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NONSMOOTH NEARLY CONVEX OPTIMIZATION PROBLEMS AND APPLICATIONS IN ROBUSTNESS

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Abstract. This paper deals with nearly convex optimization problems in the face of data uncertainty within the framework of robust optimization. Specifically, we investigate optimality conditions for nonsmooth robust optimization problems with nearly convex data. To do so, we first examine the rule for computing the subdifferential of the maximum of nearly convex functions. As an application, we study an optimization problem under uncertainty where the involved functions are nearly convex and derive necessary and sufficient conditions for solutions of the corresponding strictly robust counterpart problem. These are the main results of this paper. The paper presents several numerical examples to illustrate the results as well.

Keywords. Near convexity; Optimality conditions; Robust optimization problems; Subdifferential.
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1. INTRODUCTION

Most optimization problems involve uncertain data due to measurement errors, unknown future developments and modeling approximations. In risk theory, assets are naturally affected by uncertainty due to market changes, changing preferences of customers and unforeseeable events. Consequently, it is highly important to introduce uncertain parameters to optimization problems. Robust optimization is a field of optimization theory that studies mathematical programming problems with uncertain data. We refer to Ben-Tal et al. [4] and Kouvelis and Yu [21] for an extensive collection of results and applications for the most prominent concepts. The field of robust optimization, including both its theoretical and applied aspects, has attracted the interest of many researchers (see, e.g., [2, 4, 5, 6, 11, 15, 18, 19, 20, 22, 23, 25] and the references therein).

In the field of mathematical programming, problems that involve simultaneous minimization and maximization are known as minimax optimization problems. This is an important class

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of problems in various fields, including game theory [10, 14], Chebyshev best approximation theory [13], and multi-objective optimization [16].

One way of dealing with optimization problems under uncertainty is described in the concept of robustness: instead of assuming that all data are known, one allows different scenarios for the input parameters and looks for a solution that works well in every uncertain scenario. In all the approaches, the uncertain optimization problem is replaced by a deterministic version, called the robust counterpart of the uncertain problem. It is noteworthy that the robust counterpart is essentially a minimax optimization problem.

The concept of nearly convex sets was introduced by Minty and Rockafellar in the 1960s [26], [30]. Based on the classical notion of convexity, a set is nearly convex if it lies between a convex set and its closure in the Euclidean space $\mathbb{R}^n$ .The systematic study of these sets is strongly motivated by their crucial role in convex analysis. For instance, the domain and range of any maximal monotone mapping are nearly convex. Specifically, the subdifferential domain of a proper, lower semicontinuous convex function is always nearly convex [31, Theorem 12.41]. Furthermore, while classical convexity has been extensively studied, researchers are constantly seeking to generalize these notions for sets and functions, positioning nearly convex sets as a natural bridge beyond traditional convexity. Consequently, various properties of nearly convex sets were examined from different perspectives in [3, 7, 8, 27, 29].

Our primary objective is to establish necessary and sufficient optimality conditions for solutions to robust counterpart problems within the context of nearly convex analysis. To this end, we first develop a new rule for computing the subdifferential of the maximum of nearly convex functions. We then apply these results to an optimization problem under uncertainty involving nearly convex functions, ultimately deriving necessary and sufficient optimality conditions for the corresponding strictly robust counterpart problem. The findings are further illustrated through several numerical examples.

Our paper is organized as follows. In Section 2, we first recall the definition and properties of the subdifferential of nearly convex functions. We then present formulas for estimating the subdifferential of the maximum of nearly convex functions. In Section 3, we study optimality conditions for nearly convex optimization problems. The application of these results to strictly robust counterpart problems in optimization under uncertainty is discussed in Section 4. Finally, a conclusion is given in Section 5.

Throughout the paper, the space $\mathbb{R}^n$ is equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\|\cdot\|$. We use the notation $B(x;\delta)$ to represent the open ball centered at $x \in \mathbb{R}^n$ with a radius of $\delta > 0$. The interior (resp., closure) of D will be denoted by $\operatorname{int} D$ (resp., $\overline{D}$). Here $\operatorname{aff} D$ stands for the affine hull of a subset $D \subset \mathbb{R}^n$.

2. Subdifferential of the Maximum of Nearly Convex Functions

In this section, we present the formula for computing the subdifferential of the maximum of nearly convex functions.

Let us first recall that the *relative interior* of an arbitrary set D in $\mathbb{R}^n$ is defined by

$$\operatorname{ri} D := \{a \in D \mid \text{there exists } \delta > 0 \text{ such that } B(a;\delta) \cap \operatorname{aff} D \subseteq D\}.$$

By definition, it is easy to see that $a \in \operatorname{ri} D$ if and only if $a \in \operatorname{aff} D$ and there exists $\delta > 0$ such that

$$B(a;\delta) \cap \operatorname{aff} D \subseteq D.$$

Definition 2.1. (see [31, Theorem 12.41]) A subset D of $\mathbb{R}^n$ is said to be *nearly convex* if there exists a convex set E in $\mathbb{R}^n$ such that $E \subseteq D \subseteq \overline{E}$.

Since $\overline{\operatorname{ri} E} = \overline{E}$ for a convex set E, it follows from the definition that D is nearly convex if and only if there exists a convex set E such that $\operatorname{ri} E \subset D \subset \overline{E}$. This definition of near convexity coincides with the notion almost convexity considered in [9] (see [24, Proposition 3.3] and [17, Proposition 2.1]). Note that this definition of nearly convex does not coincide with the one provided in [7, Definition 2] and [8], where nearly convex is a generalization of midpoint convexity.

Clearly, any convex set is nearly convex and any nearly convex subset of $\mathbb{R}$ is convex. However, in $\mathbb{R}^n$ with $n \geq 2$, there are many nearly convex sets which are not convex; see, e.g., [27].

Consider a function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := [-\infty, \infty]$ with values in the extended real line. One says that φ is *proper* if the *effective domain* $\operatorname{dom} \varphi := \{x \in \mathbb{R}^n \mid \varphi(x) < \infty\}$ is nonempty and if $\varphi(x) > -\infty$ for all $x \in \mathbb{R}^n$. The set $\operatorname{epi} \varphi := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geq \varphi(x)\}$ is called the *epigraph* of φ. If $\operatorname{epi} \varphi$ is a convex (resp., nearly convex) subset of $\mathbb{R}^n \times \mathbb{R}$, then φ is said to be a *convex* (resp., *nearly convex*) function. Due to the continuity of a convex function φ in $\operatorname{ri}(\operatorname{dom} \varphi)$, it knows that the "near" convexity only happens in the boundary of its domain.

Definition 2.2. Consider a proper function $\varphi \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\bar{x} \in \operatorname{dom} \varphi$. The *lower* and *upper Dini derivatives* of φ at $\bar{x}$ in the direction $d \in \mathbb{R}^n$ are defined, respectively, by

$$\varphi^-(\bar{x}; d) := \liminf_{t \downarrow 0} \frac{\varphi(\bar{x} + td) - \varphi(\bar{x})}{t},$$

$$\varphi^+(\bar{x}; d) := \limsup_{t \downarrow 0} \frac{\varphi(\bar{x} + td) - \varphi(\bar{x})}{t}.$$

The *directional derivative* of φ at $\bar{x}$ in the direction d is given by

$$\varphi'(\bar{x}; d) := \lim_{t \downarrow 0} \frac{\varphi(\bar{x} + td) - \varphi(\bar{x})}{t}, \tag{2.1}$$

when the limit in (2.1) exists.

Definition 2.3. (see [29]) Let $\varphi \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, nearly convex function and let $\bar{x} \in \operatorname{dom} \varphi$. The *subdifferential* of φ at $\bar{x}$ is defined by $\partial \varphi(\bar{x}) := \{u \in \mathbb{R}^n \mid \langle u, x - \bar{x} \rangle \leq \varphi(x) - \varphi(\bar{x}) \text{ for all } x \in \mathbb{R}^n\}$.

