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ANALYSIS OF A STOKES VARIATIONAL-HEMIVARIATIONAL INEQUALITY FOR INCOMPRESSIBLE FLUID FLOWS WITH DAMPING

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Abstract. In this paper, a Stokes variational-hemivariational inequality is studied for incompressible viscous fluid flows subject to a damping effect and non-leak, nonsmooth slip boundary conditions of friction type. Well-posedness of the Stokes variational-hemivariational inequality is established through the consideration of an auxiliary problem, which is also a Stokes variational-hemivariational inequality. The existence of the velocity field is shown through a minimization principle for the reduced formulation of the problem in which the pressure field is eliminated. Then, the pressure field is recovered through the use of the inf-sup condition. Uniqueness of the solution is also established, as well as the continuous dependence of the solution on the source term of the Stokes equations.

Keywords. Minimization principle; Nonsmooth slip boundary conditions of friction type; Stokes equations with damping; Variational-hemivariational inequality; Well-posedness.
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1. INTRODUCTION

Hemivariational inequalities are a generalization of variational inequalities in that the nonsmooth terms in the formulations are allowed to be non-convex. More generally, an inequality problem may contain explicitly both nonsmooth convex terms and nonsmooth non-convex terms, and it is reasonable to call the problem a variational-hemivariational inequality. In this paper, we also use the term variational-hemivariational inequality for a "pure" hemivariational inequality. This area was started in early 1980s by Panagiotopoulos [40]. Early comprehensive references in the area are [38, 41]. In the recent years, modeling, mathematical analysis, numerical computations and applications of variational-hemivariational inequalities have attracted ever more attention from the research communities. Recent summarizing accounts of mathematical theories of variational-hemivariational inequalities include [2, 3, 37, 45]. Since there is no analytic solution formula for the solution of a variational-hemivariational inequality, numerical methods are needed to solve variational-hemivariational inequalities. An early comprehensive reference on the finite element method and solution algorithms for variational-hemivariational inequalities is [25]. Since the first optimal order error estimate for the numerical

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solution of variational-hemivariational inequalities is presented in [22], many papers have been published on the numerical solution and error analysis of numerical solutions of variational-hemivariational inequalities. The reader is referred to the survey papers [21] and [24] for summary accounts of numerical analysis of variational-hemivariational inequalities. The latest survey paper [19] provides a concise overview of mathematical theory and numerical analysis of stationary variational-hemivariational inequalities at an elementary level for beginners of the area.

The area of boundary value problems for incompressible viscous fluid flows subject to nonsmooth slip or leak boundary conditions of friction type has attracted steady attention from the research communities since the pioneering work of Fujita in early 1990s ([13, 14]). Numerous publications can be consulted, e.g., [15, 29, 31, 44] on the modeling and analysis of such problems, and [28, 30, 32, 33] on numerical methods for solving the problems. In these references, the slip or leak boundary conditions are modeled by monotone relations and the weak formulations are variational inequalities governed by the Stokes or Navier-Stokes equations. When the slip or leak boundary conditions involve non-monotone relations, the weak formulations of the corresponding problems become variational-hemivariational inequalities, and a few papers are devoted to this topic; see, e.g., [11, 12, 35, 36].

Damping effects are important in the modeling of fluid flows with resistance. The Stokes and Navier–Stokes equations with damping are widely employed in the resistance to the motion of the flow. It depicts various physical phenomena such as drag effects, viscous flow, and some dissipative systems ([6, 7, 43]). Analysis and numerical methods for a Stokes variational inequality for an incompressible viscous fluid with a damping effect is found in [42]. A Stokes hemivariational inequality for an incompressible viscous fluid flow with a damping effect is analyzed and numerically investigated in [23]. Analysis of a Navier–Stokes variational-hemivariational inequality for incompressible fluid flows with damping, together with its finite element approximation, is available in [47]. This paper is devoted to the study of a Stokes variational-hemivariational inequality for a viscous incompressible fluid flow with a damping effect that is subject to a nonsmooth, not necessarily monotone, slip boundary condition of friction type.

In most references on variational-hemivariational inequalities, the existence of a solution is proved through an application of an abstract surjectivity result for pseudomonotone operators. For an approach without the need of the notion and results of pseudomonotone operators, the reader is referred to the book [18] for an accessible coverage of the mathematical theory for variational-hemivariational inequalities, based on the techniques developed in [16, 17]. This approach starts with an exploration of minimization principles for special variational-hemivariational inequalities, and extends the analysis to general variational-hemivariational inequalities via fixed-point arguments. A Stokes hemivariational inequality is studied through the alternative approach in [34] in which a minimization principle is introduced for the Stokes hemivariational inequality. In this paper, we will study a Stokes variational-hemivariational inequality for an incompressible fluid flow with damping through the accessible approach. In particular, we will establish a minimization principle for the reduced formulation of the Stokes variational-hemivariational inequality in which the pressure field is eliminated.

For the formulation and study of variational-hemivariational inequalities, we need the notions of the generalized directional derivative and generalized subdifferential developed by Clarke

([8]). For a locally Lipschitz continuous functional $\Psi\colon V \to \mathbb{R}$ defined on a real Banach space V, the generalized (Clarke) directional derivative of Ψ at $u \in V$ in the direction $v \in V$ is defined by

$$\Psi^0(u;v) := \limsup_{w \to u,\, \lambda \downarrow 0} \frac{\Psi(w+\lambda v) - \Psi(w)}{\lambda},$$

where the generalized subdifferential of Ψ at $u \in V$ is

$$\partial\Psi(u) := \left\{ \eta \in V^* \mid \Psi^0(u;v) \geq \langle \eta, v \rangle \ \forall v \in V \right\}.$$

In the case where $\Psi\colon V \to \mathbb{R}$ is convex, we recall that the (convex) subdifferential of Ψ at an element $u \in V$ is defined to be the set

$$\partial\Psi(u) = \{ u^* \in V^* \mid \Psi(v) \geq \Psi(u) + \langle u^*, v-u \rangle \ \forall v \in V \}.$$

See [9] for detailed discussions of the subdifferential in the convex analysis. Basic properties of the generalized directional derivative and the generalized subdifferential can be found in [8]. We record below some of the basic properties needed in this paper.

