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COVERING CONSTANTS FOR LINEAR AND CONTINUOUS MAPPINGS WITH APPLICATIONS TO STOCHASTIC SYSTEMS OF LINEAR EQUATIONS

JINLU LI[1], ADRIAN PETRUȘEL[2], CHING-FENG WEN[3,*], JEN-CHIH YAO[4]

[1]Department of Mathematics Shawnee State University Portsmouth, Ohio 45662 USA
[2]Department of Mathematics Babes-Bolyai University Cluj-Napoca, Romania
[3]Department of Information Management, National Yunlin University of Science and Technology, Douliou, Yunlin, Taiwan
[4]Research Center for Interneural Computing, China Medical University Taichung, Taiwan

Abstract. Mordukhovich derivatives (Mordukhovich coderivatives) of set-valued mappings in Banach spaces firmly lay the foundation of the theory of generalized differentiation in set-valued analysis, which has been widely applied to varous fields. One of the most important applications of Mordukhovich derivatives is to define the covering constants for set-valued mappings in Banach spaces, which play an important role in Arutyunov Mordukhovich and Zhukovskiy Parameterized Coincidence Point theorem (AMZ theorem). In this paper, we first introduce the concept of locally or globally stochastic systems of linear equations in $\mathbb{R}^n$. We use the Mordukhovich derivatives to precisely find the covering constants for some linear and continuous mappings in $\mathbb{R}^n$. In particular, we find the precise solutions of the covering constants for linear and continuous mappings in $\mathbb{R}^2$ and find the exact points, at which the covering constant reaches. Then, by using the AMZ theorem, we prove the existence of solutions to some locally or globally stochastic systems of linear equations in $\mathbb{R}^n$. Finally, we provide some examples for finding solutions of some stochastic systems of linear equations in $\mathbb{R}^n$.

Keywords. Covering constant; Fréchet derivative; Mordukhovich derivative; Stochastic system of linear equations.

1. An Overview for Differentiability of Mappings in Banach Spaces

The continuity and differentiability of mappings from a Banach space to another Banach space are essential and of paramount importance for the analysis in Banach spaces. It is clear that the differentiability of mappings in (infinite dimensional) Banach spaces must be more complicated and more colorful than the differentiability of (real or complex valued) functions in real or complex analysis; see, e.g., [11, 12, 19, 22] for some details. When we study the differentiability of mappings in Banach spaces, there are two cases of mappings: single-valued and set-valued. For single-valued mappings, we may have Gâteaux directional differentiability, Frèchet differentiability, and strict Frèchet differentiability (see [11, 12, 19]). These differentiabilities of single-valued mappings in Banach spaces can be considered as natural extensions

*Corresponding author.
E-mail address: cfwen@kmu.edu.tw; cfwen@yuntech.edu.tw (C.F. Wen).
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of the differentiability of functions in real or complex analysis. Set-valued analysis has become an important branch in modern analysis. It is understandable that it is difficult to define the differentiability of set-valued mappings in Banach spaces. However, the Mordukhovich derivative for set-valued mappings has been developed, which firmly laid the crucial foundation of the theory of generalized differentiation of set-valued mappings in Banach spaces; see, e.g., [1, 2, 3, 4, 19, 22] and the references therein.

In this paper, unless otherwise stated, we always let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real Banach spaces with topological dual spaces $(X^*, \|\cdot\|_{X^*})$ and $(Y^*, \|\cdot\|_{Y^*})$, and with origins θ_X and θ_Y, respectively. Let (S, τ, μ) be a topological probability space, in which S is the sample space. Meanwhile, (S, τ) is a topological space where S is the sample space. The topology τ on S coincides with the σ-field of all events in S, and μ is the probability measure in S defined on τ. Let $F(\cdot) : X \rightrightarrows Y$ and $G(\cdot,\cdot) : X \times S \rightrightarrows Y$ be set-valued mappings. In set-valued and variational analysis, the following stochastic equation problems have attracted significant attention: Find an open subset $W \subset S$ and a single-valued mapping (a random variable) $\sigma : W \to X$ such that

$$\theta_Y \in F(\sigma(s)) - G(\sigma(s), s), \quad \text{for any } s \in W. \tag{1.1}$$

In order to study the existence of solutions to the stochastic equation problem (1.1), Arutyunov, Mordukhovich, and Zhukovskiy [1] studied more general cases, parameterized coincidence point problems: Find an open subset $W \subset S$ and a single-valued mapping $\sigma : W \to X$ such that

$$F(\sigma(s)) \cap G(\sigma(s), s) \neq \emptyset, \quad \text{for any } s \in W. \tag{1.2}$$

We see that stochastic equation problem (1.1) can be considered as a special case of the parameterized coincidence point problem. In particular, for special cases, we let $f : X \to Y$ and $g : X \times S \to Y$ be single-valued mappings. The parameterized (stochastic) equation problem with respect to f and g is to find an open subset $W \subset S$and a single-valued mapping $\sigma : W \to X$ such that

$$f(\sigma(s)) = g(\sigma(s), s), \quad \text{for any } s \in W. \tag{1.3}$$

An existence theorem of solutions for the above parameterized coincidence point problems was proved in [1, Theorem 3.1], which is called Arutyunov Mordukhovich Zhukovskiy Parameterized Coincidence Point Theorem. In this paper, the Arutyunov Mordukhovich Zhukovskiy Parameterized Coincidence Point Theorem play an important and is simply named as AMZ theorem. The results of this theorem have been widely applied to set-valued analysis and set-valued optimization.

In the AMZ Thereon, the concept of the covering constants for the considered set-valued mappings in Banach spaces plays a crucial important role. This is considered as one of the most important applications of the Mordukhovich derivatives to set-valued and variational analysis. In [16], the covering constants were precisely calculated for the standard metric projection operator and some linear and continuous mappings in Banach spaces. Meanwhile, (1.3) is used for studying the existence of solutions for some stochastic fixed-point problems. In [17], the AMZ Theorem is applied to prove the existence of solutions for some stochastic integral equations.

In this paper, we have more applications of the AMZ Theorem (1.2) and the special case (1.3) to some stochastic systems of linear equations. This paper is organized as follows. We overview for Frèchet and Mordukhovich differentiability of mappings in Banach spaces and overview for

some concepts of covering constants for mappings in Banach spaces and the AMZ theorem. For the considered linear and continuous mapping f, we denote $\|f\|_{\rm op}$ for the operator norm of the mapping $f:\mathbb{R}^n\to\mathbb{R}^n$. In Section 3, we concentrate on the calculation of the operator norms $\|f\|_{\rm op}$ of this linear and continuous mapping f in $\mathbb{R}^n$. In particular, for $n=2$, we find the exact points on the unit sphere of $\mathbb{R}^2$, at which $\|f\|_{\rm op}$ will reach. In Section 4, by the results obtained in Section 3, we use the AMZ theorem to prove the existence of solutions to some stochastic systems of linear equations in $\mathbb{R}^n$. Finally, in Section 5, in addition to the summaries of the results obtained in this paper, we propose a problem for interested readers to consider.

2. Preliminary

2.1. **An overview for differentiability of mappings in Banach spaces.** In this subsection, for easy reference, we first review the concepts of Gâteaux directional differentiability, Frèchet differentiability, and strict Frèchet differentiability for single-valued mappings in Banach spaces. Then, we quickly review the concepts of Mordukhovich differentiability for set-valued mappings in Banach spaces.

Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be real Banach spaces with topological dual spaces $(X^*,\|\cdot\|_{X^*})$ and $(Y^*,\|\cdot\|_{Y^*})$, respectively. Let $\langle\cdot,\cdot\rangle_X$ denote the real canonical pairing between X^* and X and$\langle\cdot,\cdot\rangle_Y$ the real canonical pairing between Y^* and Y. Let θ_X and θ_Y denote the origins in X and Y, respectively. Let $f:X\to Y$ be a single-valued mapping and let $\bar{x}\in X$.

I. (Frèchet differentiability) If there exists a linear and continuous mapping $\nabla f(\bar{x}):X\to Y$ such that
$$\lim_{x\to\bar{x}}\frac{f(x)-f(\bar{x})-\nabla f(\bar{x})(x-\bar{x})}{\|x-\bar{x}\|_X}=\theta_Y,$$
then f is said to be Frèchet differentiable and $\nabla f(\bar{x})$ is called the Frèchet derivative of f at $\bar{x}$.

II. (Strict Frèchet differentiability) If there exists a linear and continuous mapping $\nabla f(\bar{x}):X\to Y$ such that
$$\lim_{(u,v)\to(\bar{x},\bar{x})}\frac{f(u)-f(v)-\nabla f(\bar{x})(u-v)}{\|u-v\|_X}=\lim_{u\to\bar{x},v\to\bar{x}}\frac{f(u)-f(v)-\nabla f(\bar{x})(u-v)}{\|u-v\|_X}=\theta_Y,$$
then f is said to be strictly Frèchet differentiable at $\bar{x}$.

Next, we review the concepts of the Mordukhovich derivative for set-valued mappings in Banach spaces. Let Δ be a nonempty subset of X and let $F:\Delta\rightrightarrows Y$ be a set-valued mapping. The graph of F is defined by the following subset in $\Delta\times Y$: ${\rm gph}F=\{(x,y)\in\Delta\times Y:y\in F(x)\}$. For $(x,y)\in{\rm gph}F$, that is, $x\in\Delta$ and $y\in F(x)$, the Mordukhovich derivative (which is also called Mordukhovich coderivative, or coderivative) of F at point (x,y) is a set-valued mapping $\hat{D}^*F(x,y):Y^*\rightrightarrows X^*$. For any $y^*\in Y^*$, it is defined by (see [19, Definitions 1.13 and 1.32])
$$\hat{D}^*F(x,y)(y^*)\;=\;\left\{z^*\in X^*:\limsup_{(u,v)\to(x,y),(u,v)\in{\rm gph}F}\frac{\langle z^*,u-x\rangle_X-\langle y^*,v-y\rangle_Y}{\|u-x\|_X+\|v-y\|_Y}\le 0\right\}. \quad (2.1)$$
If $(x,y)\notin{\rm gph}F$, then we define, for all $y^*\in Y^*$, $\hat{D}^*F(x,y)(y^*)=\emptyset$. By definition (2.1), $\hat{D}^*F(x,y):Y^*\rightrightarrows X^*$ is a set-valued mapping, which is called the Mordukhovich coderivative (or the Mordukhovich cocoderivative) of F at (x,y). In particular, let $f:\Delta\to Y$ be a single-valued mapping. By (1.1), the Mordukhovich derivative of f at point $(x,f(x))$ is a set-valued mapping

$\hat{D}^*f(x,f(x)):Y^*\rightrightarrows X^*$. For any $y^*\in Y^*$, we denote $\hat{D}^*f(x,f(x))(y^*)$ by $\hat{D}^*f(x)(y^*)$, which is defined by

$$\begin{aligned}\hat{D}^*f(x,f(x))(y^*)&=\hat{D}^*f(x)(y^*)\\&=\left\{z^*\in X^*:\limsup_{(u,f(u))\to(x,f(x)),u\in\Delta}\frac{\langle z^*,u-x\rangle_X-\langle y^*,f(u)-f(x)\rangle_Y}{\|u-x\|_X+\|f(u)-f(x)\|_Y}\le 0\right\}.\end{aligned}$$

Furthermore, if $f:\Delta\to Y$ is a single-valued continuous mapping, then, by the above definition, the Mordukhovich derivative of f at point $(x,f(x))$ is a set-valued mapping $\hat{D}^*f(x,f(x)):Y^*\rightharpoonup X^*$ such that, for any $y^*\in Y^*$,

$$\begin{aligned}\hat{D}^*f(x,f(x))(y^*)&=\hat{D}^*f(x)(y^*)\\&=\left\{z^*\in X^*:\limsup_{u\to x,u\in\Delta}\frac{\langle z^*,u-x\rangle_X-\langle y^*,f(u)-f(x)\rangle_Y}{\|u-x\|_X+\|f(u)-f(x)\|_Y}\le 0\right\}.\end{aligned}$$

The following theorem shows the connection between Frèchet derivatives and Mordukhovich derivatives for single-valued mappings. The results of the following theorem provide a powerful tool to calculate the Mordukhovich derivatives by the Frèchet derivatives of single-valued mappings.

Theorem 2.1. [21, Theorem 1.38] *Let X be a Banach space with dual space X^*, and let $f:X\to Y$ be a single-valued mapping. Let f be Frèchet differentiable at $x\in X$ with $y=f(x)$. Then, the Mordukhovich derivative of f at x satisfies $\hat{D}^*f(x)(y^*)=\{(\nabla f(x))^*(y^*)\}$, for all $y^*\in Y^*$.*

Theorem 2.2. [21, Theorem 1.39] *(Coderivatives of linear and continuous operators A). Let the conditions in* [21, Theorem 1.38] *be fulfilled. Then, $\hat{D}^*A(x)(y^*)=\{A^*(y^*)\}$, for all $y^*\in Y^*$.*

2.2. **An overview for Arutyunov Mordukhovich Zhukovskiy parameterized coincidence point theorem.** To state the AMZ theorem, we need recall some concepts of Asplund Banach spaces, covering constants and Lipschitz-like modulus for set-valued mappings in Banach spaces; see, e.g., [5, 6, 7, 8, 9, 10, 18, 23, 24, 25] and the references therein. A Banach space Z is Asplund if every convex continuous function defined on an open convex set O in Z is Frèchet differentiable on a dense subset of O. The class of Asplund Banach spaces is very large including all reflexive Banach spaces. This implies that every uniformly convex and uniformly smooth Banach space is Asplund. Let X and Y be Banach spaces and let U and V be nonempty subsets in X and Y, respectively. Let $\Phi:X\rightrightarrows Y$ be a multifunction (a set-valued mapping). The graph of Φ is denoted by gphΦ, which is a subset of $X\times Y$ defined by gph$\Phi:=\{(x,y)\in X\times Y:y\in\Phi(x)\}$. We say that Φ enjoys the covering property with modulus $\alpha>0$ (or it has the α-covering property) on U relative to V if (see [1, (2.1)])

$$\Phi(x)\cap V+\alpha B_Y\subset\Phi(x+rB_X),\text{whenever } x+rB_X\subset U,\ \text{as } r>0, \tag{2.2}$$

where B_X and B_Y are the closed unit balls in X and Y, respectively. The supremum of all such moduli $\{\gamma\}$ in (2.2) is called the exact covering bound of Φ around $(\bar{x},\bar{y})$, which is denoted by

$$\text{cov}\Phi(\bar{x},\bar{y})=\sup\{\alpha:\alpha\text{ satisfies (2.2) for some } U\subset X,V\subset Y\}.$$