Definition 2.4. (see [29]) Let Ω be a nonempty, nearly convex subset in $\mathbb{R}^n$ and $\bar{x} \in \Omega$. The *normal cone* to Ω at $\bar{x}$ is given by $N(\bar{x}; \Omega) := \{u \in \mathbb{R}^n \mid \langle u, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega\}$.

By definition, it is clear that $N(\bar{x}; \Omega) = \partial \delta_\Omega(\bar{x})$, where δ_Ω is the *indicator function* of a set $\Omega \subset \mathbb{R}^n$. Recall that $\delta_\Omega(x) = 0$ if $x \in \Omega$ and $\delta_\Omega(x) = \infty$ if $x \notin \Omega$. Since $\operatorname{epi} \delta_\Omega = \Omega \times [0, \infty)$, we see that δ_Ω is nearly convex if and only if Ω is nearly convex.

We have the following intersection rule for computing the normal cone of nearly convex sets.

Proposition 2.1. (see, e.g. [29, Theorem 5.1] *Let* $\Omega_1, \Omega_2, ..., \Omega_m$ *be nearly convex sets. If* $\bigcap_{i=1}^m \operatorname{ri} \Omega_i \neq \emptyset$, *then, for each* $\bar{x} \in \Omega = \bigcap_{i=1}^m \Omega_i$, $N(\bar{x}; \Omega) = N(\bar{x}; \Omega_1) + N(\bar{x}; \Omega_2) + ... + N(\bar{x}; \Omega_m)$.

We now recall some properties of the subdifferentials of nearly convex functions.

Proposition 2.2. (see, e.g. [33, Proposition 2]) *Let $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, nearly convex function. Then, for every $\bar{x} \in \operatorname{ri}(\operatorname{dom}\varphi)$, the subdifferential of φ at $\bar{x}$ is nonempty, convex and closed.*

Proposition 2.3. (see [29, Corollary 5.5] *Let $\varphi_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1,2$, be two proper, nearly convex functions. If $\operatorname{ri}(\operatorname{dom}\varphi_1) \cap \operatorname{ri}(\operatorname{dom}\varphi_2) \neq \emptyset$, then $\varphi_1 + \varphi_2$ is nearly convex and*

$$\partial(\varphi_1 + \varphi_2)(\bar{x}) = \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x})$$

for all $\bar{x} \in \operatorname{dom}\varphi_1 \cap \operatorname{dom}\varphi_2$.

We now present formulas for estimating the subdifferential of the maximum function of nearly convex functions.

Consider the function

$$F(x) := \max_{\xi \in U} f(x,\xi), \tag{2.2}$$

where $f : \mathbb{R}^n \times U \to \overline{\mathbb{R}}$ and $U \subset \mathbb{R}^m$ is compact.

We assume that the following conditions are satisfied:

(A1) $f(\cdot,\xi)$ is nearly convex for all $\xi \in U$ and $\bigcap_{\xi\in U} \operatorname{ri}[\operatorname{epi} f(\cdot,\xi)] \neq \emptyset$.

(A2) $(x,\xi) \mapsto f(x,\xi)$ is upper semi-continuous (u.s.c.) at each $(x,\xi) \in \left(\bigcap_{\xi\in U} \operatorname{ri}\big(\operatorname{dom} f(\cdot,\xi)\big)\right) \times U$.

(A3) For a fixed $x \in \mathbb{R}^n$, the function f is locally Lipschitz in x uniformly for $\xi \in U$; that is, for $x \in \mathbb{R}^n$, there exist an open neighborhood $\mathcal{N}(x)$ of x and a positive constant L such that, for all $y,z \in \mathcal{N}(x)$ and $\xi \in U$, $|f(y,\xi) - f(z,\xi)| \leq L\|y-z\|$.

(A4) $f'_x(x,\xi;d)$ exists and is finite for every $x \in \operatorname{ri}(\operatorname{dom} f(\cdot,\xi))$, and the mapping $\xi \mapsto f'_x(x,\xi;d)$ is upper semi-continuous at each $\xi \in U$.

We denote by $\widehat{U}(\bar{x})$ the set of $\xi \in U$ such that $F(\bar{x}) = f(\bar{x},\xi)$. Since $f(x,\cdot)$ is upper semicontinous for all x in a certain neighborhood of $\bar{x}$ and U is compact, the set $\widehat{U}(\bar{x})$ is nonempty and compact for every x in a certain neighborhood of $\bar{x}$. Here the symbol $\partial_x f(\bar{x},\xi)$ stands for the subdifferential of the function $f(\cdot,\xi)$ at $\bar{x}$.

The main results of this section are proven by using the following lemmas.

Lemma 2.1. *Let $x \in \bigcap_{\xi\in U} \operatorname{ri}(\operatorname{dom} f(\cdot,\xi))$. Suppose that assumptions **(A1)**- **(A4)** are satisfied. Then, for every direction $d \in \mathbb{R}^n$, the directional derivative of the function $F(x) = \max_{\xi\in U} f(x,\xi)$ exists and is given by $F'(x;d) = \max_{\xi\in\widehat{U}(x)} f'_x(x,\xi;d)$.*

Proof. Fix $x \in \bigcap_{\xi\in U} \operatorname{ri}(\operatorname{dom} f(\cdot,\xi))$ and a direction $d \in \mathbb{R}^n$. For small $t > 0$, we estimate the difference

$$\begin{aligned} |F(x+td) - F(x)| &= \left|\max_{\xi\in U} f(x+td,\xi) - \max_{\xi\in U} f(x,\xi)\right| \\ &\leq \sup_{\xi\in U} |f(x+td,\xi) - f(x,\xi)| \\ &\leq Lt\|d\|, \end{aligned}$$

which implies

$$\left|\frac{F(x+td) - F(x)}{t}\right| \leq L\|d\| \quad \text{for small } t > 0.$$

Taking the upper limit as $t \downarrow 0$, we obtain

$$\limsup_{t\downarrow 0} \frac{F(x+td)-F(x)}{t} \leq L\|d\| < \infty.$$

Therefore, the upper Dini directional derivative $F^+(x;d)$ exists and is finite.

We may suppose that $0 \neq d \in \mathbb{R}^n$. Fix $(x,\xi) \in \mathbb{R}^n \times U$ and take an arbitrary sequence $t_k \downarrow 0$ such that

$$F^+(x;d) = \lim_{k\to\infty} \frac{F(x+t_kd)-F(x)}{t_k}.$$

Hence, for each k, one can find some $\xi_k \in \widehat{U}(x+t_kd)$ satisfying $F(x+t_kd) = f(x+t_kd,\xi_k)$. Since $\widehat{U}(x+t_kd) \subseteq U$, by the compactness of U, there exists a subsequence ξ_k that converges to some $\bar{\xi} \in U$. For each $\xi \in U$, we have $f(x+t_kd,\xi) \leq f(x+t_kd,\xi_k)$. Taking the upper limit as $k \to \infty$, we obtain

$$\limsup_{k\to\infty} f(x+t_kd,\xi) \leq \limsup_{k\to\infty} f(x+t_kd,\xi_k).$$

