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A NON-MONOTONE QUASI-NEWTON-TYPE APPROACH FOR UNCONSTRAINED SET-VALUED OPTIMIZATION PROBLEMS WITH FINITE CARDINALITY MAPS

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Abstract. This paper introduces a non-monotone quasi-Newton approach for a class of set optimization problems. The proposed method aims to compute weakly minimal solutions using the lower set-less ordering approach. We consider the case in which the set-valued objective mapping is defined by a finite number of twice continuously differentiable functions. This work considers three variants of non-monotone line searches. In our proposed methods, we generate a sequence of iterates that converges to a solution satisfying the necessary optimality criteria for weakly minimal points. Finally, we discuss the numerical application of the method with the help of suitable examples.

Keywords. Gerstewitz functional; Lower set less ordering; Non-monotone quasi-Newton-type method; Unconstrained set optimization; Weakly minimal point.
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1. INTRODUCTION

Set-valued optimization deals with the objective and/or the constraint mappings that are set-valued. The set-valued optimization problems that we study in this article are those of minimizing a function $\mathscr{F}$, which is a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$. The image set of such a map is ordered by a pre-ordering induced by a pointed closed convex cone $K \subset \mathbb{R}^m$. The solution concepts for such set-valued problems are mainly based on two approaches, the vector approach [1, 26] and the set-valued approach [1]. These problems have applications in various domains of finance, mathematical economics, optimal control, and differential inclusions; see, e.g., [2, 5].

The first monograph on oriented order relation for comparing sets was given by Kuroiwa et al. [34]. After that, in [33], an extension to more application-oriented order relations was given. It was observed that the vector-set criteria compare the sets corresponding to all images. An alternative solution criterion based on a set approach was thereby proposed. Initially, the authors in [32, 33, 34] considered the preorder relations for defining sets. Subsequently, Kuroiwa [32] discussed the solution concepts based on the set approach. However, the detailed characterization of the image set was still unclear. The algorithms, in [9, 10, 23], employed scalarization-based

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techniques to a particular class of set-valued mappings characterized by robust counterparts of vector optimization problems. Moreover, in sorting-type algorithms [29, 30], the forward and backward reduction techniques were applied to [18, 19] to deal with finite feasible set-valued problems. Meanwhile, Günther and Popovici [20] first evaluated the images of set-valued mappings whose values are in an increasing manner by scalarizing the mappings via a strongly monotone functional and employ a forward iteration procedure. The discussed literature on set-valued mappings suffers from several theoretical and practical difficulties. The sorting-type algorithms always require a finite feasible set. Furthermore, there is an added difficulty of large computational costs associated with comparing sets. Additionally, the existing approaches fail to recover all the solutions in set-valued optimization problems.

Recently, Bouza et al. [4] proposed the steepest descent method for the set-valued optimization problems considered in this study. It overcomes the drawbacks mentioned above and provides convergence results for the optimal points. However, this method has a slow convergence rate and is not suitable for high-dimensional problems. To overcome the drawback, Ghosh et al. [14] presented the second-order Newton method for the case of set optimization problems. Their method, which provides fast convergence, is suitable for highly non-linear objective functions, and can estimate the optimal solution in fewer iterations.

In this paper, we develop a quasi-Newton method for solving unconstrained set-valued optimization problems, where the objective mapping is represented by a finite collection of twice continuously differentiable functions. To enhance the performance of the proposed method, a non-monotone line search strategy is incorporated to compute optimal solutions. The main contributions of this work are summarized as follows:

- A quasi-Newton method incorporating a non-monotone line search strategy is proposed for solving set-valued optimization problems.
- The global convergence of the proposed method is given under suitable assumptions.
- Numerical examples and comparisons are provided to illustrate the implementation and effectiveness of the proposed approach.

This paper presents non-monotone quasi-Newton-type methods for computing weakly minimal solutions of unconstrained set-valued optimization problems. Let $\mathscr{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ denote a non-empty set-valued map. The unconstrained set optimization problem to analyze is:

$$\preceq^l\text{—}\min_{z\in\mathbb{R}^n} \mathscr{F}(z). \tag{SOP}$$

To derive a non-monotone quasi-Newton method for (SOP), we employ the following assumption.

Assumption 1.1. The map $\mathscr{F}$ in (SOP) is given by a finite collection of vector-valued functions:

$$\mathscr{F}(z) := \left\{f^1(z), f^2(z), \ldots, f^p(z)\right\}, z \in \mathbb{R}^n, \tag{1.1}$$

where $f^1, f^2, \ldots, f^p : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable on $\mathbb{R}^n$.

The contribution of work done is outlined below. In Section 2, basic notations, fundamental definitions, and some related results are given. Section 3 provides a brief overview of optimality conditions for weakly minimal solutions and investigates the relation between stationary points and weakly minimal solutions of (SOP). Section 4 formulates a non-monotone quasi-Newton method for the considered problem employing line searches of the max-type, average-type, and

modified-hybrid-type. We prove that the line search procedures (max-type, average-type, and modified-hybrid-type) are well-defined for the proposed methods. After that, Section 5 analyses the convergence of the proposed methods under suitable assumptions. In Section 6, we further illustrate the performance of the proposed methods and provide their numerical implementation. Finally, the concluding remarks are given in Section 7.

2. Notations and Definitions

First, we recall some basic definitions, notations, and preliminary results. Let $\mathbb{R}$ denote the set of real numbers. The notations $\mathbb{R}_{++}$ and $\mathbb{R}_{+}$ represent the sets of positive and non-negative real numbers, respectively. For any $k \in \mathbb{N}$, we denote $[k] := \{1,2,\ldots,k\}$. Let $\mathscr{P}(\mathbb{R}^m)$ denote the family of all non-empty subsets of $\mathbb{R}^m$. For an $A \in \mathscr{P}(\mathbb{R}^m)$, we use $\text{int}(A)$ and $|A|$ to denote its interior and cardinality, respectively. For vectors, $\|\cdot\|$ refers to the Euclidean norm, whereas for matrices it represents the spectral norm.

We consider a cone $K \subseteq \mathbb{R}^m$, which is closed, convex $(K+K=K)$, solid $(\text{int}(K) \neq \emptyset)$, and pointed $(K \cap (-K) = \{0\})$. The dual cone K^* of K is defined by $K^* := \{w \in \mathbb{R}^m : w^\top z \geq 0 \ \forall z \in K\}$. Each function $f^i : \mathbb{R}^n \to \mathbb{R}^m$ is vector-valued and can be expressed as

$$f^i(z) := \left(f^{i,1}(z), f^{i,2}(z), \ldots, f^{i,m}(z)\right)^\top.$$

The Jacobian of f^i at z is denoted by $\nabla f^i(z)$, and $\nabla^2 f^i(z)$ represents the corresponding second-order derivative matrix, defined as follows:

$$\nabla^2 f^i(z) := \left(\nabla^2 f^{i,1}(z), \nabla^2 f^{i,2}(z), \ldots, \nabla^2 f^{i,m}(z)\right)^\top.$$

We briefly discuss some fundamental results and important definitions in set optimization.