The multifunction $\Phi:X\rightrightarrows Y$ between X and Y is called Lipschitz-like on U relative to V with some modulus $\beta>0$ if we have the inclusion (see [2, (2.3)])

$$\Phi(x)\cap V\subset\Phi(y)+\beta\|x-y\|B_X,\ \text{for all } x,y\in U.$$

This is a natural extension of the Lipschitz condition from single-valued mappings to set-valued mappings. In particular, let $f : X \to Y$ be a single-valued mapping. f is said to satisfy the Lipschitz condition on U relative to V with respect to some modulus $\beta > 0$ if

$$\|f(x) - f(y)\|_Y \leq \beta \|x - y\|_X, \text{ for all } x, y \in U \text{ with } f(x) \in V.$$

Note that, in [1], the covering property of set-valued mappings is defined in normed spaces. The covering property for set-valued mappings is a very important concept, which has been widely applied to game theory, optimization theory, equilibrium theory, and variational analysis, with respect to set-valued mapping; see, e.g., [1, 2, 3, 4, 19, 20, 21, 22] for more details. One of the important applications of Mordukhovich derivatives of set-valued mappings is to define the covering constants for set-valued mappings. The covering constant for $\Phi : X \rightrightarrows Y$ at point $(\bar{x}, \bar{y}) \in \text{gph}\Phi$ is defined by (see [1, (2.6)])

$$\begin{aligned} &\hat{\alpha}(\Phi, \bar{x}, \bar{y}) \\ &:= \sup_{\eta > 0} \inf \left\{ \|z^*\|_{X^*} : z^* \in \hat{D}^* \Phi(x, y)(w^*), x \in B_X(\bar{x}, \eta), y \in \Phi(x) \cap B_Y(\bar{y}, \eta), \|w^*\|_{Y^*} = 1 \right\}. \end{aligned} \tag{2.3}$$

Here, $\|\cdot\|_{X^*}$ and $\|\cdot\|_{Y^*}$ denote the norms in X^* and Y^*, respectively, $B_X(\bar{x}, \delta)$ is the closed ball in X centered at $\bar{x}$ with radius δ, and $B_Y(\bar{y}, \eta)$ is the closed ball in Y centered at $\bar{y}$ with radius η. In particular, let $f : X \to Y$ be a single-valued mapping. For any $\bar{x}, \bar{y} \in X$ with $\bar{y} = f(\bar{x})$, (2.2) becomes

$$\hat{\alpha}(f, \bar{x}, \bar{y}) = \sup_{\eta > 0} \inf \left\{ \|z^*\|_{X^*} : z^* \in \hat{D}^* f(x, y)(w^*), x \in \mathbb{B}(\bar{x}, \eta), y \in \mathbb{B}(\bar{y}, \eta), \|w^*\|_{Y^*} = 1 \right\}. \tag{2.4}$$

The following theorem proves the covering criterion for set-valued mappings (see [1, 21]).

Theorem 2.3. [1, Theorem 2.1] *(the covering criterion) Let $\Phi : X \rightrightarrows Y$ be a set-valued mapping between normed spaces, let $(\bar{x}, \bar{y}) \in gph\,\Phi$, and let $\hat{\alpha}(\Phi, \bar{x}, \bar{y})$ be taken from (2.3). Then*

(i) The α-covering property of Φ around $(\bar{x}, \bar{y})$ implies that $\alpha \leq \hat{\alpha}(\Phi, \bar{x}, \bar{y})$.

(ii) Assume that X and Y are Asplund spaces. If $0 < \alpha < \hat{\alpha}(\Phi, \bar{x}, \bar{y})$, then Φ has the α-covering property around $(\bar{x}, \bar{y})$ provided that its graph is closed around this point.

The following theorem precisely provides some more details about the connections between the local covering property and the covering constants for mappings around a given point.

Theorem 2.4. [20, Theorem 4.1] *(neighborhood characterization of local covering) Let $F : X \rightrightarrows Y$ be a set-valued mapping between Asplund spaces. Assume that F is closed-graph around $(\bar{x}, \bar{y}) \in gph\,F$. Then the following are equivalent:*

(a) F enjoys the local covering property around $(\bar{x}, \bar{y})$ (that is, $cov\,F(\bar{x}, \bar{y}) > 0$).

(b) $\hat{\alpha}(F, \bar{x}, \bar{y}) > 0$.

Moreover, the exact covering bound of F around $(\bar{x}, \bar{y})$ is computed by $cov\,F(\bar{x}, \bar{y}) = \hat{\alpha}(F, \bar{x}, \bar{y})$.

In particular, the exact covering bounds of Frèchet differentiable single-valued mappings have been studied in [21]. For example, we have the following results from [21], which can be used in Sections 3 and 4.

Corollary 2.1. [21, Corollary 1.58] *(Covering for Linear Operators) A linear and continuous operator $A : X \to Y$ has the covering property around point $\bar{x} \in X$ with $\bar{y} = A(\bar{x})$ if and only if A is surjective and $cov\,A(\bar{x}, \bar{y}) = \inf\{\|A^* y^*\| : \|y^*\| = 1\}$, for all $\bar{x} \in X$.*

Let $F : X \rightrightarrows Y$ be a set-valued mapping. Let $\bar{x} \in X$ and $\bar{y} \in Y$ with $\bar{y} \in F(\bar{x})$. The multifunction F is said to be closed around $(\bar{x}, \bar{y})$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that $(\operatorname{gph} F) \cap (\operatorname{cl} U \times \operatorname{cl} V)$ is closed in $X \times Y$, where "cl" indicates the closure operation.

Now we state the AZM theorem ([1, Theorem 3.1]).

Theorem 2.5. *Let the Banach spaces X and Y be Asplund and let P be a topological space. Let $F : X \rightrightarrows Y$ and $G(\cdot,\cdot) : X \times P \rightrightarrows Y$ be set-valued mappings. Let $\bar{x} \in X$ and $\bar{y} \in Y$ with $\bar{y} \in F(\bar{x})$. Suppose that the following conditions are satisfied:*

(A1) The multifunction $F : X \rightrightarrows Y$ is closed around $(\bar{x}, \bar{y})$.

(A2) There exist neighborhoods $U \subset X$ of $\bar{x}$, $V \subset Y$ of $\bar{y}$, and O of $\bar{p} \in P$ as well as a number $\beta \geq 0$ such that the multifunction $G(\cdot, p) : X \rightrightarrows Y$ is Lipschitz-like on U relative to V for each $p \in O$ with the uniform modulus β, while the multifunction $p \to G(\bar{x}, p)$ is lower/inner semicontinuous at $\bar{p}$.

(A3) The Lipschitzian modulus β of $G(\cdot, p)$ is chosen as $\beta < \hat{\alpha}(F, \bar{x}, \bar{y})$, where $\hat{\alpha}(F, \bar{x}, \bar{y})$ is the covering constant of F around $(\bar{x}, \bar{y})$ taken from (2.3).

Then for each $\alpha > 0$ with $\beta < \alpha < \hat{\alpha}(F, \bar{x}, \bar{y})$, there exist a neighborhood $W \subset P$ of $\bar{p}$ and a single-valued mapping $\sigma : W \to X$ such that, whenever $p \in W$,

$$F(\sigma(p)) \cap G(\sigma(p), p) \neq \emptyset, \tag{2.5}$$

and

$$\|\sigma(p) - \bar{x}\|_X \leq \frac{dist(\bar{y}, G(\bar{x}, p))}{\alpha - \beta}. \tag{2.6}$$

In actual applications of the AMZ theorem, if at least one of the considered mappings F and G is a single-valued mapping, then we can obtain some different versions of the AMZ theorem. In these special cases, (2.5) and (2.6) in the AMZ theorem will be represented by some special formulas. [17] provides some immediate consequences (different versions) of the AMZ theorem for such special cases, which are listed as corollaries of the AMZ theorem. We recall these corollaries below for easy refer.

Corollary 2.2. [17, Corollary 2.1] *Let the Banach spaces X and Y be Asplund and let P be a topological space. Let $f : X \to Y$ be a single-valued mapping and let $G(\cdot,\cdot) : X \times P \rightrightarrows Y$ be a set-valued mapping. Let $\bar{x} \in X$ and $\bar{y} \in Y$ with $\bar{y} = f(\bar{x})$. Suppose that the following conditions are satisfied:*

(A1) the mapping $f : X \to Y$ is continuous around $(\bar{x}, \bar{y})$;

(A2) there exist neighborhoods $U \subset X$ of $\bar{x}$, $V \subset Y$ of $\bar{y}$, and O of $\bar{p} \in P$ as well as a number $\beta \geq 0$ such that the multifunction $G(\cdot, p) : X \rightrightarrows Y$ is Lipschitz-like on U relative to V for each $p \in O$ with the uniform modulus β, while the multifunction $p \to G(\bar{x}, p)$ is lower/inner semicontinuous at $\bar{p}$;

(A3) the Lipschitzian modulus β of $G(\cdot, p)$ is chosen as $\beta < \hat{\alpha}(f, \bar{x}, \bar{y})$, where $\hat{\alpha}(f, \bar{x}, \bar{y})$ is the covering constant of f around $(\bar{x}, \bar{y})$ taken from (2.4).

Then, for each $\alpha > 0$ with $\beta < \alpha < \hat{\alpha}(f, \bar{x}, \bar{y})$, there exist a neighborhood $W \subset P$ of $\bar{p}$ and a single-valued mapping $\sigma : W \to X$ such that, whenever $p \in W$, $f(\sigma(p)) \in G(\sigma(p), p)$, and

$$\|\sigma(p) - \bar{x}\|_X \leq \frac{\|G(\bar{x}, p) - \bar{y}\|_Y}{\alpha - \beta}.$$

3. Linear and Continuous Mappings in $\mathbb{R}^n$

3.1. **Mordukhovich derivatives and covering constants for linear and continuous mappings.** Let n be a positive integer. Let $(\mathbb{R}^n, \|\cdot\|_n)$ denote the n-d Euclidean space normed by the (l_2)-norm $\|\cdot\|_n$ with inner product $\langle\cdot,\cdot\rangle$ and with origin θ_n. Let $\mathbb{S}_n = \{x \in \mathbb{R}^n : \|x\|_n = 1\}$ denote the unit sphere in $\mathbb{R}^n$. Let $A = \left(a_{ij}\right)_{i,j=1}^n$ be a real $n \times n$ matrix that defines a linear and continuous mapping on $\mathbb{R}^n$. If it is not confused, this mapping induced by the matrix A is also denoted by A, such that

$$A(x) = (x_1, x_2, \ldots, x_n)A = \left(\sum_{i=1}^n a_{ij}x_i\right)_{j=1}^n, \quad \text{for any } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \tag{3.1}$$

where $x = (x_1, x_2, \ldots, x_n)$ and

$$A(x) = \left(\sum_{i=1}^n a_{i1}x_i, \sum_{i=1}^n a_{i2}x_i, \ldots, \sum_{i=1}^n a_{in}x_i\right)$$

are also considered as $1 \times n$ matrices. In (3.1), for the mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ induced by matrix A, $A(x)$ is the value of the mapping A at the point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The value is defined by $(x_1, x_2, \ldots, x_n)A$ as the product of a $1 \times n$ matrix and an $n \times n$ matrix. Let $\det(A)$ denote the $n \times n$ determinant $\left|a_{ij}\right|_{i,j=1}^n$. It is known that the adjoint operator A^* of A is its conjugate transpose A^T. Let $\|A\|_{\text{op}}$ denote the operator norm of the mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, which satisfies

$$\|A\|_{\text{op}} = \max\left\{\|A(x)\|_n : x \in \mathbb{S}_n\right\}.$$

Let $\|A\|_{\min}$ denote the minimum value of $\|A(\cdot)\|_n$ on $\mathbb{S}_n$, which is defined by

$$\|A\|_{\min} = \min\left\{\|A(x)\|_n : x \in \mathbb{S}_n\right\}.$$

For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, by Cauchy–Schwarz inequality, we have

$$\|A(x)\|_n^2 = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}x_i\right)^2 \le \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}^2 \sum_{i=1}^n x_i^2\right) = \sum_{i,j=1}^n a_{ij}^2 \|x\|_n^2.$$

This provides an estimation for $\|A\|_{\text{op}}$:

$$\|A\|_{\text{op}} \le \left(\sum_{i,j=1}^n a_{ij}^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|A^T\|_{\text{op}} \le \left(\sum_{i,j=1}^n a_{ij}^2\right)^{\frac{1}{2}}.$$

Moreover, in [9, 10, 17, 22], $\|A\|_{\text{op}}$ and $\|A\|_{\min}$ were solved:

$$\|A^T\|_{\text{op}} = \|A\|_{\text{op}} = \sqrt{\text{the largest eigenvalue of the matrix } A^TA} \tag{3.2}$$

and

$$\|A^T\|_{\min} = \|A\|_{\min} = \sqrt{\text{the smallest eigenvalue of the matrix } A^TA}. \tag{3.3}$$

In this subsection, we investigate the Fréchet and Mordukhovich derivatives of linear and continuous mappings on $\mathbb{R}^n$, which are induced by real $n \times n$ matrices.

Proposition 3.1. *Let n be a positive integer. Let $A = \left(a_{ij}\right)_{i,j=1}^n$ be a real $n \times n$ matrix that defines a linear and continuous mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ by (3.1). Then*

(i) *A is Fréchet differentiable at every point in $\mathbb{R}^n$ such that $\nabla(A)(x) = A$, for any $x \in \mathbb{R}^n$;*

(ii) *the Mordukhovich derivative of A satisfies that $\hat{D}^*(A)(x,A(x)) = A^T$, for any $x \in \mathbb{R}^n$.*

Proof. Proof of (i). Since the mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ defined by (3.1) is a linear and continuous single-valued mapping, we have

$$\lim_{u \to x} \frac{A(u) - A(x) - A(u-x)}{\|u-x\|_n} = \theta, \quad \text{for any given } x \in \mathbb{R}^n.$$

Thus this proves (i). By [15, Theorem 1.38] and (3.2), part (i) induces part (ii). □

The following results can be proved by [21, Corollary 1.58]. However, we provide a direct proof here with some more details.