Proposition 1.1. (i) *If $\Psi\colon V \to \mathbb{R}$ is locally Lipschitz continuous and convex, then the generalized subdifferential $\partial\Psi(u)$ at any $u \in V$ in the sense of Clarke coincides with the convex subdifferential $\partial\Psi(u)$.*

(ii) *Let Ψ be a locally Lipschitz function. Then $\partial(\lambda\Psi)(u) = \lambda\,\partial\Psi(u)$ for all $\lambda \in \mathbb{R}$ and all $u \in V$. Moreover, Ψ^0 is positively homogeneous and subadditive, i.e.,*

$$\Psi^0(u;\lambda v) = \lambda\,\Psi^0(u;v) \quad \forall \lambda \geq 0,\ u, v \in V,$$
$$\Psi^0(u;v_1+v_2) \leq \Psi^0(u;v_1) + \Psi^0(u;v_2) \quad \forall u, v_1, v_2 \in V.$$

(iii) *Let $\Psi_1, \Psi_2\colon V \to \mathbb{R}$ be locally Lipschitz functions. Then the inclusion*

$$\partial(\Psi_1+\Psi_2)(u) \subseteq \partial\Psi_1(u) + \partial\Psi_2(u) \quad \forall u \in V \tag{1.1}$$

holds, or equivalently, $(\Psi_1+\Psi_2)^0(u;v) \leq \Psi_1^0(u;v) + \Psi_2^0(u;v)$ for all $u, v \in V$.

Because of Proposition 1.1 (i), we use the same symbol ∂ to mean both the convex subdifferential and the Clarke subbdifferential.

The rest of the paper is organized as follows. In Section 2, we describe the physical setting of the fluid flow problem and introduce its weak formulation as a Stokes variational-hemivariational inequality. In Section 3, we analyze a related auxiliary problem, which is itself also a Stokes variational-hemivariational inequality. In the study of the auxiliary problem, we start with an analysis of its reduced form where the pressure variable is eliminated. We introduce an equivalent minimization principle for the reduced formulation based on which, we conclude the solution existence and uniqueness of the auxiliary problem. In Section 4, we present a well-posedness result on the Stokes variational-hemivariational inequality by making use of the results proved for the auxiliary problem.

2. The Stokes Variational-hemivariational Inequality

In this section, we describe the classical pointwise formulation of a Stokes fluid flow problem and introduce its weak formulation as a Stokes variational-hemivariational inequality.

The physical setting of the problem is as follows. We are interested in the velocity field u and the pressure field p of an incompressible fluid flow subject to non-leak, monotone and/or

nonmonotone nonsmooth slip boundary conditions of friction type on parts of the boundary. We take into consideration the damping effect on the fluid. The fluid occupies a Lipschitz domain $\Omega \subset \mathbb{R}^d$. For applications, the dimension $d \leq 3$. Denote the unit outward normal vector on the boundary $\Gamma = \partial\Omega$ by $\nu = (\nu_1, \cdots, \nu_d)^T$, which is defined a.e. For a vector-valued function v on the boundary, its normal and tangential components are $v_\nu = v \cdot \nu$ and $v_\tau = v - v_\nu \nu$, respectively. With the velocity field u and the pressure p, we define the deformation rate tensor $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and the stress tensor $\sigma = -pI + 2\mu\varepsilon(u)$, where $\mu > 0$ is the viscosity coefficient of the fluid, and I is the identity matrix. We call $\sigma_\nu = \nu \cdot \sigma\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$ the normal and tangential components of σ on the boundary. The boundary Γ is split into three measurable parts where different kind of boundary conditions will be imposed: $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_{S,1}} \cup \overline{\Gamma_{S,2}}$ with Γ_D, $\Gamma_{S,1}$ and $\Gamma_{S,2}$ disjoint and relatively open, $|\Gamma_D| > 0$ and $|\Gamma_{S,1}| + |\Gamma_{S,2}| > 0$. Denote $\Gamma_S = \Gamma_{S,1} \cup \Gamma_{S,2}$.

While the velocity field u takes values in $\mathbb{R}^d$, the deformation rate tensor $\varepsilon = \varepsilon(u)$ and the stress tensor σ take values in $\mathbb{S}^d$, the space of second order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order d. The canonical inner products and norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are

$$
\begin{aligned}
&u \cdot v = u_i v_i, \quad |v| = (v \cdot v)^{1/2} \quad \text{for all } u = (u_i),\ v = (v_i) \in \mathbb{R}^d,\\
&\sigma : \tau = \sigma_{ij}\tau_{ij}, \quad |\sigma| = (\sigma : \sigma)^{1/2} \quad \text{for all } \sigma = (\sigma_{ij}),\ \tau = (\tau_{ij}) \in \mathbb{S}^d.
\end{aligned}
$$

Here and below, we adopt the summation convention over a repeated index, and indices i and j run between 1 and d.

The stationary Stokes equations with damping are as follows ([6, 7, 43]):

$$
-\operatorname{div}(2\mu\varepsilon(u)) + \alpha|u|^{r-2}u + \nabla p = f \quad \text{in } \Omega, \tag{2.1}
$$

$$
\operatorname{div} u = 0 \quad \text{in } \Omega. \tag{2.2}
$$

In (2.1), f is a given force density function. The damping term is $\alpha|u|^{r-2}u$, where $\alpha > 0$ and $r \geq 2$ are two damping parameters. Equations (2.1)–(2.2) are supplemented by the following boundary conditions

$$
u = 0 \quad \text{on } \Gamma_D,
$$

$$
u_\nu = 0, \quad -\sigma_\tau \in \partial\phi(u_\tau) \quad \text{on } \Gamma_{S,1}, \tag{2.3}
$$

$$
u_\nu = 0, \quad -\sigma_\tau \in \partial\psi(u_\tau) \quad \text{on } \Gamma_{S,2}. \tag{2.4}
$$

Here, $\phi : \mathbb{R}^d \to \mathbb{R}$ is a convex and continuous potential function, and $\partial\phi$ is the convex subdifferential of ϕ; $\psi : \mathbb{R}^d \to \mathbb{R}$ is a locally Lipschitz continuous potential function, and $\partial\psi$ is the subdifferential of ψ in the sense of Clarke. The relations (2.3) and (2.4) are known as non-leak, slip boundary conditions of friction type. The first parts of the boundary conditions (2.3) and (2.4) indicate that the fluid can not pass through Γ_S outside the domain, whereas the second parts of (2.3) and (2.4) represent monotone and non-monotone friction-type conditions for the friction σ_τ in terms of the tangential velocity u_τ.

