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CONVEX REFORMULATION OF A 1R1C SMART THERMOSTAT THERMAL MODEL

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Abstract. We study the mathematical foundations of the least-squares parameter identification problem for the smart thermostat thermal model introduced by Arias et al. (*Appl. Math. Model.*, 2022). This model depends on four parameters: heat capacity, thermal resistance, effective boiler power, and a solar radiation factor. We generalize previous results on existence, stability, and differentiability from continuous to essentially bounded data by considering Carathéodory-type solutions. Furthermore, we rigorously justify the need for regularization to avoid spurious multiple solutions. To address this, we establish a reformulation in terms of three parameters that retains physical interpretability, and we prove existence, stability, and differentiability results for it. In particular, we obtain an explicit characterization of the Hessian of the reformulated objective map. As a consequence, we prove its strict convexity over a range of regularization parameters, thereby guaranteeing uniqueness for the identification problem. We rigorously show how, under a compatibility condition between both models, this result leads to uniqueness, or improved approximation when combined with a priori information, for the general four-parameter model. Finally, following a first-optimize-then-discretize approach, numerical experiments with synthetic data and with semisynthetic data based on real measurements confirm our theoretical findings and demonstrate the superior performance of the reformulated three-parameter model.

Keywords. Ordinary differential equations; Parameter estimation; Thermal building models; Thermal equivalent circuits.

1. INTRODUCTION

Building thermal models can be divided into three categories: white-box models, which provide an accurate physical description of the system; black-box models, which rely on empirical parameters; and grey-box models, the class considered here, which incorporate physical characteristics of the building through parameters estimated from measurements (see [1, 2, 13] for an overview). In grey-box models, the heat dynamics of a building can be modelled by a set of differential equations incorporating measurements of indoor and exterior temperatures, HVAC

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(Heating, Ventilation, and Air Conditioning) systems, electric appliance use, and climate conditions. Physical characteristics of the building appear as parameters to be estimated.

Following the electric-thermal analogy [15], such models can be represented by electric circuits of varying complexity (see, e.g., [4, 7, 14] and references therein). In this analogy, voltage corresponds to temperature and electric current to thermal flux, while capacitances and resistances represent the thermal capacity and resistance between different physical entities such as walls, sensors, furniture, and appliances.

In this paper, we study a variant of the simplest model of this kind, named 1R1C because its equivalent electric circuit contains one resistor and one capacitor. This model considers the building as a single entity, with no separation into different walls and rooms (see [4, 9, 17] for different applications of this model). The variant we consider was introduced in [3] for a real-world application where information comes from a single smart thermostat providing the indoor temperature and heater usage, while additional weather conditions such as exterior temperature and solar radiation are incorporated via web data servers. More specifically, our model of interest is described by the following ordinary differential equation over a time interval:

$$\begin{cases} \dfrac{dT}{dt}(t) + \dfrac{1}{RC}T(t) = \dfrac{1}{RC}T^e(t) + \dfrac{1}{C}(Pu(t) + AI^c(t)), & \text{for a.e. } t \in (t_0, t_f), \\ T(t_0) = T_0. \end{cases} \tag{1.1}$$

Here, the functions $T(\cdot)$ and $T^e(\cdot)$ [°C] represent the indoor (inside the building) and exterior temperatures, respectively, while $u(\cdot)$ [dimensionless] and $I^c(\cdot)$ [W/m^2] correspond to the heater usage time fraction and the solar radiation, respectively. The parameters C [J/°C] and R [°C/W] correspond to the heat capacity of the entire building and the thermal resistance from the interior to the ambient environment, respectively. The constant P [W] corresponds to the effective boiler power, and A [m^2] measures the impact of solar radiation on the heating of the building.

Given the simplicity of the model, one can expect significant uncertainty in its parameters. On the other hand, the availability of sufficient data allows us to consider data-driven methods. In this direction, a simulation algorithm was proposed in [3] to identify the parameters and simulate the temperature following a data-driven approach. In that paper, the parameters were statistically estimated, assuming a normal distribution. In this way, the parameters in (1.1) were treated as random variables, leading to a random ordinary differential equation (RODE) [11, 16], which was used to simulate the temperature.

In this paper, our focus is on the mathematical aspects of the identification model proposed in [3], which is a fundamental part of the prediction algorithm established in that work. Based on this analysis, we propose several improvements to the model, which are illustrated by numerical experiments. In Section 2, we study a least-squares framework, initially developed in [3, Section 3], to identify the four parameters in (1.1). We generalize the results from continuous to essentially bounded data by considering Carathéodory-type solutions. Accordingly, we extend the results of [3, Section 3] on the Lipschitz continuity and differentiability of the parameter-to-state map (Theorems 2.1 and 2.3) and on the solvability and differentiability of the identification problem (Theorems 2.2 and 2.4).

Furthermore, we add a regularization term to the model, which was not considered previously. We justify the need for regularization by showing that the unregularized model admits multiple solutions; in fact, each solution generates a one-parameter family of solutions (Proposition 2.1),

whereas, when regularization is added, the objective map has a unique minimizer within each such family (Proposition 2.2). In Section 3, we then recast the parameters to obtain a new three-parameter model and show that it possesses better mathematical properties while retaining its physical interpretation. Following a similar scheme and under the same hypotheses on the data, we prove the existence of optimal solutions for this model (Theorem 3.2). In this case, in addition to first-order differentiability (Theorem 3.3), we also study second-order derivatives (Theorem 3.4). With these results, we obtain explicit formulas for the Hessian of the objective map (Theorem 3.5). Consequently, we prove the strict convexity of the objective map and the uniqueness of the optimal parameter for a specific range of values of the regularization parameter (Theorem 3.6).

This also has consequences for the uniqueness of the four-parameter model over the same range of regularization parameters, as studied in Section 3.1. In particular, we show that, under a compatibility condition between both models and for a suitable regularization parameter, *a priori* knowledge of any one of the four parameters transfers to the other three (Proposition 3.1). Consequently, exact knowledge of one parameter determines the other three, thereby guaranteeing the uniqueness of the solution of the four-parameter model (Corollary 3.1). In Section 4, we consider two specific examples: an analytical one and a semisynthetic one, in which synthetic data samples are obtained by solving the model numerically with prescribed parameters and real thermal and meteorological data. Here, we adopt a first-optimize-then-discretize approach, as opposed to the discretize-then-optimize approach considered in [3, Section 5]. For both examples, the numerical results confirm the much better behaviour of the three-parameter model, relative to the four-parameter model, in terms of identifiability and computational efficiency, as well as the necessity of regularization in both models and the implicit connection between them. We conclude with some final remarks in Section 5.

Notation. In the sequel, for a compact interval $\mathscr{I} \subset \mathbb{R}$, we denote by $C(\mathscr{I})$ the space of continuous real-valued functions defined on $\mathscr{I}$, endowed with the supremum norm $\|f\|_{C(\mathscr{I})} = \sup_{x\in\mathscr{I}} |f(x)|$, while $L^p(\mathscr{I})$, $1 \le p \le \infty$, denotes the space of (equivalence classes of) Lebesgue measurable functions that are p-th power integrable if $1 \le p < \infty$ and essentially bounded if $p = \infty$, endowed with the norm

$$\|f\|_{L^p(\mathscr{I})} = \begin{cases} \left(\int_{\mathscr{I}} |f(s)|^p \, ds\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{s\in\mathscr{I}} |f(s)| & \text{if } p = \infty. \end{cases}$$

Finally, $\|\cdot\|_{\mathbb{R}^n}$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

2. The four-parameter (4P) model

In the following, we consider a generalization of the identification model introduced in [3]. In particular, we add a regularization term to the objective function and consider a Carathéodory-type solution [10, Section I.5], which allows for less regularity in the data. Considering specific time intervals of the form $\mathscr{I} = [t_0, t_f]$, we define the following nonlinear regularized least-squares optimization problem:

$$\text{minimize } J(R,C,P,A) = \frac{1}{2}\int_{t_0}^{t_f} (T(s) - z(s))^2 \, ds + \frac{\kappa}{2}(R^2 + C^2 + P^2 + A^2), \tag{2.1}$$

where $\kappa \geq 0$ is a regularization parameter, $z \in L^\infty(\mathscr{I})$ is a measurement of the indoor temperature, and T is the solution to the following equation in the Carathéodory sense:

$$\begin{cases} \dfrac{dT}{dt}(t) + \dfrac{1}{RC}T(t) = \dfrac{1}{RC}T^e(t) + \dfrac{1}{C}(Pu(t) + AI^c(t)), & \text{for a.e. } t \in (t_0, t_f), \\ T(t_0) = T_0. \end{cases} \tag{2.2}$$

Here, $T_0 \in \mathbb{R}$ is the initial temperature. We also assume appropriate lower and upper bounds (R^l, C^l, P^l, A^l) and (R^u, C^u, P^u, A^u), such that

$$R^l \leq R \leq R^u, \;\; C^l \leq C \leq C^u, \;\; P^l \leq P \leq P^u, \;\; A^l \leq A \leq A^u.$$

For notational simplicity, we set $\mathbf{a} = (a_1, a_2, a_3, a_4) = (R, C, P, A)$, $\mathbf{a}^l = (R^l, C^l, P^l, A^l)$, and $\mathbf{a}^u = (R^u, C^u, P^u, A^u)$, and define the feasible set $\mathbf{A} \subset \mathbb{R}^4$ by

$$\mathbf{A} = \{\mathbf{a} \in \mathbb{R}^4 \colon \mathbf{a}^l \leq \mathbf{a} \leq \mathbf{a}^u\},$$

where the symbol $\leq$ is understood componentwise. We also assume that the resistance and capacitance are strictly positive, $a_i^l > 0$ for every $i \in \{1, 2\}$, while the power and effective surface terms are non-negative, $a_i^l \geq 0$ for $i \in \{3, 4\}$. Following this notation, the state equation of our model is the following initial value problem (IVP):

$$\begin{cases} \dfrac{dT}{dt}(t) + \dfrac{1}{a_1 a_2}T(t) = \dfrac{1}{a_1 a_2}T^e(t) + \dfrac{1}{a_2}(a_3 u(t) + a_4 I^c(t)), & \text{for a.e. } t \in (t_0, t_f), \\ T(t_0) = T_0. \end{cases} \tag{2.3}$$

Assuming T^e, u, $I^c \in L^\infty(\mathscr{I})$, which is a realistic hypothesis, the Carathéodory conditions are satisfied and there exists a unique solution to (2.3), which is given by the variation of constants formula:

$$T(\mathbf{a}; t) = e^{\frac{-1}{a_1 a_2}(t - t_0)} T_0 + \int_{t_0}^{t} e^{\frac{1}{a_1 a_2}(s - t)} \left(\frac{1}{a_1 a_2} T^e(s) + \frac{1}{a_2}(a_3 u(s) + a_4 I^c(s)) \right) ds. \tag{2.4}$$

For each $\mathbf{a} \in \mathbf{A}$, the corresponding unique solution $T_{\mathbf{a}} := T(\mathbf{a}; \cdot)$ is absolutely continuous, and we can define an operator $T : \mathbf{A} \subset \mathbb{R}^4 \to C(\mathscr{I})$ such that, for every $\mathbf{a} \in \mathbf{A}$, $T(\mathbf{a}) = T_{\mathbf{a}}$ is the solution to (2.3). We can rewrite problem (2.1) as

$$\left(\mathscr{Q}^{4p}_{(t_0, t_f); \kappa; (\mathbf{a}^l, \mathbf{a}^u)} \right) : \underset{\mathbf{a} \in \mathbf{A}}{\text{minimize}} \; J^\kappa(\mathbf{a}) := \frac{1}{2} \int_{t_0}^{t_f} (T_{\mathbf{a}}(s) - z(s))^2 \, ds + \frac{\kappa}{2} \|\mathbf{a}\|^2_{\mathbb{R}^4},$$

where, for convenience, we make explicit the dependence of the problem on the time interval, regularization parameter, and bounds.

