

(b) It follows that $x_0\in \operatorname{He}(f,S,E)$ if and only if

$$\partial_\gamma\varphi(x_0)\subset\bigcup_{\substack{\mu\in\operatorname{int}\mathbb{R}^m_+\\ \sum_{i=1}^m\mu_i=1}}\ \bigcup_{\substack{\gamma_i\geq 0\\ \sum_{i=1}^{m+1}\gamma_i=\gamma}}\left\{\sum_{i=1}^m\mu_i\partial_{\frac{\gamma_i}{\mu_i}}(f_i+\varphi)(x_0)+\bigcup_{\substack{\alpha_j,\beta_j\geq 0\\ \sum_{j=1}^p\eta_j=\gamma_{m+1}}}\partial_{\beta_j}(\alpha_j g_j)(x_0)\right\},$$

for all $\gamma\geq 0$.

Proof. First, observe that, by Lemma 4.1 for $E=\mathbb{R}^m_+$, we have that $R^+_\rho\backslash\{0\}\subset \operatorname{int}\mathbb{R}^m_+$, so for any $\mu\in R^+_\rho\backslash\{0\}$ we have in particular that $\mu_i>0$, for all $i\in\{1,2,\ldots,m\}$.

On the other hand, for all $\mu\in R^+_\rho\backslash\{0\}$, with $\sum_{i=1}^m\mu_i=1$, it follows that

$$\mu\circ f+\varphi=\mu_1(f_1+\varphi)+\mu_2(f_2+\varphi)+\ldots+\mu_m(f_m+\varphi).$$

Also, since f is delta convex, we have that $f_i+\varphi$ is convex, for all $i\in\{1,2,\ldots,m\}$. Then, the proof of part (a) follows directly by Theorem 4.1 and Remark 4.2(ii), applying the sum rule for approximate subdifferentials to

$$\partial_\delta\left(\mu_1(f_1+\varphi)+\mu_2(f_2+\varphi)+\ldots+\mu_m(f_m+\varphi)\right)(x_0),$$

for some $\delta\geq 0$, and taking into account that

$$\partial_\zeta I_S(x_0)=N_\zeta(S;x_0)=\bigcup_{\substack{\alpha_j,\beta_j\geq 0\\ \sum_{j=1}^p\eta_j=\zeta}}\partial_{\beta_j}(\alpha_j g_j)(x_0),$$

where $\zeta\geq 0$, and $\eta_j:=\beta_j-\alpha_j g_j(x_0)$, $j=1,2,\ldots,p$ (see, for instance, [47, Lemma 2]). The proof of part (b) follows the same reasoning, taking into account in this case Theorem 4.2. □

Remark 4.3. Note that the characterizations obtained in Theorems 4.1 and 4.2, and in Corollary 4.1 are generalizations to the vector and, in particular, the multiobjective case of the classical characterization of approximate and exact global minima, for unconstrained scalar DC optimization problems, due to Hiriart-Urruty [1]. Moreover, our results also cover constraint optimization problems, as well as unconstrained ones.

We illustrate the results of this section with the following example.

Example 4.1. Let $f:\mathbb{R}\to\mathbb{R}^2$ be given by $f=(f_1,f_2)$, where $f_1(x)=x^2$, $f_2(x)=x-x^2$, $S=\mathbb{R}$. We are going to compute the set of exact proper efficient solutions of $(\mathscr{MCP}_{DC})$ with respect to the Pareto cone, i.e., set $\mathrm{He}(f,\mathbb{R},\mathbb{R}^2_+)$. First of all, observe that f is delta convex with associated control function $\varphi(x)=2x^2$ (see, for instance, Remark 3.4(iii)).