For all $\xi \in U$, $f(x,\xi) \leq \limsup_{k\to\infty} f(x+t_kd,\xi_k)$. Using assumption **(A2)**, we see that $f(x,\xi) \leq f(x,\bar{\xi})$. Hence $\bar{\xi} \in \widehat{U}(x)$. For each $t > 0$,

$$\frac{F(x+td)-F(x)}{t} \geq \frac{f(x+td,\bar{\xi})-f(x,\bar{\xi})}{t}.$$

Taking $t \downarrow 0$ gives

$$F^-(x;d) \geq f'_x(x,\bar{\xi};d). \tag{2.3}$$

Now, we consider the double sequence $\{\frac{f(x+t_jd,\xi_i)-f(x,\xi_i)}{t_j}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$. According to assumption **(A3)**, we obtain

$$\left|\frac{f(x+t_jd,\xi_i)-f(x,\xi_i)}{t_j}\right| \leq L\|d\|,$$

where L is the Lipschitz constant for f around x. Hence, $\{\frac{f(x+t_jd,\xi_i)-f(x,\xi_i)}{t_j}\}$ is bounded. Thus, thanks to [1, Theorem 8.39], we can find a subsequence (without relabeling) such that

$$\lim_{i,j\to\infty} \frac{f(x+t_jd,\xi_i)-f(x,\xi_i)}{t_j} = \alpha \in \mathbb{R}.$$

Moreover, we have

$$\lim_{i\to\infty}\left(\lim_{j\to\infty} \frac{f(x+t_jd,\xi_i)-f(x,\xi_i)}{t_j}\right) = \lim_{i\to\infty} f'_x(x,\xi_i;d) \leq f'_x(x,\bar{\xi};d),$$

where the last inequality follows from assumption **(A4)**. Thus, we obtain

$$\lim_{i,j\to\infty} \frac{f(x+t_jd,\xi_i)-f(x,\xi_i)}{t_j} \leq f'_x(x,\bar{\xi};d).$$

Choosing $i = j = k$, we obtain $\lim_{k\to\infty} \frac{f(x+t_k d,\xi_k)-f(x,\xi_k)}{t_k} \le f'_x(x,\bar{\xi};d)$. Thus, we conclude that

$$\begin{aligned} F^+(x;d) &= \lim_{k\to\infty} \frac{F(x+t_k d)-F(x)}{t_k} \\ &\le \lim_{k\to\infty} \frac{f(x+t_k d,\xi_k)-f(x,\xi_k)}{t_k} \\ &\le f'_x(x,\bar{\xi};d). \end{aligned}$$

So $F^+(x;d) \le f'_x(x,\bar{\xi};d)$. Combining this with (2.3), we have $F'(x;d) = f'_x(x,\bar{\xi};d)$. Moreover, for all $\xi \in \widehat{U}(x)$, $F(x) = f(x,\xi)$, and $F(x+td) \ge f(x+td,\xi)$, we obtain

$$\frac{F(x+td)-F(x)}{t} \ge \frac{f(x+td,\xi)-f(x,\xi)}{t}.$$

Taking the upper limit as $t \downarrow 0$ yields $F'(x;d) \ge f'_x(x,\xi;d)$. Therefore, we have

$$F'(x;d) = \max_{\xi\in\widehat{U}(x)} f'_x(x,\xi;d),$$

which completes the proof. □

Lemma 2.2. *Suppose that assumptions **(A1)**, **(A3)** and **(A4)** hold. Then, for any $\xi \in U$, any point $\bar{x} \in \operatorname{ri}(\operatorname{dom} f(\cdot,\xi))$, and any direction $d_0 \in \mathbb{R}^n$, the directional derivative of $f(\cdot,\xi)$ at $\bar{x}$ in the direction d_0 is given by $f'_x(\bar{x},\xi;d_0) = \max_{v\in\partial_x f(\bar{x},\xi)} \langle v, d_0\rangle$.*

Proof. First, we show that the subdifferential $\partial_x f(\bar{x},\xi)$ is compact. We aim to prove that the subdifferential $\partial_x f(\bar{x},\xi)$ is a bounded set. We argue by contradiction. Suppose that the subdifferential $\partial_x f(\bar{x},\xi)$ is unbounded. Then there exists a sequence $\{v_k\} \subset \partial_x f(\bar{x},\xi)$ such that $\|v_k\| \to \infty$ as $k \to \infty$. Define the normalized sequence $d_k := \frac{v_k}{\|v_k\|}$. Since $\|d_k\| = 1$, by compactness of the unit sphere in $\mathbb{R}^n$, we may assume (passing to a subsequence if necessary) that $d_k \to d \in \mathbb{R}^n$ with $\|d\| = 1$. By the definition of the subdifferential, for every $t > 0$, we have

$$f(\bar{x}+td_k,\xi) \ge f(\bar{x},\xi) + t\langle v_k, d_k\rangle.$$

Note that $\langle v_k, d_k\rangle = \|v_k\|$. Therefore $f(\bar{x}+td_k,\xi) \ge f(\bar{x},\xi) + t\|v_k\|$. Dividing both sides of the last inequality by t, we have

$$\frac{f(\bar{x}+td_k,\xi)-f(\bar{x},\xi)}{t} \ge \|v_k\|. \tag{2.4}$$

This implies that the directional derivative $f'_x(\bar{x},\xi;d_k) \to \infty$. However, since f is locally Lipschitz in $\bar{x}$ uniformly for $\xi \in U$, there exist a constant $L > 0$ and $\delta > 0$ such that

$$\forall x, y \in B(\bar{x};\delta) : |f(x,\xi) - f(y,\xi)| \le L\|x-y\|.$$

In particular, for sufficiently small $t > 0$,

$$\left|\frac{f(\bar{x}+td_k,\xi)-f(\bar{x},\xi)}{t}\right| \le L.$$

This contradicts with (2.4). Hence, our initial assumption must be false, and we conclude that $\partial_x f(\bar{x},\xi)$ is bounded. Moreover, by Proposition 2.2, $\partial_x f(\bar{x},\xi)$ is closed. Consequently, $\partial_x f(\bar{x},\xi)$ is a closed and bounded subset of $\mathbb{R}^n$, and hence it is compact. Let $v \in \partial_x f(\bar{x},\xi)$. By

the definition of subdifferential, we have, for all $x \in \mathbb{R}^n$, $f(x,\xi) \geq f(\bar{x},\xi) + \langle v, x-\bar{x}\rangle$. Substituting $x = \bar{x}+td_0$, for all $t>0$, we obtain $f(\bar{x}+td_0,\xi) - f(\bar{x},\xi) \geq t\langle v, d_0\rangle$. Dividing both sides by $t>0$ yields

$$\frac{f(\bar{x}+td_0,\xi)-f(\bar{x},\xi)}{t} \geq \langle v, d_0\rangle.$$

Taking the limit as $t \downarrow 0$, we obtain $f'_x(\bar{x},\xi;d_0) \geq \langle v, d_0\rangle$. Since this holds for all $v \in \partial_x f(\bar{x},\xi)$, we conclude that

$$f'_x(\bar{x},\xi;d_0) \geq \max_{v\in\partial_x f(\bar{x},\xi)} \langle v, d_0\rangle. \tag{2.5}$$

Let $t_k \downarrow 0$ and set $x_k = \bar{x}+t_k d_0$. By **(A4)**, the directional derivative exists; hence

$$f'_x(\bar{x},\xi;d_0) = \lim_{k\to\infty} \frac{f(x_k,\xi)-f(\bar{x},\xi)}{t_k}. \tag{2.6}$$

Let $v_k \in \partial_x f(x_k,\xi)$ and $v_k \to v \in \partial_x f(\bar{x},\xi)$. The subgradient inequality yields

$$\forall y \in \mathbb{R}^n : f(y,\xi) \geq f(x_k,\xi) + \langle v_k, y-x_k\rangle.$$

Choosing $y=\bar{x}$ and recalling $x_k = \bar{x}+t_k d_0$, we obtain

$$\frac{f(x_k,\xi)-f(\bar{x},\xi)}{t_k} \leq \langle v_k, d_0\rangle. \tag{2.7}$$

Taking lim sup on both sides of (2.7) and using (2.6) gives $f'_x(\bar{x},\xi;d_0) \leq \lim_{k\to\infty}\langle v_k, d_0\rangle = \langle v, d_0\rangle$. Since $\partial_x f(\bar{x},\xi)$ is a compact set, the latter means that

$$f'_x(\bar{x},\xi;d_0) \leq \max\{\langle v, d_0\rangle \mid v \in \partial_x f(\bar{x},\xi)\}. \tag{2.8}$$