Theorem 2.1. *Assume that T^e, u, $I^c \in L^\infty(\mathscr{I})$. Then, for every $\mathbf{a} \in \mathbf{A}$, there exists a unique solution $T(\mathbf{a}; \cdot)$ to (2.3) given by formula (2.4). Moreover, the parameter-to-state map $T : \mathbf{A} \subset \mathbb{R}^4 \to C(\mathscr{I})$ is Lipschitz continuous on $\mathbf{A}$; that is, there exists a constant $L_T > 0$ such that*

$$\|T(\mathbf{a}) - T(\mathbf{b})\|_{C(\mathscr{I})} \leq L_T \|\mathbf{a} - \mathbf{b}\|_{\mathbb{R}^4}, \text{ for every } \mathbf{a}, \mathbf{b} \in \mathbf{A}. \tag{2.5}$$

Proof. For a fixed $\mathbf{a} \in \mathbf{A}$, equation (2.3) corresponds to a simple nonautonomous linear equation of the form

$$T'_{\mathbf{a}}(t) = f(t, T_{\mathbf{a}}(t); \mathbf{a}), \;\; T_{\mathbf{a}}(t_0) = T_0, \tag{2.6}$$

where

$$f(t,T;\mathbf{a}) := -\frac{1}{a_1 a_2}T + \frac{1}{a_1 a_2}T^e(t) + \frac{1}{a_2}(a_3 u(t) + a_4 I^c(t)).$$

The function f satisfies the Carathéodory conditions, and by [10, Theorems I.5.1, I.5.2, I.5.3] there exists a unique solution to (2.3), which satisfies (2.4). To prove the Lipschitz continuity, we adapt the proof given in [3, Theorem 3.1]; for the sake of completeness, we give a detailed proof. To that end, we consider the equivalent integral representation of (2.6):

$$T_{\mathbf{a}}(t) = T_0 + \int_{t_0}^{t} f(s, T_{\mathbf{a}}(s);\mathbf{a})\, ds.$$

Given $\mathbf{a},\mathbf{b} \in \mathbf{A}$, by simple algebraic manipulations, we have

$$\begin{aligned} |T_{\mathbf{a}}(t) - T_{\mathbf{b}}(t)| &\le \int_{t_0}^{t} |f(s,T_{\mathbf{a}}(s);\mathbf{a}) - f(s,T_{\mathbf{b}}(s);\mathbf{b})|\, ds \\ &\le \int_{t_0}^{t} |f(s,T_{\mathbf{a}}(s);\mathbf{a}) - f(s,T_{\mathbf{a}}(s);\mathbf{b})|\, ds + \int_{t_0}^{t} |f(s,T_{\mathbf{a}}(s);\mathbf{b}) - f(s,T_{\mathbf{b}}(s);\mathbf{b})|\, ds. \end{aligned} \tag{2.7}$$

On the one hand,

$$\begin{aligned} |f(s,T_{\mathbf{a}}(s);\mathbf{a}) - f(s,T_{\mathbf{a}}(s);\mathbf{b})| \le{}& (|T_{\mathbf{a}}(s)| + |T^e(s)|)\,|h_1(\mathbf{a}) - h_1(\mathbf{b})| \\ &+ |u(s)|\,|h_2(\mathbf{a}) - h_2(\mathbf{b})| + |I^c(s)|\,|h_3(\mathbf{a}) - h_3(\mathbf{b})|, \end{aligned} \tag{2.8}$$

where

$$h_1(\mathbf{a}) := -\frac{1}{a_1 a_2},\quad h_2(\mathbf{a}) := \frac{a_3}{a_2},\quad h_3(\mathbf{a}) := \frac{a_4}{a_2}.$$

Each function h_i is continuously differentiable and thus Lipschitz continuous on the compact convex set $\mathbf{A}$ by a known result (see [18, Lemma 3.1]). Therefore, there exists a constant $L_{h_i} > 0$ such that

$$|h_i(\mathbf{a}) - h_i(\mathbf{b})| \le L_{h_i} \|\mathbf{a} - \mathbf{b}\|_{\mathbb{R}^4}, \quad \text{for every } i = 1,2,3. \tag{2.9}$$

On the other hand, the data functions T^e, u, and I^c are essentially bounded by hypothesis, and, by (2.4), the solutions $T_{\mathbf{a}}$ are bounded on $\mathscr{I}$ uniformly with respect to $\mathbf{a} \in \mathbf{A}$, since $0 < e^{\frac{1}{a_1 a_2}(s-t)} \le 1$ for $s \le t$, $a_1 a_2 \ge a_1^l a_2^l$, $a_2 \ge a_2^l$, $a_3 \le a_3^u$, and $a_4 \le a_4^u$. Hence, we can find a constant $D > 0$, independent of t and of $\mathbf{a} \in \mathbf{A}$, such that

$$\operatorname*{ess\,sup}_{t\in\mathscr{I}} \max\{|T_{\mathbf{a}}(t)| + |T^e(t)|, |u(t)|, |I^c(t)|\} \le D. \tag{2.10}$$

Therefore, using (2.9) and (2.10) in (2.8) and integrating, we have

$$\int_{t_0}^{t} |f(s,T_{\mathbf{a}}(s);\mathbf{a}) - f(s,T_{\mathbf{a}}(s);\mathbf{b})|\, ds \le M(t_f - t_0)\, \|\mathbf{a} - \mathbf{b}\|_{\mathbb{R}^4}, \tag{2.11}$$

with $M := D\,(L_{h_1} + L_{h_2} + L_{h_3})$. Furthermore,

$$|f(s,T_{\mathbf{a}}(s);\mathbf{b}) - f(s,T_{\mathbf{b}}(s);\mathbf{b})| \le \frac{1}{b_1 b_2}\,|T_{\mathbf{a}}(s) - T_{\mathbf{b}}(s)| \le \frac{1}{a_1^l a_2^l}\,|T_{\mathbf{a}}(s) - T_{\mathbf{b}}(s)|,$$

and integrating yields

$$\int_{t_0}^{t} |f(s,T_{\mathbf{a}}(s);\mathbf{b}) - f(s,T_{\mathbf{b}}(s);\mathbf{b})|\, ds \le \frac{1}{a_1^l a_2^l}\int_{t_0}^{t} |T_{\mathbf{a}}(s) - T_{\mathbf{b}}(s)|\, ds. \tag{2.12}$$

Substituting inequalities (2.11) and (2.12) into (2.7), we obtain

$$|T_{\mathbf{a}}(t)-T_{\mathbf{b}}(t)| \le M(t_f-t_0)\,\|\mathbf{a}-\mathbf{b}\|_{\mathbb{R}^4} + \frac{1}{a_1^l a_2^l}\int_{t_0}^{t} |T_{\mathbf{a}}(s)-T_{\mathbf{b}}(s)|\,ds.$$

Applying Gronwall's lemma (see [18, Lemma 3.3]), we get

$$|T_{\mathbf{a}}(t)-T_{\mathbf{b}}(t)| \le M(t_f-t_0)\exp\left[\frac{t-t_0}{a_1^l a_2^l}\right]\|\mathbf{a}-\mathbf{b}\|_{\mathbb{R}^4}.$$

By defining $L_T := M(t_f-t_0)\exp\left[\frac{t_f-t_0}{a_1^l a_2^l}\right]$, we finally have

$$\|T(\mathbf{a})-T(\mathbf{b})\|_{C(\mathscr{I})} = \max_{t\in\mathscr{I}} |T_{\mathbf{a}}(t)-T_{\mathbf{b}}(t)| \le L_T\,\|\mathbf{a}-\mathbf{b}\|_{\mathbb{R}^4}.$$

□

From the continuity of T, we infer the continuity of J^{κ}. Since $\mathbf{A}$ is nonempty and compact, the Weierstrass theorem ensures that there exists at least one solution to $\left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\mathbf{a}^l,\mathbf{a}^u)}\right)$.

Theorem 2.2. *Problem* $\left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\mathbf{a}^l,\mathbf{a}^u)}\right)$ *is solvable, that is, it admits at least one minimizer.*

Remark 2.1. Theorems 2.1 and 2.2 extend [3, Theorems 3.1 and 3.2], respectively, to L^∞ data and Carathéodory-type solutions.

2.1. **Differentiability properties.** In the following, we study the differentiability of the parameter-to-state map.

Theorem 2.3. *Assume that* T^e, u, $I^c \in L^\infty(\mathscr{I})$. *Then, for each* $\mathbf{a}\in\mathrm{int}(\mathbf{A}) = \{\mathbf{a}\in\mathbb{R}^4 : a_i^l < a_i < a_i^u,\ i=1,\dots,4\}$, *the map* $T:\mathbf{A}\subset\mathbb{R}^4\to C(\mathscr{I})$ *is differentiable at* $\mathbf{a}$. *Moreover, for every direction* $\mathbf{v}\in\mathbb{R}^4$, *the directional derivative* $x = T'(\mathbf{a};\mathbf{v})$ *is the unique solution to the following IVP:*

$$\begin{cases} x'(t)+\dfrac{1}{a_1a_2}x(t) &= -\dfrac{v_1}{a_1^2a_2}\left(T^e(t)-T(\mathbf{a};t)\right) - \dfrac{v_2}{a_1a_2^2}\left(T^e(t)-T(\mathbf{a};t)\right) \\ &\quad -\dfrac{v_2}{a_2^2}\left(a_3u(t)+a_4I^c(t)\right)+\dfrac{v_3}{a_2}u(t)+\dfrac{v_4}{a_2}I^c(t) \quad \text{for a.e. } t\in\mathscr{I}, \\ x(t_0)=0. \end{cases} \tag{2.13}$$

Proof. Let $\mathbf{a}\in\mathrm{int}(\mathbf{A})$ and $\mathbf{v}\in\mathbb{R}^4$ be arbitrary. We define an operator $g:\mathbf{A}\subset\mathbb{R}^4\to L^\infty(\mathscr{I})$ by the expression

$$g(\mathbf{b})(t) := h_1(\mathbf{b})T^e(t)+h_2(\mathbf{b})u(t)+h_3(\mathbf{b})I^c(t), \quad \text{for every } \mathbf{b}\in\mathbf{A},$$

where h_2 and h_3 are as in the proof of Theorem 2.1, while, in contrast with that proof, h_1 is now taken with a positive sign; that is, the functions $h_i:\mathbf{A}\subset\mathbb{R}^4\to\mathbb{R}$ are given by

$$h_1(\mathbf{b}) = \frac{1}{b_1b_2},\quad h_2(\mathbf{b}) = \frac{b_3}{b_2},\quad h_3(\mathbf{b}) = \frac{b_4}{b_2}.$$

The map g is differentiable at $\mathbf{a}$ with directional derivative $g'(\mathbf{a};\mathbf{v})\in L^\infty(\mathscr{I})$ given by

$$g'(\mathbf{a};\mathbf{v}) = \lim_{\varepsilon\to0^+}\frac{g(\mathbf{a}+\varepsilon\mathbf{v})-g(\mathbf{a})}{\varepsilon} = h_1'(\mathbf{a};\mathbf{v})T^e(\cdot)+h_2'(\mathbf{a};\mathbf{v})u(\cdot)+h_3'(\mathbf{a};\mathbf{v})I^c(\cdot). \tag{2.14}$$

Moreover, given a scalar $\varepsilon > 0$, we define the difference quotients

$$x_\varepsilon := \frac{T(\mathbf{a}+\varepsilon\mathbf{v}) - T(\mathbf{a})}{\varepsilon}, \quad g_\varepsilon(\mathbf{a},\mathbf{v}) := \frac{g(\mathbf{a}+\varepsilon\mathbf{v}) - g(\mathbf{a})}{\varepsilon}, \quad h_{i,\varepsilon}(\mathbf{a},\mathbf{v}) := \frac{h_i(\mathbf{a}+\varepsilon\mathbf{v}) - h_i(\mathbf{a})}{\varepsilon}.$$

Simple algebraic manipulation shows that x_ε is the solution to the following IVP:

$$\begin{cases} x_\varepsilon'(t) + \dfrac{1}{a_1 a_2} x_\varepsilon(t) = f_\varepsilon(\mathbf{a},\mathbf{v})(t) \quad \text{for a.e. } t \in \mathscr{I}, \\ x_\varepsilon(t_0) = 0, \end{cases} \tag{2.15}$$

where $f_\varepsilon(\mathbf{a},\mathbf{v}) := g_\varepsilon(\mathbf{a},\mathbf{v}) - h_{1,\varepsilon}(\mathbf{a},\mathbf{v})T(\mathbf{a}+\varepsilon\mathbf{v})$. Denoting $f_0(\mathbf{a};\mathbf{v}) := \lim_{\varepsilon\to 0^+} f_\varepsilon(\mathbf{a},\mathbf{v})$, this limit is well-defined pointwise a.e. and given by

$$f_0(\mathbf{a};\mathbf{v}) = h_1'(\mathbf{a};\mathbf{v})\,(T^e(\cdot) - T(\mathbf{a};\cdot)) + h_2'(\mathbf{a};\mathbf{v})u(\cdot) + h_3'(\mathbf{a};\mathbf{v})I^c(\cdot). \tag{2.16}$$

The solution to (2.15) is given by

$$x_\varepsilon(t) = \exp\left(-\frac{t-t_0}{a_1 a_2}\right) \int_{t_0}^{t} f_\varepsilon(\mathbf{a},\mathbf{v})(s) \exp\left(\frac{s-t_0}{a_1 a_2}\right) ds.$$

On the other hand, we define the map

$$x_0(t) = \exp\left(-\frac{t-t_0}{a_1 a_2}\right) \int_{t_0}^{t} f_0(\mathbf{a};\mathbf{v})(s) \exp\left(\frac{s-t_0}{a_1 a_2}\right) ds,$$

which corresponds to the solution of the following IVP in the Carathéodory sense:

$$\begin{cases} x_0'(t) + \dfrac{1}{a_1 a_2} x_0(t) = f_0(\mathbf{a};\mathbf{v})(t) \quad \text{for a.e. } t \in \mathscr{I}, \\ x_0(t_0) = 0. \end{cases} \tag{2.17}$$

By a straightforward computation, merging the exponentials yields

$$|x_0(t) - x_\varepsilon(t)| \le \int_{t_0}^{t} |f_0(\mathbf{a};\mathbf{v})(s) - f_\varepsilon(\mathbf{a},\mathbf{v})(s)| \exp\left(\frac{s-t}{a_1 a_2}\right) ds,$$

for any $t \in [t_0, t_f]$. Since $s \le t$ and $a_1 a_2 > 0$, the exponential term is bounded by 1, yielding

$$|x_0(t) - x_\varepsilon(t)| \le \int_{t_0}^{t_f} |f_0(\mathbf{a};\mathbf{v})(s) - f_\varepsilon(\mathbf{a},\mathbf{v})(s)|\, ds \quad \text{for every } t \in [t_0, t_f].$$

Since $\mathbf{a} \in \operatorname{int}(\mathbf{A})$, we have $\mathbf{a} + \varepsilon\mathbf{v} \in \mathbf{A}$ for $\varepsilon > 0$ small enough. For such ε, the data functions are essentially bounded, the functions $T(\mathbf{a}+\varepsilon\mathbf{v})$ are uniformly bounded (see the proof of Theorem 2.1), and the quotients $h_{i,\varepsilon}(\mathbf{a},\mathbf{v})$ are bounded by the Lipschitz continuity of the functions h_i; hence, the integrands are uniformly bounded. Since $f_\varepsilon(\mathbf{a},\mathbf{v})(\cdot) \to f_0(\mathbf{a};\mathbf{v})(\cdot)$ pointwise a.e., by the dominated convergence theorem (see [8, Theorem 4.2]), the right-hand side converges to zero:

$$\int_{t_0}^{t_f} |f_0(\mathbf{a};\mathbf{v})(s) - f_\varepsilon(\mathbf{a},\mathbf{v})(s)|\, ds \to 0,$$

proving that $\|x_0 - x_\varepsilon\|_{C(\mathscr{I})} \to 0$ as $\varepsilon \to 0^+$. This confirms that T is directionally differentiable, with the directional derivative $T'(\mathbf{a};\mathbf{v})$ given by the unique solution to equation (2.17). Therefore, (2.13) follows by expanding $f_0(\mathbf{a};\mathbf{v})$, taking into account (2.14).