Let $a\in\mathbb{R}$. By Corollary 4.1(b) it follows that $a\in\mathrm{He}(f,\mathbb{R},\mathbb{R}^2_+)$ if and only if

$$\partial_\gamma\varphi(a)\subset\bigcup_{\substack{\mu_1,\mu_2>0\\ \mu_1+\mu_2=1}}\bigcup_{\substack{\gamma_1,\gamma_2\geq 0\\ \gamma_1+\gamma_2=\gamma}}\left\{\mu_1\partial_{\frac{\gamma_1}{\mu_1}}(f_1+\varphi)(a)+\mu_2\partial_{\frac{\gamma_2}{\mu_2}}(f_2+\varphi)(a)\right\},\ \forall\gamma\geq 0,$$

which is equivalent to

$$\partial_\gamma\varphi(a)\subset\bigcup_{0<\mu_1<1}\bigcup_{0\leq\gamma_1\leq\gamma}\left\{\mu_1\partial_{\frac{\gamma_1}{\mu_1}}(f_1+\varphi)(a)+(1-\mu_1)\partial_{\frac{\gamma-\gamma_1}{1-\mu_1}}(f_2+\varphi)(a)\right\},\ \forall\gamma\geq 0.$$

Let

$$I_a(\gamma):=\bigcup_{0<\mu_1<1}\bigcup_{0\leq\gamma_1\leq\gamma}\left\{\mu_1\partial_{\frac{\gamma_1}{\mu_1}}(f_1+\varphi)(a)+(1-\mu_1)\partial_{\frac{\gamma-\gamma_1}{1-\mu_1}}(f_2+\varphi)(a)\right\}.$$

We have that $(f_1+\varphi)(x)=3x^2$ and $(f_2+\varphi)(x)=x^2+x$. Then, given $\varepsilon\geq 0$, it is not difficult to check that

$$\begin{aligned}\partial_\varepsilon\varphi(a)&=[4a-2\sqrt{2\varepsilon},4a+2\sqrt{2\varepsilon}],\\ \partial_\varepsilon(f_1+\varphi)(a)&=[6a-2\sqrt{3\varepsilon},6a+2\sqrt{3\varepsilon}],\\ \partial_\varepsilon(f_2+\varphi)(a)&=[1+2a-2\sqrt{\varepsilon},1+2a+2\sqrt{\varepsilon}].\end{aligned}$$

Therefore, given $\gamma\geq 0$, $0<\mu_1<1$ and $0\leq\gamma_1\leq\gamma$, we have that

$$\begin{aligned}&\mu_1\partial_{\frac{\gamma_1}{\mu_1}}(f_1+\varphi)(a)+(1-\mu_1)\partial_{\frac{\gamma-\gamma_1}{1-\mu_1}}(f_2+\varphi)(a)=\\ &\mu_1\left[6a-2\sqrt{3\frac{\gamma_1}{\mu_1}},6a+2\sqrt{3\frac{\gamma_1}{\mu_1}}\right]+(1-\mu_1)\left[1+2a-2\sqrt{\frac{\gamma-\gamma_1}{1-\mu_1}},1+2a+2\sqrt{\frac{\gamma-\gamma_1}{1-\mu_1}}\right]=\\ &[h_a(\mu_1)-r_a(\mu_1,\gamma_1),h_a(\mu_1)+r_a(\mu_1,\gamma_1)],\end{aligned}$$

where $h_a(\mu_1):=6a\mu_1+(1+2a)(1-\mu_1)$, and $r_a(\mu_1,\gamma_1):=2(\sqrt{3\gamma_1\mu_1}+\sqrt{(\gamma-\gamma_1)(1-\mu_1)})$. Then, for $\gamma\geq 0$, it follows that

$$\mathrm{cl}(I_a(\gamma))=[\psi_{1a}(\gamma),\psi_{2a}(\gamma)],$$

where

$$\begin{aligned}\psi_{1a}(\gamma)&:=\inf_{\substack{0<\mu_1<1\\0\leq\gamma_1\leq\gamma}}(h_a(\mu_1)-r_a(\mu_1,\gamma_1)),\\ \psi_{2a}(\gamma)&:=\sup_{\substack{0<\mu_1<1\\0\leq\gamma_1\leq\gamma}}(h_a(\mu_1)+r_a(\mu_1,\gamma_1)).\end{aligned}$$