Corollary 3.1. *Let n be a positive integer. Let $A = \left(a_{ij}\right)_{i,j=1}^n$ be a real $n \times n$ matrix that defines a linear and continuous mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ by (3.1). If $\|A\|_{\text{op}} \leq 1$, then, for any $x \in \mathbb{R}^n$,*

$$\begin{aligned} \hat{\alpha}(A,x,A(x)) &= \min\left\{\|A^T(w)\|_n : \|w\|_n = 1\right\} = \|A^T\|_{\min} \\ &= \|A\|_{\min} = \min\left\{\|A(w)\|_n : \|w\|_n = 1\right\}. \end{aligned}$$

In particular,

(a) $\det(A) = 0 \Leftrightarrow \hat{\alpha}(A,x,A(x)) = 0$, *for any $x \in \mathbb{R}^n$;*
(b) *if $\det(A) \neq 0$, then the covering constant for A is constant in $\mathbb{R}^n$ satisfying*

$$0 < \hat{\alpha}(A,x,A(x)) \leq \|A\|_{\text{op}} \leq \left(\sum_{i,j=1}^n a_{ij}^2\right)^{\frac{1}{2}} \leq 1, \quad \textit{for any } x \in \mathbb{R}^n.$$

Proof. Since A is a single-valued mapping. Note that $\|A\|_{\text{op}} \leq 1$. For any $x \in X$, let $y = A(x)$. For any $u \in \mathbb{R}^n$, we have

$$u \in \mathbb{B}(x,\eta) \implies A(u) \in \mathbb{B}(A(x),\eta), \quad \text{for any } \eta > 0.$$

For any $x \in \mathbb{R}^n$, by Proposition 3.1, we have

$$\begin{aligned} \hat{\alpha}(A,x,A(x)) &= \sup_{\eta>0} \inf\left\{\|z\|_n : z \in \hat{D}^* A(u,A(u))(w), u \in \mathbb{B}(x,\eta), A(u) \in \mathbb{B}(A(x),\eta), \|w\|_n = 1\right\} \\ &= \sup_{\eta>0} \inf\left\{\|A^T(w)\|_n : \|w\|_n = 1\right\} \\ &= \min\left\{\|A^T(w)\|_n : \|w\|_n = 1\right\} \\ &= \|A^T\|_{\min} = \|A\|_{\min} \\ &= \min\left\{\|A(w)\|_n : \|w\|_n = 1\right\}. \end{aligned}$$

Proof of (a). We provide two ways to prove (a).

Way 1. By the compactness of $\mathbb{S}_n$ and the continuity of $\|A(\cdot)\|$, for $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\hat{\alpha}(A,x,A(x)) = 0\\
\Leftrightarrow\ &\|A\|_{\min} = 0\\
\Leftrightarrow\ &\text{There is } z = (z_1, z_2, \ldots, z_n) \in \mathbb{S}_n \text{ such that } \|A(z)\|^2 = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} z_i \right)^2 = 0\\
\Leftrightarrow\ &\sum_{i=1}^{n} a_{ij} z_i = 0, \text{ for } j = 1,2,\ldots,n\\
\Leftrightarrow\ &\det(A) = 0
\end{aligned}$$

Way 2. We directly prove part (a) by using the definition of $\tilde{\alpha}(A,x,A(x))$ with similar ideas as follows.

"$\Rightarrow$" Suppose that $\det(A) = 0$. Then there exists $z \in \mathbb{R}^n$ with $\|z\|_n = 1$ such that $A^T(z) = zA^T = \theta_n$. By definition, we calculate the covering constant for A at an arbitrarily given point $x \in \mathbb{R}^n$.

$$\begin{aligned}
&\hat{\alpha}(A,x,A(x)) = \sup_{\eta>0} \inf \left\{ \|w\|_n : w \in \hat{D}^*(A)(u,A(u))(y), u \in \mathbb{B}(x,\eta), A(u) \in \mathbb{B}(A(x),\eta), \|y\|_n = 1 \right\}\\
&= \sup_{\eta>0} \inf \left\{ \|A^T(y)\|_n : \{A^T(y)\} = \hat{D}^*(A)(u,A(u))(y), u \in \mathbb{B}(x,\eta), A(u) \in \mathbb{B}(A(x),\eta), \|y\|_n = 1 \right\}\\
&\le \sup_{\eta>0} \left\{ \|A^T(z)\|_n : \{A^T(z)\} = \hat{D}^*(A)(u,A(u))(z), u \in \mathbb{B}(x,\eta), A(u) \in \mathbb{B}(A(x),\eta) \right\}\\
&= \sup_{\eta>0} \left\{ \|\theta_n\|_n : \{\theta_n\} = \hat{D}^*(A)(u,A(u))(z), u \in \mathbb{B}(x,\eta), A(u) \in \mathbb{B}(A(x),\eta) \right\}\\
&= \sup_{\eta>0} 0\\
&= 0.
\end{aligned}$$

"$\Leftarrow$" Suppose that $\hat{\alpha}(A,x,A(x)) = 0$, for some $x \in \mathbb{R}^n$. The condition $\hat{\alpha}(A,x,A(x)) = 0$ implies that $0 = \hat{\alpha}(A,x,A(x)) = \min \left\{ \|A^T(w)\|_n : \|w\|_n = 1 \right\}$. Since the set $\{y \in \mathbb{R}^n : \|y\|_n = 1\}$ is a compact subset in $\mathbb{R}^n$ and $\|A^T(y)\|_n$ is a continuous function of $y \in \mathbb{R}^n$, this implies that there exists $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ with $\|z\|_n = 1$ such that $\|A^T(z)\|_n = 0$. From (3.1), we have

$$A^T(z) = zA^T = \left(\sum_{j=1}^{n} a_{ij} z_j \right)_{i=1}^{n},$$

which implies that

$$\|A^T(z)\|_n = 0 \Leftrightarrow \sum_{j=1}^{n} a_{ij} z_j = 0, \text{ for } i = 1,2,\ldots,n.$$

Hence, $\|A^T(z)\|_n = 0$ implies $A^T(z) = zA^T = \theta_n$. By $\|z\|_n = 1$, we obtain $\det(A^T) = 0$. That is, $\det(A) = 0$.

Proof of (b). Suppose that $\det(A) \neq 0$. By part (a), we have $\hat{\alpha}(A,x,A(x)) > 0$, for all $x \in \mathbb{R}^n$. By (3.2) and (3.3), we obtain

$$0 < \hat{\alpha}(A,x,A(x)) = \|A^T\|_{\min} \le \|A^T\|_{\text{op}} \le \left(\sum_{i,j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}} \le 1$$

□

In Corollary 3.1, for a given linear and continuous mapping A in $\mathbb{R}^n$, the covering constant for A is characterized by the nonzero property of $\det(A)$. However, in the cases of Hilbert spaces with infinite dimension, this is impossible to do. Hence, for the purpose of extension of the results Corollary 3.1 from $\mathbb{R}^n$ to l_p, which will be studied in the follow-up articles., we rewrite parts (a, b) of Corollary 3.1 as follows (the partial of the results can be directly obtained by [21, Corollary 1.58].

Corollary 3.2. *Let n be a positive integer. Let $A = (a_{ij})_{i,j=1}^n$ be a real $n \times n$ matrix that defines a linear and continuous mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ by (3.1). Then, The system of linear equations $zA = \theta_n$ only has zero solution $\theta_n \Leftrightarrow \hat{\alpha}(A,x,A(x)) > 0$, for any $x \in \mathbb{R}^n$.*

Proof. Note that the system of linear equations $zA = \theta_n$ only has zero solution θ_n if and only if $\det(A) \neq 0$. □

In particular, if $A = (a_{ij})_{i,j=1}^n$ is a real $n \times n$ diagonal matrix satisfying $a_{ij} = 0$, for any $i \neq j$, then the linear and continuous mapping A becomes a pointwise multiplication operator on $\mathbb{R}^n$, which satisfies

$$A(x) = (x_1, x_2, \ldots, x_n)A = (a_{11}x_1, a_{22}x_2, \ldots, a_{nn}x_n), \quad \text{for any } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n. \quad (3.4)$$

By Proposition 3.1 and Corollary 3.1, we have the following results.

Corollary 3.3. *Let n be a positive integer. Let $A = (a_{ij})_{i,j=1}^n$ be a real $n \times n$ diagonal matrix that defines a linear and continuous pointwise multiplication operator $A : \mathbb{R}^n \to \mathbb{R}^n$ by (3.4). If A satisfies $\max\{|a_{ii}| : i = 1,2,\ldots,n\} \leq 1$, then*

(a) $\prod_{i=1}^n a_{ii} = 0 \quad \Leftrightarrow \quad \hat{\alpha}(A,x,A(x)) = 0$, *for any $x \in \mathbb{R}^n$;*

(b) *if $\prod_{i=1}^n a_{ii} \neq 0$, then the covering constant for A is positive constant in $\mathbb{R}^n$ satisfying*

$$0 < \min\{|a_{ii}| : i = 1,2,\ldots,n\} = \hat{\alpha}(A,x,A(x)) \leq \|A\|_{\text{op}} = \max\{|a_{ii}| : i = 1,2,\ldots,n\},$$

for any $x \in \mathbb{R}^n$.

Proof. For the diagonal matrix A, we have $A^T = A$ and $\det(A) = \det(A^T) = \prod_{i=1}^n a_{ii}$. Then this corollary follows from Corollary 3.1 immediately. □

3.2. **Operator norms and covering constants of linear and continuous mappings in $\mathbb{R}^2$.** For a linear and continuous mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ induced by a real $n \times n$ matrix, (3.2) and (3.3) give the solutions for $\|A\|_{\text{op}}$ and $\|A\|_{\min}$. We consider the following two topics

(1) Find the largest and smallest eigenvalues of $A^T A$ that solve $\|A\|_{\text{op}}$ and $\|A\|_{\min}$;

(2) Find the points on $\mathbb{S}_n$, at which $\|A\|_{\text{op}}$ and $\|A\|_{\min}$ respectively reach their values.

In Corollary 3.3, we consider some pointwise multiplication operators, which are considered as special cases of linear and continuous mappings in $\mathbb{R}^n$. From this corollary, we can find the covering constants and operator norms of any pointwise multiplication operators. However, in general, we understand that it is not simple to precisely find these values. In rest of this subsection, we concentrate to $\mathbb{R}^2$ and we use the method of Lagrange Multiplier to solve the following two problems

(a) The formulas to precisely calculate the operator norms and covering constants $\|A\|_{\text{op}}$ and $\|A\|_{\min}$ for linear and continuous mappings in $\mathbb{R}^2$.

(b) The exact points on $\mathbb{S}_2$, at which $\|A\|_{\text{op}}$ and $\|A\|_{\min}$ respectively reach their values.

Meanwhile, we see that the topic (b) is a complicated optimization problem, and the solutions are complex, in general. Let $A = \left(a_{ij}\right)_{i,j=1}^{2}$ be a real 2×2 matrix. We write some notations that depend on A

$$a = a_{11}^2 + a_{12}^2, \quad b = a_{11}a_{21} + a_{12}a_{22} \quad \text{and} \quad c = a_{21}^2 + a_{22}^2,$$
$$\lambda_k = \frac{-(a+c) + (-1)^k\sqrt{(a-c)^2+4b^2}}{2}, \quad \text{for } k = 1,2. \tag{3.5}$$

Then, for any $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\begin{aligned}\|A(x)\|_2^2 = \|xA\|_2^2 &= (a_{11}^2 + a_{12}^2)x_1^2 + 2(a_{11}a_{21} + a_{12}a_{22})x_1x_2 + (a_{21}^2 + a_{22}^2)x_2^2\\ &:= ax_1^2 + 2bx_1x_2 + cx_2^2.\end{aligned}$$

Lemma 3.1. *Let $A = \left(a_{ij}\right)_{i,j=1}^{2}$ be a real 2×2 matrix that defines a linear and continuous operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ by (3.1).*

(i) *If $b = 0$, then*

(a) *$\|A\|_{op}$ is attained at $(\pm 1, 0)$ or $(0, \pm 1)$ such that*

$$\|A\|_{\text{op}} = \|A(\pm 1, 0)\|_2 \,(\text{or } \|A(0,\pm 1)\|_2) = \max\{\sqrt{a}, \sqrt{c}\}.$$

(b) *$\|A\|_{\min}$ is attained at $(\mp 1, 0)$ or $(0, \mp 1)$ such that*

$$\|A\|_{\min} = \|A(\mp 1, 0)\|_2 \,(\text{or } \|A(0,\mp 1)\|_2) = \min\{\sqrt{a}, \sqrt{c}\}.$$

(ii) *If $b \neq 0$, then*

(a) *$\|A\|_{\text{op}}$ is attained at two points*

$$\left(\pm\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}, \mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right)$$

with

$$\begin{aligned}\|A\|_{\text{op}} &= \left\|A\left(\pm\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}, \mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right)\right\|_2\\ &= \sqrt{\frac{(a-c)^2 + (a+c)\sqrt{(a-c)^2+4b^2} + 4b^2}{2\sqrt{(a-c)^2+4b^2}}}\\ &= \frac{1}{\sqrt{2}}\sqrt{a+c+\sqrt{(a-c)^2+4b^2}}\\ &= \frac{1}{\sqrt{2}}\sqrt{a+c+\sqrt{a^2+c^2+2b^2-2(\det(A))^2}}\end{aligned}$$

(b) *$\|A\|_{\min}$ is attained at two points*

$$\left(\pm\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}, \mp\frac{a+\lambda_2}{b}\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}\right)$$

with

$$\begin{aligned}\|A\|_{\min} &= \left\|A\left(\pm\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}, \mp\frac{a+\lambda_2}{b}\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}\right)\right\|_2 \\ &= \sqrt{\frac{-(a-c)^2+(a+c)\sqrt{(a-c)^2+4b^2}-4b^2}{2\sqrt{(a-c)^2+4b^2}}} \\ &= \frac{1}{\sqrt{2}}\sqrt{a+c-\sqrt{(a-c)^2+4b^2}} \\ &= \frac{1}{\sqrt{2}}\sqrt{a+c-\sqrt{a^2+c^2+2b^2-2(\det(A))^2}}\end{aligned}$$

Furthermore, in addition to $b \neq 0$*, if* $\det(A) \neq 0$*, then* $\|A\|_{\min}$ *is positive. That is,*

$$\det(A) \neq 0 \implies \|A\|_{\min} = \sqrt{\frac{-(a-c)^2+(a+c)\sqrt{(a-c)^2+4b^2}-4b^2}{2\sqrt{(a-c)^2+4b^2}}} > 0 \tag{3.6}$$

(iii) *In particular, if* $b \neq 0$ *and* $a = c$*, then* $\lambda_k = -a+(-1)^k|b|$*, for* $k = 1,2$*, such that*

(a) $\|A\|_{\text{op}}$ *is attained at* $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$ *(with* $\lambda_1 = -a-|b|$*) satisfying*

$$\|A\|_{\text{op}} = \left\|A\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)\right\|_2 = \sqrt{a+|b|}.$$

(b) $\|A\|_{\min}$ *is attained at* $(\pm\frac{1}{\sqrt{1}}, \mp\frac{1}{\sqrt{2}})$ *(with* $\lambda_2 = -a+|b|$*) satisfying*

$$\|A\|_{\min} = \left\|A\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right)\right\|_2 = \sqrt{a-|b|}.$$

Proof. Proof of (i). If $b = 0$, then $\|A(x)\|_2^2 = ax_1^2 + cx_2^2$, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then the proof of (i) is similar to the proof of Corollary 3.2 and the proof of part (iii) in Proposition 3.1 and it is omitted here.