Finally, since $|h_i'(\mathbf{a};\mathbf{v})| \le L\|\mathbf{v}\|_{\mathbb{R}^4}$ for every $i \in \{1,2,3\}$ and some $L > 0$, it follows that $|f_0(\mathbf{a};\mathbf{v})(s)| \le K\|\mathbf{v}\|_{\mathbb{R}^4}$ for a.e. $s \in \mathscr{I}$ and some constant $K > 0$. Consequently, the operator

$DT(\mathbf{a}) : \mathbb{R}^4 \to C(\mathscr{I})$ defined by $DT(\mathbf{a})(\mathbf{v}) := T'(\mathbf{a};\mathbf{v})$ is linear in $\mathbf{v}$, since $f_0(\mathbf{a};\mathbf{v})$ is linear in $\mathbf{v}$, and bounded, since

$$\begin{aligned}\|DT(\mathbf{a})(\mathbf{v})\|_{C(\mathscr{I})} &= \max_{t\in\mathscr{I}} |DT(\mathbf{a})(\mathbf{v})(t)| \\ &\leq \max_{t\in\mathscr{I}} \int_{t_0}^{t} |f_0(\mathbf{a};\mathbf{v})(s)| \exp\left(\frac{s-t}{a_1 a_2}\right) ds \\ &\leq K(t_f - t_0)\, \|\mathbf{v}\|_{\mathbb{R}^4}.\end{aligned}$$

Since the domain is finite-dimensional and T is Lipschitz continuous by Theorem 2.1, it follows by known results that T is also (Fréchet) differentiable at $\mathbf{a}$ (see, for example, [6, Section 2.2.1]). The proof is complete. □

Remark 2.2. Theorem 2.3 corrects the derivative formula given in [3, Theorem 3.3], which contained an error in the derivatives with respect to the parameters R and C, and extends it to L^∞ data and Carathéodory-type solutions.

As a consequence of Theorem 2.3, we can characterize the gradient of T at any point $\mathbf{a} = (R,C,P,A) \in \mathrm{int}(\mathbf{A})$. For notational simplicity, we denote

$$T_R = \frac{\partial T}{\partial R}, \quad T_C = \frac{\partial T}{\partial C}, \quad T_P = \frac{\partial T}{\partial P}, \quad T_A = \frac{\partial T}{\partial A},$$

so that

$$\nabla T(\mathbf{a}) = (T_R(\mathbf{a}), T_C(\mathbf{a}), T_P(\mathbf{a}), T_A(\mathbf{a})),$$

where the sensitivity functions are the solutions to the following IVPs:

$$\begin{cases} \dfrac{dT_R}{dt}(t) + \dfrac{1}{RC} T_R(t) = -\dfrac{1}{R^2 C}\left(T^e(t) - T(\mathbf{a};t)\right), & \text{for a.e. } t \in \mathscr{I}, \\ T_R(t_0) = 0, \end{cases}$$

$$\begin{cases} \dfrac{dT_C}{dt}(t) + \dfrac{1}{RC} T_C(t) = -\dfrac{1}{RC^2}\left(T^e(t) - T(\mathbf{a};t)\right) - \dfrac{1}{C^2}\left(Pu(t) + AI^c(t)\right), & \text{for a.e. } t \in \mathscr{I}, \\ T_C(t_0) = 0, \end{cases}$$

$$\begin{cases} \dfrac{dT_P}{dt}(t) + \dfrac{1}{RC} T_P(t) = \dfrac{1}{C} u(t), & \text{for a.e. } t \in \mathscr{I}, \\ T_P(t_0) = 0, \end{cases}$$

$$\begin{cases} \dfrac{dT_A}{dt}(t) + \dfrac{1}{RC} T_A(t) = \dfrac{1}{C} I^c(t), & \text{for a.e. } t \in \mathscr{I}, \\ T_A(t_0) = 0. \end{cases} \tag{2.18}$$

Let us denote by

$$J(\mathbf{a}) = \frac{1}{2}\int_{t_0}^{t_f} (T(\mathbf{a};s) - z(s))^2\, ds$$

the fitting term in $\left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\mathbf{a}^l,\mathbf{a}^u)}\right)$, so that $J^\kappa(\mathbf{a}) = J(\mathbf{a}) + \dfrac{\kappa}{2}\|\mathbf{a}\|^2_{\mathbb{R}^4}$. By the chain rule, the map J is differentiable at any $\mathbf{a} = (R,C,P,A) \in \mathrm{int}(\mathbf{A})$. Following the previous notation, the gradient

of J at $\mathbf{a}$ is denoted by $\nabla J(\mathbf{a}) = (J_R(\mathbf{a}), J_C(\mathbf{a}), J_P(\mathbf{a}), J_A(\mathbf{a}))$, and its partial derivatives are given by

$$\begin{aligned} J_R(\mathbf{a}) &= \int_{t_0}^{t_f} (T(\mathbf{a};s) - z(s)) T_R(\mathbf{a};s)\, ds, \\ J_C(\mathbf{a}) &= \int_{t_0}^{t_f} (T(\mathbf{a};s) - z(s)) T_C(\mathbf{a};s)\, ds, \\ J_P(\mathbf{a}) &= \int_{t_0}^{t_f} (T(\mathbf{a};s) - z(s)) T_P(\mathbf{a};s)\, ds, \\ J_A(\mathbf{a}) &= \int_{t_0}^{t_f} (T(\mathbf{a};s) - z(s)) T_A(\mathbf{a};s)\, ds. \end{aligned} \tag{2.19}$$

Theorem 2.4. *Assume that T^e, u, I^c, $z \in L^\infty(\mathscr{I})$. Then the map J^κ is differentiable at any $\mathbf{a} = (R,C,P,A) \in \operatorname{int}(\mathbf{A})$, with gradient given by $\nabla J^\kappa(\mathbf{a}) = \nabla J(\mathbf{a}) + \kappa \mathbf{a}$, where $\nabla J(\mathbf{a})$ is given by (2.19).*

2.2. **Identifiability properties.** In this section, we show that the unregularized model is, in general, not uniquely solvable, and we analyse the effect of a positive regularization parameter. Assume that $\bar{\mathbf{a}} = (\bar{R}, \bar{C}, \bar{P}, \bar{A})$ is a solution to problem $\left(\mathscr{Q}^{4p}_{(t_0,t_f);0;(\mathbf{a}^l,\mathbf{a}^u)}\right)$. Given $\alpha > 0$, we can define another parameter by

$$\bar{\mathbf{a}}^\alpha := (\bar{R}^\alpha, \bar{C}^\alpha, \bar{P}^\alpha, \bar{A}^\alpha) := \left(\alpha \bar{R}, \frac{\bar{C}}{\alpha}, \frac{\bar{P}}{\alpha}, \frac{\bar{A}}{\alpha}\right) \tag{2.20}$$

such that the products of parameters remain invariant,

$$\bar{R}^\alpha \bar{C}^\alpha = \bar{R}\bar{C},\ \bar{R}^\alpha \bar{P}^\alpha = \bar{R}\bar{P},\ \bar{R}^\alpha \bar{A}^\alpha = \bar{R}\bar{A},$$

and hence so do the ratios $\bar{P}^\alpha/\bar{C}^\alpha = \bar{P}/\bar{C}$ and $\bar{A}^\alpha/\bar{C}^\alpha = \bar{A}/\bar{C}$. Consequently, $T(\bar{\mathbf{a}}) = T(\bar{\mathbf{a}}^\alpha)$, and thus $J(\bar{\mathbf{a}}) = J(\bar{\mathbf{a}}^\alpha)$. On the other hand, to have feasibility, we require $\mathbf{a}^l \le \bar{\mathbf{a}}^\alpha \le \mathbf{a}^u$, which is equivalent to

$$\frac{R^l}{\bar{R}} \le \alpha \le \frac{R^u}{\bar{R}}, \frac{\bar{C}}{C^u} \le \alpha \le \frac{\bar{C}}{C^l}, \frac{\bar{P}}{P^u} \le \alpha \le \frac{\bar{P}}{P^l}, \frac{\bar{A}}{A^u} \le \alpha \le \frac{\bar{A}}{A^l},$$

with the convention that $\bar{P}/P^l = +\infty$ if $P^l = 0$, and $\bar{A}/A^l = +\infty$ if $A^l = 0$. Defining

$$\alpha^l := \max\left\{\frac{R^l}{\bar{R}}, \frac{\bar{C}}{C^u}, \frac{\bar{P}}{P^u}, \frac{\bar{A}}{A^u}\right\},\ \alpha^u := \min\left\{\frac{R^u}{\bar{R}}, \frac{\bar{C}}{C^l}, \frac{\bar{P}}{P^l}, \frac{\bar{A}}{A^l}\right\}, \tag{2.21}$$

we have by definition that $\alpha \in [\alpha^l, \alpha^u]$, with $0 < \alpha^l \le 1 \le \alpha^u$ (note that $\alpha^l \ge R^l/\bar{R} > 0$). If $\alpha^l < \alpha^u$, we can take $\alpha \ne 1$ in $[\alpha^l, \alpha^u]$; then $\bar{\mathbf{a}} \ne \bar{\mathbf{a}}^\alpha$, and $\bar{\mathbf{a}}^\alpha \in \mathbf{A}$ is another solution to the problem. Hence, $\left(\mathscr{Q}^{4p}_{(t_0,t_f);0;(\mathbf{a}^l,\mathbf{a}^u)}\right)$ is not uniquely solvable; instead, it has a continuum of solutions $\bar{\mathbf{A}}^\alpha := \{\bar{\mathbf{a}}^\alpha \in \mathbb{R}^4 : \alpha^l \le \alpha \le \alpha^u\}$. Furthermore, we can measure the impact on the identification error of obtaining $\bar{\mathbf{a}}^\alpha$ as a solution instead of $\bar{\mathbf{a}}$. By a direct computation, whenever $\bar{P}, \bar{A} > 0$, we have

$$\left|\frac{\bar{R}^\alpha - \bar{R}}{\bar{R}}\right| = |1 - \alpha|, \tag{2.22a}$$

$$\left|\frac{\bar{C}^\alpha - \bar{C}}{\bar{C}}\right| = \left|\frac{\bar{P}^\alpha - \bar{P}}{\bar{P}}\right| = \left|\frac{\bar{A}^\alpha - \bar{A}}{\bar{A}}\right| = \frac{|1-\alpha|}{\alpha}. \tag{2.22b}$$

Since $|1-\alpha| \leq \max\{\alpha^u-\alpha, \alpha-\alpha^l\} \leq \alpha^u-\alpha^l$, and similarly $\dfrac{|1-\alpha|}{\alpha} \leq \dfrac{\alpha^u-\alpha^l}{\alpha^l}$, we have the following estimates:

$$\begin{aligned} \max_{\alpha^l\leq\alpha\leq\alpha^u} \left|\frac{\bar{R}^\alpha-\bar{R}}{\bar{R}}\right| &\leq \alpha^u-\alpha^l, \\ \max_{\alpha^l\leq\alpha\leq\alpha^u} \left\{\left|\frac{\bar{C}^\alpha-\bar{C}}{\bar{C}}\right|, \left|\frac{\bar{P}^\alpha-\bar{P}}{\bar{P}}\right|, \left|\frac{\bar{A}^\alpha-\bar{A}}{\bar{A}}\right|\right\} &\leq \frac{\alpha^u-\alpha^l}{\alpha^l}. \end{aligned} \tag{2.23}$$

We summarize the above discussion in the following result.

Proposition 2.1. *Let α^l and α^u be as defined in (2.21), and let $\bar{\mathbf{a}}$ be a solution to problem $\left(\mathscr{Q}^{4p}_{(t_0,t_f);0;(\mathbf{a}^l,\mathbf{a}^u)}\right)$. If $\alpha^l<\alpha^u$, then every element of the set $\bar{\mathbf{A}}^\alpha$ is also a solution to the problem, and estimates (2.22) (whenever $\bar{P},\bar{A}>0$) and (2.23) hold.*

In particular, if there exists a solution in the interior of the feasible set, $\bar{\mathbf{a}} \in \mathrm{int}(\mathbf{A})$, then by definition $\alpha^l<1<\alpha^u$, and the problem is not uniquely solvable.

Corollary 2.1. *If problem $\left(\mathscr{Q}^{4p}_{(t_0,t_f);0;(\mathbf{a}^l,\mathbf{a}^u)}\right)$ has a solution in the interior of the feasible set, then it is not uniquely solvable.*

If $\kappa>0$, the situation improves. Although, in general, we cannot establish the uniqueness of the solution, we can at least prove that J^κ has a unique minimizer within $\bar{\mathbf{A}}^\alpha$. Indeed,

$$J^\kappa(\bar{\mathbf{a}}^\alpha) = J(\bar{\mathbf{a}}^\alpha) + \frac{\kappa}{2}\|\bar{\mathbf{a}}^\alpha\|^2_{\mathbb{R}^4} = J(\bar{\mathbf{a}}) + \frac{\kappa}{2}\left(\alpha^2\bar{a}_1^2 + \frac{1}{\alpha^2}\sum_{j=2}^4 \bar{a}_j^2\right) \quad \text{for every } \alpha^l\leq\alpha\leq\alpha^u. \tag{2.24}$$

By direct analysis, we can prove the following result.