Then, ψ_{1a} and ψ_{2a} are obtained by solving a standard constrained scalar optimization problem with two variables, depending on parameter $a\in\mathbb{R}$. To avoid increasing the length of the paper and making it tedious, we only show the final result for $I_a(\gamma)$. Indeed, let $\Delta:=\frac{(4a-1)^2}{4}$. Then,

$$I_a(\gamma)=\begin{cases}(6a-2\sqrt{3\gamma},1+2a+2\sqrt{\gamma}) & 0\leq\gamma\leq\Delta\\ \left(6a-2\sqrt{3\gamma},\frac{3}{2}+\frac{2\gamma}{1-4a}\right] & \Delta<\gamma<3\Delta\\ (6a-2\sqrt{3\gamma},6a+2\sqrt{3\gamma}) & \gamma\geq 3\Delta\end{cases},\qquad \text{if } a<\frac{1}{4}$$

$$I_a(\gamma)=\begin{cases}\left(\frac{3}{2}-2\sqrt{3\gamma},\frac{3}{2}+2\sqrt{3\gamma}\right) & \gamma>0\\ \left\{\frac{3}{2}\right\} & \gamma=0\end{cases},\qquad \text{if } a=\frac{1}{4}$$

$$I_a(\gamma)=\begin{cases}(1+2a-2\sqrt{\gamma},6a+2\sqrt{3\gamma}) & 0\leq\gamma\leq\Delta\\ \left[\frac{3}{2}-\frac{2\gamma}{4a-1},6a+2\sqrt{3\gamma}\right) & \Delta<\gamma<3\Delta\\ (6a-2\sqrt{3\gamma},6a+2\sqrt{3\gamma}) & \gamma\geq 3\Delta\end{cases},\qquad \text{if } a>\frac{1}{4}$$

It follows that $\partial_\gamma\varphi(a)\subset I_a(\gamma)$, for all $\gamma\geq 0$ if and only if $a<0$. The calculus to check the assertion above are easy but too long, so we do not show them here (as a small sample, for instance, if $a=0$, then $\partial_0\varphi(0)\not\subset I_0(0)$; if $a=2$ then $\partial_{14}\varphi(2)\not\subset I_2(14)$). Thus, by virtue of Corollary 4.1(b), we deduce that $\mathrm{He}(f,\mathbb{R},\mathbb{R}^2_+)=(-\infty,0)$, as it can be easily seen following the definition of exact proper efficiency in the sense of Henig.

5. Conclusions

We introduced several variants of the most known concepts of vector DC mapping defined on Banach spaces, and we studied their main properties and connections between them. A DC decomposition associated with each one of these variants is easier to handle and thus more interesting computationally. After that, we provided a characterization of approximate and exact proper efficient solutions, respectively, in the sense of Henig, of a constrained vector DC optimization problem, whose objective mapping satisfies one of these variants of DC condition. These results were derived through linear scalarization and in terms of the classical subdifferential of Convex Analysis, extending to the vector case the celebrated characterization of approximate and exact global minima of unconstrained scalar DC optimization problems due to Hiriart-Urruty [1]. The results were particularized to the finite dimensional setting, as well, where the feasible set is given by inequality constraints.

In a forthcoming paper, we will study alternative characterizations of efficient solutions of a vector DC optimization problem, through nonlinear scalarization. Moreover, in the finite dimensional setting, we will also focus on the case when one of the mappings of a DC decomposition is polyhedral, which is one of the most interesting cases in DC (and DCA) Optimization, due to its great practicality.

Acknowledgments

The authors are very grateful to the anonymous referees for their valuable comments and suggestions, which considerably improved the paper. This work was partially supported by Grant PID2024-156273NA-I00 funded by MICIU/AEI/10.13039/501100011033/ERDF, EU.