Combining inequalities (2.5) and (2.8), we conclude $f'_x(\bar{x},\xi;d_0) = \max_{v\in\partial_x f(\bar{x},\xi)}\langle v, d_0\rangle$. □

Lemma 2.3. *Consider the maximum function F given in (2.2). Under assumptions **(A1)–(A3)**, the following properties hold:*

(i) *The function F is proper and nearly convex.*
(ii) *The subdifferential $\partial F(x)$ is nonempty, compact for every $x \in \operatorname{ri}(\bigcap_{\xi\in U} \operatorname{dom} f(\cdot,\xi))$.*

Proof. (i) First, we show that the maximum function F given in (2.2) is nearly convex. Indeed, we note that

$$\operatorname{epi} F = \bigcap_{\xi\in U} \operatorname{epi} f(\cdot,\xi).$$

Since $\bigcap_{\xi\in U} \operatorname{ri}[\operatorname{epi} f(\cdot,\xi)] \neq \emptyset$, it follows from [9, Theorem 2.1] that $\bigcap_{\xi\in U}\operatorname{epi} f(\cdot,\xi)$ is a nearly convex set. Consequently, $\operatorname{epi} F$ is a nearly convex set. Thus F is a nearly convex function. The proper property of F holds due to **(A2)**. The proof of (i) is complete.

(ii) Our first step is to prove that $\partial F(x)$ is bounded. Suppose, for the sake of contradiction, that $\partial F(x)$ is unbounded. Then there exists a sequence $\{v_k\} \subset \partial F(x)$ such that $\|v_k\| \to \infty$ as $k \to \infty$. Define the unit vectors $d_k := \frac{v_k}{\|v_k\|}$. Since $\{v_k\} \subset \partial F(x)$, by the definition of subdifferential, we have $F(x+td_k) \geq F(x) + t\langle v_k, d_k\rangle$ for all $t>0$. Observe that $\langle v_k, d_k\rangle = \|v_k\| \to \infty$. Then

$$F(x+td_k) \geq F(x) + t\|v_k\|.$$

Dividing both sides of the latter by t, and taking the lower limit as $t \downarrow 0$, we obtain

$$\liminf_{t\downarrow 0} \frac{F(x+td_k)-F(x)}{t} \geq \|v_k\| \to \infty.$$

This implies that the one-sided directional derivatives $F^-(x;d_k) \to \infty$, which contradicts the assumption that F is locally Lipschitz near x, since Lipschitz functions have finite directional derivatives. Hence, $\partial F(x)$ must be bounded. As $\operatorname{dom} F = \bigcap_{\xi\in U}(\operatorname{dom} f(\cdot,\xi))$, it follows that

$$\operatorname{ri}(\operatorname{dom} F) = \operatorname{ri} \bigcap_{\xi\in U} (\operatorname{dom} f(\cdot,\xi)).$$

By Proposition 2.2, $\partial F(x)$ is nonempty and closed. Combining with its boundedness in $\mathbb{R}^n$ implies that $\partial F(x)$ is a compact subset of $\mathbb{R}^n$. □

We now describe the subdifferential of F at the point $\bar{x} \in \operatorname{ri} \bigcap_{\xi\in U}(\operatorname{dom} f(\cdot,\xi))$.

Theorem 2.1. *Let assumptions **(A1)–(A4)** be satisfied. For every $\bar{x} \in \operatorname{ri} \bigcap_{\xi\in U} (\operatorname{dom} f(\cdot,\xi))$, the subdifferential $\partial F(\bar{x})$ is given by*

$$\partial F(\bar{x}) = \operatorname{co} \bigcup_{\xi\in\widehat{U}(\bar{x})} \partial_x f(\bar{x},\xi). \tag{2.9}$$

Proof. Define $\Lambda := \operatorname{co}\bigcup_{\xi\in\widehat{U}(\bar{x})} \partial_x f(\bar{x},\xi)$. First, we prove $\partial F(\bar{x}) \supseteq \Lambda$. Take any $v \in \partial_x f(\bar{x},\xi)$ for some $\xi \in \widehat{U}(\bar{x})$. Then, for every x',

$$F(x') = \max_{\xi\in\widehat{U}(\bar{x})} f(x',\xi) \geq f(x',\xi) \geq f(\bar{x},\xi) + \langle v, x'-\bar{x}\rangle.$$

So $v \in \partial F(\bar{x})$. Moreover, by Lemma 2.3 (i), function F is proper, nearly convex. Hence, it follows from Proposition 2.2 that the subdifferential of F is convex. Thus $\Lambda \subseteq \partial F(\bar{x})$. To prove the converse $\partial F(\bar{x}) \subseteq \Lambda$, we first show that Λ is closed. Let $\{v_k\} \subseteq \Lambda$ be a sequence such that $v_k \to v$ as $k \to \infty$. We next show that $v \in \Lambda$. By Carathéodory's theorem (see [12, Proposition 2.6]), for each k, there exist $\lambda_i^k \geq 0$ $(i = 1,\ldots,n+1)$, with $\sum_{i=1}^{n+1} \lambda_i^k = 1$, and points $w_i^k \in \partial_x f(\bar{x},\xi_i^k)$ for some $\xi_i^k \in \widehat{U}(\bar{x})$ such that $v_k = \sum_{i=1}^{n+1} \lambda_i^k w_i^k$. Since $\widehat{U}(\bar{x})$ is compact and $\lambda_i^k \in [0,1]$, by passing to a subsequence if necessary, we may assume that

$$\lambda_i^k \to \lambda_i \geq 0, \quad i = 1,\ldots,n+1, \quad \text{with } \sum_{i=1}^{n+1} \lambda_i = 1,$$
$$\xi_i^k \to \xi_i \in \widehat{U}(\bar{x}), \quad i = 1,\ldots,n+1.$$

Moreover, from assumption **(A3)**, one sees that $x \mapsto \partial_x f(x,\xi)$ is locally bounded around $\bar{x}$. Hence, the sets $\partial_x f(\bar{x},\xi_i^k)$ are uniformly bounded. This yields that $\{w_i^k\}$ is bounded for each i, therefore, $w_i^k \to w_i, \quad i = 1,\ldots,n+1$. Note that the upper semi-continuity of the mapping $\xi \mapsto \partial_x f(\bar{x},\xi)$ from the assumption **(A4)** ensures that $w_i \in \partial_x f(\bar{x},\xi_i)$, $i = 1,\ldots,n+1$. Taking the limit in the convex representation of v_k, we have $v = \lim_{k\to\infty} v_k = \sum_{i=1}^{n+1} \lambda_i w_i$. Since each $w_i \in \partial_x f(\bar{x},\xi_i)$ with $\xi_i \in \widehat{U}(\bar{x})$, and $\sum_{i=1}^{n+1} \lambda_i = 1$, $\lambda_i \geq 0$, we deduce that v is a convex combination of points from $\bigcup_{\xi\in\widehat{U}(\bar{x})} \partial_x f(\bar{x},\xi)$. Thus

$$v \in \operatorname{co}\Big(\bigcup_{\xi\in\widehat{U}(\bar{x})} \partial_x f(\bar{x},\xi)\Big) = \Lambda.$$

Therefore, Λ is closed.