Proposition 2.2. *If $\kappa>0$, then J^κ has a unique minimizer on $\bar{\mathbf{A}}^\alpha$. More precisely, defining $\alpha^* := \left(\sum_{j=2}^4 \bar{a}_j^2\right)^{\frac{1}{4}} \bar{a}_1^{-\frac{1}{2}}$, we have the following possibilities:*

- *If $\alpha^* \in [\alpha^l,\alpha^u]$, the unique minimizer of J^κ on $\bar{\mathbf{A}}^\alpha$ is $\bar{\mathbf{a}}^{\alpha^*}$.*
- *If $\alpha^*<\alpha^l$, the unique minimizer of J^κ on $\bar{\mathbf{A}}^\alpha$ is $\bar{\mathbf{a}}^{\alpha^l}$.*
- *If $\alpha^*>\alpha^u$, the unique minimizer of J^κ on $\bar{\mathbf{A}}^\alpha$ is $\bar{\mathbf{a}}^{\alpha^u}$.*

Proof. Let us define $g(\alpha) := \dfrac{\kappa}{2}\left(\alpha^2\bar{a}_1^2 + \dfrac{1}{\alpha^2}\sum_{j=2}^4 \bar{a}_j^2\right)$, which is the regularization term in (2.24) and the only term depending on α. Since $\alpha \mapsto \bar{\mathbf{a}}^\alpha$ is injective, minimizing J^κ over $\bar{\mathbf{A}}^\alpha$ is equivalent to

$$\underset{\alpha^l\leq\alpha\leq\alpha^u}{\text{minimize}}\ g(\alpha).$$

By a direct computation,

$$g'(\alpha) = \kappa\left(\alpha\bar{a}_1^2 - \frac{1}{\alpha^3}\sum_{j=2}^4 \bar{a}_j^2\right) = \frac{\kappa}{\alpha^3}\left(\alpha^4\bar{a}_1^2 - \sum_{j=2}^4 \bar{a}_j^2\right).$$

As $\alpha > 0$ by definition, we have $\dfrac{\kappa}{\alpha^3} > 0$ and $g'(\alpha) = 0 \Leftrightarrow \alpha^4 \bar{a}_1^2 - \sum_{j=2}^{4} \bar{a}_j^2 = 0$; thus, α^* is the unique critical point of g in $(0,\infty)$. Furthermore, $g'(\alpha) > 0$ (resp. $g'(\alpha) < 0$) if and only if $\alpha > \alpha^*$ (resp. $\alpha < \alpha^*$). Consequently, if $\alpha^* \in [\alpha^l, \alpha^u]$, then α^* is the unique global minimizer of g on $[\alpha^l, \alpha^u]$. If $\alpha^* < \alpha^l$, then g' is strictly positive on $[\alpha^l, \alpha^u]$, so g is strictly increasing there, and the unique minimizer is $\alpha = \alpha^l$. Similarly, if $\alpha^* > \alpha^u$, then g is strictly decreasing on $[\alpha^l, \alpha^u]$, and the unique minimizer is $\alpha = \alpha^u$. □

3. THE THREE-PARAMETER (3P) MODEL

In this section, we study a reduced (sub)model with three parameters given by

$$\mathbf{q} = (q_1, q_2, q_3) = \left(\frac{1}{RC}, \frac{P}{C}, \frac{A}{C}\right). \tag{3.1}$$

These parameters can also be expressed in physical terms. The first parameter $q_1 = 1/RC$ [s^{-1}] represents the inverse thermal time constant of the building, measuring how quickly the indoor temperature reacts to changes in external conditions, while $q_2 = P/C$ [°C/s] (heating rate) and $q_3 = A/C$ [m^2 °C/J] (area per heat capacity) indicate how effectively the heating system and the solar radiation, respectively, can change the indoor temperature. In this case, the feasible set $\mathbf{Q} \subset \mathbb{R}^3$ is given by

$$\mathbf{Q} = \{\mathbf{q} \in \mathbb{R}^3 : \mathbf{q}^l \leq \mathbf{q} \leq \mathbf{q}^u\},$$

where we assume $q_1^l > 0$ and $q_2^l, q_3^l \geq 0$. For a regularization parameter $\kappa \geq 0$, the corresponding optimization problem is given by

$$\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\mathbf{q}^l,\mathbf{q}^u)}\right) : \underset{\mathbf{q}\in\mathbf{Q}}{\text{minimize}}\ \hat{J}^{\kappa}(\mathbf{q}) := \frac{1}{2}\int_{t_0}^{t_f} (T_{\mathbf{q}}(s) - z(s))^2\, ds + \frac{\kappa}{2}\|\mathbf{q}\|^2_{\mathbb{R}^3},$$

where $T_{\mathbf{q}} = T_{\mathbf{q}}(t)$ is the solution to the following initial value problem:

$$\begin{cases} \dfrac{dT_{\mathbf{q}}}{dt}(t) + q_1 T_{\mathbf{q}}(t) = q_1 T^e(t) + q_2 u(t) + q_3 I^c(t), & \text{for a.e. } t \in (t_0, t_f), \\ T_{\mathbf{q}}(t_0) = T_0. \end{cases} \tag{3.2}$$

For this model, we can establish results analogous to those of the four-parameter model. Using the variation of constants formula, for every $\mathbf{q} \in \mathbf{Q}$, there exists a unique solution $T_{\mathbf{q}} = \hat{T}(\mathbf{q};\cdot)$ to (3.2), given by

$$\hat{T}(\mathbf{q};t) = e^{-q_1(t-t_0)} T_0 + \int_{t_0}^{t} e^{q_1(s-t)} \left(q_1 T^e(s) + q_2 u(s) + q_3 I^c(s)\right) ds \tag{3.3}$$

for every $t \in \mathscr{I}$. This allows us to define the parameter-to-state operator $\hat{T} : \mathbf{Q} \subset \mathbb{R}^3 \to C(\mathscr{I})$ such that, for every $\mathbf{q} \in \mathbf{Q}$, $\hat{T}(\mathbf{q}) = \hat{T}(\mathbf{q};\cdot)$ is the solution to (3.2). By reasoning analogous to the proof of Theorem 2.1, we can prove that this operator is also Lipschitz continuous on the feasible set.

Theorem 3.1. *Assume that T^e, u, $I^c \in L^\infty(\mathscr{I})$. Then, for every $\mathbf{q} \in \mathbf{Q}$, there exists a unique solution $\hat{T}(\mathbf{q})$ to (3.2). Moreover, the parameter-to-state map $\hat{T} : \mathbf{Q} \subset \mathbb{R}^3 \to C(\mathscr{I})$ is Lipschitz continuous on $\mathbf{Q}$; that is, there exists a constant $L_{\hat{T}} > 0$ such that*

$$\left\|\hat{T}(\mathbf{q}) - \hat{T}(\mathbf{r})\right\|_{C(\mathscr{I})} \leq L_{\hat{T}} \|\mathbf{q} - \mathbf{r}\|_{\mathbb{R}^3}, \text{ for every } \mathbf{q}, \mathbf{r} \in \mathbf{Q}. \tag{3.4}$$

As a consequence, by the Weierstrass theorem, we obtain the solvability of the problem.

Theorem 3.2. *Problem* $\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\mathbf{q}^l,\mathbf{q}^u)}\right)$ *is solvable, that is, it admits at least one minimizer.*

As before, we denote

$$\hat{T}_{q_1}(\mathbf{q};\cdot)=\frac{\partial \hat{T}}{\partial q_1}(\mathbf{q};\cdot),\quad \hat{T}_{q_2}(\mathbf{q};\cdot)=\frac{\partial \hat{T}}{\partial q_2}(\mathbf{q};\cdot),\quad \hat{T}_{q_3}(\mathbf{q};\cdot)=\frac{\partial \hat{T}}{\partial q_3}(\mathbf{q};\cdot).$$

In the following, we prove the differentiability of the operator $\hat{T}$ and calculate its partial derivatives by using formula (3.3).

Theorem 3.3. *Assume that* T^e, u, $I^c \in L^\infty(\mathscr{I})$. *Then, the map* $\hat{T}:\mathbf{Q}\subset\mathbb{R}^3\to C(\mathscr{I})$ *is continuously differentiable on* $\operatorname{int}(\mathbf{Q})$, *with partial derivatives determined by*

$$\frac{d\hat{T}_{q_1}}{dt}(\mathbf{q};t)+q_1\hat{T}_{q_1}(\mathbf{q};t)=T^e(t)-\hat{T}(\mathbf{q};t),\ \text{a.e. on } (t_0,t_f),\ \hat{T}_{q_1}(\mathbf{q};t_0)=0, \tag{3.5a}$$

$$\frac{d\hat{T}_{q_2}}{dt}(\mathbf{q};t)+q_1\hat{T}_{q_2}(\mathbf{q};t)=u(t),\ \text{a.e. on } (t_0,t_f),\ \hat{T}_{q_2}(\mathbf{q};t_0)=0, \tag{3.5b}$$

$$\frac{d\hat{T}_{q_3}}{dt}(\mathbf{q};t)+q_1\hat{T}_{q_3}(\mathbf{q};t)=I^c(t),\ \text{a.e. on } (t_0,t_f),\ \hat{T}_{q_3}(\mathbf{q};t_0)=0. \tag{3.5c}$$

Proof. In this case, we prove the result by differentiating formula (3.3) directly. Since the derivatives with respect to q_i of the integrands in (3.3) are bounded uniformly in $s,t\in\mathscr{I}$ and in $\mathbf{q}$ on bounded sets, the corresponding difference quotients converge uniformly in $t\in\mathscr{I}$, so the expressions below are partial derivatives of $\hat{T}$ as a map into $C(\mathscr{I})$. For the second and third parameters, a straightforward computation gives

$$\hat{T}_{q_2}(\mathbf{q};t)=\int_{t_0}^{t}e^{q_1(s-t)}u(s)\,ds, \tag{3.6}$$

$$\hat{T}_{q_3}(\mathbf{q};t)=\int_{t_0}^{t}e^{q_1(s-t)}I^c(s)\,ds, \tag{3.7}$$

which correspond to equations (3.5b) and (3.5c), respectively. For the first parameter, the calculation is a little more involved:

$$\begin{aligned}
\hat{T}_{q_1}(\mathbf{q};t)&=-(t-t_0)\,e^{-q_1(t-t_0)}T_0+\int_{t_0}^{t}(s-t)\,e^{q_1(s-t)}\left(q_1T^e(s)+q_2u(s)+q_3I^c(s)\right)ds\\
&\quad+\int_{t_0}^{t}e^{q_1(s-t)}T^e(s)\,ds\\
&=-t\left(e^{-q_1(t-t_0)}T_0+\int_{t_0}^{t}e^{q_1(s-t)}\left(q_1T^e(s)+q_2u(s)+q_3I^c(s)\right)ds\right)\\
&\quad+e^{-q_1(t-t_0)}t_0T_0+\int_{t_0}^{t}se^{q_1(s-t)}\left(q_1T^e(s)+q_2u(s)+q_3I^c(s)\right)ds\\
&\quad+\int_{t_0}^{t}e^{q_1(s-t)}T^e(s)\,ds\\
&=-t\hat{T}(\mathbf{q};t)+L(t)+\int_{t_0}^{t}e^{q_1(s-t)}T^e(s)\,ds,
\end{aligned} \tag{3.8}$$

where

$$L(t) := e^{-q_1(t-t_0)}t_0T_0 + \int_{t_0}^{t} se^{q_1(s-t)}\left(q_1T^e(s) + q_2u(s) + q_3I^c(s)\right)ds.$$

By the variation of constants formula, the function L satisfies the following equation:

$$\frac{dL}{dt}(t) + q_1L(t) = t\left(q_1T^e(t) + q_2u(t) + q_3I^c(t)\right), \text{ a.e. on } (t_0,t_f),\ L(t_0) = t_0T_0,$$

which, by (3.2), can be equivalently written as

$$\frac{dL}{dt}(t) + q_1L(t) = t\left(\frac{d\hat{T}}{dt}(\mathbf{q};t) + q_1\hat{T}(\mathbf{q};t)\right), \text{ a.e. on } (t_0,t_f),\ L(t_0) = t_0T_0.$$

Now, if we define the function $K(t) := t\hat{T}(\mathbf{q};t)$, applying the product rule and using the previous expression yields

$$\begin{aligned}\frac{dK}{dt}(t) + q_1K(t) &= \hat{T}(\mathbf{q};t) + t\frac{d\hat{T}}{dt}(\mathbf{q};t) + q_1t\hat{T}(\mathbf{q};t) = \hat{T}(\mathbf{q};t) + t\left(\frac{d\hat{T}}{dt}(\mathbf{q};t) + q_1\hat{T}(\mathbf{q};t)\right)\\ &= \hat{T}(\mathbf{q};t) + \frac{dL}{dt}(t) + q_1L(t),\end{aligned}$$

from which we obtain

$$\frac{d(K-L)}{dt}(t) + q_1(K-L)(t) = \hat{T}(\mathbf{q};t), \text{ a.e. on } (t_0,t_f),\ (K-L)(t_0) = 0.$$

From the variation of constants formula,

$$(K-L)(t) = \int_{t_0}^{t} e^{q_1(s-t)}\hat{T}(\mathbf{q};s)\,ds.$$

Therefore, using this expression in (3.8),

$$\begin{aligned}\hat{T}_{q_1}(\mathbf{q};t) &= -(K-L)(t) + \int_{t_0}^{t} e^{q_1(s-t)}T^e(s)\,ds\\ &= -\int_{t_0}^{t} e^{q_1(s-t)}\hat{T}(\mathbf{q};s)\,ds + \int_{t_0}^{t} e^{q_1(s-t)}T^e(s)\,ds,\end{aligned}$$

and from this,

$$\hat{T}_{q_1}(\mathbf{q};t) = \int_{t_0}^{t} e^{q_1(s-t)}\left(T^e(s) - \hat{T}(\mathbf{q};s)\right)ds, \tag{3.9}$$

which corresponds to equation (3.5a):

$$\frac{d\hat{T}_{q_1}}{dt}(\mathbf{q};t) + q_1\hat{T}_{q_1}(\mathbf{q};t) = T^e(t) - \hat{T}(\mathbf{q};t), \text{ a.e. on } (t_0,t_f),\ \hat{T}_{q_1}(\mathbf{q};t_0) = 0.$$