Proof of (ii). For any $x = (x_1, x_2) \in \mathbb{R}^2$, with $\|x\|_2 = 1$, we have

$$\|A(x)\|_2^2 = \|xA\|_2^2 = (a_{11}^2 + a_{12}^2)x_1^2 + 2(a_{11}a_{21} + a_{12}a_{22})x_1x_2 + (a_{21}^2 + a_{22}^2)x_2^2$$

By $\|A\|_{ap} = \max\{\|A(x)\|_2 : x \in \mathbb{S}_2\}$, to prove (ii), we need to solve the following constrained optimization problem under the condition that $b = a_{11}a_{21} + a_{12}a_{22} \neq 0$:

$$\begin{array}{ll} \text{optimize with } x_1, x_2 & \|(x_1,x_2)A\|_2^2 = ax_1^2 + 2bx_1x_2 + cx_2^2 \\ \text{subject to} & g(x_1,x_2) := x_1^2 + x_2^2 - 1 = 0 \end{array}$$

We write the corresponding Lagrange multiplier by $\mathscr{L}(x_1, x_2, \lambda) = \|(x_1,x_2)A\|_2^2 + \lambda g(x_1,x_2)$. Then, we solve the following system of equations:

$$\begin{cases} (a+\lambda)x_1 + bx_2 = 0 \\ bx_1 + (c+\lambda)x_2 = 0 \\ x_1^2 + x_2^2 = 1 \end{cases} \tag{3.7}$$

The third equation of the above system implies that this system has nonzero solution(s). Under the condition that $b \neq 0$, the following inequalities and equality must be satisfied:

$$a+\lambda \neq 0, \quad c+\lambda \neq 0, \quad \text{and} \quad b^2-(a+\lambda)(c+\lambda)=0 \tag{3.8}$$

This equation is converted to $\lambda^2+(a+c)\lambda+ac-b^2=0$. Solving for λ from the above equation, we obtain

$$\lambda = \frac{-(a+c)\pm\sqrt{(a+c)^2-4ac+4b^2}}{2} = \frac{-(a+c)\pm\sqrt{(a-c)^2+4b^2}}{2}. \tag{3.9}$$

We write for these two solutions by

$$\lambda_1 = \frac{-(a+c)-\sqrt{(a-c)^2+4b^2}}{2} \quad \text{and} \quad \lambda_2 = \frac{-(a+c)+\sqrt{(a-c)^2+4b^2}}{2},$$

which implies that

$$a+\lambda = \frac{a-c\pm\sqrt{(a-c)^2+4b^2}}{2} \quad \text{and} \quad c+\lambda = \frac{-(a-c)\pm\sqrt{(a-c)^2+4b^2}}{2}.$$

Notice that condition $b \neq 0$ implies that $a>0$ and $c>0$. Once again, we obtain that $a+\lambda \neq 0$ and $c+\lambda \neq 0$. From the first equation in (3.7), we have $x_2 = -\frac{a+\lambda}{b}x_1$. By the subject condition that $x_1^2+x_2^2=1$, this implies

$$\left(\left(\frac{a+\lambda}{b}\right)^2+1\right)x_1^2 = 1.$$

Solving for x_1 from the above equation, by (3.8) and by $x_2 = -\frac{a+\lambda}{b}x_1$, we get the solutions of (3.7):

$$\begin{cases} x_1 = \pm\sqrt{\frac{b^2}{(a+\lambda_k)^2+b^2}} \\ x_2 = \mp\frac{a+\lambda_k}{b}\sqrt{\frac{b^2}{(a+\lambda_k)^2+b^2}} \end{cases} \quad \text{for } k=1,2. \tag{3.10}$$

Then, function $\|A(\cdot)\|_2^2$ is attained at its extrema values at the following points

$$\left(\pm\sqrt{\frac{b^2}{(a+\lambda_k)^2+b^2}}, \mp\frac{a+\lambda_k}{b}\sqrt{\frac{b^2}{(a+\lambda_k)^2+b^2}}\right), \quad \text{for } k=1,2,$$

where λ_k is given in (3.9). For $k=1,2$, the extrema values are

$$\begin{aligned}
&\left\|A\left(\pm\sqrt{\frac{b^2}{(a+\lambda_k)^2+b^2}},\mp\frac{a+\lambda_k}{b}\sqrt{\frac{b^2}{(a+\lambda_k)^2+b^2}}\right)\right\|_2^2\\
&=a\frac{b^2}{(a+\lambda_k)^2+b^2}-2b\frac{a+\lambda_k}{b}\frac{b^2}{(a+\lambda_k)^2+b^2}+c\left(\frac{a+\lambda_k}{b}\right)^2\frac{b^2}{(a+\lambda_k)^2+b^2}\\
&=\frac{b^2}{(a+\lambda_k)^2+b^2}\left(a-2(a+\lambda_k)+c\left(\frac{a+\lambda_k}{b}\right)^2\right)\\
&=\frac{b^2}{(a+\lambda_k)^2+b^2}\left(a-2(a+\lambda_k)+c\frac{a+\lambda_k}{c+\lambda_k}\right)\\
&=\frac{c+\lambda_k}{a+\lambda_k+c+\lambda_k}\left(a-2(a+\lambda_k)+c\frac{a+\lambda_k}{c+\lambda_k}\right)\\
&=\frac{a(c+\lambda_k)-2b^2+c(a+\lambda_k)}{a+\lambda_k+c+\lambda_k}\\
&=\frac{a\left(-\frac{-(a-c)+(-1)^k\sqrt{(a-c)^2+4b^2}}{2}\right)-2b^2+c\left(\frac{(a-c)+(-1)^k\sqrt{(a-c)^2+4b^2}}{2}\right)}{(-1)^k\sqrt{(a-c)^2+4b^2}}\\
&=\frac{a\left(-(a-c)+(-1)^k\sqrt{(a-c)^2+4b^2}\right)-4b^2+c\left((a-c)+(-1)^k\sqrt{(a-c)^2+4b^2}\right)}{2(-1)^k\sqrt{(a-c)^2+4b^2}}\\
&=\frac{-(a-c)^2+(a+c)(-1)^k\sqrt{(a-c)^2+4b^2}-4b^2}{2(-1)^k\sqrt{(a-c)^2+4b^2}}
\end{aligned}$$

This implies that, at points

$$\left(\pm\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}},\mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right),$$

the maximum value of $\|A(\cdot)\|_2^2$ on $\mathbb{S}_2$ is

$$\begin{aligned}
&\left\|A\left(\pm\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}},\mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right)\right\|_2^2\\
&=\frac{(a-c)^2+(a+c)\sqrt{(a-c)^2+4b^2}+4b^2}{2\sqrt{(a-c)^2+4b^2}}.
\end{aligned}$$

Hence, the operator norm of A is attained at two points

$$\left(\pm\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}},\mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right)$$

with

$$\begin{aligned}\|A\|_{\text{op}} &= \left\|A\left(\pm\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}},\mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right)\right\|_2\\ &= \sqrt{\frac{(a-c)^2+(a+c)\sqrt{(a-c)^2+4b^2}+4b^2}{2\sqrt{(a-c)^2+4b^2}}}.\end{aligned}$$

To complete the proof of (a) in part (ii), we have

$$\begin{aligned}&\sqrt{\frac{(a-c)^2+(a+c)\sqrt{(a-c)^2+4b^2}+4b^2}{2\sqrt{(a-c)^2+4b^2}}}\\ &= \frac{1}{\sqrt{2}}\sqrt{a+c+\sqrt{a^2+c^2+2b^2-2(\det(A))^2}}.\end{aligned}$$

Since

$$\frac{(a-c)^2+(a+c)\sqrt{(a-c)^2+4b^2}+4b^2}{2\sqrt{(a-c)^2+4b^2}} = \frac{(a+c)+\sqrt{(a-c)^2+4b^2}}{2},$$

we only need to calculate $(a-c)^2+4b^2$

$$\begin{aligned}(a-c)^2+4b^2 &= a^2+c^2+2b^2-2\left(\left(a_{11}^2+a_{12}^2\right)\left(a_{21}^2+a_{22}^2\right)-(a_{11}a_{21}+a_{12}a_{22})^2\right)\\ &= a^2+c^2+2b^2-2(\det(A))^2.\end{aligned}$$

On $\mathbb{S}_2$, $\|A(\cdot)\|_2$ is attained at its minimum value at two points

$$\left(\pm\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}},\mp\frac{a+\lambda_2}{b}\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}\right),$$

and the minimum value is

$$\begin{aligned}\|A\|_{\min} &= \left\|A\left(\pm\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}},\mp\frac{a+\lambda_2}{b}\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}\right)\right\|_2\\ &= \sqrt{\frac{-(a-c)^2+(a+c)\sqrt{(a-c)^2+4b^2}-4b^2}{2\sqrt{(a-c)^2+4b^2}}}\\ &= \frac{1}{\sqrt{2}}\sqrt{a+c-\sqrt{a^2+c^2+2b^2-2(\det(A))^2}}.\end{aligned}$$

In addition, suppose that $\det(A)\neq 0$. Then the equation $xA=\theta_2$ has no nonzero solution. Since the norm function $\|A(\cdot)\|_2$ on the compact subset $\mathbb{S}_2$ is continuous, it yields that the minimum value of $\|A(\cdot)\|_2$ on $\mathbb{S}_2$ must be positive. However, we give a direct proof of (3.6)

$$-(a-c)^2+(a+c)\sqrt{(a-c)^2+4b^2}-4b^2>0$$

if and only if $(a+c)\sqrt{(a-c)^2+4b^2}>(a-c)^2+4b^2$. That is,

$$a+c>\sqrt{(a-c)^2+4b^2}\Leftrightarrow(a+c)^2>(a-c)^2+4b^2\Leftrightarrow ac>b^2.$$

The last inequality is by $\det(A) \neq 0$, which implies that the vectors (a_{11}, a_{12}) and (a_{21}, a_{22}) are not linearly independent. Hence $b^2 = (a_{11}a_{21} + a_{12}a_{22})^2 < \left(a_{11}^2 + a_{12}^2\right)\left(a_{21}^2 + a_{22}^2\right) = ac$.

Proof of (iii). If $b \neq 0$ and $a = c$, by part (ii), we have

$$\lambda_k = \frac{-(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2} = -a + (-1)^k |b|, \quad \text{for } k = 1, 2,$$

such that $\|A\|_{\text{op}}$ is attained at two points

$$\left(\pm \sqrt{\frac{b^2}{(a+\lambda_1)^2 + b^2}}, \mp \frac{a+\lambda_1}{b} \sqrt{\frac{b^2}{(a+\lambda_1)^2 + b^2}} \right) = \left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}} \right)$$

(with $\lambda_1 = -a - |b|$) satisfying

$$\|A\|_{\text{op}} = \left\| A\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}} \right) \right\|_2 = \sqrt{\frac{(a-c)^2 + (a+c)\sqrt{(a-c)^2 + 4b^2} + 4b^2}{2\sqrt{(a-c)^2 + 4b^2}}} = \sqrt{a + |b|}.$$

$\|A\|_{\text{min}}$ is attained at two points

$$\left(\pm \sqrt{\frac{b^2}{(a+\lambda_2)^2 + b^2}}, \mp \frac{a+\lambda_2}{b} \sqrt{\frac{b^2}{(a+\lambda_2)^2 + b^2}} \right) = \left(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}} \right)$$

(with $\lambda_2 = -a + |b|$) such that

$$\|A\|_{\text{min}} = \left\| A\left(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}} \right) \right\|_2 = \sqrt{\frac{-(a-c)^2 + (a+c)\sqrt{(a-c)^2 + 4b^2} - 4b^2}{2\sqrt{(a-c)^2 + 4b^2}}} = \sqrt{a - |b|}.$$

□

Let $A = \left(a_{ij}\right)_{i,j=1}^2$ be a real 2×2 matrix with adjoint operator A^T. We study the connections of the extremes between A and A^T restricted on $\mathbb{S}_2$. From Lemma 3.1, we see that λ defined in (3.5) is the multiplier value in the processes to precisely find the operator norms and covering constants for linear and continuous mapping A in $\mathbb{R}^2$, which uses the method of Lagrange Multiplier. Regarding to A^T, we also write the corresponding notations, which depend on A

$$a' = a_{11}^2 + a_{21}^2, \quad b' = a_{11}a_{12} + a_{21}a_{22}, \quad \text{and} \quad c' = a_{12}^2 + a_{22}^2,$$

$$\lambda_k' = \frac{-(a'+c') + (-1)^k \sqrt{(a'-c')^2 + 4(b')^2}}{2}, \quad \text{for } k = 1, 2, \tag{3.11}$$

$$\|xA^T\|_2^2 = (a_{11}^2 + a_{21}^2)x_1^2 + 2(a_{11}a_{12} + a_{21}a_{22})x_1x_2 + (a_{12}^2 + a_{22}^2)x_2^2 := a'x_1^2 + 2b'x_1x_2 + c'x_2^2.$$

For a given real 2×2 matrix $A = \left(a_{ij}\right)_{i,j=1}^2$, we provide some properties of the values $a, b, c, \lambda, a', b', c'$ and λ', which will be used for study the connections of the extremes between A and A^T restricted on $\mathbb{S}_2$.

Lemma 3.2. *Let $A = \left(a_{ij}\right)_{i,j=1}^2$ be a real 2×2 matrix. Both A and A^T define linear and continuous operators on $\mathbb{R}^2$ by (3.1). Then,*

(i) $a + c = a' + c'$;
(ii) $(a-c)^2 + 4b^2 = (a'-c')^2 + 4(b')^2$;
(iii) $\lambda_k = \lambda_k'$, *for* $k = 1, 2$.