Now, assume by contradiction that there exists a point $v \in \partial F(\bar{x}) \setminus \Lambda$. We note that Λ is convex and closed, and $\partial F(\bar{x})$ is a nonempty compact set by Lemma 2.3. Therefore, by a classical separation theorem for convex sets (see [28, Theorem 2.35]), there exist a nonzero vector $d_0 \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $\langle v, d_0\rangle > \alpha \geq \langle v', d_0\rangle$ for all $v' \in \Lambda$. Thanks to Lemma 2.1, we have

$$\max_{\xi \in \widehat{U}(\bar{x})} f'_x(\bar{x}, \xi; d_0) = F'(\bar{x}; d_0) \geq \langle v, d_0\rangle > \alpha \geq \langle v', d_0\rangle \ \text{ for all } \ v' \in \Lambda.$$

On the other hand, there exists $\hat{\xi} \in \widehat{U}(\bar{x})$ such that $F'(\bar{x}; d_0) = f'_x(\bar{x}, \hat{\xi}; d_0)$, and there exists $\hat{v} \in \partial_x f(\bar{x}, \hat{\xi})$ such that $\langle \hat{v}, d_0\rangle = f'_x(\bar{x}, \hat{\xi}; d_0)$. Combining these observations, we have

$$f'_x(\bar{x}, \hat{\xi}; d_0) > \alpha \geq \langle \hat{v}, d_0\rangle = f'_x(\bar{x}, \hat{\xi}; d_0)$$

which is a contradiction. The poof is complete. □

Let us consider an illustrative example for Theorem 2.1.

Example 2.1. Let $U := \{-1, 1\}$ and define the function $f : \mathbb{R}^2 \times U \to \mathbb{R}$ by

$$f((x,y), \xi) := \begin{cases} |x - \xi| + |y| & \text{if } (x,y) \in [-1,1] \times [-1,1] \setminus \{(0,1)\}, \\ \infty & \text{otherwise.} \end{cases}$$

Consider the max-function

$$F(x,y) := \max_{\xi \in U} f((x,y), \xi).$$

We now check the assumptions of Theorem 2.1.

(A1): For each $\xi \in U$, function $f(\cdot, \xi)$ is nearly convex on its effective domain $[-1,1] \times [-1,1] \setminus \{(0,1)\}$. The epigraph of the function $f(\cdot, \xi)$ is defined as:

$$\operatorname{epi}(f(\cdot, \xi)) := \left\{((x,y), t) \in \mathbb{R}^2 \times \mathbb{R} \mid f((x,y), \xi) \leq t\right\}.$$

Considering the point $(x,y) = (0,0)$, we have

$$f((0,0), \xi) = |0 - \xi| + |0| = |\xi| = 1 \quad \text{for both } \xi = -1 \text{ and } \xi = 1.$$

Hence, $((0,0), 2)$ belongs to the relative interior of the epigraph, i.e.,

$$((0,0), 2) \in \operatorname{ri} \operatorname{epi}(f(\cdot, \xi)) \quad \text{for both } \xi = -1 \text{ and } \xi = 1.$$

It follows that $\bigcap\limits_{\xi \in U} \operatorname{ri}[\operatorname{epi} f(\cdot, \xi)] \neq \emptyset$. Thus assumption **(A1)** holds.

(A2): It is easy to check that the function $((x,y), \xi) \mapsto f((x,y), \xi)$ is upper semi-continuous on $\left(\bigcap_{\xi \in U} \operatorname{ri}\left(\operatorname{dom} f(\cdot, \xi)\right)\right) \times U$.

(A3): We show that f is locally Lipschitz in $(\bar{x}, \bar{y}) = (0,0)$ uniformly for $\xi \in U$. Let $(x,y), (x',y')$ be two nearby points of $(\bar{x}, \bar{y})$ in the domain $[-1,1] \times [-1,1] \setminus \{(0,1)\}$. Then

$$\begin{aligned} |f((x,y), \xi) - f((x',y'), \xi)| &= \big| |x - \xi| - |x' - \xi| + |y| - |y'| \big| \\ &\leq \big| |x - \xi| - |x' - \xi| \big| + \big| |y| - |y'| \big| \\ &\leq |x - x'| + |y - y'| \\ &= \|(x,y) - (x',y')\|. \end{aligned}$$

Therefore, f is locally Lipschitz in $(\bar{x}, \bar{y})$ with the Lipschitz constant $L = 1$ that does not depend on ξ.

(A4): We consider the function

$$f((x,y),\xi) = |x-\xi| + |y|, \text{ with } \xi \in U := \{-1,1\}.$$

We analyze the directional derivative along a direction $d = (d_1,d_2) \in \mathbb{R}^2$. Since f is a sum of absolute value functions, the directional derivative exists and is finite at every point in the effective domain.

We see that $f((x,y),\xi) = g(x) + h(y)$, where $g(x) = |x-\xi|$ and $h(y) = |y|$. Both g and h are convex and Lipschitz continuous on $\mathbb{R}$, and thus their one-sided directional derivatives exist and are finite at every point. The directional derivatives of the absolute value functions at the non-differentiable points $x = \xi$ or $y = 0$ are given by:

$$g'(\xi;d_1) = |d_1| \quad \text{and} \quad h'(0;d_2) = |d_2|.$$

The function $f(\cdot,\xi)$ is convex and Lipschitz continuous, which guarantees the upper semi-continuity of the directional derivative $f'(\cdot,\xi;d)$ with respect to (x,y). Moreover, because $U = \{-1,1\}$ is a finite set, any function from $\xi \in U$ into $\mathbb{R}$ is automatically continuous (and thus upper semicontinuous). Hence, condition **(A4)** is satisfied.

Consider the function

$$F(x,y) := \max_{\xi \in U} f((x,y),\xi) = \max\{|x+1|+|y|, |x-1|+|y|\}.$$

We analyze the case at the point $(\bar{x},\bar{y}) = (0,0)$. Observe that, for all $\xi \in \{-1,1\}$, $f((0,0),\xi) = |\xi| + 0 = 1$. Hence, $\widehat{U}(0,0) = \{-1,1\}$. The subdifferential of f with respect to the variable (x,y) is given by

$$\partial_{(x,y)} f((x,y),\xi) = \left(\partial_x f((x,y),\xi),\ \partial_y f((x,y),\xi)\right),$$

where the partial subdifferentials are

$$\partial_x f((x,y),\xi) = \begin{cases} \{1\} & \text{if } x > \xi, \\ \{-1\} & \text{if } x < \xi, \\ [-1,1] & \text{if } x = \xi, \end{cases} \text{ and } \partial_y f((x,y),\xi) = \begin{cases} \{1\} & \text{if } y > 0, \\ \{-1\} & \text{if } y < 0, \\ [-1,1] & \text{if } y = 0. \end{cases}$$

We compute the subdifferential of f at $(0,0)$ for each ξ as follows

$$\partial_{(x,y)} f((0,0),-1) = \partial_{(x,y)}(|x+1|+|y|)\Big|_{(0,0)} = (1,[-1,1]),$$

$$\partial_{(x,y)} f((0,0),1) = \partial_{(x,y)}(|x-1|+|y|)\Big|_{(0,0)} = (-1,[-1,1]).$$

Taking the union over $\xi \in \widehat{U}(0,0)$, we have

$$\bigcup_{\xi \in \widehat{U}(0,0)} \partial_{(x,y)} f((0,0),\xi) = \{(1,a),(-1,b) \mid a,b \in [-1,1]\}.$$

Taking the convex hull of this set yields all convex combinations of the form

$$\lambda(1,a) + (1-\lambda)(-1,b) = (\lambda - (1-\lambda), \lambda a + (1-\lambda)b)$$

for $\lambda \in [0,1]$ and $a,b \in [-1,1]$, which gives

$$\operatorname{co}\left(\bigcup_{\xi \in \widehat{U}(0,0)} \partial_{(x,y)} f((0,0),\xi)\right) = [-1,1] \times [-1,1].$$

On the other hand, we have $F(x,y) := \varphi(x) + |y|$, where $\varphi(x) := \max\{|x+1|, |x-1|\}$. Then

$$\partial F(x,y) = \partial\varphi(x) \times \partial|y|$$

and, hence $\partial F(0,0) = \partial\varphi(0) \times \partial|0| = [-1,1] \times [-1,1]$. Therefore,

$$\partial F(0,0) = \operatorname{co}\left(\bigcup_{\xi \in \widehat{U}(0,0)} \partial_{(x,y)} f((0,0),\xi) \right),$$

which verifies (2.9).