Finally, let us prove that the partial derivatives are continuous. Focusing on $\hat{T}_{q_1}$, we consider the integral expression (3.9). Let $\{\mathbf{q}^n\} \subset \text{int}(\mathbf{Q})$ be a sequence with $\mathbf{q}^n \to \mathbf{q} \in \text{int}(\mathbf{Q})$. By simple manipulation, we have

$$\begin{aligned}\left|\hat{T}_{q_1}(\mathbf{q}^n;t) - \hat{T}_{q_1}(\mathbf{q};t)\right| \le &\int_{t_0}^{t}\left|e^{q_1^n(s-t)} - e^{q_1(s-t)}\right|\left|T^e(s) - \hat{T}(\mathbf{q}^n;s)\right|ds\\ &+ \int_{t_0}^{t} e^{q_1(s-t)}\left|\hat{T}(\mathbf{q}^n;s) - \hat{T}(\mathbf{q};s)\right|ds.\end{aligned}$$

Since $s-t \leq 0$ for $s \in [t_0,t]$ and $q_1^n, q_1 > 0$, the mean value theorem gives

$$\left| e^{q_1^n(s-t)} - e^{q_1(s-t)} \right| \leq (t-s)\,|q_1^n - q_1| \leq (t_f - t_0)\,|q_1^n - q_1|\,,$$

and $e^{q_1(s-t)} \leq 1$. Moreover, by Theorem 3.1, $\sup_n \|\hat{T}(\mathbf{q}^n)\|_{C(\mathscr{I})} < \infty$. Hence, for every $t \in \mathscr{I}$,

$$\begin{aligned} \left|\hat{T}_{q_1}(\mathbf{q}^n;t) - \hat{T}_{q_1}(\mathbf{q};t)\right| \leq (t_f-t_0)^2 \left(\|T^e\|_{L^\infty(\mathscr{I})} + \sup_n \left\|\hat{T}(\mathbf{q}^n)\right\|_{C(\mathscr{I})} \right) |q_1^n - q_1| \\ + (t_f - t_0) \left\|\hat{T}(\mathbf{q}^n) - \hat{T}(\mathbf{q})\right\|_{C(\mathscr{I})}. \end{aligned} \tag{3.10}$$

Taking the supremum over $t \in \mathscr{I}$ in (3.10) and using Theorem 3.1 again, we obtain

$$\lim_{n\to\infty} \left\|\hat{T}_{q_1}(\mathbf{q}^n;\cdot) - \hat{T}_{q_1}(\mathbf{q};\cdot)\right\|_{C(\mathscr{I})} = 0.$$

This proves that $\dfrac{\partial \hat{T}}{\partial q_1} : \mathrm{int}(\mathbf{Q}) \subset \mathbb{R}^3 \to C(\mathscr{I})$ is a continuous operator. In the same way, we can prove the continuity of $\dfrac{\partial \hat{T}}{\partial q_2}$ and $\dfrac{\partial \hat{T}}{\partial q_3}$ by applying similar reasoning to (3.6) and (3.7). By a known result (see [12, Chap. XIII, Theorem 7.1]), this proves that $\hat{T}$ is continuously differentiable. □

The second-order derivatives, denoted in this case by

$$\hat{T}_{q_iq_j}(\mathbf{q};\cdot) = \frac{\partial^2 \hat{T}}{\partial q_i \partial q_j}(\mathbf{q};\cdot) \text{ for every } i,j \in \{1,2,3\},$$

can be calculated from the integral expressions (3.6), (3.7), and (3.9). In this sense, we have the following result.

Theorem 3.4. *Under the assumptions of Theorem 3.3, the map $\hat{T}$ is twice continuously differentiable on* $\mathrm{int}(\mathbf{Q})$. *Its second-order partial derivatives are given by*

$$\frac{d\hat{T}_{q_1q_1}}{dt}(\mathbf{q};t) + q_1 \hat{T}_{q_1q_1}(\mathbf{q};t) = -2\hat{T}_{q_1}(\mathbf{q};t), \text{ a.e. on } (t_0,t_f),\ \hat{T}_{q_1q_1}(\mathbf{q};t_0) = 0, \tag{3.11a}$$

$$\frac{d\hat{T}_{q_1q_2}}{dt}(\mathbf{q};t) + q_1 \hat{T}_{q_1q_2}(\mathbf{q};t) = -\hat{T}_{q_2}(\mathbf{q};t), \text{ a.e. on } (t_0,t_f),\ \hat{T}_{q_1q_2}(\mathbf{q};t_0) = 0, \tag{3.11b}$$

$$\frac{d\hat{T}_{q_1q_3}}{dt}(\mathbf{q};t) + q_1 \hat{T}_{q_1q_3}(\mathbf{q};t) = -\hat{T}_{q_3}(\mathbf{q};t), \text{ a.e. on } (t_0,t_f),\ \hat{T}_{q_1q_3}(\mathbf{q};t_0) = 0, \tag{3.11c}$$

while the remaining second-order derivatives are zero:

$$\hat{T}_{q_2q_2}(\mathbf{q};\cdot) = \hat{T}_{q_2q_3}(\mathbf{q};\cdot) = \hat{T}_{q_3q_3}(\mathbf{q};\cdot) = 0.$$

Proof. By a direct computation on (3.9) and following similar reasoning as in the proof of Theorem 3.3,

$$\begin{aligned}\hat{T}_{q_1q_1}(\mathbf{q};t) &= \int_{t_0}^{t}(s-t)e^{q_1(s-t)}(T^e(s)-\hat{T}(\mathbf{q};s))\,ds - \int_{t_0}^{t}e^{q_1(s-t)}\hat{T}_{q_1}(\mathbf{q};s)\,ds \\ &= -t\int_{t_0}^{t}e^{q_1(s-t)}(T^e(s)-\hat{T}(\mathbf{q};s))\,ds + \int_{t_0}^{t}se^{q_1(s-t)}(T^e(s)-\hat{T}(\mathbf{q};s))\,ds \\ &\quad - \int_{t_0}^{t}e^{q_1(s-t)}\hat{T}_{q_1}(\mathbf{q};s)\,ds \\ &= -K(t)+L(t)-\int_{t_0}^{t}e^{q_1(s-t)}\hat{T}_{q_1}(\mathbf{q};s)\,ds, \end{aligned} \tag{3.12}$$

where $K(t) := t\hat{T}_{q_1}(\mathbf{q};t)$, and $L(t) := \int_{t_0}^{t}se^{q_1(s-t)}(T^e(s)-\hat{T}(\mathbf{q};s))\,ds$ is the solution to the following ODE:

$$\frac{dL}{dt}(t)+q_1L(t) = t(T^e(t)-\hat{T}(\mathbf{q};t)),\ \text{a.e. on } (t_0,t_f),\ L(t_0)=0.$$

Therefore, applying (3.5a),

$$\frac{dL}{dt}(t)+q_1L(t) = t\left(\frac{d\hat{T}_{q_1}}{dt}(\mathbf{q};t)+q_1\hat{T}_{q_1}(\mathbf{q};t)\right).$$

On the other hand, applying the product rule gives

$$\frac{dK}{dt}(t)+q_1K(t) = \hat{T}_{q_1}(\mathbf{q};t)+t\left(\frac{d\hat{T}_{q_1}}{dt}(\mathbf{q};t)+q_1\hat{T}_{q_1}(\mathbf{q};t)\right) = \hat{T}_{q_1}(\mathbf{q};t)+\frac{dL}{dt}(t)+q_1L(t),$$

and from this, $K-L$ satisfies the following ODE:

$$\frac{d(K-L)}{dt}(t)+q_1(K-L)(t) = \hat{T}_{q_1}(\mathbf{q};t),\ \text{a.e. on } (t_0,t_f),\ (K-L)(t_0)=0,$$

or, equivalently,

$$(K-L)(t) = \int_{t_0}^{t}e^{q_1(s-t)}\hat{T}_{q_1}(\mathbf{q};s)\,ds.$$

Plugging this expression into (3.12), we get

$$\hat{T}_{q_1q_1}(\mathbf{q};t) = -2\int_{t_0}^{t}e^{q_1(s-t)}\hat{T}_{q_1}(\mathbf{q};s)\,ds,$$

which is equivalent to equation (3.11a). From the integral expressions (3.6) and (3.7), it is clear that

$$\hat{T}_{q_2q_2}(\mathbf{q};\cdot) = \hat{T}_{q_2q_3}(\mathbf{q};\cdot) = \hat{T}_{q_3q_3}(\mathbf{q};\cdot) = 0,$$

and by a direct computation, we can prove that

$$\hat{T}_{q_1q_2}(\mathbf{q};t) = -\int_{t_0}^{t}e^{q_1(s-t)}\hat{T}_{q_2}(\mathbf{q};s)\,ds, \tag{3.13}$$

$$\hat{T}_{q_1q_3}(\mathbf{q};t) = -\int_{t_0}^{t}e^{q_1(s-t)}\hat{T}_{q_3}(\mathbf{q};s)\,ds, \tag{3.14}$$

which correspond to equations (3.11b) and (3.11c), respectively.

Finally, the continuity of the second-order partial derivatives with respect to $\mathbf{q}$ can be shown as in the proof of Theorem 3.3, and the same standard result yields that $\hat{T}$ is twice continuously differentiable. □

Let us denote by

$$\hat{J}(\mathbf{q}) = \frac{1}{2}\int_{t_0}^{t_f} (\hat{T}(\mathbf{q};s) - z(s))^2\, ds$$

the fitting term in $\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\mathbf{q}^l,\mathbf{q}^u)}\right)$, so that $\hat{J}^\kappa(\mathbf{q}) = \hat{J}(\mathbf{q}) + \frac{\kappa}{2}\|\mathbf{q}\|^2_{\mathbb{R}^3}$. As before, the map $\hat{J}$ is differentiable at any $\mathbf{q} \in \mathrm{int}(\mathbf{Q})$, with partial derivatives given by

$$\hat{J}_{q_i}(\mathbf{q}) = \int_{t_0}^{t_f} (\hat{T}(\mathbf{q};s) - z(s))\hat{T}_{q_i}(\mathbf{q};s)\, ds, \quad \text{for every } i \in \{1,2,3\}. \tag{3.15}$$

From this, we can derive a formula for the Hessian. To shorten the notation, we write the $L^2(t_0,t_f)$ inner product and its associated norm as

$$\langle f,g\rangle_2 = \int_{t_0}^{t_f} f(s)g(s)\, ds, \quad \|f\|_2 = \sqrt{\langle f,f\rangle_2}, \quad \text{for all } f,g \in L^2(t_0,t_f).$$

Theorem 3.5. *Assume that T^e, u, I^c, $z \in L^\infty(\mathscr{I})$. Then the map $\hat{J}^\kappa$ is twice continuously differentiable on* $\mathrm{int}(\mathbf{Q})$*, with gradient $\nabla\hat{J}^\kappa(\mathbf{q}) = \nabla\hat{J}(\mathbf{q}) + \kappa\mathbf{q}$, where $\nabla\hat{J}(\mathbf{q})$ is given by (3.15), and Hessian*

$$\nabla^2\hat{J}^\kappa(\mathbf{q}) = \nabla^2\hat{J}(\mathbf{q}) + \kappa I, \tag{3.16}$$

where I is the 3×3 identity matrix, and $\nabla^2\hat{J}(\mathbf{q}) \in \mathbb{R}^{3\times 3}$ is the symmetric matrix given by

$$\nabla^2\hat{J}(\mathbf{q}) = \begin{pmatrix} \left\|\hat{T}_{q_1}(\mathbf{q})\right\|_2^2 + \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_1}(\mathbf{q})\rangle_2 & \cdots & \cdots \\ \langle \hat{T}_{q_1}(\mathbf{q}), \hat{T}_{q_2}(\mathbf{q})\rangle_2 + \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_2}(\mathbf{q})\rangle_2 & \left\|\hat{T}_{q_2}(\mathbf{q})\right\|_2^2 & \cdots \\ \langle \hat{T}_{q_1}(\mathbf{q}), \hat{T}_{q_3}(\mathbf{q})\rangle_2 + \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_3}(\mathbf{q})\rangle_2 & \langle \hat{T}_{q_2}(\mathbf{q}), \hat{T}_{q_3}(\mathbf{q})\rangle_2 & \left\|\hat{T}_{q_3}(\mathbf{q})\right\|_2^2 \end{pmatrix}.$$

Proof. From (3.15), by a direct computation and applying Theorem 3.4,

$$\hat{J}_{q_1q_1}(\mathbf{q}) = \frac{\partial}{\partial q_1}\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1}(\mathbf{q})\rangle_2 = \left\|\hat{T}_{q_1}(\mathbf{q})\right\|_2^2 + \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_1}(\mathbf{q})\rangle_2,$$

$$\hat{J}_{q_1q_i}(\mathbf{q}) = \frac{\partial}{\partial q_i}\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1}(\mathbf{q})\rangle_2 = \langle \hat{T}_{q_1}(\mathbf{q}), \hat{T}_{q_i}(\mathbf{q})\rangle_2 + \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_i}(\mathbf{q})\rangle_2, \forall i \in \{2,3\},$$

and

$$\hat{J}_{q_2q_2}(\mathbf{q}) = \frac{\partial}{\partial q_2}\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_2}(\mathbf{q})\rangle_2 = \left\|\hat{T}_{q_2}(\mathbf{q})\right\|_2^2 + \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_2q_2}(\mathbf{q})\rangle_2 = \left\|\hat{T}_{q_2}(\mathbf{q})\right\|_2^2,$$

$$\hat{J}_{q_3q_3}(\mathbf{q}) = \frac{\partial}{\partial q_3}\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_3}(\mathbf{q})\rangle_2 = \left\|\hat{T}_{q_3}(\mathbf{q})\right\|_2^2 + \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_3q_3}(\mathbf{q})\rangle_2 = \left\|\hat{T}_{q_3}(\mathbf{q})\right\|_2^2,$$

$$\begin{aligned}\hat{J}_{q_3q_2}(\mathbf{q}) &= \frac{\partial}{\partial q_2}\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_3}(\mathbf{q})\rangle_2 = \langle \hat{T}_{q_2}(\mathbf{q}), \hat{T}_{q_3}(\mathbf{q})\rangle_2 + \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_3q_2}(\mathbf{q})\rangle_2 \\ &= \langle \hat{T}_{q_2}(\mathbf{q}), \hat{T}_{q_3}(\mathbf{q})\rangle_2.\end{aligned}$$

□

Under certain conditions, we can prove that the Hessian $\nabla^2\hat{J}^\kappa(\cdot)$ is positive definite, and thus $\hat{J}^\kappa$ is strictly convex on $\mathbf{Q}$. As a consequence, the solution is unique.