Definition 2.1. [16] Let K be the cone on $\mathbb{R}^m$. The partial $(\preceq)$ and a strict order $(\prec)$ generated by K on $\mathbb{R}^m$ is defined as follows: for any $z', \bar{z} \in \mathbb{R}^m$,

$$z' \preceq \bar{z} \iff \bar{z} - z' \in K \ \text{ and } \ z' \prec \bar{z} \iff \bar{z} - z' \in int(K).$$

Definition 2.2. A function $\mathscr{F} : S \to \mathbb{R}^m$ is said to be Lipschitz continuous on a non-empty subset $S \subseteq \mathbb{R}^n$ if there exists $L > 0$ satisfying $\|\mathscr{F}(z) - \mathscr{F}(\bar{z})\| \leq L\|z - \bar{z}\|$ for all $z, \bar{z} \in S$.

Definition 2.3. [25] Let $A \in \mathscr{P}(\mathbb{R}^m)$. An element $a \in A$ is called a minimal element if $(a - K) \cap A = \{a\}$, and $a \in A$ is called a weakly minimal element if $(a - int(K)) \cap A = \emptyset$.

Further, the set of all minimal and weakly minimal elements of a set A is denoted by $Min(A,K)$ and $WMin(A,K)$, respectively.

Proposition 2.1. [25] *Let $A \in \mathscr{P}(\mathbb{R}^m)$ be a compact set. Then, A satisfies the domination property with respect to K, i.e., $A + K = \text{Min}(A,K) + K$.*

In this study, we use the Gerstewitz scalarizing function $\Omega_e : \mathbb{R}^m \to \mathbb{R}$ (see [12]) for performing fundamental results of the paper, which is

$$\Omega_e(y) := \min\{t \in \mathbb{R} \ : \ te \in y + K\}, \ e \in \ \text{int}(K), \ y \in \mathbb{R}^m.$$

Next, some important properties of Ω_e are discussed.

Proposition 2.2. [28] *For $e \in \text{int}(K)$, the function $\Omega_e : \mathbb{R}^m \to \mathbb{R}$ satisfies*

(i) $\Omega_e(y+\bar{y}) \leq \Omega_e(y) + \Omega_e(\bar{y})$ *and* $\Omega_e(y) - \Omega_e(\bar{y}) \leq \Omega_e(y-\bar{y}), \ \forall \ y, \bar{y} \in \mathbb{R}^m$.

(ii) *If* $y \preceq \bar{y}$, *then* $\Omega_e(y) \leq \Omega_e(\bar{y})$; *moreover if* $y \prec \bar{y}$, *then* $\Omega_e(y) < \Omega_e(\bar{y})$.
(iii) Ω_e *is Lipschitz continuous on* $\mathbb{R}^m$.
(iv) $\Omega_e(y+te) = \Omega_e(y) + t$ *for all* $y \in \mathbb{R}^m$.
(v) $-\text{int}(K) = \{y \in \mathbb{R}^m : \Omega_e(y) < 0\}$ *and* $-K = \{y \in \mathbb{R}^m : \Omega_e(y) \leq 0\}$.

Definition 2.4. The mapping $\mathscr{F}$ is said to be K-convex on $\mathbb{R}^n$ if

$$\mathscr{F}(\lambda z + (1-\lambda)z') \preceq \lambda \mathscr{F}(z) + (1-\lambda)\mathscr{F}(z')$$

for all $z, z' \in \mathbb{R}^n$ and $\lambda \in (0,1)$. If the above inequality is strict for all $z \neq z'$ and $\lambda \in (0,1)$, then $\mathscr{F}$ is called strictly K-convex. Further, the function $\mathscr{F}$ is said to be strongly K-convex on $\mathbb{R}^n$ if there exists a vector $\hat{y} \in int(K)$ such that

$$\mathscr{F}(\lambda z + (1-\lambda)z') \preceq \lambda \mathscr{F}(z) + (1-\lambda)\mathscr{F}(z') - \tfrac{1}{2}\lambda(1-\lambda)\|z-z'\|^2\hat{y}$$

holds for all $z, z' \in \mathbb{R}^n$ and $\lambda \in (0,1)$.

Now, for comparing sets and finding the minimal solution to (SOP), we introduce the set ordering relations.

Definition 2.5. [34] The lower set less relation, denoted by $\preceq^l$, for a given cone K on $\mathscr{P}(\mathbb{R}^m)$ is given by $A \preceq^l E \iff E \subseteq A + K$. Similarly, strict lower set less relation, denoted by $\prec^l$, for a given cone K on $\mathscr{P}(\mathbb{R}^m)$ is given by $A \prec^l E \iff E \subseteq A + int(K)$.

Next, to derive the optimality conditions for (SOP), we need to discuss the criteria for weakly minimal solutions of (SOP).

Definition 2.6. [4] A point $\bar{z} \in \mathbb{R}^n$ is weakly minimal solution of *(SOP)* if there does not exist any $z \in \mathbb{R}^n$ that satisfies $\mathscr{F}(z) \prec^l \mathscr{F}(\bar{z})$.

3. Optimality Conditions

In this section, we present the optimality conditions for weakly minimal solutions of (SOP) under Assumption 1.1. These results form the theoretical foundation for developing the proposed non-monotone quasi-Newton method, which aims to capture weakly minimal solutions of (SOP). The main contributions of this section are organized into two subsections:

(i) To generate the sequence of iterates for the proposed method, we introduce a family of auxiliary vector optimization problems constructed via the notion of partition sets at a given point. Based on this framework, we define the concept of a stationary point for (SOP) and establish its relationship with weakly minimal solutions.
(ii) We establish a necessary optimality condition for weakly minimal solutions of (SOP). To achieve this, we utilize BFGS-type updates to approximate the Hessian matrices associated with each component function $f^1, f^2, \ldots, f^p$ within the vector optimization framework (see [31, 37, 38]).

3.1. Family of vector optimization problems to find weakly minimal solutions of (SOP).

We begin by discussing some index-related set-valued mappings as given in [4].