3. Optimality Conditions for Solutions of Nearly Convex Optimization Problems

In this section, we study the optimality conditions for nearly convex optimization problems by using the subdifferential. In Section 4, we apply these results to derive necessary optimality conditions for solutions of a robust counterpart problem to an optimization problem under uncertainty.

Consider the following optimization problem

$$\min \varphi(x) \ \text{ subject to } \ g_i(x) \le 0, \ i = 1,2,...,m, \tag{P}$$

where $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1,2,...,m$ are nearly convex functions and $\inf\limits_{x\in\mathbb{R}^n} g_i(x) < 0$. Put

$$S := \{x \in \mathbb{R}^n \mid g_i(x) \le 0, \ i = 1,2,...,m\}. \tag{3.1}$$

We observe that S can be rewritten by

$$S = \bigcap_{i=1}^{m} \operatorname{lev}_{\le 0} g_i,$$

where $\operatorname{lev}_{\le\alpha} f := \{x \in \mathbb{R}^n \mid f(x) \le \alpha\}$ is the lower level set of the function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, with respect to the level α.

In order to characterize the optimality conditions, we need the following lemma.

Lemma 3.1. *Let $g_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1,2,...,m$ be nearly convex functions and $\inf\limits_{x\in\mathbb{R}^n} g_i(x) < 0$. Suppose that the following condition*

$$\bigcap_{i=1}^{m} \operatorname{ri}\left(\operatorname{lev}_{\le 0} g_i\right) \neq \emptyset \tag{3.2}$$

is satisfied. Then S given by (3.1) is a nearly convex set and

$$\operatorname{ri} S = \operatorname{ri}\left(\bigcap_{i=1}^{m} \operatorname{lev}_{\le 0} g_i \right) = \bigcap_{i=1}^{m} \left(\operatorname{ri} \operatorname{lev}_{\le 0} g_i\right).$$

Proof. Since $g_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1,2,...,m$ are nearly convex functions and $\inf\limits_{x\in\mathbb{R}^n} g_i(x) < 0$, by [24, Proposition 4.6] the set $\operatorname{lev}_{\le 0} g_i$ is nearly convex for $i = 1,2,...,m$. In addition, if (3.2) is

satisfied, thanks to [9, Theorem 2.1], we obtain the near convexity of S and

$$\operatorname{ri} S = \operatorname{ri}\left(\bigcap_{i=1}^{m} \operatorname{lev}_{\leq 0} g_i\right) = \bigcap_{i=1}^{m} \left(\operatorname{ri} \operatorname{lev}_{\leq 0} g_i\right).$$

The proof is complete. □

We now present the result on the necessary optimality condition for the nearly convex optimization problem.

Theorem 3.1. *Suppose that $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1,2,\ldots,m$, are nearly convex functions, $\inf_{x\in\mathbb{R}^n} g_i(x) < 0$, for all $i = 1,\ldots,m$, and (3.2) is satisfied. Further suppose that*

$$\operatorname{ri}(\operatorname{dom}\varphi) \cap \operatorname{ri}\left(\bigcap_{i=1}^{m} \operatorname{lev}_{\leq 0} g_i\right) \neq \emptyset. \tag{3.3}$$

Then $\bar{x}$ is a solution of (P) if and only if

$$0 \in \partial\varphi(\bar{x}) + N(\bar{x}; \operatorname{lev}_{\leq 0}\, g_1) + N(\bar{x}; \operatorname{lev}_{\leq 0}\, g_2) + \ldots + N(\bar{x}; \operatorname{lev}_{\leq 0}\, g_m).$$

Proof. We first observe that problem (P) can be rewritten as $\min\{\varphi(x) \mid x \in S\}$, where S is given by (3.1). So $\bar{x}$ is a solution of (P) if and only if

$$0 \in \partial(\varphi + \delta_S)(\bar{x}). \tag{3.4}$$

Assumption (3.3) ensures that $\operatorname{ri}(\operatorname{dom}\varphi) \cap \operatorname{ri} S \neq \emptyset$. Using Proposition 2.3, we have

$$\partial(\varphi + \delta_S)(\bar{x}) = \partial\varphi(\bar{x}) + N(\bar{x}; S). \tag{3.5}$$

Moreover, under condition (3.2) and by applying Proposition 2.1, we obtain

$$N(\bar{x}; S) = \sum_{i=1}^{m} N(\bar{x}; \operatorname{lev}_{\leq 0} g_i). \tag{3.6}$$

Hence, from (3.4), (3.5), and (3.6), $\bar{x}$ is a solution of (P) if and only if

$$0 \in \partial\varphi(\bar{x}) + \sum_{i=1}^{m} N(\bar{x}; \operatorname{lev}_{\leq 0} g_i).$$

This completes the proof. □

We now provide an illustrative example for Theorem 3.1.

Example 3.1. Consider the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\varphi(x) := |x_1| + |x_2|$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and the constraint function $g : \mathbb{R}^2 \to \overline{\mathbb{R}}$ given by

$$g(x) := x_2 + \delta_{[-1,1]^2 \setminus \{(0,1)\}}(x),\ x = (x_1, x_2) \in \mathbb{R}^2$$

where $\delta_A(\cdot)$ denotes the indicator function of the set $A \subset \mathbb{R}^2$. Problem (P) now becomes

$$\min_{x\in\mathbb{R}^2} \varphi(x) \quad \text{subject to} \quad g(x) \leq 0.$$

Clearly, both functions φ and g are proper and nearly convex on $\mathbb{R}^2$. Note that the feasible set is $S = \{x \in \mathbb{R}^2 \mid x_2 \leq 0\} \cap \{[-1,1]^2 \setminus \{(0,1)\}\}$. We claim that $\bar{x} = (0,0)$ is a (global) solution of the problem in question. Indeed, we have $\varphi(0,0) = |0| + |0| = 0$. Since $\varphi(x) \geq 0$ for all

$x \in \mathbb{R}^2$, it follows that $\bar{x}$ minimizes φ over S. Let us verify the necessary optimality condition $0 \in \partial\varphi(\bar{x}) + N(\bar{x}; S)$. Since $\varphi(x) = |x_1| + |x_2|$, we have

$$\partial\varphi(\bar{x}) = \partial|x_1|(0) \times \partial|x_2|(0) = [-1,1] \times [-1,1].$$

In addition, we have $N((0,0);S) = \{(0,\alpha) \mid \alpha \geq 0\}$. Hence,

$$\partial\varphi(\bar{x}) + N(\bar{x};S) = \{(u_1,u_2) + (0,\alpha) \mid u_1,u_2 \in [-1,1],\ \alpha \geq 0\}.$$

It follows that the first coordinate ranges over $[-1,1]$, while the second coordinate is of the form $u_2 + \alpha$ with $u_2 \in [-1,1]$ and $\alpha \geq 0$, which implies that it takes all values greater than or equal to -1. Consequently, $\partial\varphi(\bar{x}) + N(\bar{x};S) = [-1,1] \times [-1,\infty)$. Since $0 = (0,0) \in [-1,1] \times [-1,\infty)$, the necessary optimality condition $0 \in \partial\varphi(\bar{x}) + N(\bar{x};S)$ is satisfied.