Theorem 3.6. *For each regularization parameter satisfying*

$$\kappa > 2c, \tag{3.17}$$

where $c := \max_{\mathbf{q}\in\mathbf{Q},\, j\in\{1,2,3\}} \left\|\hat{T}(\mathbf{q}) - z\right\|_2 \left\|\hat{T}_{q_1q_j}(\mathbf{q})\right\|_2$, *the map* $\hat{J}^\kappa$ *is strictly convex on* $\mathbf{Q}$. *Consequently, problem* $\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\mathbf{q}^l,\mathbf{q}^u)}\right)$ *has a unique solution.*

Proof. First, note that formula (3.3) defines $\hat{T}(\mathbf{q})$ for every $\mathbf{q} \in \mathbb{R}^3$ with $q_1 > 0$, and that the computations in the proofs of Theorems 3.3, 3.4, and 3.5 remain valid on this open set, which contains $\mathbf{Q}$ since $q_1^l > 0$. Hence, (3.16) holds for every $\mathbf{q} \in \mathbf{Q}$, including its boundary, and $c < \infty$ by the continuity of the maps involved on the compact set $\mathbf{Q}$.

Let $\mathbf{q} \in \mathbf{Q}$. From (3.16), we have the decomposition

$$\nabla^2 \hat{J}^\kappa(\mathbf{q}) = G + H + \kappa I,$$

where $G := \left(\langle \hat{T}_{q_i}(\mathbf{q}), \hat{T}_{q_j}(\mathbf{q})\rangle_2\right)_{i,j=1}^3$ and

$$H + \kappa I := \begin{pmatrix} \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_1}(\mathbf{q})\rangle_2 + \kappa & \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_2}(\mathbf{q})\rangle_2 & \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_3}(\mathbf{q})\rangle_2 \\ \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_2}(\mathbf{q})\rangle_2 & \kappa & 0 \\ \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_3}(\mathbf{q})\rangle_2 & 0 & \kappa \end{pmatrix}.$$

The matrix G is a Gram matrix and is therefore positive semi-definite. By Sylvester's criterion, $H + \kappa I$ is positive definite if its leading principal minors are strictly positive:

$$\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_1}(\mathbf{q})\rangle_2 + \kappa > 0, \tag{3.18}$$

$$\left(\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_1}(\mathbf{q})\rangle_2 + \kappa\right)\kappa - \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_2}(\mathbf{q})\rangle_2^2 > 0, \tag{3.19}$$

$$\kappa \begin{vmatrix} \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_1}(\mathbf{q})\rangle_2 + \kappa & \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_2}(\mathbf{q})\rangle_2 \\ \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_2}(\mathbf{q})\rangle_2 & \kappa \end{vmatrix} + \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_3}(\mathbf{q})\rangle_2 \begin{vmatrix} 0 & \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_3}(\mathbf{q})\rangle_2 \\ \kappa & 0 \end{vmatrix} > 0. \tag{3.20}$$

Expanding the determinants, condition (3.20) is equivalent to

$$\kappa\left[\left(\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_1}(\mathbf{q})\rangle_2 + \kappa\right)\kappa - \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_2}(\mathbf{q})\rangle_2^2 - \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_3}(\mathbf{q})\rangle_2^2\right] > 0. \tag{3.21}$$

On the other hand, applying the Cauchy–Schwarz inequality, we can estimate

$$\left|\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_j}(\mathbf{q})\rangle_2\right| \le \left\|\hat{T}(\mathbf{q}) - z\right\|_2 \left\|\hat{T}_{q_1q_j}(\mathbf{q})\right\|_2 \le c, \quad j \in \{1,2,3\}.$$

Now, let $\kappa > 2c$. If $c = 0$, then $H = 0$ and $H + \kappa I = \kappa I$ is trivially positive definite. If $c > 0$, conditions (3.18), (3.19), and (3.21) are satisfied. In the first place,

$$\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_1}(\mathbf{q})\rangle_2 + \kappa > -c + 2c = c > 0,$$

and, using this bound together with $\kappa > 2c$,

$$\left(\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_1}(\mathbf{q})\rangle_2 + \kappa\right)\kappa - \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_2}(\mathbf{q})\rangle_2^2 > 0,$$
$$\left(\langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_1}(\mathbf{q})\rangle_2 + \kappa\right)\kappa - \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_2}(\mathbf{q})\rangle_2^2 - \langle \hat{T}(\mathbf{q}) - z, \hat{T}_{q_1q_3}(\mathbf{q})\rangle_2^2 >= 0.$$

Consequently, if (3.17) holds, then $H + \kappa I$ is positive definite, and $\nabla^2 \hat{J}^\kappa(\mathbf{q})$ is positive definite as the sum of a positive semi-definite and a positive definite matrix. Because $\mathbf{q} \in \mathbf{Q}$ is arbitrary

and $\mathbf{Q}$ is convex, $\hat{J}^\kappa$ is strictly convex on $\mathbf{Q}$, and problem $\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\mathbf{q}^l,\mathbf{q}^u)}\right)$ is uniquely solvable by standard results (see [5, Propositions 1.2.6 and 2.1.2]). □

Remark 3.1. Some comments about Theorem 3.6:

- Since c is a worst-case bound over the whole feasible set $\mathbf{Q}$, it is small when the bounds $\mathbf{q}^l$, $\mathbf{q}^u$ are chosen so that the model fits the data reasonably well for every $\mathbf{q} \in \mathbf{Q}$, that is, when the fitting error $\left\|\hat{T}(\mathbf{q}) - z\right\|_2$ is small on $\mathbf{Q}$. In that case, uniqueness holds even for small regularization parameters κ.
- Under the same hypotheses as Theorem 3.6, $\nabla^2 \hat{J}^\kappa(\mathbf{q}) - \alpha I = G + H + (\kappa - \alpha) I$ is positive definite for every $0 < \alpha < \kappa - 2c$, since then $\kappa - \alpha > 2c$; this implies that $\hat{J}^\kappa$ is strongly convex on $\mathbf{Q}$ with modulus α.

3.1. **Some consequences for the 4p model.** Uniqueness for the 3p model has consequences for the 4p model under some hypotheses. Considering reference bounds $(\check{\mathbf{a}}^l, \check{\mathbf{a}}^u)$ for the four-parameter model, we denote by $(\check{\mathbf{q}}^l, \check{\mathbf{q}}^u)$ the associated bounds for the three-parameter model:

$$\check{q}^l_1 = \frac{1}{\check{a}^u_1 \check{a}^u_2}, \quad \check{q}^u_1 = \frac{1}{\check{a}^l_1 \check{a}^l_2}, \quad \check{q}^l_i = \frac{\check{a}^l_{i+1}}{\check{a}^u_2}, \quad \check{q}^u_i = \frac{\check{a}^u_{i+1}}{\check{a}^l_2} \quad \text{for every } i \in \{2,3\},$$

so that, for every feasible element $\mathbf{a}$ of $\left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\check{\mathbf{a}}^l,\check{\mathbf{a}}^u)}\right)$, the vector

$$\mathbf{q}(\mathbf{a}) := \left(\frac{1}{a_1 a_2}, \frac{a_3}{a_2}, \frac{a_4}{a_2}\right) \tag{3.22}$$

is feasible for $\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\check{\mathbf{q}}^l,\check{\mathbf{q}}^u)}\right)$. Moreover, since $T(\mathbf{a}) = \hat{T}(\mathbf{q}(\mathbf{a}))$, the fitting terms of both problems coincide, $J(\mathbf{a}) = \hat{J}(\mathbf{q}(\mathbf{a}))$. However, their regularization terms differ, so the image under (3.22) of a solution to the 4p problem need not solve the 3p problem. Hence, in this subsection, we assume the following compatibility condition between both models:

$$\mathbf{a}^\kappa \text{ solves } \left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\check{\mathbf{a}}^l,\check{\mathbf{a}}^u)}\right) \implies \mathbf{q}(\mathbf{a}^\kappa) \text{ solves } \left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\check{\mathbf{q}}^l,\check{\mathbf{q}}^u)}\right). \tag{3.23}$$

Let $\mathbf{a}^\kappa = (R^\kappa, C^\kappa, P^\kappa, A^\kappa)$ be a generic solution of $\left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\check{\mathbf{a}}^l,\check{\mathbf{a}}^u)}\right)$. By (3.23), $\mathbf{q}^\kappa := \mathbf{q}(\mathbf{a}^\kappa)$ solves $\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\check{\mathbf{q}}^l,\check{\mathbf{q}}^u)}\right)$, and the following relation holds:

$$\frac{1}{a^\kappa_1 a^\kappa_2} = q^\kappa_1, \quad \frac{a^\kappa_3}{a^\kappa_2} = q^\kappa_2, \quad \frac{a^\kappa_4}{a^\kappa_2} = q^\kappa_3. \tag{3.24}$$

If κ satisfies (3.17), then $\mathbf{q}^\kappa$ is uniquely determined by Theorem 3.6, and it is therefore the same for every solution $\mathbf{a}^\kappa$ of the 4p problem. From this, if one of the parameters in $\mathbf{a}^\kappa$ is uniquely determined, that is, if this parameter takes the same value for every solution, the remaining parameters are uniquely determined by (3.24). For example, if $a^\kappa_1 = \bar{a}_1$ is known, then a^κ_2, a^κ_3, and a^κ_4 are given by

$$a^\kappa_2 = \frac{1}{\bar{a}_1 q^\kappa_1}, \quad a^\kappa_3 = \frac{q^\kappa_2}{\bar{a}_1 q^\kappa_1}, \quad a^\kappa_4 = \frac{q^\kappa_3}{\bar{a}_1 q^\kappa_1}.$$

We can consider a more general hypothesis and assume that we have an a priori error estimate for one of the parameters. Formally, we consider the following condition: for some $\varepsilon > 0$, there

exist a solution $\bar{\mathbf{a}}^{\kappa,\varepsilon}$ of $\left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\check{\mathbf{a}}^l,\check{\mathbf{a}}^u)}\right)$ and $\beta \in \{1,\dots,4\}$ such that

$$\mathbf{a}^\kappa \text{ solves } \left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\check{\mathbf{a}}^l,\check{\mathbf{a}}^u)}\right) \implies \left|a^\kappa_\beta - \bar{a}^{\kappa,\varepsilon}_\beta\right| \le \varepsilon. \tag{3.25}$$

In this case, the error estimate retains the same accuracy, up to a constant, for the other three parameters, as we show in the following result.

Proposition 3.1. *Assume that κ satisfies (3.17) for problem $\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\check{\mathbf{q}}^l,\check{\mathbf{q}}^u)}\right)$, and that (3.23) and (3.25) hold. Then there exists $\tilde{c} > 0$ such that*

$$\|\mathbf{a}^\kappa - \bar{\mathbf{a}}^{\kappa,\varepsilon}\|_\infty = \max_{j\in\{1,\dots,4\}} \left|a^\kappa_j - \bar{a}^{\kappa,\varepsilon}_j\right| \le \tilde{c}\,\varepsilon \tag{3.26}$$

for every solution $\mathbf{a}^\kappa$ of problem $\left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\check{\mathbf{a}}^l,\check{\mathbf{a}}^u)}\right)$.