Definition 3.1. [4]
(i) Active index set relative to $Min(\mathscr{F}(z), K)$ is $I(z) := \{i \in [p] : f^i(z) \in Min(\mathscr{F}(z), K)\}$.
(ii) Index set relative to $v \in \mathbb{R}^m$ is $I_v(z) := \{i \in I(z) : f^i(z) = v\}$.

Also, for $z \in \mathbb{R}^n$, we have

$$I(z) = \bigcup_{v \in Min(\mathscr{F}(z),K)} I_v(z)$$
$$\text{and} \quad I_v(z) = \emptyset, \text{ when } v \notin Min(\mathscr{F}(z),K).$$

Definition 3.2. [4] The cardinality function $\omega : \mathbb{R}^n \to \mathbb{N} \cup \{0\}$ for the set $Min(\mathscr{F}(z),K)$ is given by $\omega(z) := |Min(\mathscr{F}(z),K)|$. For $\bar{z} \in \mathbb{R}^n$, we assume $\omega_k = \omega(z_k)$.

Definition 3.3. [4] Let $z \in \mathbb{R}^n$ and $\{r_1^z, r_2^z, \ldots, r_{\omega(z)}^z\}$ be an enumeration of $Min(\mathscr{F}(z),K)$. Then, the partition set associated with z is defined by $P_z = \prod_{j=1}^{\omega(z)} I_{r_j^z}(z)$.

Let $\text{Min}(\mathscr{F}(z_k),K) = \{r_1^{z_k}, r_2^{z_k}, \ldots, r_{\omega(z_k)}^{z_k}\}$. For each iteration point $z_k \in \mathbb{R}^n$, we denote an element of P_{z_k} by a^i, and the components of a^i by a^i_j, where $j \in [\omega(z_k)]$. Specifically, if $|P_{z_k}| = p_k$, then, for each $i \in [p_k]$,

$$P_{z_k} = \{a^1, a^2, \ldots, a^{p_k}\},$$
$$\text{and} \quad a^i = \left(a^i_1, a^i_2, \ldots, a^i_{\omega(z_k)}\right),\ a^i_j \in I_{r_j^{z_k}},\ j \in [\omega(z_k)].$$

Next, we introduce a family of vector optimization problems that aid in determining the weakly minimal solutions to (SOP). At a given iteration, one of the vector optimization problems from this family is solved, and the solution is checked for optimality. If the optimality condition is reached, then we stop. Otherwise, we solve another vector optimization problem from the family to progress the iteration. Towards finding a stopping condition, we study the key relation from [4].

Theorem 3.1. [4] *Let $P_{\bar{z}}$ denote the partition set at $\bar{z}$. Assume that*

(i) *$\widetilde{K} \in \mathscr{P}(\mathbb{R}^{m\bar{\omega}})$ is the cone defined by $\widetilde{K} := \prod_{j=1}^{\bar{\omega}} K$,*

(ii) *$\preceq_{\widetilde{K}}$ is the partial order on $\mathbb{R}^{m\bar{\omega}}$ induced by the cone $\widetilde{K}$, and*

(iii) *for each $a := (a_1, a_2, \ldots, a_{\bar{\omega}}) \in P_{\bar{z}}$, define the mapping $\widetilde{f}^a : \mathbb{R}^n \to \prod_{j=1}^{\bar{\omega}} \mathbb{R}^m$ by*

$$\widetilde{f}^a(z) := \left(f^{a_1}(z), f^{a_2}(z), \ldots, f^{a_{\bar{\omega}}}(z)\right)^\top.$$

Then, for any $a \in P_{\bar{z}}$, the point $\bar{z}$ is a local weakly minimal solution of (SOP) if and only if it is a local weakly minimal solution of the vector optimization problem

$$\preceq_{\widetilde{K}} - \min_{z \in \mathbb{R}^n} \widetilde{f}^a(z). \tag{VOP}$$

To characterize the conditions for weak minimality, we next focus on the concept of a stationary point for (SOP).

Definition 3.4. [4] A point $\bar{z}$ is a stationary point for *(SOP)* if, for any given $a \in P_{\bar{z}}$ and each $u \in \mathbb{R}^n$, one can find $j \in [\bar{\omega}]$ that satisfies

$$\nabla f^{a_j}(\bar{z})^\top u \notin -\,int(K), \text{ i.e., } \Omega_e(\nabla f^{a_j}(\bar{z})^\top u) \geq 0. \tag{3.1}$$

Lemma 3.1. [15] *A necessary and sufficient condition for $\bar{z} \in \mathbb{R}^n$ to be a stationary point for* (SOP) *is that it is a stationary point for* (VOP) *corresponding to every $a \in P_{\bar{z}}$.*

Lemma 3.2. [14] *A point $\bar{z}$ is a stationary point for* (SOP) *if and only if $\bar{z}$ is a local weakly minimal solution of* (SOP).

3.2. **Hessian approximation and necessary optimality conditions for weakly minimal solutions of (SOP).** To construct a quadratic approximation of the functions $f^1, f^2, \ldots, f^p$, we utilize a positive definite approximation of their Hessian by using the Broyden-Fletcher-Goldfarb-Shanno method (see [31]).

Starting from an initial point z_0, for the function f^i, $i \in [p]$, we generate a sequence of symmetric positive definite matrices $\{B^i(z_k)\}_{k\geq 0}$ starting with an initial symmetric positive definite matrix $B^i(z_0)$. For every iteration k and each $i \in [p]$, the quadratic approximation of each f^i about z_k is given by

$$Q^i(z) := f^i(z_k) + \nabla f^i(z_k)^\top (z - z_k) + \tfrac{1}{2}(z - z_k)^\top B^i(z_k)(z - z_k).$$

We choose $\{B^i(z_k)\}$ that satisfies the quasi-Newton equation $B^i(z_{k+1})\ s_k = y_k^i$, $k = 0, 1, 2, \ldots$, where $s_k := z_{k+1} - z_k$ and $y_k^i := \nabla f^i(z_{k+1}) - \nabla f^i(z_k)$. To guarantee that each matrix in the sequence $\{B^i(z_k)\}$ remains symmetric and positive definite, we begin with an initial symmetric positive definite matrix $B^i(z_0)$. Subsequently, at every iteration point z_k, the matrices are updated using the BFGS formula

$$B^i(z_{k+1}) := B^i(z_k) - \frac{B^i(z_k) s_k s_k^\top B^i(z_k)}{s_k^\top B^i(z_k) s_k} + \frac{y_k^i y_k^{i^\top}}{s_k^\top y_k^i}. \tag{3.2}$$

As noted in [41, Section 6.1], for each $k \in \mathbb{N} \cup \{0\}$, the positive definiteness of $B^i(z_k)$ ensures that the updated matrix $B^i(z_{k+1})$ also remains positive definite.