References

[1] J.B. Hiriart-Urruty, Generalized differentiability, duality and optimization for problems dealing with differences of convex functions, In: J. Ponstein, (eds) Convexity and Duality in Optimization. Lecture Notes in Econom. and Math. Systems, vol. 256, pp. 37-70, Springer, Berlin, Heidelberg, 1985.

[2] R. Horst, N.V. Thoai, DC programming: Overview, J. Optim. Theory Appl. 103 (1999), 1-43.

[3] W. Oliveira, The ABC of DC programming, Set-Valued Var. Anal. 28 (2020), 679-706.

[4] H.A. Le Thi, P.D. Tao, DC programming and DCA: thirty years of developments, Math. Program. Ser. B. 169 (2018), 5-68.

[5] H.A. Le Thi, P.D. Tao, H.M. Le, X.T. Vo, DC approximation approaches for sparse optimization, Eur. J. Oper. Res. 244 (2015), 26-46.

[6] R.T. Rockafellar, Favorable classes of Lipschitz continuous functions in subgradient optimization, In: Non-differentiable Optimization, Nurminski E. (ed.), Pergamon Press, New York, 1982.

[7] A. Daniilidis, J. Malick, Filling the gap between lower-C^1 and lower-C^2 functions, J. Convex. Anal. 12 (2005), 315-329.

[8] F. Aragón-Artacho, P.T. Vuong, The boosted difference of convex functions algorithm for nonsmooth functions, SIAM J. Optim. 30 (2020), 980–1006.

[9] R. Correa, M.A. López, P. Pérez-Aros, Necessary and sufficient optimality conditions in DC Semi-infinite programming, SIAM J. Optim. 31 (2021), 837-865.

[10] H. Dong, M. Tao, On the linear convergence to weak/standard d-stationary points of DCA-based algorithms for structured nonsmooth dc programming, J. Optim. Theory. Appl. 189 (2021), 190-220.
[11] M.D. Fajardo, J. Vidal-Núñez, Lagrange duality on DC evenly convex optimization problems via a generalized conjugation scheme, J. Optim. Lett. 19 (2025), 1217–1238.
[12] A. Griewank, A. Walther, Polyhedral DC decomposition and DCA optimization of piecewise linear functions, Algorithms (Basel) 13 (2020), 166.
[13] V.T. Huong, D.T.K. Huyen, N.D. Yen, Generalized polyhedral DC optimization problems, J. Optim. Theory Appl. 207 (2025), 11.
[14] G. Li, Y. Xu, Z. Qin, Fenchel-Lagrange duality for DC infinite programs with inequality constraints, J. Comput. Appl. Math. 391 (2021), 113426.
[15] A. Löhne, A. Wagner, Solving DC programs with a polyhedral component utilizing a multiple objective linear programming solver, J. Global Optim. 69 (2017), 369-385.
[16] F.M. Pirani, F. Ulus, Global solution algorithms for DC programming via polyhedral approximations of convex functions, J. Global Optim. 93 (2025), 335–357.
[17] J.F. Toland, Duality in nonconvex optimization, J. Math. Anal. Appl. 66 (1978), 399-415.
[18] I. Singer, A Fenchel-Rockafellar type duality theorem for maximization, Bull. Australian Math. Soc. 20 (1979), 193-198.
[19] F. Flores-Bazán, On minima of the difference of functions, J. Optim. Theory Appl. 93 (1997), 525-531.
[20] R. Horst, P.M. Pardalos, N.V. Thoai, Introduction to Global Optimization, Kluwer Academic Publishers, Dordrecht, 1995.
[21] J.E. Martínez-Legaz, A. Seeger, A formula on the approximate subdifferential of the difference of convex functions, Bull. Austral. Math. Soc. 45 (1992), 37-41.
[22] P. Hartman, On functions representable as a difference of convex functions, Pacific. J. Math. 9 (1959), 707-713.
[23] L. Veselý, L. Zajíček, Delta-convex mappings between Banach spaces and applications, Dissertationes Math. (Rozprawy Mat.). 289 (1989), 52 pp.
[24] L. Veselý, L. Zajíček, On D.C. mappings and differences of convex operators, Acta Univ. Carolin. Math. Phys. 42 (2001), 89-97.
[25] L. Veselý, L. Zajíček, On connections between delta-convex mappings and convex operators, Proc. Edinb. Math. Soc. 49 (2006), 739-751.
[26] N. Gadhi, A. Metrane, Sufficient optimality condition for vector optimization problems under D.C. data, J. Global Optim. 28 (2004), 55-66.
[27] X.L. Guo, S.J. Li, Optimality conditions for vector optimization problems with difference of convex maps, J. Optim. Theory Appl. 162 (2014), 821-844.
[28] M.I. Henig, Proper efficiency with respect to cones, J. Optim. Theory Appl. 36 (1982), 387-407.
[29] A. Shafie, F. Bozorgnia, A note on the paper "Optimality conditions for vector optimization problems with difference of convex maps", J. Optim. Theory Appl. 182 (2019), 837-849.
[30] A. Shafie, Necessary ans sufficient optimality conditions for D.C. vector optimization, Acta Universitatis Apulensis 51 (2017), 41-52.