4. Application for Strictly Robust Counterpart Problems to Optimization Problems under Uncertainty

Consider an *optimization problem under uncertainty* of the form:

$$\min f_0(x,\xi_0) \ \text{ subject to } \ f_i(x,\xi_i) \leq 0,\ i = 1,...,m, \tag{UP}$$

where $f_0 : \mathbb{R}^n \times U_0 \to \mathbb{R}$, $x \in \mathbb{R}^n$ is the vector of decision variables, $\xi_i \in U_i$, $i = 0,1,..,m$, are uncertain parameters and they belong to the specified compact uncertainty sets $U_i \subset \mathbb{R}^{q_i}$, respectively, and $f_i : \mathbb{R}^n \times U_i \to \mathbb{R}$, $i = 1,2,...,m$, are given functions. Here, uncertainty can be interpreted in a manner that the parameter vectors $\xi_i, i = 0,1,...,m$, are not known exactly at the time of the decision. Throughout this section, we assume that the function $(x,\xi_0) \mapsto f_0(x,\xi_0)$ is upper semi-continuous at each point in $\big(\bigcap_{\xi \in U_0} \operatorname{ri}\big(\operatorname{dom} f_0(\cdot,\xi)\big)\big) \times U_0$, and function $f_i(x,\cdot)$ is upper semi-continuous for each $i = 1,\ldots,m$.

To investigate problem (UP), one usually associates a so-called robust counterpart problem with it. We use the concept of strict robustness introduced by Soyster [32] (and extensively studied; see Ben-Tal, Nemirovski [5] and Ben-Tal, El Ghaoui, Nemirovski [4]):

$$\min_{x \in \mathbb{R}^n} \max_{\xi_0 \in U_0} f_0(x,\xi_0) \ \text{ subject to } \ x \in S, \tag{RCUP}$$

where the feasible set S is defined by

$$S := \{x \in \mathbb{R}^n \mid \forall \xi_i \in U_i :\ f_i(x,\xi_i) \leq 0,\ i = 1,2,\ldots,m\}.$$

Put $F(x) = \max\limits_{\xi_0 \in U_0} f_0(x,\xi_0)$. Then problem (RCUP) can be expressed as

$$\min_{x \in \mathbb{R}^n} F(x) \ \text{ subject to } \ x \in S. \tag{RP1}$$

One calls F the objective function and S the constraint set of (RP1).

We now focus on deriving necessary and sufficient optimality conditions for solutions to the robust counterpart problem (RCUP) under the assumption of nearly convex data.

Theorem 4.1. (Nearly Convex Objective Function and Constraint Set) *Let $\bar{x} \in \mathbb{R}^n$. Assume that the following conditions are satisfied:*

(i) *For every $\xi_0 \in U_0$, the function $f_0(\cdot,\xi_0)$ is nearly convex and $\bigcap_{\xi_0 \in U_0} \operatorname{ri}\big(\operatorname{epi} f_0(\cdot,\xi_0)\big) \neq \emptyset$.*

(ii) *For every $i = 1,2,...,m$ and $\xi_i \in U_i$, $f_i(\cdot,\xi_i)$ is nearly convex, $\inf_{x \in \mathbb{R}^n} \max_{\xi_i \in U_i} f_i(x,\xi_i) < 0$, and $\bigcap_{\xi_i \in U_i} \operatorname{ri}\big(\operatorname{epi} f_i(\cdot,\xi_i)\big) \neq \emptyset$.*

(iii) *The function f_0 is locally Lipschitz in $\bar{x}$ uniformly for $\xi_0 \in U_0$.*

(iv) *For every* $x \in \mathrm{ri}\,(\mathrm{dom}\, f_0(\cdot,\xi_0))$, *the directional derivative* $f'_{0,x}(x,\xi_0;d)$ *exists and is finite for all* $d \in \mathbb{R}^n$, *and the mapping* $\xi_0 \mapsto f'_{0,x}(x,\xi_0;d)$ *is upper-semicontinuous on* U_0.

(v) *The qualification condition*

$$\mathrm{ri}\,(\mathrm{dom} \max_{\xi_0 \in U_0} f_0(\cdot,\xi_0)) \cap \mathrm{ri}\left(\bigcap_{i=1}^{m} \mathrm{lev}_{\leq 0} \max_{\xi_i \in U_i} f_i(\cdot,\xi_i)\right) \neq \emptyset$$

holds.

Then, $\bar{x} \in \mathbb{R}^n$ *is an optimal solution to the robust counterpart problem* (RCUP) *if and only if*

$$0 \in \mathrm{co}\left(\bigcup_{\xi_0 \in \widehat{U_0}(\bar{x})} \partial_x f_0(\bar{x},\xi_0)\right) + \sum_{i=1}^{m} N\Big(\bar{x}; \mathrm{lev}_{\leq 0} f_i(\cdot,\xi_i)\Big). \tag{4.1}$$

Proof. Under the assumption (i), (iii) and (iv), by Theorem 2.1, we obtain

$$\partial F(\bar{x}) = \mathrm{co}\left(\bigcup_{\xi_0 \in \widehat{U_0}(\bar{x})} \partial_x f_0(\bar{x},\xi_0)\right). \tag{4.2}$$

Observe that $S = \{x \in \mathbb{R}^n \mid \max_{\xi_i \in U_i} f_i(x,\xi_i) \leq 0,\ i = 1,2,\ldots,m\}$. From assumption (ii), we see that S is a nearly convex set by Lemma 3.1. Since (v) holds, it follows $\mathrm{ri}\,\mathrm{dom}\, F \cap \mathrm{ri}\, S \neq \emptyset$ immediately. According to Theorem 3.1, $\bar{x}$ is an optimal solution to (RCUP) if and only if

$$0 \in \partial\big(F + \delta_S(\cdot)\big)(\bar{x}) = \partial F(\bar{x}) + N(\bar{x};S).$$

It follows that

$$0 \in \mathrm{co}\left(\bigcup_{\xi_0 \in \widehat{U_0}(\bar{x})} \partial_x f_0(\bar{x},\xi_0)\right) + N(\bar{x};S), \tag{4.3}$$

due to (4.2). Moreover, under condition (v), by applying Proposition 2.1, we obtain

$$N(\bar{x};S) = \sum_{i=1}^{m} N\Big(\bar{x}; \mathrm{lev}_{\leq 0} f_i(\cdot,\xi_i)\Big). \tag{4.4}$$

Hence, $\bar{x} \in \mathbb{R}^n$ be an optimal solution to robust optimization problem (RCUP) if and only if

$$0 \in \mathrm{co}\left(\bigcup_{\xi_0 \in \widehat{U_0}(\bar{x})} \partial_x f_0(\bar{x},\xi_0)\right) + \sum_{i=1}^{m} N\Big(\bar{x}; \mathrm{lev}_{\leq 0} f_i(\cdot,\xi_i)\Big),$$

which verifies (4.1). The proof is complete. □

Theorem 4.2. (Nearly Convex Objective Function and Convex Constraint Set) *Suppose that the following conditions are satisfied:*

(i) *For every* $\xi_0 \in U_0$, $f_0(\cdot,\xi_0)$ *is nearly convex and* $\bigcap_{\xi_0 \in U_0} \mathrm{ri}\,(\mathrm{epi}\, f_0(\cdot,\xi_0)) \neq \emptyset$.

(ii) *The function* f_0 *is locally Lipschitz in* $\bar{x}$ *uniformly for* $\xi_0 \in U_0$.

(iii) *For every* $x \in \mathrm{ri}\,(\mathrm{dom}\, f_0(\cdot,\xi_0))$, *the directional derivative* $f'_{0,x}(x,\xi_0;d)$ *exists and is finite for all* $d \in \mathbb{R}^n$, *and the mapping* $\xi_0 \mapsto f'_{0,x}(x,\xi_0;d)$ *is upper semicontinuous on* U_0.

(iv) *For all* $\xi_i \in U_i$, $i = 1,2,\ldots,m$, $f_i(\cdot,\xi_i)$ *are convex and continuous at* $\bar{x}$ *and there exists* $x_0 \in \mathbb{R}^n$ *such that* $f_i(x_0,\xi_i) < 0$.