The problem (2.1)–(2.4) will be studied in its weak form. We will use the following function spaces for the velocity variable and for the pressure variable:

$$V = \left\{ v \in H^1(\Omega;\mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_D,\, v_\nu = 0 \text{ on } \Gamma_S \right\},$$
$$Q = \left\{ q \in L^2(\Omega) \mid (q,1)_{L^2(\Omega)} = 0 \right\}.$$

By assumption, $|\Gamma_D| > 0$. This allows us to apply Korn's inequality: there exists a constant $c > 0$ depending only on Ω and Γ_D such that $\|v\|_{H^1(\Omega;\mathbb{R}^d)} \le c\,\|\varepsilon(v)\|_{L^2(\Omega;\mathbb{S}^d)}$ for all $v \in V$. A proof of this result can be found in numerous publications; see, e.g., [39, p. 79]. Consequently, $\|\varepsilon(\cdot)\|_{L^2(\Omega;\mathbb{S}^d)}$ defines a norm over the space V and this norm is equivalent to the standard $H^1(\Omega;\mathbb{R}^d)$-norm over the space V. We use the norm $\|\cdot\|_V = \|\varepsilon(\cdot)\|_{L^2(\Omega;\mathbb{S}^d)}$ on V. The following trace inequality holds:

$$\|v_\tau\|_{L^2(\Gamma_{S,2};\mathbb{R}^d)} \le \lambda_0^{-1/2}\|v\|_V \quad \forall v \in V, \tag{2.5}$$

where $\lambda_0 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$u \in V, \quad \int_\Omega \varepsilon(u) : \varepsilon(v)\,dx = \lambda \int_{\Gamma_{S,2}} u_\tau \cdot v_\tau\,ds \quad \forall v \in V.$$

Define

$$a(w;u,v) = a_0(u,v) + a_1(w;u,v), \quad \forall w,u,v \in V,$$

where

$$a_0(u,v) = 2\mu \int_\Omega \varepsilon(u) : \varepsilon(v)\,dx,\ a_1(w;u,v) = \alpha \int_\Omega |w|^{r-2}u\cdot v\,dx \quad \forall w,u,v \in V.$$

Define $b(v,q) = -\int_\Omega q\,\mathrm{div} v\,dx$ for all $v \in V$, $q \in Q$. Assume $f \in V^*$ and

$$2 \le r < \infty \text{ if } d = 2, \quad 2 \le r \le 6 \text{ if } d = 3, \tag{2.6}$$

which ensures $H^1(\Omega) \hookrightarrow L^r(\Omega)$ by the Sobolev embedding theorem. We will use this embedding property needed in our study of the weak formulations.

Concerning the potentials ϕ and ψ, we assume the following:

$H(\phi)$. $\phi : \mathbb{R}^d \to \mathbb{R}$ is convex and continuous.

$H(\psi)$. $\psi : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz, and there exist constants $\alpha_\psi, c_0, c_1 \ge 0$ such that

$$\begin{aligned} &\psi^0(\xi_1;\xi_2-\xi_1) + \psi^0(\xi_2;\xi_1-\xi_2) \le \alpha_\psi|\xi_1-\xi_2|^2 \quad \forall\, \xi_1,\xi_2 \in \mathbb{R}^d, \\ &|\eta| \le c_0 + c_1|\xi| \quad \forall \xi \in \mathbb{R}^d, \eta \in \partial\psi(\xi). \end{aligned} \tag{2.7}$$

It is known ([37]) that (2.7) is equivalent to

$$(\eta_1-\eta_2)\cdot(\xi_1-\xi_2) \ge -\alpha_\psi|\xi_1-\xi_2|^2 \quad \forall \xi_i \in \mathbb{R}^d, \eta_i \in \partial\psi(\xi_i),\ i = 1,2.$$

We notice that in the literature (e.g., [13, 14, 15, 32, 33]), a popular choice of the convex function satisfying $H(\phi)$ is $\phi(\xi) = g|\xi|$, $g > 0$, $\xi \in \mathbb{R}^d$. Some examples of non-convex functions ψ satisfying $H(\psi)$ can be found in [24, pp. 186–187].

The next result summarizes some elementary properties (cf. [5, Section 5.3]).

Lemma 2.1. *For $r \ge 2$,*

$$\begin{aligned} &\left(|\eta|^{r-2}\eta - |\xi|^{r-2}\xi\right)\cdot(\eta-\xi) \ge 0 \quad \forall \xi,\eta \in \mathbb{R}^d, \\ &\left||\eta|^{r-2}\eta - |\xi|^{r-2}\xi\right| \le c\,(|\eta|+|\xi|)^{r-2}|\eta-\xi| \quad \forall \xi,\eta \in \mathbb{R}^d. \end{aligned} \tag{2.8}$$

We will use some elementary properties of the form $a_1(\cdot;\cdot,\cdot)$ given next ([23, Lemma 2.2]).