Proof. Proof of (i). $a+c=a'+c'=a_{11}^2+a_{21}^2+a_{12}^2+a_{22}^2$. Next, we prove (ii) by calculating

$$\begin{aligned}
&\left(a'-c'\right)^2+4\left(b'\right)^2\\
&=\left(a_{11}^2+a_{21}^2\right)^2+\left(a_{12}^2+a_{22}^2\right)^2-2\left(a_{11}^2+a_{21}^2\right)\left(a_{12}^2+a_{22}^2\right)+4\left(a_{11}a_{12}+a_{21}a_{22}\right)^2\\
&=a_{11}^4+a_{21}^4+2a_{11}^2a_{21}^2+a_{12}^4+a_{22}^4+2a_{12}^2a_{22}^2-2a_{11}^2a_{12}^2-2a_{11}^2a_{22}^2-2a_{12}^2a_{21}^2-2a_{21}^2a_{22}^2\\
&\quad+4a_{11}^2a_{12}^2+8a_{11}a_{12}a_{21}a_{22}+4a_{21}^2a_{22}^2\\
&=\left(a_{11}^4+2a_{11}^2a_{12}^2+a_{12}^4\right)+\left(a_{21}^4+2a_{21}^2a_{22}^2+a_{22}^4\right)-2a_{11}^2a_{22}^2-2a_{12}^2a_{21}^2-2a_{11}^2a_{21}^2-2a_{12}^2a_{22}^2\\
&\quad+4a_{11}^2a_{21}^2+4a_{12}^2a_{22}^2+8a_{11}a_{12}a_{21}a_{22}\\
&=(a-c)^2+4b^2.
\end{aligned}$$

Proof of (iii). By the definitions (3.5), (3.11) and parts (i) and (ii), we obtain

$$\lambda=\frac{-(a+c)\pm\sqrt{(a-c)^2+4b^2}}{2}=\frac{-\left(a'+c'\right)\pm\sqrt{\left(a'-c'\right)^2+4\left(b'\right)^2}}{2}=\lambda'.$$

□

From Lemma 3.2, we find that when we use the method of Lagrange Multiplier to precisely find the operator norms and covering constants for linear and continuous mapping A in $\mathbb{R}^2$ restricted on $\mathbb{S}_2$, the multiplier values λ_k and λ_k', for $k=1,2$, are exact same.

Proposition 3.2. *Let $A=\left(a_{ij}\right)_{i,j=1}^2$ be a real 2×2 matrix. Both A and A^T define linear and continuous operators on $\mathbb{R}^2$ by (3.1). Then,*

(i) $\left\|A^T\right\|_{\text{op}}=\|A\|_{\text{op}}$.

(a) *If $b'=0$, then $\left\|A^T\right\|_{\text{op}}$ is attained at $(\pm1,0)$ or $(0,\pm1)$ such that*

$$\left\|A^T\right\|_{\text{op}}=\left\|A^T(\pm1,0)\right\|_2\left(\text{or }\left\|A^T(0,\pm1)\right\|_2\right)=\max\left\{\sqrt{a'},\sqrt{c'}\right\}.$$

(b) *If $b'\neq 0$, then $\left\|A^T\right\|_{\text{op}}$ is attained at*

$$\left(\pm\sqrt{\frac{\left(b'\right)^2}{\left(a'+\lambda_1'\right)^2+\left(b'\right)^2}},\mp\frac{a'+\lambda_1'}{b'}\sqrt{\frac{\left(b'\right)^2}{\left(a'+\lambda_1'\right)^2+\left(b'\right)^2}}\right)$$

with

$$\left\|A^T\right\|_{\text{op}}=\sqrt{\frac{a+c+\sqrt{(a-c)^2+4b^2}}{2}}.$$

(c) *If $b\neq 0$, then $\|A\|_{\text{op}}$ is attained at*

$$\left(\pm\sqrt{\frac{b^2}{\left(a+\lambda_1\right)^2+b^2}},\mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{\left(a+\lambda_1\right)^2+b^2}}\right)$$

with

$$\|A\|_{\text{op}}=\sqrt{\frac{a+c+\sqrt{(a-c)^2+4b^2}}{2}}$$

(ii) $\left\|A^T\right\|_{\min} = \|A\|_{\min}$.

(a) *If* $b' = 0$, $\left\|A^T\right\|_{\min}$ *is attained at* $(\pm 1, 0)$ *or* $(0, \pm 1)$ *such that*

$$\left\|A^T\right\|_{\min} = \left\|A^T(\pm 1, 0)\right\|_2 \left(or\ \left\|A^T(0, \pm 1)\right\|_2\right) = \min\left\{\sqrt{a'}, \sqrt{c'}\right\}.$$

(b) *If* $b' \neq 0$, *then* $\left\|A^T\right\|_{\min}$ *is attained at*

$$\left(\pm\sqrt{\frac{(b')^2}{(a'+\lambda_2')^2+(b')^2}}, \mp\frac{a'+\lambda_2'}{b'}\sqrt{\frac{(b')^2}{(a'+\lambda_2')^2+(b')^2}}\right)$$

with

$$\left\|A^T\right\|_{\min} = \sqrt{\frac{a+c-\sqrt{(a-c)^2+4b^2}}{2}}.$$

(c) *If* $b \neq 0$, *then* $\|A\|_{\min}$ *is attained at*

$$\left(\pm\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}, \mp\frac{a+\lambda_2}{b}\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}\right)$$

with

$$\|A\|_{\min} = \sqrt{\frac{a+c-\sqrt{(a-c)^2+4b^2}}{2}}.$$

Proof. We prove the facts $\left\|A^T\right\|_{\text{op}} = \|A\|_{\text{op}}$ and $\left\|A^T\right\|_{\min} = \|A\|_{\min}$, which is divided into the following 5 cases. Rest proof of this proposition follows from Lemma 3.1.

Case 1. A is symmetric. This case is clear.

Case 2. A is not symmetric, $b = 0$ and $b' = 0$. Since A is not symmetric, then at least one of the two elements a_{12} and a_{21} is not zero. Assume $a_{12} \neq 0$. Conditions $b = 0$ and $b' = 0$ imply that $a_{11}a_{21} + a_{12}a_{22} = 0$ and $a_{11}a_{12} + a_{21}a_{22} = 0$. This implies that

$$a_{22} = -\frac{a_{21}}{a_{12}}a_{11} \quad \text{and} \quad a_{11} = -\frac{a_{21}}{a_{12}}a_{22}. \tag{3.12}$$

Subcase 2.1. Suppose $a_{22} = 0$. Note $a_{22} = 0 \Leftrightarrow a_{11} = 0$. In this case, we have

$$A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \quad \text{and} \quad A^T = \begin{pmatrix} 0 & a_{21} \\ a_{12} & 0 \end{pmatrix}$$

Then, $\left\|A^T\right\|_{\text{op}} = \|A\|_{\text{op}} = \max\{|a_{12}|, |a_{21}|\}$ and $\left\|A^T\right\|_{\min} = \|A\|_{\min} = \min\{|a_{12}|, |a_{21}|\}$.

Subcase 2.2. Suppose $a_{22} \neq 0$. Note that $a_{22} \neq 0 \Leftrightarrow a_{11} \neq 0$. By (3.12), we have $a_{22} = \frac{a_{21}}{a_{12}}\frac{a_{21}}{a_{12}}a_{22}$ and $a_{11} = \frac{a_{21}}{a_{12}}\frac{a_{21}}{a_{12}}a_{11}$. In this case, by the assumption that A is not symmetric, we have $a_{21} = -a_{12}$. By (3.12), $a_{11} = a_{22}$, $A = \begin{pmatrix} a_{11} & -a_{21} \\ a_{21} & a_{11} \end{pmatrix}$ and $A^T = \begin{pmatrix} a_{11} & a_{21} \\ -a_{21} & a_{11} \end{pmatrix}$ This implies that $a = a'$, $b = b' = 0$ and $c = c'$. Then, this Proposition is proved by (i) in Lemma 3.1.

Case 3. $b = 0$ and $b' \neq 0$. Since $b = 0$, by (i) in Lemma 3.1 to A, we have that $\|A\|_{\text{op}}$ is attained at points $(\pm 1, 0)$ or $(0, \pm 1)$ such that

$$\|A\|_{\text{op}} = \max\{\sqrt{a}, \sqrt{c}\}. \tag{3.13}$$

By $b' \neq 0$, $\|A^T\|_{\text{op}}$ is attained at $\left(\pm\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}}, \mp\frac{a'+\lambda_1'}{b'}\sqrt{\frac{(b')^2}{(a'+\lambda_2')^2+(b')^2}}\right)$. By Lemma 3.2 and $b=0$,

$$\begin{aligned}
\|A^T\|_{\text{op}} &= \left\|A^T\left(\pm\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}}, \mp\frac{a'+\lambda_1'}{b'}\sqrt{\frac{(b')^2}{(a'+\lambda_2')^2+(b')^2}}\right)\right\|_2 \\
&= \frac{1}{\sqrt{2}}\sqrt{a'+c'+\sqrt{(a'-c')^2+4(b')^2}} \\
&= \sqrt{\frac{a+c+|a-c|}{2}} \\
&= \max\{\sqrt{a}, \sqrt{c}\} = \|A\|_{\text{op}}. \qquad (3.14)
\end{aligned}$$

Case 3 is proved by (3.13) and (3.14).

Case 4. $b \neq 0$ and $b' = 0$. By $(A^T)^T = A$, this case is same with case 3.

Case 5. $b \neq 0$ and $b' \neq 0$. By part (ii) in Lemma 3.1 for A and A^T, respectively, we have that $\|A\|_{\text{op}}$ is attained at $\left(\pm\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}, \mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right)$ with $\|A\|_{\text{op}} = \sqrt{\frac{a+c+\sqrt{(a-c)^2+4b^2}}{2}}$. Meanwhile, by Lemma 3.2, $\|A^T\|_{\text{op}}$ is attained at $\left(\pm\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}}, \mp\frac{a'+\lambda_1'}{b'}\sqrt{(a'+\lambda_1')^2+(b')^2}\right)$ with

$$\begin{aligned}
\|A^T\|_{\text{op}} &= \sqrt{\frac{a'+c'+\sqrt{(a')^2+(c')^2+2(b')^2-2(\det(A^T))^2}}{2}} \\
&= \sqrt{\frac{a+c+\sqrt{(a-c)^2+4b^2}}{2}} = \|A\|_{\text{op}}.
\end{aligned}$$

□

In Proposition 3.2, we proved $\|A\|_{\text{op}} = \|A^T\|_{\text{op}} \cdot \|A\|_{\text{op}}$ and $\|A^T\|_{\text{op}}$ respectively reach at $\left(\pm\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}, \mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right)$ and $\left(\pm\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}}, \mp\frac{a'+\lambda_1'}{b'}\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}}\right)$. Since b' and b may be different and a' and a may be different, hence these two points may be different. Similarly, we proved $\|A\|_{\min} = \|A^T\|_{\min} \cdot \|A\|_{\min}$ and $\|A^T\|_{\min}$ respectively reach at $\left(\pm\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}, \mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right)$ and $\left(\pm\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}}, \mp\frac{a'+\lambda_1'}{b'}\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}}\right)$, which may be different. Let l_1, l_2 be line segments in $\mathbb{R}^2$. If l_1 and l_2 are perpendicular to each other, then we denote $l_1 \perp l_2$. The following corollary shows the perpendicular property of the two lines passing through the extrema of linear and continuous mappings in $\mathbb{R}^2$. This perpendicular property of linear and continuous mappings in $\mathbb{R}^2$ was raised by Professor Robert Mendris.

Corollary 3.4. *Let $A = (a_{ij})_{i,j=1}^2$ be a real 2×2 matrix. Both A and A^T define linear and continuous operators on $\mathbb{R}^2$ by (3.1). Let l_{op} be the vector passing through the two points, at*

which $\|A\|_{\text{op}}$ *is attained at its value and let* $l_{\min}$ *be the vector passing through the two points, at which* $\|A\|_{\min}$ *is attained at its value. Then,* $l_{\text{op}} \perp l_{\min}$.

Proof. If $b = 0$, by Proposition 3.2, then this corollary clearly holds. Let $b \neq 0$. Recall that

$$a+\lambda_1 = \frac{(a-c)-\sqrt{(a-c)^2+4b^2}}{2} \quad \text{and} \quad a+\lambda_2 = \frac{(a-c)+\sqrt{(a-c)^2+4b^2}}{2}$$

Then by Proposition 3.2, we have

$$l_{\text{op}} = 2\left(\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}, -\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right)$$

and

$$l_{\min} = 2\left(\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}, -\frac{a+\lambda_2}{b}\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}\right).$$

We calculate

$$\begin{aligned}
&\left\langle \left(\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}, -\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right), \right. \\
&\qquad \left. \left(\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}, -\frac{a+\lambda_2}{b}\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}\right)\right\rangle \\
&= \sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}\left(1+\frac{(a+\lambda_1)(a+\lambda_2)}{b^2}\right) \\
&= \sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\sqrt{\frac{b^2}{(a+\lambda_2)^2+b^2}}\left(1+\frac{-b^2}{b^2}\right) \\
&= 0.
\end{aligned}$$

□

Now, we apply the results of Propositions 3.1 and 3.2 to the calculations of the covering constants for linear and continuous operators on $\mathbb{R}^2$.

Proposition 3.3. *Let* $A = \left(a_{ij}\right)_{i,j=1}^{2}$ *be a real* 2×2 *matrix. Both* A *and* A^T *define linear and continuous operators on* $\mathbb{R}^2$ *by (3.1). Then*

(a) *If* $b' = 0$, *then* $\hat{\alpha}(A,x,A(x))$ *is attained at* $(\pm 1,0)$ *or* $(0,\pm 1)$ *such that* $\hat{\alpha}(A,x,A(x)) = \min\left\{\sqrt{a'},\sqrt{c'}\right\}$, *for all* $x \in \mathbb{R}^2$;

(b) *If* $b' \neq 0$, *then* $\hat{\alpha}(A,x,A(x))$ *is attained at*

$$\left(\pm\sqrt{\frac{(b')^2}{(a'+\lambda_2')^2+(b')^2}}, \mp\frac{a'+\lambda_2'}{b'}\sqrt{\frac{(b')^2}{(a'+\lambda_2')^2+(b')^2}}\right)$$

with

$$\hat{\alpha}(A,x,A(x)) = \sqrt{\frac{a+c-\sqrt{(a-c)^2+4b^2}}{2}}, \quad \text{for all } x \in \mathbb{R}^2$$

Proof. By definition and Proposition 3.1, we have $\hat{\alpha}(A,x,A(x)) = \left\|A^T\right\|_{\min}$, for all $x \in \mathbb{R}^2$. Then, this proposition follows from Proposition 3.2 immediately.