For the rest of the analysis, unless explicitly mentioned, for every $z \in U$, where U is a nonempty subset of $\mathbb{R}^n$, each f^i is twice continuously differentiable and $B^i(z) \succeq 0$ (positive definite) for all $z \in \mathbb{R}^n$, $a \in P_z$, and $j \in [\omega(z)]$.

We now proceed to establish a necessary condition for weakly minimal solutions of (SOP). To this end, we first present the following lemma.

Lemma 3.3. [15] *For any given $z \in \mathbb{R}^n$, $a \in P_z$ and $j \in [\omega(z)]$, let $g : \mathbb{R}^n \to \mathbb{R}$ be a function defined by $g(u) := \Omega_e\left(\nabla f^{a_j}(z)^\top u + \frac{1}{2} u^\top B^{a_j}(z) u\right)$, where $B^{a_j}(z) \in \mathbb{R}^{n\times n}$ is symmetric and positive definite matrix. Then, g is strongly convex on $\mathbb{R}^n$.*

Next, from [14], we recall a necessary condition for weak minimality in (SOP). A point $\bar{z}$ is a stationary point of (SOP) iff, for each $a \in P_{\bar{z}}$, for any $u \in \mathbb{R}^n$, one can find a_j satisfying

$$\Omega_e(\nabla f^{a_j}(\bar{z})^\top u) \geq 0 \tag{3.3}$$

$$\text{and} \quad \Omega_e\left(\nabla f^{a_j}(\bar{z})^\top u + u^\top B^{a_j}(\bar{z}) u\right) \geq \Omega_e\left(\nabla f^{a_j}(\bar{z})^\top u\right) + \tfrac{1}{2}\rho \|u\|^2 \geq 0, \tag{3.4}$$

where $\rho := \min\{\rho_1, \rho_2, \ldots, \rho_p\}$ and each ρ_i, $i \in [p]$, is a positive constant as given in [7, Corollary 2.2]. Moreover, from [14], we recall the functions $\xi_z : P_z \times \mathbb{R}^n \to \mathbb{R}$ and $\Phi : \mathbb{R}^n \to \mathbb{R}$, given by

$$\xi_z(a,u) := \max_{j\in[\omega(z)]} \left\{\Omega_e(\nabla f^{a_j}(z)^\top u + \tfrac{1}{2} u^\top B^{a_j}(z) u)\right\},\ a \in P_z,\ u \in \mathbb{R}^n,\ \text{for any } z \in \mathbb{R}^n, \tag{3.5}$$

$$\text{and} \quad \Phi(z) := \min_{(a,u)\in P_z\times\mathbb{R}^n} \xi_z(a,u). \tag{3.6}$$

Then, by (3.4), at a stationary point $\bar{z}$, we have

(i) $\min_{u\in\mathbb{R}^n} \xi_{\bar{z}}(a,u) = 0$ for all $a \in P_{\bar{z}}$.

(ii) Let $a \in P_{\bar{z}}$ and $\bar{u}_{a,\bar{z}} \in \mathbb{R}^n$ be such that $\xi_{\bar{z}}(a,\bar{u}_{a,\bar{z}}) = \min_{u\in\mathbb{R}^n} \xi_{\bar{z}}(a,u)$. Then, $\xi_{\bar{z}}(a,\bar{u}_{a,\bar{z}}) = 0$ if and only if $\bar{u}_{a,\bar{z}} = 0$.

(iii) $\Phi(z) = \min\limits_{(a,u)\in P_z\times\mathbb{R}^n} \xi_z(a,u) \leq \xi_z(a,0) = 0$ for any $z \in \mathbb{R}^n$.

(iv) At $z = \bar{z}$, it holds that $\Phi(\bar{z}) = 0$ and $\bar{u} = 0$.

Proofs of these statements can be found in [14].

Proposition 3.1. [14] *Let $\bar{z}$ be a weakly minimal point of (SOP) and $\bar{a} \in P_{\bar{z}}$ and $\bar{u} \in \mathbb{R}^n$ be such that $\Phi(\bar{z}) = \xi_{\bar{z}}(\bar{a},\bar{u})$, where $\xi_{\bar{z}}$ and Φ are as defined in (3.5) and (3.6), respectively. Then, $\bar{u} = 0$.*

Below, we discuss the stationarity points and boundedness of sequences of (SOP) in terms of ξ and Φ.

Theorem 3.2. [14] *Consider the functionals $\xi_{\bar{z}}$ and Φ as defined in (3.5) and (3.6), respectively. Let $(\bar{a},\bar{u}) \in P_{\bar{z}} \times \mathbb{R}^n$ satisfy $\Phi(\bar{z}) = \xi_{\bar{z}}(\bar{a},\bar{u})$. Then, the following statements are equivalent:*

(i) *The point $\bar{z}$ is not a stationary point of* (SOP).

(ii) $\Phi(\bar{z}) < 0$.

(iii) $\bar{u} \neq 0$.

4. Non-monotone Quasi-Newton Methods

In this section, we present the solution approach for the class of set optimization problems. We discuss three non-monotone line searches: the average-type, the max-type, and the non-monotone-type. In monotone line search, the step length t_k is chosen in the descent direction u_k such that $f^{a_j^k}(z_k + t_k u_k) \prec f^{a_j^k}(z_k)$, for all $a^k \in P_k$ and $j \in [\omega_k]$. The non-monotone line search strategy permits an occasional increase in the function value, that is, we take a step length $t_k > 0$ in the direction u_k, satisfying $f^{a_j^k}(z_k + t_k u_k) \preceq C^{a_j^k}(z_k) + \beta t_k \nabla f^{a_j^k}(z_k)^\top u_k$ for all $j \in [\omega_k]$, with $\beta \in (0,1)$. The detailed discussion on term $C^{a_j^k}$ is given in Subsections 4.1, 4.2, and 4.3.

4.1. Max-type non-monotone line search for set optimization problem (SOP).

We begin with the max-type non-monotone line search, originally introduced by Grippo et al. [17]. This strategy selects the maximum among a set of recent objective function values. This line search procedure for (SOP) is given in Algorithm 1.

Next, the lemma below discusses that the condition in Step 7 of Algorithm 1 is well-defined.

Lemma 4.1. *For each iteration k and $a^k \in P_{z_k}$ in Algorithm 1, it holds that $f^{a_j^k}(z_k) \preceq C^{a_j^k}(z_k)$ for all $j \in [\omega_k]$.*

Proof. The proof follows from the definition of $C^{a_j^k}$ in Step 7 of Algorithm 1. □

4.2. Average-type non-monotone line search for set optimization problems (SOP).

This line search was proposed by Zhang and Hager [42] for scalar optimization. In this line search, we consider the average of recent function values. The average-type non-monotone line search procedure for (SOP) is given in Algorithm 2.