Proof. Let $\mathbf{a}^\kappa$ be an arbitrary solution to $\left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\check{\mathbf{a}}^l,\check{\mathbf{a}}^u)}\right)$, and let $\mathbf{q}^\kappa$ be the unique solution to $\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\check{\mathbf{q}}^l,\check{\mathbf{q}}^u)}\right)$, which exists by Theorem 3.6. By (3.23), $\mathbf{q}(\mathbf{a}^\kappa) = \mathbf{q}(\bar{\mathbf{a}}^{\kappa,\varepsilon}) = \mathbf{q}^\kappa$. We give the proof for $\beta = 1$; the remaining cases are similar. From (3.24),

$$a^\kappa_2 = \frac{1}{q^\kappa_1}\frac{1}{a^\kappa_1}, \quad a^\kappa_3 = \frac{q^\kappa_2}{q^\kappa_1}\frac{1}{a^\kappa_1}, \quad a^\kappa_4 = \frac{q^\kappa_3}{q^\kappa_1}\frac{1}{a^\kappa_1}, \tag{3.27}$$

and similarly,

$$\bar{a}^{\kappa,\varepsilon}_2 = \frac{1}{q^\kappa_1}\frac{1}{\bar{a}^{\kappa,\varepsilon}_1}, \quad \bar{a}^{\kappa,\varepsilon}_3 = \frac{q^\kappa_2}{q^\kappa_1}\frac{1}{\bar{a}^{\kappa,\varepsilon}_1}, \quad \bar{a}^{\kappa,\varepsilon}_4 = \frac{q^\kappa_3}{q^\kappa_1}\frac{1}{\bar{a}^{\kappa,\varepsilon}_1}.$$

By a direct manipulation,

$$\left|\frac{1}{a^\kappa_1} - \frac{1}{\bar{a}^{\kappa,\varepsilon}_1}\right| = \frac{1}{a^\kappa_1 \bar{a}^{\kappa,\varepsilon}_1}\left|a^\kappa_1 - \bar{a}^{\kappa,\varepsilon}_1\right| \le \frac{1}{(\check{a}^l_1)^2}\varepsilon.$$

Using these estimates and the bounds,

$$\begin{aligned}
\left|a^\kappa_2 - \bar{a}^{\kappa,\varepsilon}_2\right| &= \frac{1}{q^\kappa_1}\left|\frac{1}{a^\kappa_1} - \frac{1}{\bar{a}^{\kappa,\varepsilon}_1}\right| \le \frac{1}{\check{q}^l_1}\frac{1}{(\check{a}^l_1)^2}\varepsilon,\\
\left|a^\kappa_3 - \bar{a}^{\kappa,\varepsilon}_3\right| &= \frac{q^\kappa_2}{q^\kappa_1}\left|\frac{1}{a^\kappa_1} - \frac{1}{\bar{a}^{\kappa,\varepsilon}_1}\right| \le \frac{\check{q}^u_2}{\check{q}^l_1}\frac{1}{(\check{a}^l_1)^2}\varepsilon,\\
\left|a^\kappa_4 - \bar{a}^{\kappa,\varepsilon}_4\right| &= \frac{q^\kappa_3}{q^\kappa_1}\left|\frac{1}{a^\kappa_1} - \frac{1}{\bar{a}^{\kappa,\varepsilon}_1}\right| \le \frac{\check{q}^u_3}{\check{q}^l_1}\frac{1}{(\check{a}^l_1)^2}\varepsilon,
\end{aligned}$$

and (3.26) holds with $\tilde{c} = \max\left\{1, \frac{1}{\check{q}^l_1}\frac{1}{(\check{a}^l_1)^2}, \frac{\check{q}^u_2}{\check{q}^l_1}\frac{1}{(\check{a}^l_1)^2}, \frac{\check{q}^u_3}{\check{q}^l_1}\frac{1}{(\check{a}^l_1)^2}\right\}$. □

In particular, if we consider $\varepsilon = 0$, then condition (3.25) corresponds to a priori knowledge of one of the parameters: there exists $\beta \in \{1,\dots,4\}$ with $\bar{a}_\beta \in [\check{a}^l_\beta, \check{a}^u_\beta]$ such that

$$\mathbf{a}^\kappa \text{ solves } \left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\check{\mathbf{a}}^l,\check{\mathbf{a}}^u)}\right) \implies a^\kappa_\beta = \bar{a}_\beta. \tag{3.28}$$

In this case, we obtain a uniqueness result for the four-parameter model as a direct consequence of the previous proposition.

Corollary 3.1. *Assume that* κ *satisfies* (3.17) *for problem* $\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\check{\mathbf{q}}^l,\check{\mathbf{q}}^u)}\right)$, *and that* (3.23) *and* (3.28) *hold. Then problem* $\left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\check{\mathbf{a}}^l,\check{\mathbf{a}}^u)}\right)$ *is uniquely solvable.*

4. Numerical examples

In this section, we consider two examples: a fully synthetic one and a semisynthetic one.

Example 4.1. In the first one, we consider a fully synthetic example where the data are prescribed by analytical functions and the solution can be computed exactly. We consider $[t_0,t_f] = [0,10]$, $T_0 = 17$, $T^e(t) = 15$, and

$$u(t) = \begin{cases} \frac{t}{5} & \text{if } 0 \le t \le 5, \\ 2 - \frac{t}{5} & \text{if } 5 \le t \le 10, \end{cases} \qquad I^c(t) = \begin{cases} t(5-t) & \text{if } 0 \le t \le 5, \\ 0 & \text{otherwise,} \end{cases}$$

with reference parameters $(\bar{R},\bar{C},\bar{P},\bar{A}) = (5,2,3,4)$. Using the variation of constants formula, the solution is given by

$$T(t) = 17e^{-\frac{1}{RC}(t-t_0)} + 15\left(1 - e^{-\frac{1}{RC}(t-t_0)}\right) + \frac{P}{C}\Phi_u(t) + \frac{A}{C}\Phi_{I^c}(t), \tag{4.1}$$

where the integrals

$$\Phi_u(t) = \int_{t_0}^{t} e^{\frac{1}{RC}(s-t)}u(s)\,ds, \quad \Phi_{I^c}(t) = \int_{t_0}^{t} e^{\frac{1}{RC}(s-t)}I^c(s)\,ds$$

can be computed explicitly. See Figure 1a for a graphical representation of the data for this example, where we compare the solution obtained by formula (4.1) with the simulated temperature obtained by solving the ODE model with an explicit Runge–Kutta method of order 5(4) (RK45); see the numerical details below.

Example 4.2. In the second example, we consider a semisynthetic example, where synthetic data samples are obtained by solving the model numerically with prescribed parameters and real thermal and meteorological data. These data are provided by the company Valor, Desarrollo e Innovación S.L. (Asturias, Spain), using its own smart thermostat technology. They consist of hourly time series of heater usage for a specific building, while meteorological data (exterior temperature and cloud cover) are used to obtain the exterior temperature and to compute the corrected solar radiation as described in [3]. We consider a specific home located in Canet de Mar (Barcelona, Spain) over a four-day period, with prescribed parameters $(\bar{R},\bar{C},\bar{P},\bar{A}) = (9,9,4,2)$; these parameters are in the range of those obtained in [3] for real temperature data. See Figure 1b.

4.1. A first-optimize-then-discretize approach.

In contrast to the discretize-then-optimize approach followed in [3], where the optimization problem was discretized and derivatives were obtained subsequently, we follow a first-optimize-then-discretize approach. Although a discretize-then-optimize technique could also be applied to both models, here we consider a direct discretization of the continuous derivative formulas for both the four- and three-parameter identification models. For this, we consider a grid

$$t_i = t_0 + i\Delta t \quad \text{for every } i = 0,\dots,N,$$

of $N+1$ points, where the grid points t_i correspond to the measurement times and $\Delta t = \frac{t_f - t_0}{N}$ to the measurement period. All numerical experiments were coded in Python; ODE sensitivities were computed using `solve_ivp` in SciPy with a standard explicit Runge–Kutta method of order 5(4) (RK45), while integrals were numerically calculated using a direct implementation of the trapezoidal rule. For the first example, we consider $N = 100$, while for the second we consider $N = 96$, corresponding to hourly measurements over the four-day period.

Numerical tests confirm the sensitivity formulas for both models. For the four-parameter model, the discretization of the gradient of T at a point $\mathbf{a} \in \mathbb{R}^4$ is compared with a fourth-order centered finite difference approximation with step size $h > 0$,

$$\nabla T(\mathbf{a}) \approx \left(T_R^h(\mathbf{a}), T_C^h(\mathbf{a}), T_P^h(\mathbf{a}), T_A^h(\mathbf{a})\right) = \left(T_1^h(\mathbf{a}), T_2^h(\mathbf{a}), T_3^h(\mathbf{a}), T_4^h(\mathbf{a})\right),$$

where

$$T_i^h(\mathbf{a}) := \frac{-T(\mathbf{a}+2h\mathbf{e}^i) + 8T(\mathbf{a}+h\mathbf{e}^i) - 8T(\mathbf{a}-h\mathbf{e}^i) + T(\mathbf{a}-2h\mathbf{e}^i)}{12h}$$

for every $i \in \{1,\dots,4\}$, and $\mathbf{e}^i$ denotes the i-th canonical basis vector of $\mathbb{R}^4$. As seen in Figures 2a and 2b, the analytical and numerical gradients evaluated at $\mathbf{a} = (1,4,4,1)$ visually coincide in both examples. Following a similar technique, we numerically tested the gradient of the three-parameter temperature $\hat{T}$ at $\mathbf{q} = (1,4,4)$ in both examples (see Figures 3a and 3b). Similar results were obtained for the second-order derivative formulas established in (3.11), as shown in Figures 4a and 4b.

4.2. **Identification error performance of the 4p and 3p models.** In this subsection, we study the identification error, the impact of regularization, and the dependence on the initial point for the four- and three-parameter models in the two examples. For this purpose, we consider a finite sequence of regularization parameters:

$$\kappa \in \{1, 10^{-1}, 5\times10^{-2}, 10^{-2}, 5\times10^{-3}, 10^{-3}, 5\times10^{-4}, 10^{-4}, 5\times10^{-5}, 10^{-6}, 0\}.$$

For each parameter value, we randomly generate twenty initial points in the feasible set and apply a numerical optimization algorithm to solve the corresponding optimization problem $\left(\mathscr{Q}^{4p}_{(t_0,t_f);\kappa;(\mathbf{a}^l,\mathbf{a}^u)}\right)$ or $\left(\mathscr{Q}^{3p}_{(t_0,t_f);\kappa;(\mathbf{q}^l,\mathbf{q}^u)}\right)$, and then we study the distribution of the relative error.

We use the relative error to measure the error in each component. For example, for the resistance, we measure $\frac{|R^h - \bar{R}|}{|\bar{R}|}$, where R^h is the computed value and $\bar{R}$ is the exact value.

All computational experiments were carried out on a computer with a 13th Gen Intel Core i7-1355U CPU at 1.70 GHz and 16 GB of memory, using a Python implementation in a Jupyter Notebook. For both optimization models, we solved the optimization problem using the quasi-Newton L-BFGS-B algorithm implemented in the Python library SciPy [19].

In both examples, we consider $\mathbf{a}^l = (1,1,0,0.1)$ and $\mathbf{a}^u = (10,10,15,8)$ as lower and upper bounds. For the 3p model, we take the corresponding bounds, i.e., $\mathbf{q}^l = (0.01,0,0.01)$ and $\mathbf{q}^u = (1,15,8)$.

For Example 4.1, the numerical distribution of the identification error with respect to the regularization parameter is presented in the boxplots in Figures 5a and 6a; the corresponding results for Example 4.2 are shown in Figures 5b and 6b.

In both examples, we observe that the 3p model clearly outperforms the 4p model in terms of identification error. The error of the 4p model stagnates at a level between 10^{-2} and 10^{-1}

for the first example and just above 10^{-1} for the second example. In contrast, for the 3p model, the error is empirically observed to decrease approximately linearly as κ decreases, becoming negligible. This trend is more clearly visible in the mean error; see Figures 7a and 8a for the first example, and Figures 7b and 8b for the second example. All these numerical results are consistent with the convexity of the 3p model predicted by the theory.

In general, this behaviour is stable above a certain threshold of the regularization parameter. Below this threshold, the error distribution becomes significantly more volatile and the error increases. See, for example, Figure 8a, where the 3p model in the first example exhibits a more volatile error distribution below 5×10^{-5}, even though the errors remain negligible. This empirically confirms the necessity of regularization for both the 4p and 3p models.

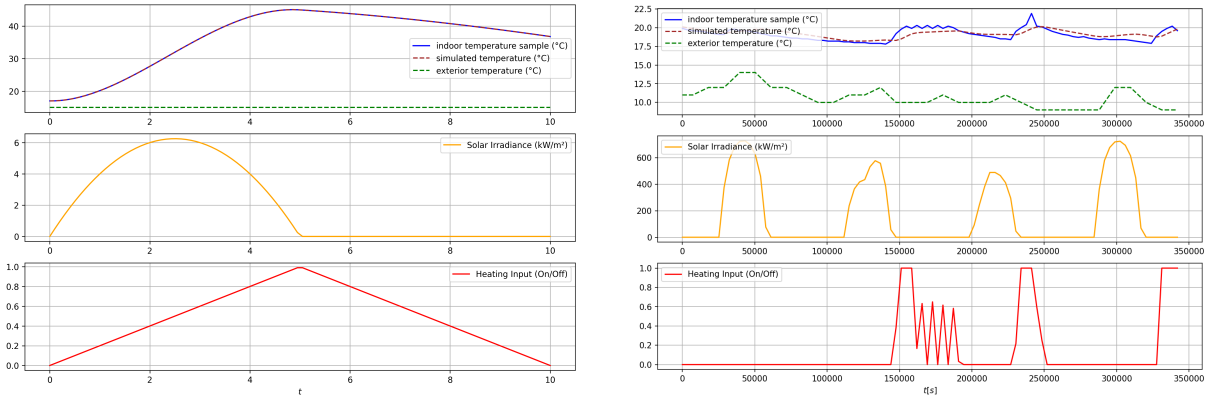
In terms of computational cost, the 4p model requires more CPU time than the 3p model. Moreover, the typical increase in computational cost and volatility for decreasing regularization parameters further underlines the necessity of regularization; for example, compare Figures 9a and 9b for Example 4.1. We note that these results depend heavily on the implementation of the optimization routine, which we do not study here. It is also interesting to note that the 4p model implicitly provides a good estimation of the three parameters of the 3p model; see Figure 10 for the mean identification of the inverse thermal time constant $q_1 = 1/(RC)$. This is consistent with the identifiability analysis of Section 2.2: the 4p model identifies the parameter combinations $\mathbf{q}$ well, but not the individual parameters, contrary to the 3p model.