Next, we provide some examples to demonstrate the results of Propositions 3.1 and 3.2. From the following examples, we see that, for any real 2×2 matrix $A = (a_{ij})_{i,j=1}^2$, $\|A\|_{\mathrm{op}} = \left\|A^T\right\|_{\mathrm{op}}$ and $\|A\|_{\min} = \left\|A^T\right\|_{\min}$. However, they may reach at different points in $\mathbb{S}_2$. □

Example 3.1. Let $A = \begin{pmatrix} 3 & -2 \\ 4 & 6 \end{pmatrix}$. Then $a = 9+4 = 13$, $c = 52$ and $b = 0$. By part (i) in Proposition 3.2, $\|A\|_{\mathrm{op}} = \|A(0,\pm 1)\|_2 = \max\{\sqrt{a},\sqrt{c}\} = \sqrt{c} = \sqrt{52}$ and $\|A\|_{\min} = \|A(\pm 1,0)\|_2 = \min\{\sqrt{a},\sqrt{c}\} = \sqrt{a} = \sqrt{13}$. Meanwhile, by $A^T = \begin{pmatrix} 3 & 4 \\ -2 & 6 \end{pmatrix}$, we have $a' = 25$, $c' = 40$ and $b' = -6+24 = 18 \neq 0$. We calculate λ_k' by

$$\lambda_k' = \frac{-65+(-1)^k\sqrt{1521}}{2} = \frac{-65+(-1)^k 39}{2}, \quad \text{for } k = 1,2.$$

This implies $\lambda_1' = \frac{-65-39}{2} = -52$ and $\lambda_2' = \frac{-65+39}{2} = -13$. By Proposition 3.2, $\|A^T\|_{\mathrm{op}}$ is attained at points

$$\left(\pm\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}}, \mp\frac{a'+\lambda_1'}{b'}\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}}\right) = \left(\pm\frac{18}{\sqrt{1053}}, \pm\frac{27}{\sqrt{1053}}\right).$$

This implies that

$$\begin{aligned}\|A^T\|_{\mathrm{op}} &= \sqrt{\frac{(a'-c')^2+(a'+c')\sqrt{(a'-c')^2+4(b')^2}+4(b')^2}{2\sqrt{(a'-c')^2+4(b')^2}}} \\ &= \sqrt{\frac{a'+c'+\sqrt{(a'-c')^2+4(b')^2}}{2}} = \sqrt{\frac{25+40+\sqrt{(25-40)^2+4(18)^2}}{2}} = \|A\|_{\mathrm{op}}.\end{aligned}$$

Note that $\|A^T\|_{\mathrm{op}} = \left\|\left(\pm\frac{18}{\sqrt{1053}}, \pm\frac{27}{\sqrt{1053}}\right)\begin{pmatrix} 3 & 4 \\ -2 & 6 \end{pmatrix}\right\|_2 = \sqrt{52}$. Thus $\|A^T\|_{\mathrm{op}} = \|A\|_{\mathrm{op}}$, but they reach at different points in $\mathbb{S}^2$. $\bar{\alpha}(A,x,A(x)) = \|A^T\|_{\min}$ is attained at

$$\left(\pm\sqrt{\frac{(b')^2}{(a'+\lambda_2')^2+(b')^2}}, \mp\frac{a'+\lambda_2'}{b'}\sqrt{\frac{(b')^2}{(a'+\lambda_2')^2+(b')^2}}\right) = \left(\pm\frac{18}{\sqrt{468}}, \mp\frac{12}{\sqrt{468}}\right).$$

For any $x \in \mathbb{R}^2$, we have

$$\begin{aligned}\hat{\alpha}(A,x,A(x)) &= \left\|A^T\left(\pm\frac{18}{\sqrt{468}}, \mp\frac{12}{\sqrt{468}}\right)\right\|_2 \\ &= \sqrt{\frac{-(a'-c')^2+(a'+c')\sqrt{(a'-c')^2+4(b')^2}-4(b')^2}{2\sqrt{(a'-c')^2+4(b')^2}}} \\ &= \sqrt{\frac{-(25-40)^2+(25+40)\sqrt{(25-40)^2+4(18^2)}-4(18^2)}{2\sqrt{(25-40)^2+4(18^2)}}} \\ &= \sqrt{13}.\end{aligned}$$

We may directly check that, for any $x \in \mathbb{R}^2$,

$$\hat{\alpha}(A,x,A(x)) = \left\| \left(-\frac{18}{\sqrt{468}}, \frac{12}{\sqrt{468}} \right) \begin{pmatrix} 3 & 4 \\ -2 & 6 \end{pmatrix} \right\|_2 = \sqrt{13},$$

which implies that $\hat{\alpha}(A,x,A(x)) = \left\|A^T\right\|_{\min} = \|A\|_{\min} = \sqrt{13}$, for all $x \in \mathbb{R}^2$. Note that $\left\|A^T\right\|_{\min}$ and $\|A\|_{\min}$ reach at different points in $\mathbb{S}_2$. One can check that $\begin{pmatrix} 3 & 4 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 25 & 18 \\ 18 & 40 \end{pmatrix}$ has two eigenvalues 52 and 13.

Example 3.2. Let $A = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}$. We have $a = 9+16 = 25$, $c = 0$ and $b = 0$. By Propositions 3.1 and 3.2,

$$\|A\|_{\text{op}} = \|A(\pm 1, 0)\|_2 = \max\{\sqrt{a}, \sqrt{c}\} = \sqrt{a} = 5,$$

and

$$\|A\|_{\min} = \|A(0, \pm 1)\|_2 = \min\{\sqrt{a}, \sqrt{c}\} = \sqrt{c} = 0.$$

Meanwhile, by $A^T = \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}$, we have $a' = 9$, $c' = 16$ and $b' = 12 \neq 0$. We calculate λ_1':

$$\lambda_1' = \frac{-(a'+c') - \sqrt{(a'-c')^2 + 4(b')^2}}{2} = \frac{-(9+16) - \sqrt{(9-16)^2 + 4(12^2)}}{2} = -25.$$

Then, $\left\|A^T\right\|_{\text{op}}$ is attained at points

$$\left(\pm \sqrt{\frac{12^2}{(9-25)^2 + 12^2}}, \mp \frac{9-25}{12} \sqrt{\frac{12^2}{(9-25)^2 + 12^2}} \right) = \left(\pm \frac{3}{5}, \pm \frac{4}{5} \right).$$

$$\left\|A^T\right\|_{\text{op}} = \left\| A^T \left(\pm \frac{3}{5}, \pm \frac{4}{5} \right) \right\|_2 = \sqrt{\frac{(a'-c')^2 + (a'+c')\sqrt{(a'-c')^2 + 4(b')^2}\, 4(b')^2}{2\sqrt{(a'-c')^2 + 4(b')^2}}} = 5.$$

We may directly check that

$$\left\|A^T\right\|_{\text{op}} = \left\| \left(\pm \frac{3}{5}, \pm \frac{4}{5} \right) \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix} \right\|_2 = 5$$

This implies that $\|A\|_{\text{op}} = \left\|A^T\right\|_{\text{op}} = 5$. But $\left\|A^T\right\|_{\text{op}}$ and $\|A\|_{\text{op}}$ reach at different points in $\mathbb{S}_2$. Then, we calculate λ_2'

$$\lambda_2' = \frac{-(a'+c') + \sqrt{(a'-c')^2 + 4(b')^2}}{2} = \frac{-(9+16) + \sqrt{(9-16)^2 + 4(12^2)}}{2} = 0.$$

We calculate the corresponding critical points

$$\left(\pm \sqrt{\frac{(b')^2}{(a'+\lambda_2')^2 + (b')^2}}, \mp \frac{a'+\lambda_2'}{b'} \sqrt{\frac{(b')^2}{(a'+\lambda_2')^2 + (b')^2}} \right) = \left(\pm \frac{4}{5}, \mp \frac{4}{5} \right).$$

By Proposition 3.2, $\hat{\alpha}(A,x,A(x)) = \|A^T\|_{\min}$ is attained in $\mathbb{S}_2$ at points $\left(\pm\frac{4}{5},\mp\frac{3}{5}\right)$. For any $x \in \mathbb{R}^2$, we have

$$\hat{\alpha}(A,x,A(x)) = \|A^T\|_{\min} = \left\|A^T\left(\pm\frac{12}{15},\mp\frac{9}{15}\right)\right\|_2$$
$$= \sqrt{\frac{-(a'-c')^2+(a'+c')\sqrt{(a'-c')^2+4(b')^2}-4(b')^2}{2\sqrt{(a'-c')^2+4(b')^2}}} = 0.$$

We may directly check that, for any $x \in \mathbb{R}^2$,

$$\hat{\alpha}(A,x,A(x)) = \|A^T\|_{\min} = \left\|\left(\pm\frac{4}{5},\mp\frac{3}{5}\right)\begin{pmatrix}3 & 0\\ 4 & 0\end{pmatrix}\right\|_2 = 0.$$

Hence, $\tilde{\alpha}(A,x,A(x)) = \|A^T\|_{\min} = \|A\|_{\min}$. But $\|A^T\|_{\min}$ and $\|A\|_{\min}$ reach at different points in $\mathbb{S}_2$. One can check that

$$\begin{pmatrix}3 & 0\\ 4 & 0\end{pmatrix}\begin{pmatrix}3 & 4\\ 0 & 0\end{pmatrix} = \begin{pmatrix}9 & 12\\ 12 & 16\end{pmatrix}$$

has two eigenvalues 25 and 0.

Example 3.3. Let $A = \begin{pmatrix}3 & 4\\ 4 & 3\end{pmatrix}$. This is a symmetric matrix with $A^T = A$. We have $a = c = a' = c' = 25$ and $b = b' = 24$. By Proposition 3.2, we have

(I) $\|A\|_{\text{op}}$ is attained at two points $\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)$ (with $\lambda_1 = -a-|b|$) satisfying

$$\|A\|_{\text{op}} = \left\|A\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)\right\|_2 = \sqrt{a+|b|} = \sqrt{25+24} = 7.$$

(II) $\|A(\cdot)\|_2$ is attained at its minimum value on $\mathbb{S}$ at two points $\left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{\sqrt{2}}\right)$ (with $\lambda_2 = -a+|b|$) and the minimum value coincides with $\hat{\alpha}(A,x,A(x))$, that is,

$$\hat{\alpha}(A,x,A(x)) = \|A\|_{\min} = \left\|A\left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{\sqrt{2}}\right)\right\|_2 = \sqrt{a-|b|} = 1.$$

We may directly check that

$$\|A\|_{\text{op}} = \left\|\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\begin{pmatrix}3 & 4\\ 4 & 3\end{pmatrix}\right\|_2 = \left\|\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\begin{pmatrix}3 & 4\\ 4 & 3\end{pmatrix}\right\|_2 = 7,$$

and

$$\hat{\alpha}(A,x,A(x)) = \|A\|_{\min} = \left\|\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\begin{pmatrix}3 & 4\\ 4 & 3\end{pmatrix}\right\|_2 = \left\|\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\begin{pmatrix}3 & 4\\ 4 & 3\end{pmatrix}\right\|_2 = 1.$$

One can check that

$$\begin{pmatrix}3 & 4\\ 4 & 3\end{pmatrix}\begin{pmatrix}3 & 4\\ 4 & 3\end{pmatrix} = \begin{pmatrix}25 & 24\\ 24 & 25\end{pmatrix}$$

has two eigenvalues 49 and 1.

Example 3.4. Let $A=\begin{pmatrix}3 & 4\\ 0 & 5\end{pmatrix}$. We have $a=c=25$ and $b=20$. Meanwhile, by $A^T=\begin{pmatrix}3 & 0\\ 4 & 5\end{pmatrix}$, we have $a'=9$, $c'=41$ and $b'=12\neq 0$. We calculate $\lambda_k=\lambda_k'$:

$$\lambda_k=\lambda_k'=\frac{-(a'+c')+(-1)^k\sqrt{(a'-c')^2+4(b')^2}}{2}=\frac{-50+(-1)^k 40}{2},\quad \text{for } k=1,2.$$

This implies $\lambda_1=\lambda_1'=-45$ and $\lambda_2=\lambda_2'=-5$. By Proposition 3.2, $\|A\|_{\text{op}}$ is attained at the following points

$$\left(\pm\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}},\mp\frac{a+\lambda_1}{b}\sqrt{\frac{b^2}{(a+\lambda_1)^2+b^2}}\right)=\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right).$$

By (iii) in Lemma 3.1, we have

$$\|A\|_{\text{op}}=\left\|A\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)\right\|_2=\sqrt{\frac{a+c+\sqrt{(a-c)^2+4b^2}}{2}}=\sqrt{45}.$$

We may directly check that

$$\|A\|_{\text{op}}=\left\|\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)\begin{pmatrix}3 & 4\\ 0 & 5\end{pmatrix}\right\|_2=\sqrt{45}.$$

$\left\|A^T\right\|_{\text{op}}$ is attained

$$\left(\pm\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}},\mp\frac{a'+\lambda_1'}{b'}\sqrt{\frac{(b')^2}{(a'+\lambda_1')^2+(b')^2}}\right)=\left(\pm\frac{1}{\sqrt{10}},\pm\frac{3}{\sqrt{10}}\right),$$

which implies that

$$\left\|A^T\right\|_{\text{op}}=\left\|A^T\left(\pm\frac{1}{\sqrt{10}},\pm\frac{3}{\sqrt{10}}\right)\right\|_2=\sqrt{\frac{a'+c'+\sqrt{(a'-c')^2+4(b')^2}}{2}}=\sqrt{45}=\|A\|_{\text{op}}.$$

We may directly check that $\left\|A^T\right\|_{\text{op}}=\sqrt{45}$. We have $\left\|A^T\right\|_{\text{op}}=\|A\|_{\text{op}}$. But they reach at different points in $\mathbb{S}_2$. We may check

$$\left\|A^T\right\|_{\text{min}}=\|A\|_{\text{min}}=\left\|A\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)\right\|_2=\sqrt{\frac{a+c-\sqrt{(a-c)^2+4b^2}}{2}}=\sqrt{5}.$$

One can check that

$$\begin{pmatrix}3 & 0\\ 4 & 5\end{pmatrix}\begin{pmatrix}3 & 4\\ 0 & 5\end{pmatrix}=\begin{pmatrix}9 & 12\\ 12 & 41\end{pmatrix}$$

has two eigenvalues 45 and 5.