We initialize the process by assuming $C^{a_j^k}(z_0) = f^{a_j^k}(z_0)$ for the iteration $k = 0$, and for the $(k+1)$-th iteration such that $a^{k+1} \in P_{z_{k+1}}$ and $j \in [\omega_{k+1}]$, the term $C^{a_j^{k+1}}$ is a combination of $f^{a_j^{k+1}}(z_0), f^{a_j^{k+1}}(z_1), \ldots, f^{a_j^{k+1}}(z_k), f^{a_j^{k+1}}(z_{k+1})$ for every $j \in [\omega_{k+1}]$. Observe that if $\eta_k = 0$ holds

Algorithm 1 Max-type non-monotone quasi-Newton method for (SOP)

Step 1 **Initial data.** Define the objective function $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^p(z)\}$, where each function $f^1, f^2, \ldots, f^p$ is assumed to be twice continuously differentiable.

Step 2 **Initialization.** Choose an initial point $z_0 \in \mathbb{R}^n$. Let $\beta \in (0,1)$ be the step-size parameter, $\nu \in (0,1)$ a positive constant $e \in \text{int}(K)$, and M a non-negative integer. Set $C^i := f^i(z_0)$ for all $i = 1,2,\ldots,p$. Further, select an initial symmetric positive definite matrix $B^i \in \mathbb{R}^{n\times n}$, for each $i \in [p]$. Initialize the iteration counter with $k = 0$, and fix a tolerance level $\varepsilon > 0$ as the stopping criterion.

Step 3 **Evaluate the minimal set and the partition set.**

Determine $M_k := \text{Min}(\mathscr{F}(z_k), K) = \{r_1^{z_k}, r_2^{z_k}, \ldots, r_{\omega_k}^{z_k}\}$ and denote $\omega_k := |\text{Min}(\mathscr{F}(z_k), K)|$.

Construct $P_k := P_{z_k} = I_{r_1^{z_k}} \times I_{r_2^{z_k}} \times \cdots \times I_{r_{\omega_k}^{z_k}}$, $p_k := |P_{z_k}|$, and $P_{z_k} := \{a^1, a^2, \ldots, a^{p_k}\}$, and for each $i \in [p_k]$, $a^i = (a_1^i, a_2^i, \ldots, a_{\omega_k}^i) \in P_{z_k}$, $a_j^i \in I_{r_j^{z_k}}$, $j \in [\omega_k]$.

Step 4 **Descent direction.**

For $\xi_{z_k}(a,u) := \max\limits_{j\in[\omega_k]} \Omega_e \left(\nabla f^{a_j}(z_k)^\top u + \frac{1}{2} u^\top B^{a_j}(z_k) u\right)$, where $\xi_{z_k} : P_{z_k} \times \mathbb{R}^n \to \mathbb{R}$, compute $(a^k, u_k) \in \underset{(a,u)\in P_k\times\mathbb{R}^n}{\text{argmin}} \xi_{z_k}(a,u)$.

Step 5 **Stopping criterion.** Terminate the procedure if $\|u_k\| < \varepsilon$; otherwise, proceed to Step 6.

Step 6 **Compute a step length.** Determine the smallest integer q such that ν^q satisfies the required condition on t_k given by

$$t_k := \max_{q\in\mathbb{N}\cup\{0\}} \left\{ \nu^q \ : f^{a_j^k}(z_k + \nu^q u_k) \preceq C^{a_j^k}(z_k) + \beta \nu^q \nabla f^{a_j^k}(z_k)^\top u_k \ \forall\ j \in [\omega_k] \right\},$$

and evaluate $z_{k+1} \leftarrow z_k + t_k u_k$.

Step 7 **Update the non-monotone parameter.** Evaluate the value of non-monotone parameter C^i at z_{k+1}. For this, let $m(k) := \min\{k, M\}$ and $e \in \text{int}(K)$, and then we use

$$C^i(z_{k+1}) := \max_{0\le l\le m(k)} \left\{ w^\top f^i(z_{k-l}) \ : \ w \in K^* \text{ and } \|w\| = 1 \right\} e \text{ for all } i = 1,2,\ldots,p.$$

Step 8 **Update the iterate and approximation.**

Compute $s_k := z_{k+1} - z_k$ and $y_k^i := \nabla f^i(z_{k+1}) - \nabla f^i(z_k)$ for all $i \in [p]$.

Compute the subsequent symmetric positive definite matrix according to the following update formula:

$$B^i(z_{k+1}) := B^i(z_k) - \frac{B^i(z_k) s_k s_k^\top B^i(z_k)}{s_k^\top B^i(z_k) s_k} + \frac{y_k^i y_k^{i^\top}}{s_k^\top y_k^i}.$$

Update $k \leftarrow k+1$, and then return to Step 3.

for every k in Algorithm 2, then $C^{a_j^k} = f^{a_j^k}(z_k)$. This corresponds to the standard monotone line search. Moreover, if $\eta_k = 1$ for every k, then $C^{a_j^k}(z_k) = T^{a_j^k}(z_k)$, where

$$T^{a_j^k}(z_k) := \frac{f^{a_j^k}(z_0) + f^{a_j^k}(z_1) + \cdots + f^{a_j^k}(z_k)}{k+1}$$

is the vector formed by averaging the function values, with η_k as a parameter controlling the non-monotone behavior of the search.

Next, we prove that, for any choice of $\eta_k \in [0,1]$ and $a^k \in P_{z_k}$, $C^{a^k_j}(z_k)$ lies between $f^{a^k_j}(z_k)$ and $T^{a^k_j}(z_k)$. Hence, the line search is consistently defined for all $z \in \mathbb{R}^n$.

Algorithm 2 Average-type non-monotone quasi-Newton method for (SOP)

Step 1 **Initial data.** Define the objective function $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^p(z)\}$, where each function $f^1, f^2, \ldots, f^p$ is assumed to be twice continuously differentiable.

Step 2 **Initialization.** Select an initial point $z_0 \in \mathbb{R}^n$, and choose parameters satisfying $0 \leq \eta_{\min} \leq \eta_{\max} \leq 1$, along with $\beta, \nu \in (0,1)$. For each $i \in [p]$, provide an initial symmetric positive definite matrix $B^i \in \mathbb{R}^{n\times n}$. Give tolerance precision $\varepsilon > 0$. Set $k = 0$, define $C^i := f^i(z_0)$, and initialize $q_0 = 1$.