Lemma 2.2. *For r in the range specified in (2.6),*

$$|a_1(u;u,v)| \le c\,\|u\|^{r-1}_{L^r(\Omega)^d}\|v\|_V \quad \forall u,v \in V, \tag{2.9}$$

$$|a_1(u;u,w)-a_1(v;v,w)| \le c\left(\|u\|^{r-2}_{L^r(\Omega)^d}+\|v\|^{r-2}_{L^r(\Omega)^d}\right)\|u-v\|_V\|w\|_V \quad \forall u,v,w \in V. \tag{2.10}$$

Due to the Sobolev embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$, we can replace (2.10) by

$$|a_1(u;u,w)-a_1(v;v,w)| \le c\left(\|u\|^{r-2}_{V}+\|v\|^{r-2}_{V}\right)\|u-v\|_V\|w\|_V \quad \forall u,v,w \in V.$$

Through a standard procedure (cf. [20] for a similar problem without the damping term), the following mixed weak formulation can be derived for problem (2.1)–(2.4).

Problem 2.1. Find $(u,p) \in V \times Q$ such that

$$\begin{aligned} &a(u;u,v-u)+b(v-u,p)+\int_{\Gamma_{S,1}}[\phi(v_\tau)-\phi(u_\tau)]\,ds+\int_{\Gamma_{S,2}}\psi^0(u_\tau;v_\tau-u_\tau)\,ds \\ &\qquad \ge \langle f, v-u\rangle \quad \forall v \in V, \end{aligned} \tag{2.11}$$

$$b(u,q)=0 \quad \forall q \in Q. \tag{2.12}$$

In general, the weak formulation of problem (2.1)–(2.4) is a mixed variational-hemivariational inequality (2.11)–(2.12). If $|\Gamma_{S,2}|=0$, we have a special case of a mixed variational inequality for which (2.11) is reduced to

$$a(u;u,v-u)+b(v-u,p)+\int_{\Gamma_{S,1}}[\phi(v_\tau)-\phi(u_\tau)]\,ds \ge \langle f, v-u\rangle \quad \forall v \in V.$$

If $|\Gamma_{S,1}|=0$, we have a special case of a "pure" mixed hemivariational inequality for which (2.11) is reduced to

$$a(u;u,v)+b(v,p)+\int_{\Gamma_{S,2}}\psi^0(u_\tau;v_\tau)\,ds \ge \langle f, v\rangle \quad \forall v \in V.$$

3. Analysis of an Auxiliary Stokes Variational-Hemivariational Inequality

In this section, we introduce and analyze an auxiliary mixed variational-hemivariational inequality for Problem 2.1.

We introduce two functionals $\Phi,\Psi\colon V \to \mathbb{R}$. Similar to $H(\phi)$ and $H(\psi)$, we make the following assumptions on Φ and Ψ.

$H(\Phi)$. $\Phi\colon V \to \mathbb{R}$ is convex and continuous.

$H(\Psi)$. $\Psi\colon V \to \mathbb{R}$ is locally Lipschitz, and there exist constants $\alpha_\Psi, C_0, C_1 \ge 0$ such that

$$\begin{aligned} &\Psi^0(v_1;v_2-v_1)+\Psi^0(v_2;v_1-v_2) \le \alpha_\Psi|v_1-v_2|^2 \quad \forall\, v_1,v_2 \in V, \\ &|\eta| \le C_0+C_1|v| \quad \forall v \in V,\ \eta \in \partial\Psi(v). \end{aligned} \tag{3.1}$$

Then, we introduce the following auxiliary Stokes variational-hemivariational inequality.

Problem 3.1. Find $(u,p) \in V \times Q$ such that

$$\begin{aligned} &a(u;u,v-u)+b(v-u,p)+\Phi(v)-\Phi(u)+\Psi^0(u;v-u) \\ &\qquad \ge \langle f, v-u\rangle \quad \forall v \in V, \end{aligned} \tag{3.2}$$

$$b(u,q)=0 \quad \forall q \in Q. \tag{3.3}$$

Problem 3.1 will be studied in two steps. In the first step, we consider a reduced problem for the velocity variable u by eliminating the pressure variable p in Problem 3.1. In the second step, we recover the pressure variable p with the help of an inf-sup property.

Define a subspace of V by $\tilde{V} = \{v \in V \mid b(v,q) = 0 \ \forall q \in Q\}$. Observe that (3.3) implies $u \in \tilde{V}$. By restricting v to be in $\tilde{V}$ in (3.2), we obtain the following reduced formulation of Problem 3.1.

Problem 3.2. Find $u \in \tilde{V}$ such that

$$a(u;u,v-u) + \Phi(v) - \Phi(u) + \Psi^0(u;v-u) \geq \langle f, v-u\rangle \quad \forall v \in \tilde{V}. \tag{3.4}$$

We will study Problem 3.2 through an equivalent minimization reformulation. We introduce an energy functional

$$E(v) = \frac{1}{2}a_0(v,v) + \frac{\alpha}{r}\int_\Omega |v|^r dx + \Phi(v) + \Psi(v) - \langle f, v\rangle, \quad v \in \tilde{V}, \tag{3.5}$$

and consider the following minimization problem.

Problem 3.3. Find $u \in \tilde{V}$ such that $E(u) = \inf\{E(v) \mid v \in \tilde{V}\}$.

We will show that Problem 3.3 has a unique solution under certain assumptions. Let us recall two results, the first one being from [10, Proposition 3.1] whereas the second being proved in [16].

Lemma 3.1. *Let V be a real Banach space and let $g: V \to \mathbb{R}$ be locally Lipschitz continuous. Then g is strongly convex on V with a constant $\kappa > 0$, i.e.,*

$$g(\lambda u + (1-\lambda)v) \leq \lambda g(u) + (1-\lambda)g(v) - \kappa\lambda(1-\lambda)\|u-v\|_V^2 \quad \forall u,v \in V, \ \forall \lambda \in [0,1],$$

if and only if ∂g is strongly monotone on V with a constant 2κ, i.e.,

$$\langle \xi - \eta, u - v\rangle \geq 2\kappa\|u-v\|_V^2 \quad \forall u,v \in V, \ \xi \in \partial g(u), \ \eta \in \partial g(v).$$

Proposition 3.1. *Let V be a real Hilbert space and let $g: V \to \mathbb{R}$ be a locally Lipschitz continuous and strongly convex functional on V with a constant $\kappa > 0$. Then there exist two constants $\bar{c}_0$ and $\bar{c}_1$ such that $g(v) \geq \kappa\|v\|_V^2 + \bar{c}_0 + \bar{c}_1\|v\|_V \quad \forall v \in V$. In particular, $g(\cdot)$ is coercive on V.*

For convenience, define operators $A_0, A_1 : V \to V^*$ by

$$\langle A_0 u, v\rangle = a_0(u,v) \quad \forall u,v \in V,$$

$$\langle A_1 v, w\rangle = \alpha\int_\Omega |v|^{r-2}v\cdot w\,dx \quad \forall v,w \in V.$$

Note that

$$\langle A_0 v, v\rangle = a_0(v,v) = 2\mu\|v\|_V^2 \quad \forall v \in V. \tag{3.6}$$

The main result on Problem 3.3 is the following.