[31] A. Taa, Optimality conditions for vector optimization problems of a difference of convex mappings, J. Global Optim. 31 (2005), 421-436.
[32] J.B. Hiriart-Urruty, Tangent cones, generalized gradients and mathematical programming in Banach spaces, Math. Oper. Res. 4 (1979), 79-97.
[33] N. Gadhi, M. Laghdir, A. Metrane, Optimality conditions for D.C. vector optimization problems under reverse convex constraints, J. Global Optim. 33 (2005), 527-540.
[34] C. Zălinescu, Convex Analysis in General Vector Spaces, World Scientific, New Jersey, 2002.
[35] J. Jahn, Vector optimization. Theory, applications, and extensions, Springer-Verlag, Berlin, 2011.
[36] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu, Variational Methods in Partially Ordered Spaces, CMS Books in Mathematics, Springer-Verlag, New York, 2003.
[37] J.M. Borwein, D. Zhuang, Super efficiency in vector optimization, Trans. Amer. Math. Soc. 338 (1993), 105-122.
[38] L. Huerga, B. Jiménez, V. Novo, New notions of proper efficiency in set optimization with the set criterion, J. Optim. Theory Appl. 195 (2022), 878-902.
[39] C. Gutiérrez, B. Jiménez, V. Novo, A unified approach and optimality conditions for approximate solutions of vector optimization problems, SIAM J. Optim. 17 (2006), 688-710.
[40] C. Gutiérrez, L. Huerga, B. Jiménez, V. Novo, Henig approximate proper efficiency and optimization problems with difference of vector mappings, J. Convex. Anal. 23 (2016), 661-690.
[41] A.M. Geoffrion, Proper efficiency and the theory of vector maximization, J. Math. Anal. Appl. 22 (1968), 618-630.
[42] M. Ehrgott, Multicriteria Optimization, Springer, Berlin-Heidelberg New York, 2005.
[43] Y. Sawaragi, H. Nakayama, T. Tanino, Theory of Multiobjective Optimization, Academic Press, Orlando, 1985.
[44] A.A. Khan, C. Tammer, C. Zălinescu, Set-valued Optimization. An Introduction with Applications, Vector Optimization, Springer, Heidelberg, 2015.
[45] C. Gutiérrez, L. Huerga, V. Novo, M. Sama, Limit behaviour of approximate proper solutions in vector optimization, SIAM J. Optim. 29 (2019), 2677-2696.
[46] J.B. Hiriart-Urruty, From convex optimization to nonconvex optimization. Necessary and sufficient conditions for global optimality, In: F.H. Clarke et al. (eds.) Nonsmooth Optimization and Related Topics, pp. 219-239, Springer Science+Business Media, New York, 1989.
[47] C. Gutiérrez, L. Huerga, B. Jiménez, V. Novo, Optimality conditions for approximate proper solutions in multiobjective optimization with polyhedral cones, TOP 28 (2020), 526-544.