Step 3 **Evaluate the minimal set and the partition set.** Evaluate

$$M_k := \text{Min}\,(F(z_k), K),\ P_k := P_{z_k},\ \omega_k := |\text{Min}\,(F(z_k), K)|.$$

Step 4 **Descent direction.**

Define $\xi_{z_k}(a,u) := \max\limits_{j\in[\omega_k]} \Omega_e\left(\nabla f^{a_j}(z_k)^\top u + \frac{1}{2}u^\top B^{a_j}(z_k)u\right)$, where $\xi_{z_k} : P_{z_k} \times \mathbb{R}^n \to \mathbb{R}$.

Then, compute $(a^k, u_k) \in \underset{(a,u)\in P_k\times\mathbb{R}^n}{\text{argmin}}\ \xi_{z_k}(a,u)$.

Step 5 **Stopping criterion.** Terminate if $\|u_k\| < \varepsilon$; otherwise, proceed to Step 6.

Step 6 **Compute a step length.** Determine the step length t_k such that it satisfies the non-monotone Armijo condition. Specifically, let q be the smallest non-negative integer for which the following holds:

$$t_k := \max_{q\in\mathbb{N}\cup\{0\}} \left\{\nu^q \ :\ \forall\ j \in [\omega_k] : f^{a^k_j}(z_k + \nu^q u_k) \preceq C^{a^k_j}(z_k) + \beta\nu^q \nabla f^{a^k_j}(z_k)^\top u_k\right\}, \tag{4.1}$$

and evaluate $z_{k+1} := z_k + t_k u_k$.

Step 7 **Update the non-monotone parameter.** Choose $\eta_k \in [\eta_{\min}, \eta_{\max}]$, and update

$$q_{k+1} := \eta_k q_k + 1 \text{ and } C^i(z_{k+1}) := \frac{\eta_k q_k C^i(z_k) + f^i(z_{k+1})}{q_{k+1}}.$$

Step 8 **Update the iterate and approximation.** Compute $s_k := z_{k+1} - z_k$ with $y^i_k := \nabla f^i(z_{k+1}) - \nabla f^i(z_k)$, $i \in [p]$. Then, obtain the updated symmetric positive definite matrix using the following formula:

$$B^i(z_{k+1}) := B^i(z_k) - \frac{B^i(z_k)s_k s_k^\top B^i(z_k)}{s_k^\top B^i(z_k) s_k} + \frac{y^i_k y^{i\top}_k}{s_k^\top y^i_k} \text{ for all } i \in [p].$$

Update $k \leftarrow k+1$ and return to Step 3.

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Lemma 4.2. *Let $U \subseteq \mathbb{R}^n$ be a non-empty set, and let $\{z_k\}$ be a sequence of non-stationary points of (SOP). Then, for each iteration k of Algorithm 2 and $a^k \in P_{z_k}$, $f^{a^k_j}(z_k) \preceq C^{a^k_j}(z_k) \preceq T^{a^k_j}(z_k)$ for all $j \in [\omega_k]$.*

Proof. First, we prove the lower bound for $C^{a_j^k}$. Note that, for $k=0$, the result is trivial. Now, for a partition element $a^k \in P_{z_k}$, let us define a function $D^{a_j^k} : \mathbb{R} \to \mathbb{R}^m$ by

$$D^{a_j^k}(t) := \frac{tC^{a_j^k}(z_k) + f^{a_j^k}(z_{k+1})}{t+1} \text{ and } \frac{\partial D^{a_j^k}(t)}{\partial t} = \frac{C^{a_j^k}(z_k) - f^{a_j^k}(z_{k+1})}{(t+1)^2} \text{ for all } j \in [\omega_k].$$

We show that the function $D^{a_j^k}$ is non-decreasing for every $t \geq 0$. Observe that $\{z_k\}$ is a sequence of non-stationary points of (SOP). Therefore, in view of Step 7 of Algorithm 2, for all $j \in [\omega_k]$, we can deduce

$$\begin{aligned} & f^{a_j^k}(z_k + t_k u_k) \preceq C^{a_j^k}(z_k) + \beta t_k \nabla f^{a_j^k}(z_k)^\top u_k \\ \text{or,} \quad & f^{a_j^k}(z_k + t_k u_k) - C^{a_j^k}(z_k) \preceq \beta t_k \nabla f^{a_j^k}(z_k)^\top u_k \\ \text{or,} \quad & \Omega_e(f^{a_j^k}(z_k + t_k u_k) - C^{a_j^k}(z_k)) \leq \Omega_e(\beta t_k \nabla f^{a_j^k}(z_k)^\top u_k) \text{ from Proposition 2.2(ii).} \end{aligned} \tag{4.2}$$

Since z_k is non-stationary, one sees that there exists $u_k \in \mathbb{R}^n$ such that $\nabla f^{a_j^k}(z_k)^\top u_k \in -\text{int}(K)$. Also, $\beta t_k \nabla f^{a_j^k}(z_k)^\top u_k \in -\text{int}(K)$. So, we have

$$\begin{aligned} \Omega_e(\beta t_k \nabla f^{a_j^k}(z_k)^\top u_k) < 0 \quad \text{or,} \quad & \Omega_e(f^{a_j^k}(z_k + t_k u_k) - C^{a_j^k}(z_k)) < 0 \text{ from (4.2)} \\ \text{or,} \quad & f^{a_j^k}(z_k + t_k u_k) - C^{a_j^k}(z_k) \in -\text{int}(K) \text{ from Proposition 2.2(v)} \\ \text{or,} \quad & f^{a_j^k}(z_{k+1}) \prec C^{a_j^k}(z_k) \\ \text{or,} \quad & 0 \prec \frac{\partial D^{a_j^k}(t)}{\partial t}. \end{aligned}$$

Therefore, $D^{a_j^k}$ is a non-decreasing function for all $j \in [\omega(z_k)]$. Thus, for all $t > 0$ and $j \in [\omega(z_k)]$, we have

$$\begin{aligned} & f^{a_j^k}(z_k) = D^{a_j^k}(0) \prec D^{a_j^k}(t) \\ \implies \quad & f^{a_j^k}(z_k) = D^{a_j^k}(0) \prec D^{a_j^k}(\eta_k q_k) = C^{a_j^k}(z_k) \text{ taking } t = \eta_k q_k > 0, \end{aligned}$$

where q_k and η_k are defined in Algorithm 2. Thus, we get the lower bound for $C^{a_j^k}$.