Theorem 3.4. *Assume $H(\Phi)$, $H(\Psi)$ and $\alpha_\Psi < 2\mu$. Then for any $f \in V^*$, Problem 3.3 has a unique solution.*

Proof. Let us show that $E(\cdot)$ is strongly convex on $\tilde{V}$. Evidently, $E: \tilde{V} \to \mathbb{R}$ is locally Lipschitz continuous since each term in the definition (3.5) of $E(\cdot)$ is locally Lipschitz continuous. For its generalized subdifferential, we apply (1.1) and Proposition 1.1 (i) on Φ to obtain

$$\partial E(v) \subset A_0 v + A_1 v + \partial\Phi(v) + \partial\Psi(v) - f, \tag{3.7}$$

Let $v_1, v_2 \in \tilde{V}$ and let $\zeta_1 \in \partial E(v_1)$, $\zeta_2 \in \partial E(v_2)$. From the inclusion (3.7), we know that for some elements $\xi_i \in \partial\Phi(v_i)$, $\eta_i \in \partial\Psi(v_i)$, $i = 1,2$,

$$\zeta_1 = A_0 v_1 + A_1 v_1 + \xi_1 + \eta_1 - f,\ \zeta_2 = A_0 v_2 + A_1 v_2 + \xi_2 + \eta_2 - f.$$

Then,

$$\begin{aligned}\langle \zeta_1 - \zeta_2, v_1 - v_2\rangle &= a_0(v_1 - v_2, v_1 - v_2) + \langle A_1 v_1 - A_1 v_2, v_1 - v_2\rangle \\ &\quad + \langle \xi_1 - \xi_2, v_1 - v_2\rangle + \langle \eta_1 - \eta_2, v_1 - v_2\rangle.\end{aligned}$$

By (3.6), we have $a_0(v_1 - v_2, v_1 - v_2) = 2\mu \|v_1 - v_2\|_V^2$. In view of (2.8), we obtain $\langle A_1 v_1 - A_1 v_2, v_1 - v_2\rangle \geq 0$. Since Φ is convex, we arrive at $\langle \xi_1 - \xi_2, v_1 - v_2\rangle \geq 0$. By (4.4), we conclude $\langle \eta_1 - \eta_2, v_1 - v_2\rangle \geq -\alpha_\Psi \|v_1 - v_2\|_V^2$. Hence, $\langle \zeta_1 - \zeta_2, v_1 - v_2\rangle \geq (2\mu - \alpha_\Psi)\|v_1 - v_2\|_V^2$. Since $\alpha_\Psi < 2\mu$, by Lemma 3.1, we see that energy functional E is strongly convex on $\tilde{V}$. Applying Proposition 3.1, we deduce that E is coercive on $\tilde{V}$. From a well-known result on convex minimization [1, Subsection 3.3.2], we see that E has a unique minimizer u on $\tilde{V}$, i.e., Problem 3.3 has a unique solution. □

Then we study Problem 3.2 by relating it to Problem 3.3.

Theorem 3.5. *Assume $H(\Phi)$, $H(\Psi)$, $\alpha_\Psi < 2\mu$, and $f \in V^*$. Then, Problem 3.2 has a unique solution which is also the unique solution of Problem 3.3.*

Proof. We already know from Theorem 3.4 that, under the stated assumptions, Problem 3.3 has a unique solution $u \in \tilde{V}$. Since $E(\cdot)$ is strongly convex, the solution $u \in \tilde{V}$ of Problem 3.3 is characterized by the inclusion $0 \in \partial E(u)$. Using inclusion property (3.7), $0 \in A_0 u + A_1 u + \partial\Phi(u) + \partial\Psi(u) - f$. Thus the solution $u \in \tilde{V}$ of Problem 3.3 satisfies

$$\langle A_0 u, v - u\rangle + \langle A_1 u, v - u\rangle + \Phi(v) - \Phi(u) + \Psi^0(u; v - u) \geq \langle f, v - u\rangle \quad \forall v \in \tilde{V}.$$

Hence, the solution of Problem 3.3 is also a solution to Problem 3.2.

It remains to show the uniqueness of a solution to Problem 3.2. Suppose that $u_1, u_2 \in \tilde{V}$ are two solutions of Problem 3.2: for all $v \in \tilde{V}$,

$$a(u_1; u_1, v - u_1) + \Phi(v) - \Phi(u_1) + \Psi^0(u_1; v - u_1) \geq \langle f, v - u_1\rangle, \tag{3.8}$$

$$a(u_2; u_2, v - u_2) + \Phi(v) - \Phi(u_2) + \Psi^0(u_2; v - u_2) \geq \langle f, v - u_2\rangle. \tag{3.9}$$

Taking $v = u_2$ in (3.8), $v = u_1$ in (3.9), and adding the two inequalities yield

$$\begin{aligned}a_0(u_1 - u_2, u_1 - u_2) &\leq a_1(u_1; u_1, u_2 - u_1) + a_1(u_2; u_2, u_1 - u_2) \\ &\quad + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2).\end{aligned} \tag{3.10}$$

By (3.6), we have $a_0(u_1 - u_2, u_1 - u_2) = 2\mu \|u_1 - u_2\|_V^2$. By (2.8), we obtain

$$a_1(u_1; u_1, u_2 - u_1) + a_1(u_2; u_2, u_1 - u_2) \leq 0.$$

It follows from (3.1) that

$$\Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2) \leq \alpha_\Psi \|u_1 - u_2\|_V^2.$$

Then, we derive from (3.10) that $2\mu \|u_1 - u_2\|_V^2 \leq \alpha_\Psi \|u_1 - u_2\|_V^2$. Since $\alpha_\Psi < 2\mu$ by assumption, we deduce from the above inequality that $u_1 = u_2$. Hence, Problem 3.2 and Problem 3.3 are equivalent, each problem admits a unique solution, and the solutions of the two problems are the same. □

Now we proceed to the second step in the analysis of Problem 3.1 by recovering the pressure variable p.