Example 3.5. Let $A=\begin{pmatrix}0.2 & 0\\ 0 & 0.5\end{pmatrix}$. Then $\|A\|_{\text{op}}=\left\|A^T\right\|_{\text{op}}=0.5$ and $\left\|A^T\right\|_{\text{min}}=\|A\|_{\text{min}}=0.2$. Meanwhile, $\det(A)=0.1$.

Remark 3.1. To compare $\|A\|_{\text{op}}$, $\|A\|_{\text{min}}$ and $\det(A)$, by summarizing the above examples, in general, we have

(i) Neither $\|A\|_{\text{op}}\leq|\det(A)|$, nor $\|A\|_{\text{op}}\geq|\det(A)|$, for any real 2×2 matrix A;

(ii) Neither $\|A\|_{\min} \leq |\det(A)|$, nor $\|A\|_{\min} \geq |\det(A)|$, for any real 2×2 matrix A.

4. Solvability of Stochastic Systems of Linear Equations in $\mathbb{R}^n$

In this section, we introduce the concepts of locally or globally stochastic system of linear equations in $\mathbb{R}^n$. Then, we use the results from the previous sections about the operator norms and covering constants for linear and continuous mappings in $\mathbb{R}^n$ and by applying the Arutyunov Mordukhovich Zhukovskiy Parameterized Coincidence Theorem to prove the existence of solutions to some locally or globally stochastic systems of linear equations in $\mathbb{R}^n$. As stated in Section 1, let (S,τ,μ) be a topological probability space, in which S is the sample space, the topology τ on S coincides with the σ-field of all events in S, and μ is the probability measure in S defined on τ. Let n be a positive integer. Let $\omega : S \to \mathbb{R}^n$ be a τ-measurable single-valued mapping (that is considered as a noise). Let $\bar{s} \in S$. If there exist a neighborhood (an event) $W \subset S$ of $\bar{s}$ and a single-valued mapping $\sigma : W \to \mathbb{R}^n$ (σ is not required to be τ-measurable) such that $A(\sigma(s)) = B(\sigma(s)) + \omega(s)$, for all $s \in W$, then we say that this single-valued mapping $\sigma : W \to \mathbb{R}^n$ is a *local solution* to the stochastic system of linear equations on W around the possible outcome $\bar{s}$, with respect to the matrices A, B and the noise ω. In particular, in the above inclusion property, if the neighborhood W of $\bar{s}$ is the whole space S; that is, if

$$A(\sigma(s)) = B(\sigma(s)) + \omega(s), \quad \text{for any } s \in S, \tag{4.1}$$

then σ is called a *global solution* to the stochastic system of linear equations on S around the possible outcome $\bar{s}$ and with respect to the matrices A, B and the noise ω.

Now, we use the AMZ theorem and the results in the previous subsection to prove the existence of solutions to the locally stochastic system of linear equations on a topological probability space S around any given possible outcome $\bar{s}$ and with respect to matrices A, B and a noise ω.

Theorem 4.1. *Let n be a positive integer and let (S,τ,μ) be a topological probability space. Let $A = \left(a_{ij}\right)_{i,j=1}^{n}$ and $B = \left(b_{ij}\right)_{i,j=1}^{n}$ be real $n \times n$ matrices. Let $\omega : S \to \mathbb{R}^n$ be a τ-measurable single-valued mapping. Let $\bar{s} \in S$. Suppose that A, B and ω satisfy the following conditions*

(a1) $0 < \|B\|_{\text{op}} < \hat{\alpha}(A,x,A(x)) \leq \|A\|_{\text{op}} \leq 1$, *for every $x \in \mathbb{R}^n$;*

(a2) *The function $s \mapsto \omega(s)$ is lower semicontinuous at $\bar{s}$.*

Then, for any λ, α with $\hat{\alpha}(A,x,A(x)) \geq \lambda > \alpha > \|B\|_{\text{op}}$, there exist a neighborhood $W_{\lambda\alpha} \subset S$ of $\bar{s}$ and a single-valued mapping $\sigma_{\lambda\alpha} : W_{\lambda\alpha} \to \mathbb{R}^n$ such that $A\left(\sigma_{\lambda\alpha}(s)\right) = B\left(\sigma_{\lambda\alpha}(s)\right) + \omega(s)$, for every $s \in W_{\lambda\alpha}$ and

$$\|\sigma_{\lambda\alpha}(s) - x\| \leq \frac{\|(B(x) + \omega(s)) - A(x)\|_n}{\alpha - \|B\|_{\text{op}}}, \quad \text{for any } s \in W_{\lambda\alpha}, \text{ for every } x \in \mathbb{R}^n. \tag{4.2}$$

Proof. By part (iii) of Proposition 3.1, the condition in (a1), for every $x \in \mathbb{R}^n$, $\hat{\alpha}(A,x,A(x)) > 0$ implies that $\det(A) \neq 0$. In this case, by (b) in part (iii) of Proposition 3.1, the covering constant for A is constant in $\mathbb{R}^n$ satisfying $\hat{\alpha}(A,x,A(x)) = \min\left\{\|A^T(y)\|_n : y \in \mathbb{R}^n, \|y\|_n = 1\right\}$ and

$$0 < \hat{\alpha}(A,x,A(x)) \leq \|A\|_{\text{op}} \leq \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{\frac{1}{2}} \leq 1, \quad \text{for all } x \in \mathbb{R}^n.$$

In particular, in AMZ theorem, we take both of the Banach spaces X and Y to be $\mathbb{R}^n$. That is, let $X = Y = \mathbb{R}^n$. Since $\mathbb{R}^n$ is reflexive; we have that $\mathbb{R}^n$ is Asplund. Let $F = A$ and define

$G(x,s) = B(x) + \omega(s)$ for all $(x,\tilde{s}) \in \mathbb{R}^n \times S$. Then, we check that F and G satisfy all conditions (A1), (A2) and (A3) in the AMZ theorem. Since $F = A$ is a linear and continuous single-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, then $F : \mathbb{R}^n \to \mathbb{R}^n$ is closed around point $(x, A(x)) \in \mathbb{R}^n \times \mathbb{R}^n$, for any point $x \in \mathbb{R}^n$. Hence, the condition (A1) in AMZ theorem is satisfied with respect to every point $x \in \mathbb{R}^n$. In the condition (A2) in AMZ theorem, take all neighborhoods $U \subset \mathbb{R}^n$ of x, $V \subset \mathbb{R}^n$ of $A(x)$ to be $\mathbb{R}^n$. That is, let $U = V = \mathbb{R}^n$ in AMZ theorem. For any fixed $s \in S$, by the continuity of B, we have

$$\|G(u,s) - G(v,s)\|_n = \|(B(u) + \omega(s)) - (B(v) + \omega(s))\|_n \leq \|B\|_{\text{op}} \|u - v\|_n, \text{ for all } u, v \in \mathbb{R}^n.$$

This implies that, for any $s \in S$, the function $G(\cdot, s) : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz-like on $U = \mathbb{R}^n$ relative to $V = \mathbb{R}^n$ with the uniform modulus $\|B\|_{\text{op}}$. From the condition (a1) in this theorem, it satisfies $0 < \|B\|_{\text{op}} < \tilde{\alpha}(A, x, A(x)) \leq 1$, for all $x \in \mathbb{R}^n$. From condition (a2), we see that $s \mapsto \omega(s)$ is lower/inner semicontinuous at $\bar{s}$. This implies that, for any given $x \in \mathbb{R}^n$, $G(x,s) = B(x) + \omega(s) : S \to \mathbb{R}^n$ is lower semicontinuous at $\bar{s}$. Hence, the condition (A2) in AMZ theorem is satisfied. Using condition (a2) again, we see that the Lipschitzian modulus $\|B\|_{\text{op}}$ of $G(\cdot, s)$ satisfies $0 < \|B\|_{\text{op}} < \hat{\alpha}(F, \bar{x}, F(\bar{y}))$. This implies that the condition (A3) in AMZ theorem is satisfied. Hence, for any λ, α with $\hat{\alpha}(A, x, A(x)) \geq \lambda > \alpha > \|B\|_{\text{op}}$, there exist a neighborhood $W_{\lambda\alpha} \subset S$ of $\bar{s}$ and a single-valued mapping $\sigma_{\lambda\alpha} : W_{\lambda\alpha} \to \mathbb{R}^n$ such that, for any $x \in \mathbb{R}^n$,

$$A\big(\sigma_{\lambda\alpha}(s)\big) \cap \big(B(\sigma_{\lambda\alpha}(s)) + \omega(s)\big) \neq \emptyset, \quad \text{for every } s \in W_{\lambda\alpha} \tag{4.3}$$

and

$$\|\sigma_{\lambda\alpha}(s) - x\|_n \leq \frac{\text{dist}\big(A(x), B(x) + \omega(s)\big)}{\alpha - \|B\|_{\text{op}}}, \quad \text{for any } s \in W_{\lambda\alpha} \text{ and for any } x \in \mathbb{R}^n.$$

Since mappings A, B and ω are single-valued, then (4.3) becomes

$$A\big(\sigma_{\lambda\alpha}(s)\big) = B\big(\sigma_{\lambda\alpha}(s)\big) + \omega(s), \quad \text{for every } s \in W_{\lambda\alpha}$$

and

$$\text{dist}\big(A(x), B(x) + \omega(s)\big) = \big\|(B(x) + \omega(s)) - A(x)\big\|_n,$$

which proves (4.2). □

Remark 4.1. In Theorem 4.1, two matrices $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$ are supposed to satisfy condition (a1): $0 < \|B\|_{\text{op}} < \hat{\alpha}(A, x, A(x)) \leq \|A\|_{\text{op}} \leq 1$, for every $x \in \mathbb{R}^n$. From this condition, that $\hat{\alpha}(A, x, A(x)) > 0$, we must have

(a) $\det(A) \neq 0$,
(b) $\det(B)$ is not required to be nonzero.

Corollary 4.1. *Let n be a positive integer and let (S, τ, μ) be a topological probability space. Let $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$ be real $n \times n$ diagonal matrices. Let $\omega : S \to \mathbb{R}^n$ be a τ-measurable single-valued mapping. Let $\bar{s} \in S$. Suppose that A, B and ω satisfy the following conditions.*

(a1) $0 < \max\{|b_{ii}| : i = 1, 2, \ldots, n\} = \|B\|_{\text{op}} < \min\{|a_{ii}| : i = 1, 2, \ldots, n\} \leq \|A^T\|_{\text{op}} \leq 1$;
(a2) *The function $s \mapsto \omega(s)$ is lower semicontinuous at $\bar{s}$.*

Then, for any λ, α *with* $\min\{|a_{ii}| : i = 1,2,\ldots,n\} \geq \lambda > \alpha > \max\{|b_{ii}| : i = 1,2,\ldots,n\} = \|B\|_{\text{op}}$, *there exist a neighborhood* $W_{\lambda\alpha} \subset S$ *of* $\bar{s}$ *and a single-valued mapping* $\sigma_{\lambda\alpha} : W_{\lambda\alpha} \to \mathbb{R}^n$ *such that* $A\big(\sigma_{\lambda\alpha}(s)\big) = B\big(\sigma_{\lambda\alpha}(s)\big) + \omega(s)$ *for all* $s \in W_{\lambda\alpha}$ *and*

$$\|\sigma_{\lambda\alpha}(s) - x\| \leq \frac{\big\|(B(x) + \omega(s)) - A(x)\big\|_n}{\alpha - \|B\|_{\text{op}}}, \quad \text{for any } s \in W_{\lambda\alpha} \text{ and for any } x \in \mathbb{R}^n.$$

Proof. From Corollary 3.1, we have $\min\{|a_{ii}| : i = 1,2,\ldots,n\} = \hat{\alpha}(A,x,A(x)) \leq \|A^T\|_{\text{op}}$ for all $x \in \mathbb{R}^n$ and $\max\{|b_{ii}| : i = 1,2,\ldots,n\} = \|B\|_{\text{op}}$. Then, this corollary follows from Theorem 4.1 immediately. □

Example 4.1. Let (S, τ, μ) be a topological probability space. Let $\omega : S \to \mathbb{R}^2$ be a single-valued mapping defined by $\omega(s) = (\omega_1(s), \omega_2(s))$, which is lower semicontinuous, for any $s \in S$. Let

$$A = \begin{pmatrix} 0.3 & -0.4 \\ 0.4 & 0.3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0.2 & -0.3 \\ 0.3 & 0.2 \end{pmatrix}.$$

Then $A(x) : \mathbb{R}^2 \to \mathbb{R}^2$ and $B(x) + \omega(s) : \mathbb{R}^2 \times S \to \mathbb{R}^2$ are single-valued mappings. By part (i) in Proposition 3.2,$\hat{\alpha}(A,x,A(x)) = \|A\|_{\min} = \sqrt{0.25} = 0.5$, for any $x \in \mathbb{R}^2$ and the uniform modulus β of the mapping $B(x) + \omega(s)$ is $\beta = \|B\|_{\text{op}} = \sqrt{0.13}$. The mappings A, B and ω satisfy all conditions in Theorem 4.1. At first, this implies that the globally stochastic systems of linear equations with respect to A, B and ω has a global solution, which is written as $\sigma(s) = (\sigma_1(s), \sigma_2(s))$ satisfying (4.1). This solution is calculated as follows

$$\sigma(s) = (\sigma_1(s), \sigma_2(s)) = (\omega_1(s), \omega_2(s)) \begin{pmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{pmatrix}^{-1}, \text{ for all } s \in S.$$

This is equivalent to $\sigma(s)(A - B) = \omega(s)$ for all $s \in S$, which implies

$$\sigma(s) = \sigma(s)BA^{-1} + \omega(s)A^{-1}, \text{ for all} s \in S. \tag{4.4}$$

Next, we show that this solution satisfies the inequality (4.2) in Theorem 4.1. Note that $A^{-1} = \begin{pmatrix} 1.2 & 1.6 \\ -1.6 & 1.2 \end{pmatrix}$. By part (i) in Proposition 3.2, $\|A^{-1}\|_{\text{op}} = 2$. Then, for $x = (x_1, x_2) \in \mathbb{R}^2$, for any α with $\sqrt{0.13} < \alpha < 0.5 = \hat{\alpha}(A,x,A(x))$, we have by (4.4) that

$$\begin{aligned} \|\sigma(s) - x\|_2 &= \big\|(\sigma(s)B + \omega(s) - xA)A^{-1}\big\|_2 \\ &\leq 2\|\sigma(s)B + \omega(s) - xA\|_2 \\ &\leq \frac{\|B(\sigma(s)) - B(x)\|_2 + \|A(x) - (B(x) + \omega(s))\|_2}{0.5} \\ &\leq \frac{\sqrt{0.13}\|\sigma(s) - x\|_2 + \|A(x) - (B(x) + \omega(s))\|_2}{0.5}. \end{aligned}$$

This implies that

$$\|\sigma(s) - x\|_2 \left(1 - \frac{\sqrt{0.13}}{0.5}\right) \leq \frac{\|A(x) - (B(x) + \omega(s))\|_2}{0.5}.$$

We obtain that, for any α with $\sqrt{0.13} < \alpha < 0.5 = \hat{\alpha}(A,x,A(x))$,

$$\|\sigma(s) - x\|_2 \leq \frac{\|A(x) - (B(x) + \omega(s))\|_2}{0.5 - \sqrt{0.13}} \leq \frac{\|A(x) - (B(x) + \omega(s))\|_2}{\alpha - \sqrt{0.13}}, \text{ for all } s \in S.$$

This shows that the solution σ of the globally stochastic systems of linear equations with respect to A, B and ω satisfies the inequality (4.2) in Theorem 4.1.