Next, we prove an upper bound $C^{a_j^k} \preceq T^{a_j^k}$ for all $j \in [\omega_k]$ with the help of induction. Towards this, observe that, for $k=0$, the result holds by initialization assumption $C^{a_j^k}(z_0) = f^{a_j^k}(z_0) = T^{a_j^k}(z_0)$ for all $j \in [\omega_k]$. Further, assume that, for an iteration k, $a^k \in P_k$, the relation holds for $(k-1)$ many points, i.e., $C^{a_j^k}(z_{k-1}) \preceq T^{a_j^k}(z_{k-1})$ for all $j \in [\omega_k]$. Since $q_0 = 1$ and $\eta_k \in [0,1]$, we write by Step 7 of Algorithm 2

$$q_{k+1} = 1 + \sum_{i=0}^{k} \prod_{l=0}^{i} \eta_{k-l} \leq k+2 \text{ for each } k. \tag{4.3}$$

Therefore, $0 \leq q_{k-1} \leq k$ holds. Thus, in view of Step 7 of Algorithm 2, we have

$$\begin{aligned} & C^{a_j^k}(z_k) = \frac{\eta_{k-1} q_{k-1} C^{a_j^k}(z_{k-1}) + f^{a_j^k}(z_k)}{q_k} \text{ for all } j \in [\omega(z_k)] \\ \text{or,} \quad & C^{a_j^k}(z_k) = \frac{\eta_{k-1} q_{k-1} C^{a_j^k}(z_{k-1}) + f^{a_j^k}(z_k)}{\eta_{k-1} q_{k-1} + 1} \text{ for all } j \in [\omega(z_k)]. \end{aligned}$$

Note that $D^{a_j^k}$ is non-decreasing for all $j \in [\omega_k]$. Therefore, in view of the above relation, we arrive at

$$C^{a_j^k}(z_k) = D^{a_j^k}(\eta_{k-1}q_{k-1}) = D^{a_j^k}(q_k - 1) \preceq D^{a_j^k}(k),\ j \in [\omega(z_k)]. \tag{4.4}$$

Now, by the induction, we have

$$D^{a_j^k}(k) = \frac{kC^{a_j^k}(z_{k-1}) + f^{a_j^k}(z_k)}{k+1} \preceq \frac{kT^{a_j^k}(z_{k-1}) + f^{a_j^k}(z_k)}{k+1} = T^{a_j^k}(z_k). \tag{4.5}$$

From (4.4) and (4.5), we obtain the required upper bound for C_j^k. □

Lemma 4.3. *Let $\beta \in (0,1)$ be fixed, and consider the functionals $\xi_{\bar{z}}$ and Φ defined in (3.5) and (3.6), respectively. Assume that $\{z_k\}$ is a sequence of non-stationary points of* (SOP)*. Then, there exists a step-size $t_k > 0$ of Algorithms 1 and 2 that satisfies the non-monotone Armijo's condition in the line search procedure, i.e.,*

$$\text{for all } t \in (0,t_k],\ j \in [\omega_k] : f^{a_j^k}(z_k + tu_k) \preceq C^{a_j^k}(z_k) + \beta t \nabla f^{a_j^k}(z_k)^\top u_k, \tag{4.6}$$

where $C^{a_j^k}$ is defined in Algorithms 1 and 2.

Proof. Let $\{z_k\}$ be a sequence of non-stationary points of (SOP). With the help of induction, we show that there exists t_k satisfying the relation (4.6). First, for $k = 0$, we have

$$\begin{aligned} & f^{a_j^k}(z_0 + t_0u_0) \preceq C^{a_j^k}(z_0) + \beta t_0 \nabla f^{a_j^k}(z_0)^\top u_0 \text{ for all } j \in [\omega(z_0)] \\ \text{or,}\quad & f^{a_j^k}(z_0 + t_0u_k) \preceq f^{a_j^k}(z_0) + \beta t_0 \nabla f^{a_j^k}(z_0)^\top u_0 \text{ for all } j \in [\omega(z_0)] \text{ from Lemma 4.2.} \end{aligned}$$

Next, proceeding in a similar manner as in Lemma 4.2, we have $f^{a_j^k}(z_k) \preceq C^{a_j^k}(z_k)$ for all $j \in [\omega(z_k)]$. Now, the Taylor's expansion of $f^{a_j^k}$ at point z_k is given by

$$f^{a_j^k}(z_k + tu_k) = f^{a_j^k}(z_k) + t\nabla f^{a_j^k}(z_k)^\top u_k + o(t) \text{ for all } j \in [\omega(z_k)],$$

where $\lim\limits_{t \to 0+} \frac{o(t)}{t} = 0$. So, there exists t_k such that, for all $t \in (0,t_k]$,

$$f^{a_j^k}(z_k + tu_k) \preceq C^{a_j^k}(z_k) + \beta t \nabla f^{a_j^k}(z_k)^\top u_k + t\left[(1-\beta)\nabla f^{a_j^k}(z_k)^\top u_k + \frac{o(t)}{t}\right].$$

Observe that $\nabla f^{a_j^k}(z_k)^\top u_k \prec 0$ for every $j \in [\omega_k]$. Consequently, for sufficiently small $t \in (0,t_k]$, the last term on the right-hand side of the above inequality becomes non-positive. Hence, the desired conclusion follows for all sufficiently small $t \in (0,t_k]$, i.e., there exists $t_k > 0$ such that, for all $t \in (0,t_k]$ and $j \in [\omega_k]$, $f^{a_j^k}(z_k + tu_k) \preceq C^{a_j^k}(z_k) + \beta t \nabla f^{a_j^k}(z_k)^\top u_k$, which is the required result. □

4.3. **A modified-hybrid-type non-monotone line search for set optimization problems** (SOP) **and its convergence.** This line search was proposed by Ou and Liu [36]. The modified-hybrid-type non-monotone line search procedure for (SOP) is given in Algorithm 3. For given $\sigma_1 \in (0,1), \sigma_2 > 0$, and $\beta \in (0,1)$, set

$$s_k := -\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|^2}.$$

Then, we choose $t_k := \max\{s_k, s_k\beta, s_k\beta^2, \ldots\}$ satisfying

$$f^{a_j^k}(z_k + t_k u_k) \preceq C^{a_j^k}(z_k) + \sigma_1 t_k \nabla f^{a_j^k}(z_k)^\top u_k - \sigma_2 t_k^2 \|u_k\|^2 e \text{ for all } j \in [\omega_k], \tag{4.7}$$

where

$$C^{a_j^k}(z_k) := \begin{cases} f^{a_j^k}(z_0) & \text{if } k = 0 \\ \theta_k C^{a_j^k}(z_{k-1}) + (1-\theta_k) f^{a_j^k}(z_k) & \text{if } k \geq 1 \end{cases} \tag{4.8}$$

with $0 \leq \theta_k \leq \theta_{\max} < 1$. It is worth noting that when $\sigma_2 = 0$ and $s_k = s$ for all k, the non-monotone line search in (4.7) simplifies to the non-monotone line search scheme given by

$$f^{a_j^k}(z_k + t_k u_k) \preceq C^{a_j^k}(z_k) + \beta t_k \nabla f^{a_j^k}(z_k)^\top u_k, \text{ for all } j \in [\omega_k],$$

where $C^{a_j^k}(z_k)$ is defined in (4.8). If $\sigma_1 = 0$ and $s_k = s$ for all k, then the non-monotone line search (4.7) reduces to the monotone line search scheme proposed in [15].