Theorem 3.6. *Let $H(\Phi)$, $H(\Psi)$ and $\alpha_\Psi < 2\mu$. Then, for any $f \in V^*$, Problem 3.1 has a unique solution $(u,p) \in V \times Q$, and for a constant $c > 0$,*

$$\|u\|_V \le c\,(1 + \|f\|_{V^*})\,, \tag{3.11}$$

$$\|u\|_{L^r(\Omega)^d} \le c\left(1 + \|f\|_{V^*}^{2/r}\right), \tag{3.12}$$

$$\|p\|_Q \le c\left(1 + \|f\|_{V^*}^{2(r-1)/r}\right). \tag{3.13}$$

Moreover, $u \in V$ depends Lipschitz continuously on $f \in V^*$, and $p \in Q$ depends locally Lipschitz continuously on $f \in V^*$.

Proof. Let $u \in \tilde{V}$ be the unique solution of Problem 3.2. Let $V_0 = H_0^1(\Omega)^d$ and

$$\tilde{V}_0 = \{v \in V_0 \mid b(v,q) = 0\ \forall q \in Q\}\,.$$

From (3.4), we have

$$a(u;u,v) = \langle f, v\rangle \quad \forall v \in \tilde{V}_0. \tag{3.14}$$

Thanks to the inf-sup property ([27, Theorem 3.46], [46]),

$$\beta_0 \|q\|_Q \le \sup_{v \in V_0} \frac{b(v,q)}{\|v\|_V} \quad \forall q \in Q, \tag{3.15}$$

we derive from (3.14) the existence of an element $p \in Q$ such that

$$a(u;u,v) + b(v,p) = \langle f, v\rangle \quad \forall v \in V_0. \tag{3.16}$$

For any $v \in V$, due to the inf-sup condition (3.15), we can find an element $v_1 \in V_0$ such that

$$b(v_1,q) = b(v,q) \quad \forall q \in Q. \tag{3.17}$$

Define $v_2 = v - v_1$. Then, $v_2 \in V$ and $b(v_2,q) = 0$ for all $q \in Q$. Thus $v_2 \in \tilde{V}$. We use $v_2 = v - v_1$ in (3.4) to obtain

$$a(u;u,v-v_1-u) + \Phi(v-v_1) - \Phi(u) + \Psi^0(u;v-v_1-u) \ge \langle f, v-v_1-u\rangle.$$

Since $v_1 = 0$ on Γ_S, we can rewrite the above inequality as

$$a(u;u,v-u) + \Phi(v) - \Phi(u) + \Psi^0(u;v-u) \ge \langle f, v-u\rangle + a(u;u,v_1) - \langle f, v_1\rangle.$$

By (3.16) and (3.17) in turn, one finds $a(u;u,v_1) - \langle f, v_1\rangle = -b(v_1,p) = -b(v,p)$. Thus

$$a(u;u,v-u) + b(v,p) + \Phi(v) - \Phi(u) + \Psi^0(u;v-u) \ge \langle f, v-u\rangle.$$

Note that $u \in \tilde{V}$, $b(u,p) = 0$. Hence,

$$a(u;u,v-u) + b(v-u,p) + \Phi(v) - \Phi(u) + \Psi^0(u;v-u) \ge \langle f, v-u\rangle,$$

i.e., (3.2) holds. Since $u \in \tilde{V}$, (3.3) also holds.

To prove the uniqueness of a solution $(u,p) \in V \times Q$ of Problem 3.1, we note that $u \in \tilde{V}$ is a solution of Problem 3.2. Thus, u is unique. To prove the uniqueness of p, suppose $(u,\tilde{p})$

is another solution of Problem 3.1. Then, from (3.16) for p and for $\tilde{p}$, $b(v, p-\tilde{p}) = 0$ for all $v \in V_0$. Applying the inf-sup condition (3.15), we have

$$\|p-\tilde{p}\|_Q \le \frac{1}{\beta_0} \sup_{v\in V_0} \frac{b(v, p-\tilde{p})}{\|v\|_V} = 0.$$

Thus $\tilde{p} = p$ and p is unique.

Now we prove the bounds (3.11)–(3.13). Takie $v = 0$ in (3.2) to obtain

$$2\mu \int_\Omega |\varepsilon(u)|^2 dx + \alpha \int_\Omega |u|^r dx \le \Phi(0) - \Phi(u) + \Psi^0(u; -u) + \langle f, u\rangle. \tag{3.18}$$

The convex continuous functional Φ is bounded below by a continuous affine functional ([1, Lemma 11.3.5]): $\Phi(v) \ge \ell(v) + c$ for all $v \in V$, where $\ell \in V^*$ and $c \in \mathbb{R}$. Thus, for some constants c_3 and c_4, not necessarily positive, we have $-\Phi(u) \le c_3 + c_4\|u\|_V$. Write

$$\Psi^0(u; -u) = \left[\Psi^0(u; -u) + \Psi^0(0; u)\right] - \Psi^0(0; u)$$

and use $H(\Psi)$ to find $\Psi^0(u; -u) \le \alpha_\Psi \|u\|_V^2 + c\,\|u\|_V$. Thus, we derive from (3.18) that, for some constant $c \ge 0$,

$$2\mu\,\|u\|_V^2 + \alpha\,\|u\|_{L^r(\Omega)^d}^r \le \alpha_\Psi \|u\|_V^2 + c\,(1 + \|u\|_V) + \|f\|_{V^*}\|u\|_V.$$