5. Concluding remarks

In this work, we have presented a rigorous mathematical study of the identifiability properties of the four-parameter thermal model presented in [3] for essentially bounded data. Our main result is a convex reformulation of the problem in terms of three parameters, which has better mathematical properties. These theoretical and numerical results have direct practical implications for future studies and applications. We highlight three topics of particular interest:

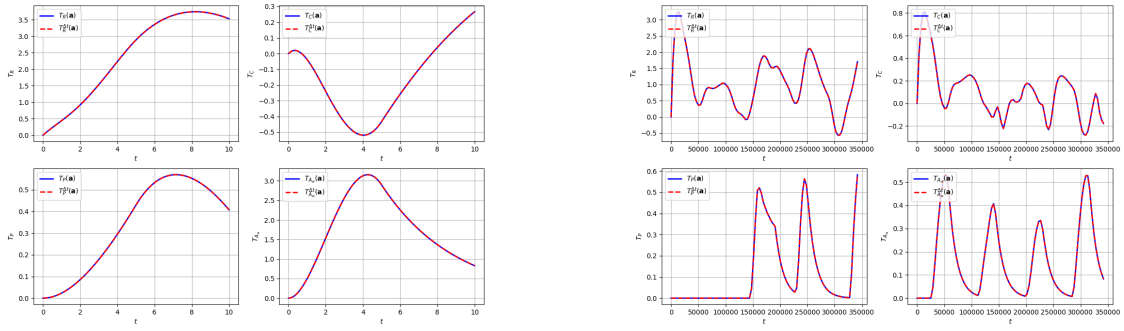
- Regularization parameter selection. The use of regularization, which was not considered in [3, Sections 6 and 7], has a significant impact on the quality of the identification and on the computational time. How to choose the regularization parameter efficiently is a problem of high practical relevance.
- New discretization techniques. The development of a functional approach to parameter identification in the thermal model—which is not common in the specialized literature, as it is generally more focused on statistical approaches or discrete models—leads to new numerical methods that we expect to be better adapted to the natural continuous least-squares formulation. In this sense, the discretization can lead to efficient alternative numerical approaches like the one proposed here. The continuous formulas are suitable for online calculation, since incorporating new samples only requires updating numerical integrals, which is straightforward to implement. Exploring these ideas is a worthwhile direction, and extending the study to more general models constitutes a natural next step.
- Algorithm design and numerical validation in applied models. The convex reformulated model provides a natural foundation for algorithm development in terms of identifiability and computational efficiency. A key question is how to design an efficient algorithm that connects the three-parameter model with the four-parameter one; once an optimal

three-parameter vector has been obtained, one must determine how to choose the most suitable four-parameter counterpart. In the context of the original temperature prediction application considered in [3, Sections 6 and 7], we anticipate that such an algorithm will offer greater gains in computational efficiency than in accuracy. However, the true value of identifiability becomes evident in applications where parameter values play a more critical role—for instance, in control problems where heater usage serves as a control variable and directly affects economic cost, or in other applied 1R1C models, where parameters carry a more defined physical meaning and must be reliably estimated for robust decision-making.



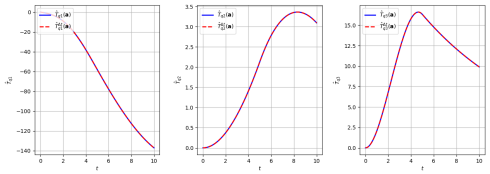
(A) Ex. 1. Data. (B) Ex. 2. Data.

FIGURE 1. Data used in Examples 1 and 2.

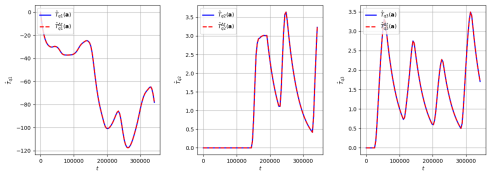


(A) Ex. 1. 4p model. Temperature gradient at $\mathbf{a} = (1,4,4,1)$. (B) Ex. 2. 4p model. Temperature gradient at $\mathbf{a} = (1,4,4,1)$.

FIGURE 2. Temperature gradient, 4p model.

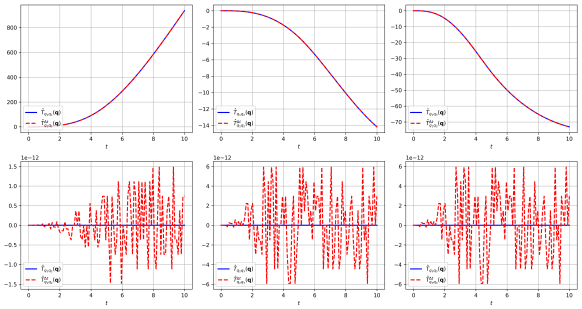


(A) Ex. 1. 3p model. Temperature gradient at $\mathbf{q} = (1,4,1)$.

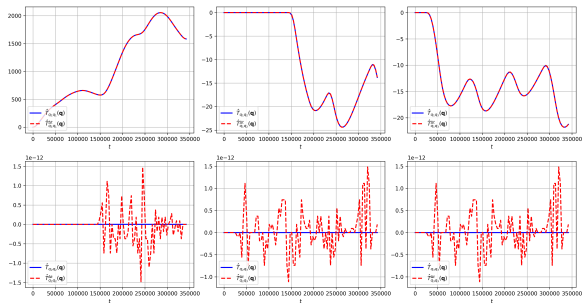


(B) Ex. 2. 3p model. Temperature gradient at $\mathbf{q} = (1,4,1)$.

FIGURE 3. Temperature gradient, 3p model.

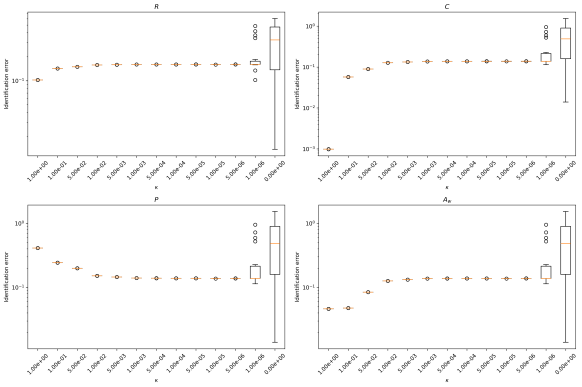


(A) Ex. 1. 3p model. Temperature Hessian at $\mathbf{q} = (1,4,1)$.

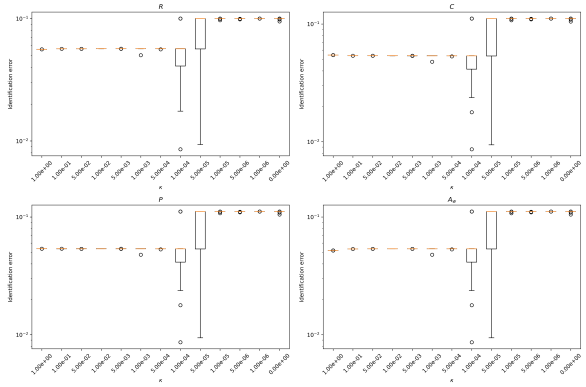


(B) Ex. 2. 3p model. Temperature Hessian at $\mathbf{q} = (1,4,1)$.

FIGURE 4. Temperature Hessian, 3p model.

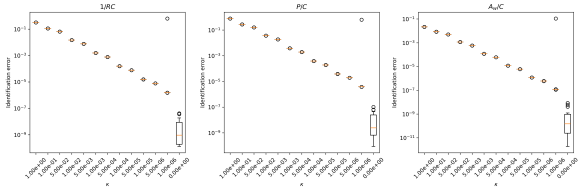


(A) Ex. 1. 4p model. Boxplot of identification error versus regularization parameter.

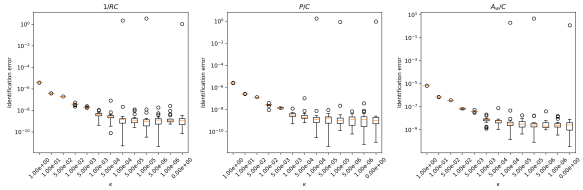


(B) Ex. 2. 4p model. Boxplot of identification error versus regularization parameter.

FIGURE 5. Boxplot of identification error vs. regularization parameter, 4p model.

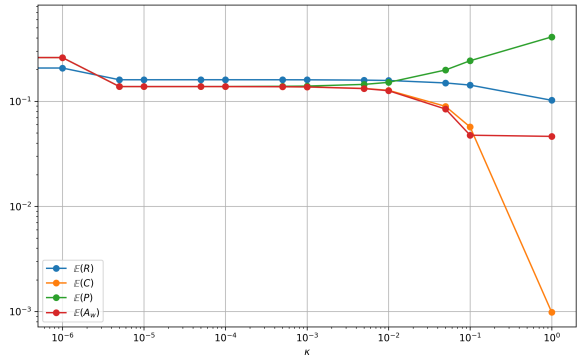


(A) Ex. 1. 3p model. Boxplot of identification error versus regularization parameter.

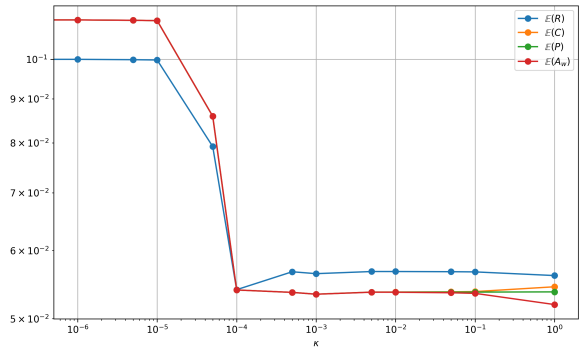


(B) Ex. 2. 3p model. Boxplot of identification error versus regularization parameter.

FIGURE 6. Boxplot of identification error vs. regularization parameter, 3p model.

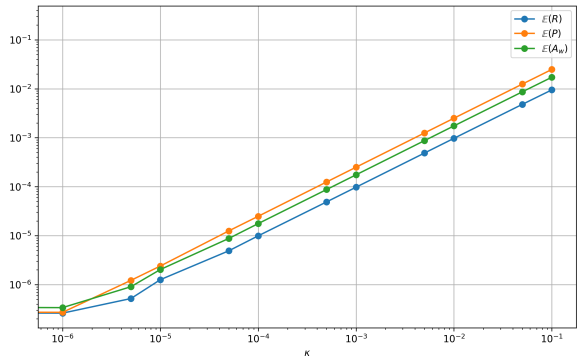


(A) Ex. 1. 4p model. Mean identification error versus regularization parameter.

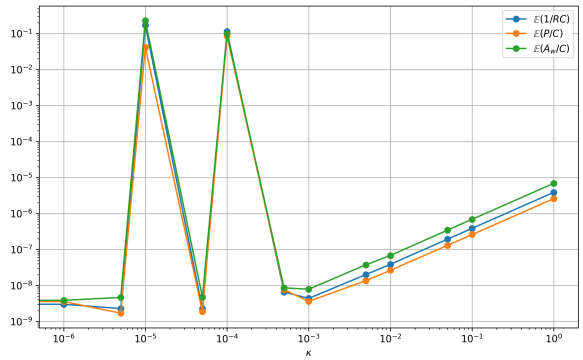


(B) Ex. 2. 4p model. Mean identification error versus regularization parameter.

FIGURE 7. Mean identification error vs. regularization parameter, 4p model.

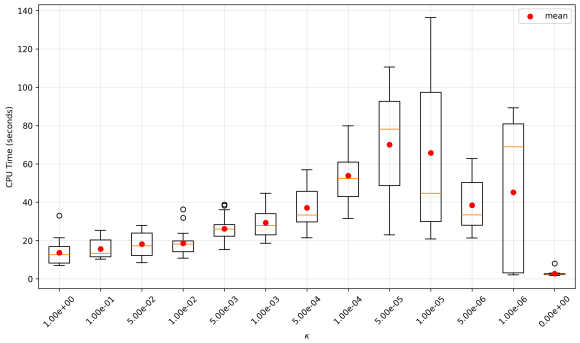


(A) Ex. 1. 3p model. Mean identification error versus regularization parameter.

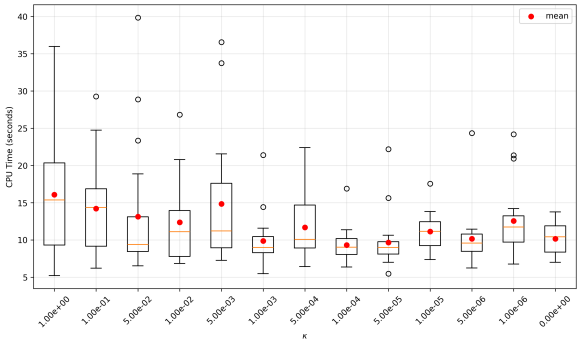


(B) Ex. 2. 3p model. Mean identification error versus regularization parameter.

FIGURE 8. Mean identification error vs. regularization parameter, 3p model.



(A) Ex. 1. 4p model. Boxplot of CPU time versus regularization parameter.



(B) Ex. 1. 3p model. Boxplot of CPU time versus regularization parameter.

FIGURE 9. CPU time, Ex. 1, 4p vs. 3p model.

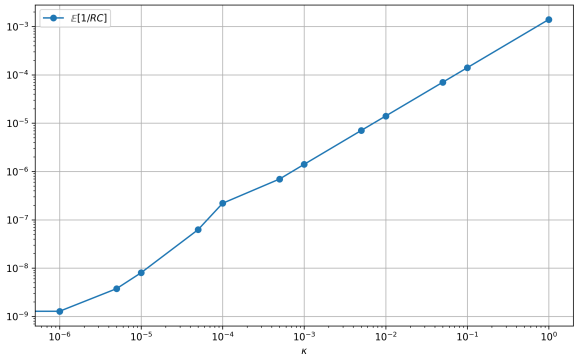


FIGURE 10. Ex. 2. 4p model. Mean identification of the inverse thermal time constant $q_1 = 1/(RC)$.

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