For given A, B and ω as in Theorem 4.1, we see that the conditions (a2) in Theorem 4.1 are sufficient conditions for the existence of solutions for the locally or globally stochastic systems of linear equations with respect to A, B and ω. They may not be necessary conditions. However, we give an example below to show that if A, B and ω do not satisfy the conditions (a2) in Theorem 4.1, then the considered locally or globally stochastic systems of linear equations with respect to A, B and ω does not have a solution. We just modify Example 4.1 to get the following example.

Example 4.2. Let (S, τ, μ) be a topological probability space with $S = [0,1]$. Let $\omega : S \to \mathbb{R}^2$ be defined by $\omega(s) = \left(s, s^2\right)$, for any $s \in S$, which is a continuous single-valued mapping. Let

$$A = \begin{pmatrix} 0.3 & 0.4 \\ 0.4 & 0.3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0.2 & 0.3 \\ 0.3 & 0.2 \end{pmatrix}.$$

Then $A(x) : \mathbb{R}^2 \to \mathbb{R}^2$ and $B(x) + \omega(s) : \mathbb{R}^2 \times S \to \mathbb{R}^2$ are single-valued mappings. By part (i) in Proposition 3.2, we have $\hat{\alpha}(A, x, A(x)) = \|A\|_{\min} = \sqrt{0.25 - 0.24} = 0.1$, for all $x \in \mathbb{R}^2$. The uniform modulus β of the mapping $B(x) + \omega(s)$ is $\beta = \|B\|_{\text{op}} = \sqrt{0.09 + 0.04 + 2(0.12)} = \sqrt{0.37}$, which implies that the condition $\beta < \hat{\alpha}(A, x, A(x))$ does not hold. Then, for any mapping $\sigma : S \to \mathbb{R}^2$, which is written as $\sigma(s) = (\sigma_1(s), \sigma_2(s))$, we have $\sigma(s)A = \sigma(s)B + \omega(s)$ for some $s \in S$ if and only if

$$\sigma(s)(A - B) = \sigma(s) \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix} + \left(s, s^2\right), \quad \text{for some } s \in S. \tag{4.5}$$

For any σ, equation (4.5) only has two possible solutions in $S = [0,1]$, which are 0 and 1. For any point $\bar{s} \in S$, there is no neighborhood W of $\bar{s}$ such that (4.5) holds for every $s \in W$. Hence, the locally stochastic system of linear equations with respect to A, B and ω does not have a solution.

Example 4.3. Let n be a positive integer. Let $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$ be real $n \times n$ diagonal matrices. Let (S, τ, μ) be a topological probability space with $S = [0,1]$. Let $\omega : S \to \mathbb{R}^n$ be a single-valued mapping defined by $\omega(s) = (\omega_1(s), \omega_2(s), \ldots, \omega_n(s))$, for any $s \in S$, which is a continuous single-valued mapping. Suppose that A and B satisfy

$$0 < \max\{|b_{ii}| : i = 1, 2, \ldots, n\} < \min\{|a_{ii}| : i = 1, 2, \ldots, n\} \leq \max\{|a_{ii}| : i = 1, 2, \ldots, n\} \leq 1.$$

That is,

$$0 < \|B\|_{\text{op}} < \hat{\alpha}(A, x, A(x)) = \|A\|_{\min} \leq \|A\|_{\text{op}} \leq 1, \quad \text{for all } x \in \mathbb{R}^n,$$

which implies that A, B and ω satisfy all the conditions of Theorem 4.1. Hence, the globally stochastic system of linear equations with respect to A, B and ω has a global solution written as $\sigma(s) = (\sigma_1(s), \sigma_2(s))$ that satisfies (4.1), $\sigma(s)A = \sigma(s)B + \omega(s)$, for any $s \in S$. Next, with $\|B\|_{\text{op}}$ being the uniform modulus β of the mapping $B(x) + \omega(s)$, we show that the solution σ of the globally stochastic system of linear equations with respect to A, B and ω satisfies the inequality (4.2) in Theorem 4.1. Since A is a diagonal matrix, then A^{-1} is also a diagonal matrix, which satisfies

$$\left\|A^{-1}\right\|_{\text{op}} = \frac{1}{\|A\|_{\min}} = \frac{1}{\hat{\alpha}(A, x, A(x))}, \quad \text{for all } x \in \mathbb{R}^n. \tag{4.6}$$

Then, for any $x \in \mathbb{R}^n$ and α with $\|B\|_{\mathrm{op}} < \alpha < 0.5 = \hat{\alpha}(A,x,A(x))$, we obtain by (4.6) that

$$\begin{aligned}\|\sigma(\mathrm{s}) - x\|_2 &= \left\|(\sigma(\mathrm{s})B + \omega(\mathrm{s}) - xA)A^{-1}\right\|_2 \\ &\leq \frac{\|\sigma(\mathrm{s})B + \omega(\mathrm{s}) - xA\|_2}{\hat{\alpha}(A,x,A(x))} \\ &\leq \frac{\|\sigma(\mathrm{s})B - xB\|_2 + \|A(x) - (B(x) + \omega(s))\|_2}{\hat{\alpha}(A,x,A(x))} \\ &\leq \frac{\|B\|_{\mathrm{op}}\|\sigma(\mathrm{s}) - x\|_2 + \|A(x) - (B(x) + \omega(s))\|_2}{\hat{\alpha}(A,x,A(x))}.\end{aligned}$$

By $0 < \|B\|_{\mathrm{op}} < \hat{\alpha}(A,x,A(x))$, one has

$$\|\sigma(\mathrm{s}) - x\|_2 \left(1 - \frac{\|B\|_{\mathrm{op}}}{\hat{\alpha}(A,x,A(x))}\right) \leq \frac{\|A(x) - (B(x) + \omega(s))\|_2}{\bar{\alpha}(A,x,A(x))}$$

For any α with $\|B\|_{\mathrm{op}} < \alpha < \hat{\alpha}(A,x,A(x))$, one sees that

$$\|\sigma(\mathrm{s}) - x\|_2 \leq \frac{\|A(x) - (B(x) + \omega(s))\|_2}{\hat{\alpha}(A,x,A(x)) - \|B\|_{\mathrm{op}}} \leq \frac{\|A(x) - (B(x) + \omega(s))\|_2}{\alpha - \|B\|_{\mathrm{op}}}, \quad \text{for all } s \in S$$

This shows that the solution σ of the globally stochastic system of linear equations with respect to A, B and ω satisfies the inequality (4.2) in Theorem 4.1.

Next, by the properties of covering constants and operator norms of linear and continuous mappings, without applying the AMZ theorem, we directly prove the existence of solutions to some globally stochastic system of linear equations (4.1) in $\mathbb{R}^n$. However, it is not guaranteed for the solution to satisfy inequality (4.2) in Theorem 4.1.

Proposition 4.1. *Let n be a positive integer and let (S,τ,μ) be a probability space. Let $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$ be real $n \times n$ matrices. Let $\omega : S \to \mathbb{R}^n$ be a single-valued mapping. Let $\bar{s} \in S$. Suppose that A and B satisfy $\|B\|_{\mathrm{op}} < \bar{\alpha}(A,x,A(x))$, for all $x \in \mathbb{R}^n$. Then, the globally stochastic systems of linear equations with respect to A, B and ω has a global solution σ satisfying (4.1).*

Proof. For every $x \in \mathbb{R}^n$, by Proposition 3.1, we calculate

$$\min\{\|(A-B)(u)\|_n : \|u\|_n = 1\} \geq \hat{\alpha}(A,x,A(x)) - \|B\|_{\mathrm{op}} > 0,$$

which implies that $\det(A-B) \neq 0$. Letting $\sigma(s) = \omega(s)(A-B)^{-1}$ for all $s \in S$, we obtain that $A(\sigma(s)) = B(\sigma(s)) + \omega(s)$, for all $s \in S$. □

Similarly, to Example 4.3, we provide a counterexample.

Example 4.4. Let n be a positive integer. Let $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$ be real $n \times n$ diagonal matrices. Let (S,τ,μ) be a topological probability space with $S = [0,1]$. Let $\omega : S \to \mathbb{R}^n$ be a single-valued mapping defined by $\omega(s) = \big(\omega_1(s), \omega_2(s), \ldots, \omega_n(s)\big)$, for all $s \in S$, which is a continuous single-valued mapping. Suppose that A and B satisfy

$$\begin{aligned}0 &< \min\{|a_{ii}| : i = 1,2,\ldots,n\} \leq \max\{|a_{ii}| : i = 1,2,\ldots,n\} \\ &< \min\{|b_{ii}| : i = 1,2,\ldots,n\} \leq \min\{|b_{ii}| : i = 1,2,\ldots,n\} \leq 1.\end{aligned}$$

That is, $0 < \hat{\alpha}(A,x,A(x)) \leq \|A\|_{\mathrm{op}} < \|B\|_{\min} \leq \|B\|_{\mathrm{op}} \leq 1$, which implies that A, B do not satisfy all of the conditions in Theorem 4.1. However, $A - B$ is also an $n \times n$ diagonal matrix with all

elements $a_{ii}-b_{ii}\neq 0$, for $i=1,2,\ldots,n$. Hence the linear and continuous mapping $A-B$ is a pointwise multiplication operator on $\mathbb{R}^n$. For $\omega(s)=\big(\omega_1(s),\omega_2(s),\ldots,\omega_n(s)\big)$, we obtain a global solution $\sigma(s)=\big(\sigma_1(s),\sigma_2(s),\ldots,\sigma_n(s)\big)$ such that

$$\sigma_i(s)=\frac{1}{a_{ii}-b_{ii}}\,\omega_i(s),\quad \text{for all } i=1,2,\ldots,n, s\in S.$$

That is, $\sigma(s)=(A-B)^{-1}\omega(s)$, for every $s\in S$, which implies $A(\sigma(s))=B(\sigma(s))+\omega(s)$ for all $s\in S$. Hence, σ is a solution to the globally stochastic systems of linear equations with respect to A,B and ω. Next, with $\|B\|_{\rm op}$ being the uniform modulus β of the mapping $B(x)+\omega(s)$, we show that the solution σ does not satisfy the inequality (4.2) in Theorem 4.1. Similarly, to Example 4.3, for any $x\in\mathbb{R}^n$, by $\hat{\alpha}(A,x,A(x))<\|B\|_{\rm op}$, we have

$$\begin{aligned}\|\sigma(\mathrm{s})-x\|_2 &\leq \left\|A^{-1}\right\|_{\rm op}\|\sigma(\mathrm{s})B+\omega(\mathrm{s})-xA\|_2\\ &=\frac{\|\sigma(\mathrm{s})B+\omega(\mathrm{s})-xA\|_2}{\hat{\alpha}(A,x,A(x))}\not\leq\frac{\|A(x)-(B(x)+\omega(s))\|_2}{\hat{\alpha}(A,x,A(x))-\|B\|_{\rm op}},\quad \text{for all } s\in S.\end{aligned}$$

5. Conclusions and Remarks

In this paper, we used the AMZ theorem to prove the existence of some locally or globally stochastic systems of linear equations with respect to some linear and continuous mappings in $\mathbb{R}^n$ and some noise $\omega:S\to\mathbb{R}^n$. The considered linear and continuous mappings are induced by some $n\times n$ matrices. In general, we find that when we apply the AMZ theorem to some Asplund Banach spaces, it is difficult to calculate the covering constant of the considered mapping F and the uniform modulus β for the considered mapping G. However, in particular, in the special space $\mathbb{R}^n$, if both A and B are linear and continuous mappings, then we proved that the covering constant of the mapping A is $\bar{\alpha}(A,x,A(x))=\|A\|_{\min}=\sqrt{\text{the smallest eigenvalue of } A^TA}$ for all $x\in\mathbb{R}^n$. Meanwhile, the uniform modulus β for the mapping $B+\omega$ is

$$\beta=\|B\|_{\rm op}=\sqrt{\text{the largest eigenvalue of } B^TB}.$$

Based on the method of Lagrange Multiplier, we obtained

(a) The formulas to precisely calculate $\|A\|_{\rm op}$ and $\|A\|_{\min}$ in $\mathbb{R}^2$.
(b) The exact points on $\mathbb{S}_2$, at which $\|A\|_{\rm op}$ and $\|A\|_{\min}$ respectively reach their values.

In $\mathbb{R}^n$, it is very difficult to calculate the eigenvalues of A^TA for $n\times n$ matrix A. In comparing the case $\mathbb{R}^2$, we have the following questions for interested authors to consider.

Question Q1. Let n be a positive integer. Let $A=\big(a_{ij}\big)_{i,j=1}^n$ be a real $n\times n$ matrix that defines a linear and continuous operator $A:\mathbb{R}^n\to\mathbb{R}^n$ by (3.1). Find

(a) The formulas to precisely calculate $\|A\|_{\rm op}$ and $\|A\|_{\min}$ in $\mathbb{R}^n$.
(b) The exact points on $\mathbb{S}_n$, at which $\|A\|_{\rm op}$ and $\|A\|_{\min}$ respectively reach their values.

This is indeed an optimization problem in $\mathbb{R}^n$ for linear and continuous mappings. We believe that these formulas are useful in optimization theory.

Question Q2. For any linear and continuous mapping A in $\mathbb{R}^2$, from Proposition 3.2, the two points on the unit sphere in $\mathbb{R}^2$ at which $\|A\|_{\rm op}$ reaches are symmetric to the origin. It is also true for $\|A\|_{\min}$. Does this property hold in $\mathbb{R}^n$, in general?

Question Q3. Does the perpendicular property of linear and continuous mappings, which was proved in Corollary 3.4, hold in $\mathbb{R}^n$, in general?

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