Recalling the smallness condition $\alpha_\Psi < 2\mu$, we deduce from the above inequality that

$$\|u\|_V^2 + \|u\|_{L^r(\Omega)^d}^r \le c\left(1 + \|f\|_{V^*}^2\right).$$

This implies both (3.11) and (3.12). Combining (3.15) and (3.16), we have

$$\beta_0 \|p\|_Q \le \sup_{v\in V_0} \frac{1}{\|v\|_V} \left[\langle f, v\rangle - a_0(u, v) - a_1(u; u, v)\right].$$

By the boundedness of the bilinear form $a_0(\cdot,\cdot)$ and the bound (2.9), we derive from the above inequality that $\|p\|_Q \le c\left(\|f\|_{V^*} + \|u\|_V + \|u\|_{L^r(\Omega)^d}^{r-1}\right)$, which, together with (3.11) and (3.12), leads to

$$\|p\|_Q \le c\left(1 + \|f\|_{V^*} + \|f\|_{V^*}^{2(r-1)/r}\right).$$

Since $2(r-1)/r \ge 1$, the above inequality implies (3.13).

Finally, we prove the Lipschitz continuous dependence of the solution on the right hand side. Let $f_1, f_2 \in V^*$. For $i = 1, 2$, let $(u_i, p_i) \in V \times Q$ be the solution of Problem 2.1 for $f = f_i$. Then, for all $v \in \tilde{V}$,

$$a(u_1; u_1, v - u_1) + \Phi(v) - \Phi(u_1) + \Psi^0(u_1; v - u_1) \ge \langle f_1, v - u_1\rangle, \tag{3.19}$$

$$a(u_2; u_2, v - u_2) + \Phi(v) - \Phi(u_2) + \Psi^0(u_2; v - u_2) \ge \langle f_2, v - u_2\rangle. \tag{3.20}$$

Similar to the argument on the solution uniqueness of Problem 4.3 in the proof of Theorem 3.5, we derive from (3.19) and (3.20) that

$$\begin{aligned} a_0(u_1 - u_2, u_1 - u_2) \le\ & a_1(u_1; u_1, u_2 - u_1) + a_1(u_2; u_2, u_1 - u_2) \\ & + \Psi^0(u_1; u_2 - u_1) + \Psi^0(u_2; u_1 - u_2) + \langle f_1 - f_2, u_1 - u_2\rangle. \end{aligned}$$

Then, $2\mu\,\|u_1 - u_2\|_V^2 \le \alpha_\Psi \|u_1 - u_2\|_V^2 + \|f_1 - f_2\|_{V^*}\|u_1 - u_2\|_V$. Hence,

$$\|u_1 - u_2\|_V \le \frac{1}{2\mu - \alpha_\Psi} \|f_1 - f_2\|_{V^*}. \tag{3.21}$$

Therefore, the solution component u depends Lipschitz continuously on f. From (3.16), we have

$$a(u_1;u_1,v)+b(v,p_1)=\langle f_1,v\rangle \quad \forall v\in V_0,$$
$$a(u_2;u_2,v)+b(v,p_2)=\langle f_2,v\rangle \quad \forall v\in V_0.$$

Then,

$$b(v,p_1-p_2)=\langle f_1-f_2,v\rangle-a_0(u_1-u_2,v)-[a_1(u_1;u_1,v)-a_1(u_2;u_2,v)] \quad \forall v\in V_0.$$

Applying (2.10) to bound the last term in the above equality

$$\begin{aligned}-[a_1(u_1;u_1,v)-a_1(u_2;u_2,v)] &\le c\left(\|u_1\|^{r-2}_{L^r(\Omega)^d}+\|u_2\|^{r-2}_{L^r(\Omega)^d}\right)\|u_1-u_2\|_V\|v\|_V\\ &\le c\left(\|u_1\|_{L^r(\Omega)^d}+\|u_2\|_{L^r(\Omega)^d}\right)^{r-2}\|u_1-u_2\|_V\|v\|_V,\end{aligned}$$

we find the bound

$$\begin{aligned}b(v,p_1-p_2)\le{}&\|f_1-f_2\|_{V^*}\|v\|_V+2\mu\,\|u_1-u_2\|_V\|v\|_V\\ &+c\left(\|u_1\|_{L^r(\Omega)^d}+\|u_2\|_{L^r(\Omega)^d}\right)^{r-2}\|u_1-u_2\|_V\|v\|_V.\end{aligned}$$

It follows from (3.12) that

$$\left(\|u_1\|_{L^r(\Omega)^d}+\|u_2\|_{L^r(\Omega)^d}\right)^{r-2}\le c\,(1+\|f_1\|_{V^*}+\|f_2\|_{V^*})^{2(r-2)/r}.$$

Hence,

$$\begin{aligned}b(v,p_1-p_2)\le{}&\|f_1-f_2\|_{V^*}\|v\|_V+2\mu\,\|u_1-u_2\|_V\|v\|_V\\ &+c\,(1+\|f_1\|_{V^*}+\|f_2\|_{V^*})^{2(r-2)/r}\|u_1-u_2\|_V\|v\|_V.\end{aligned}$$

Now applying the inf-sup condition (3.15) and (3.21), we have

$$\|p_1-p_2\|_Q\le\frac{1}{\beta_0}\sup_{v_0\in V_0}\frac{b(v,p_1-p_2)}{\|v\|_V}\le c\,(1+\|f_1\|_{V^*}+\|f_2\|_{V^*})^{2(r-2)/r}\|f_1-f_2\|_{V^*}. \tag{3.22}$$

Therefore, the solution component $p\in Q$ depends locally Lipschitz continuously on $f\in V^*$. □

4. Well-posedness of the Stokes Variational-hemivariational Inequality

In this section, we turn to an analysis of Problem 2.1. We first introduce an intermediate mixed variational-hemivariational inequality for Problem 2.1.