(v) *The qualification condition*

$$\operatorname{ri}(\operatorname{dom}\max_{\xi_0\in U_0} f_0(\cdot,\xi_0))\cap \operatorname{ri}\left(\bigcap_{i=1}^{m}\operatorname{lev}_{\le 0}\max_{\xi_i\in U_i} f_i(\cdot,\xi_i)\right)\neq\emptyset$$

is satisfied.

Then, $\bar{x}\in\mathbb{R}^n$ is an optimal solution to the robust counterpart problem (RCUP) *if and only if*

$$0\in\operatorname{co}\left(\bigcup_{\xi_0\in\widehat{U_0}(\bar{x})}\partial_x f_0(\bar{x},\xi_0)\right)+\bigcup_{\xi_i\in U_i,\lambda_i\ge 0}\left\{\sum_{i=1}^{m}\lambda_i a_i, a_i\in\partial_x f_i(\bar{x},\xi_i), \lambda_i f_i(\bar{x},\xi_i)=0\right\}. \tag{4.5}$$

Proof. We note that every convex function is a nearly convex function. Under assumptions (i), (ii), (iii), and (v), we derive from Theorem 4.1 that $\bar{x}$ is an optimal solution to (RCUP) if and only if (4.3) holds. We now establish the expression for $N(\bar{x};S)$:

$$N(\bar{x};S)=\bigcup_{\xi_i\in U_i,\lambda_i\ge 0}\left\{\sum_{i=1}^{m}\lambda_i a_i, a_i\in\partial_x f_i(\bar{x},\xi_i), \lambda_i f_i(\bar{x},\xi_i)=0\right\}. \tag{4.6}$$

The latter is done by using the scheme in the proof of [18, Proposition 2.1] for the convex setting. Combining (4.3) and (4.6), we have (4.5) immediately. □

Remark 4.1. In the case, where the objective function F and the constraint set S are convex, necessary and sufficient conditions for a robust solution have been proved in [18, Proposition 2.1].

We finish this section by an example to illustrative Theorem 4.2.

Example 4.1. Consider $D:=\{x=(x_1,x_2)\in\mathbb{R}^2 \mid \|x\|_\infty\le 1\}\setminus\{(0,1)\}=([-1,1]\times[-1,1])\setminus\{(0,1)\}$, which is nearly convex, and note that $\bar{x}=(0,0)\in\operatorname{int}D$. Let the uncertainty sets be $U_0=\{-1,1\}$, $U_1=[-1,1]$, and define the objective and constraint functions by

$$f_0(x,\xi_0)=\max\{\,|x_1-\xi_0|,\ 0.5|x_1|+0.8|x_2|\,\}+\delta_D(x),\qquad \xi_0\in U_0,$$

$$f_1(x,\xi_1)=x_1+\xi_1 x_2,\qquad \xi_1\in U_1,$$

where $\delta_D(\cdot)$ denotes the indicator function of D. The robust counterpart problem takes the form

$$\min_{x\in\mathbb{R}^2}\max_{\xi_0\in U_0} f_0(x,\xi_0)\qquad\text{subject to}\qquad \max_{\xi_1\in U_1} f_1(x,\xi_1)\le 0.$$

We proceed to verify that the assumptions of Theorem 4.2 are satisfied at $\bar{x}=(0,0)$.

(i) For each $\xi_0\in U_0$, the function $x\mapsto\max\{|x_1-\xi_0|,\ 0.5|x_1|+0.8|x_2|\}$ is convex; hence $f_0(\cdot,\xi_0)$ is nearly convex due to the nearly convexity of the indicator $\delta_D(\cdot)$. Moreover, since $\bar{x}\in\operatorname{int}D$, each $f_0(\cdot,\xi_0)$ is continuous at $\bar{x}$ and $\bigcap_{\xi_0\in U_0}\operatorname{ri}(\operatorname{epi}f_0(\cdot,\xi_0))\neq\emptyset$.

(ii) Each $f_0(\cdot,\xi_0)$ is locally Lipschitz in $\bar{x}$ uniformly for $\xi_0\in U_0$, since absolute values are Lipschitz and $\delta_D(\cdot)$ vanishes near $\bar{x}$.

(iii) For each ξ_0, the directional derivatives $f'_{0,x}(x,\xi_0;d)$ exist and are finite at $\bar{x}$; since U_0 is finite, the mapping $\xi_0\mapsto f'_{0,x}(\bar{x},\xi_0;d)$ is trivially upper-semicontinuous.

(iv) For every $\xi_1\in U_1$, $f_1(\cdot,\xi_1)$ is affine and hence convex and continuous. The mapping $\xi_1\mapsto f_1(x,\xi_1)$ is continuous (thus u.s.c.) on the compact set U_1. Choosing $x^0=(-1,0)\in D$, we have $f_1(x^0,\xi_1)=-1<0$ for all $\xi_1\in U_1$, so the Slater condition holds.

(v) The effective domain of the objective is $\operatorname{dom}\left(\max_{\xi_0\in U_0} f_0(x,\xi_0)\right) = D$. Hence

$$\operatorname{ri}\left(\operatorname{dom}\max_{\xi_0\in U_0} f_0(x,\xi_0)\right) = \operatorname{int} D.$$

Moreover, $\max_{\xi_1\in[-1,1]} f_1(x,\xi_1) = \max_{\xi_1\in[-1,1]}(x_1+\xi_1 x_2) = x_1+|x_2|$, so the lower level set is

$$\operatorname{lev}_{\le 0}\max_{\xi_1\in[-1,1]} f_1(x,\xi_1) = \{(x_1,x_2)\in\mathbb{R}^2 \mid x_1+|x_2|\le 0\},$$

with relative interior

$$\operatorname{ri}\left(\operatorname{lev}_{\le 0}\max_{\xi_1\in[-1,1]} f_1(x,\xi_1)\right) = \{(x_1,x_2)\in\mathbb{R}^2 \mid x_1+|x_2|< 0\}.$$

Thus,

$$\operatorname{ri}\left(\operatorname{dom}\max_{\xi_0\in U_0} f_0(x,\xi_0)\right)\cap \operatorname{ri}\left(\operatorname{lev}_{\le 0}\max_{\xi_1\in U_1} f_1(\cdot,\xi_1)\right)\neq\emptyset.$$

On D, the indicator $\delta_D(\cdot)$ vanishes. At $\bar{x}$, we have

$$f_0(\bar{x},1) = \max\{|0-1|,0\} = 1, \qquad f_0(\bar{x},-1) = \max\{|0+1|,0\} = 1,$$

so $\widehat{U}_0(\bar{x}) = \{-1,1\}$. On the one hand, we have

$$\begin{aligned} F(x) &= \max\{\max(|x_1-1|, 0.5|x_1|+0.8|x_2|),\ \max(|x_1+1|, 0.5|x_1|+0.8|x_2|)\} \\ &= \max\{|x_1-1|,\ |x_1+1|\} = 1+|x_1|. \end{aligned}$$

Hence $\partial F(\bar{x}) = \partial|x_1|(0)\times\{0\} = [-1,1]\times\{0\}$. On the other hand, $\max_{\xi_1\in[-1,1]}(x_1+\xi_1 x_2) = x_1+|x_2|$. Hence, $S = \{x\in\mathbb{R}^2 \mid x_1+|x_2|\le 0\}$. By the definition, we have

$$N(\bar{x};S) = \{(v_1,v_2)\in\mathbb{R}^2 \mid v_1\ge 0, |v_2|\le v_1\}.$$

Consequently, $0\in\partial F(\bar{x})+N(\bar{x};S)$, which verifies Theorem 4.2.

5. Conclusions

In this paper, we derived necessary and sufficient optimality conditions for solutions of nearly convex optimization problems and applied these results to strictly robust counterpart problems in optimization under uncertainty.

For further research, it is of interest to consider other concepts for robust counterpart problems to an optimization problem under uncertainty and to derive corresponding optimality conditions using the unifying approaches to robustness introduced in [19] and [20].

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