Define functionals Φ and Ψ by the formulas

$$\Phi(v)=\int_{\Gamma_{S,1}}\phi(v_\tau)\,ds,\quad v\in V, \tag{4.1}$$

$$\Psi(v)=\int_{\Gamma_{S,2}}\psi(v_\tau)\,ds,\quad v\in V. \tag{4.2}$$

From the proof of Theorem 4.20 in [37], we have the following result.

Lemma 4.1. *Assume that $\psi\colon\mathbb{R}^d\to\mathbb{R}$ has the properties $H(\psi)$. Then the functional Ψ defined by (4.2) is locally Lipschitz on V and*

$$\Psi^0(u;v)\le\int_{\Gamma_{S,2}}\psi^0(u_\tau;v_\tau)\,ds\quad\forall u,v\in V. \tag{4.3}$$

Combining (2.7) and (4.3), we have, for $v_1, v_2 \in V$,

$$\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \int_{\Gamma_{S,2}} \left[\psi^0(v_{1,\tau}; v_{2,\tau} - v_{1,\tau}) + \psi^0(v_{2,\tau}; v_{1,\tau} - v_{2,\tau})\right] ds$$
$$\leq \alpha_\psi \int_{\Gamma_{S,2}} |v_{1,\tau} - v_{2,\tau}|^2 ds.$$

From (2.5), we obtain $\Psi^0(v_1; v_2 - v_1) + \Psi^0(v_2; v_1 - v_2) \leq \alpha_\psi \lambda_0^{-1} \|v_1 - v_2\|_V^2$ for all $v_1, v_2 \in V$. Equivalently, we have

$$\langle \eta_1 - \eta_2, v_1 - v_2 \rangle \geq -\alpha_\psi \lambda_0^{-1} \|v_1 - v_2\|_V^2 \quad \forall v_1, v_2 \in V,\ \eta_1 \in \partial\Psi(v_1),\ \eta_2 \in \partial\Psi(v_2). \tag{4.4}$$

Now we replace the term $\int_{\Gamma_{S,2}} \psi^0(u_\tau; v_\tau - u_\tau)\, ds$ in (2.11) by $\Psi^0(u; v - u)$ and form the intermediate problem for Problem 2.1.

Problem 4.1. Find $(u, p) \in V \times Q$ such that

$$a(u; u, v - u) + b(v - u, p) + \Phi(v) - \Phi(u) + \Psi^0(u; v - u) \geq \langle f, v - u \rangle \quad \forall v \in V,$$
$$b(u, q) = 0 \quad \forall q \in Q.$$

Problem 4.1 is simply Problem 3.1 with Φ and Ψ defined in (4.1) and (4.2), respectively. Note that $H(\phi)$ implies $H(\Phi)$, and $H(\psi)$ implies $H(\Psi)$ with $\alpha_\Psi = \alpha_\psi \lambda_0^{-1}$. Thus, we can apply Theorem 3.6 to conclude that under the assumptions $H(\phi)$, $H(\psi)$ and $\alpha_\psi < 2\mu\lambda_0$, for any $f \in V^*$, Problem 4.1 has a unique solution $(u, p) \in V \times Q$, the bounds (3.11)–(3.13) hold, and $u \in V$ depends Lipschitz continuously on $f \in V^*$, and $p \in Q$ depends locally Lipschitz continuously on $f \in V^*$.

A relation between Problem 2.1 and Problem 4.1 is stated next, which is a consequence of the property (4.3).

Proposition 4.1. *If $(u, p) \in V \times Q$ is a solution of Problem 4.1, then it is a solution of Problem 2.1.*

Theorem 4.2. *Assume $H(\phi)$, $H(\psi)$ and $\alpha_\psi < 2\mu\lambda_0$. Then, for any $f \in V^*$, Problem 2.1 has a unique solution $(u, p) \in V \times Q$, and for a constant $c > 0$,*

$$\|u\|_V \leq c\,(1 + \|f\|_{V^*}),\ \|u\|_{L^r(\Omega)^d} \leq c\left(1 + \|f\|_{V^*}^{2/r}\right),\ \|p\|_Q \leq c\left(1 + \|f\|_{V^*}^{2(r-1)/r}\right).$$

Moreover, $u \in V$ depends Lipschitz continuously on $f \in V^*$, and $p \in Q$ depends locally Lipschitz continuously on $f \in V^*$.

Proof. We see from Proposition 4.1 that the unique solution $(u, p) \in V \times Q$ of Problem 4.1 is also a solution to Problem 2.1. Thus, the stated results on Problem 2.1 are valid once we have proved the uniqueness of a solution of Problem 2.1. However, the solution uniqueness of Problem 2.1 can be proved by an argument similar to the one used in the proof of Theorem 3.6, and the detail is omitted here. □

When $r = 2$, we see from (3.22) that $p \in Q$ depends Lipschitz continuously on $f \in V^*$. Similar to Problem 3.2, we can consider a reduced formulation of Problem 2.1.

Problem 4.3. Find $u \in \tilde{V}$ such that

$$a(u; u, v - u) + \Phi(v) - \Phi(u) + \int_{\Gamma_{S,2}} \psi^0(u_\tau; v_\tau - u_\tau) \geq \langle f, v - u \rangle \quad \forall v \in \tilde{V}.$$

Under the assumptions stated in Theorem 4.2, Problem 4.3 has a unique solution.

Moreover, we have a minimization principle associated with Problem 4.3, and under the assumptions stated in Theorem 4.2, there is a unique minimizer.

Problem 4.4. Find $u \in \tilde{V}$ such that $E(u) = \inf\{E(v) \mid v \in \tilde{V}\}$, where the energy functional

$$E(v) = \frac{1}{2}a_0(v,v) + \frac{\alpha}{r}\int_\Omega |v|^r dx + \Phi(v) + \int_{\Gamma_{S,2}} \psi(v_\tau)\, ds - \langle f, v\rangle, \quad v \in \tilde{V}.$$

We end the paper by commenting that minimization principles such as Problem 4.4 are of independent interest. The availability of such minimization principles allows the possibility of developing machine learning techniques to solve the Stokes variational-hemivariational inequality. For some references on machine learning methods to solve variational inequalities or hemivariational inequalities, one is referred to [4] and [26].

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