Next, we prove that the line search procedure defined above is well-defined for all $z \in \mathbb{R}^n$.

Lemma 4.4. *Let $U \subseteq \mathbb{R}^n$ be a non-empty subset and $\{z_k\}$ be the sequence of non-stationary points generated by Algorithm 3. Then, for each iteration k, $a^k \in P_{z_k}$, $f^{a_j^k}(z_k) \preceq C^{a_j^k}(z_k)$ for all j in $[\omega_k]$.*

Proof. First, in view of relation (4.8), for any k and for all $j \in [\omega_k]$, we note that

$$f^{a_j^k}(z_k) - C^{a_j^k}(z_k) = \theta_k (f^{a_j^k}(z_k) - C^{a_j^k}(z_{k-1})). \tag{4.9}$$

We prove the required result with the help of induction. At $k = 0$, the claim follows. Now, let the statement hold for some $k \geq 1$. Observe that $\{z_k\}$ is a sequence of non-stationary points of (SOP). Therefore, from Step 6 of Algorithm 3, for all $j \in [\omega_k]$, we have

$$\begin{aligned} & f^{a_j^k}(z_{k-1} + t_{k-1}u_{k-1}) \preceq C^{a_j^k}(z_{k-1}) + \sigma_1 t_{k-1} \nabla f^{a_j^k}(z_{k-1})^\top u_{k-1} - \sigma_2 t_{k-1}^2 \|u_k\|^2 e \\ \text{or,} \quad & f^{a_j^k}(z_{k-1} + t_{k-1}u_{k-1}) \preceq C^{a_j^k}(z_{k-1}) + \sigma_1 t_{k-1} \nabla f^{a_j^k}(z_{k-1})^\top u_{k-1} \\ \text{or,} \quad & \Omega_e(f^{a_j^k}(z_{k-1} + t_{k-1}u_{k-1}) - C^{a_j^k}(z_{k-1})) \leq \Omega_e(\sigma_1 t_{k-1} \nabla f^{a_j^k}(z_{k-1})^\top u_{k-1}), \end{aligned} \tag{4.10}$$

by Proposition 2.2(ii). Now, since z_{k-1} is non-stationary, then there exists $u_{k-1} \in \mathbb{R}^n$ such that $\nabla f^{a_j^k}(z_{k-1})^\top u_{k-1} \in -\text{int}(K)$. Also, $\sigma_1 t_{k-1} \nabla f^{a_j^k}(z_{k-1})^\top u_{k-1} \in -\text{int}(K)$ gives

$$\begin{aligned} & \Omega_e(\sigma_1 t_{k-1} \nabla f^{a_j^k}(z_{k-1})^\top u_{k-1}) < 0 \\ \text{or,} \quad & \Omega_e(f^{a_j^k}(z_{k-1} + t_{k-1}u_{k-1}) - C^{a_j^k}(z_{k-1})) < 0 \quad \text{from (4.10)} \\ \text{or,} \quad & \Omega_e(f^{a_j^k}(z_k) - C^{a_j^k}(z_k)) < 0 \quad \text{from (4.9) and } \theta \in (0,1) \\ \text{or,} \quad & f^{a_j^k}(z_k) - C^{a_j^k}(z_k) \in -\text{int}(K) \quad \text{from Proposition 2.2(v)} \\ \text{or,} \quad & f^{a_j^k}(z_k) \prec C^{a_j^k}(z_k), \end{aligned}$$

which is the required relation. □

Theorem 4.1. [14] *Let $\{z_k\}$ be a convergent sequence consisting of non-stationary points. Then, the corresponding sequence of directions $\{u_k\}$, generated by Algorithms 1, 2, and 3 is bounded.*

Now, we assume that, at any given point z_k, the search direction u_k satisfies some important conditions discussed below.

Algorithm 3 Modified non-monotone quasi-Newton method for (SOP)

Step 1 **Initial data.** Define the objective mapping $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^p(z)\}$, where each function $f^1, f^2, \ldots, f^p$ satisfies Assumption 1.1.

Step 2 **Initialization.** Choose an initial point $z_0 \in \mathbb{R}^n$, a trial step length $\sigma_1 \in (0,1)$, and a parameter $\sigma_2 > 0$. Let $\nu \in (0,1)$ $e \in \text{int}(K)$, and choose parameters satisfying $0 \leq \theta_k \leq \theta_{\max} < 1$. At $k = 0$, specify an initial symmetric positive definite matrix $B^i \in \mathbb{R}^{n\times n}$ and set $C^i = f^i(z_0)$ for all $i \in [p]$. Finally, fix a tolerance level $\varepsilon > 0$ to serve as the stopping criterion.

Step 3 **Evaluate the minimal set and the partition set.**
Compute $M_k := \text{Min}\,(\mathscr{F}(z_k), K) = \{r_1, r_2, \ldots, r_{\omega_k}\}$ and $\omega_k := |\text{Min}\,(\mathscr{F}(z_k), K)|$.
Find $P_k := P_{z_k} = I_{r_1} \times I_{r_2} \times \cdots \times I_{r_{\omega_k}}$, $p_k = |P_{z_k}|$, and $P_{z_k} = \{a^1, a^2, \ldots, a^{p_k}\}$,
and for each $i \in [p_k]$, $a^i = (a^i_1, a^i_2, \ldots, a^i_{\omega_k}) \in P_{z_k}$, $a^i_j \in I^{z_k}_{r_j}$, $j \in [\omega_k]$.

Step 4 **Descent direction.** Compute

$$(a^k, u_k) \in \underset{(a,u)\in P_k\times\mathbb{R}^n}{\text{argmin}} \max_{j\in[\omega_k]} \{\Omega_e(\nabla f^{a_j}(z_k)^\top u + \tfrac{1}{2}u^\top B^{a_j}(z_k)u)\}.$$

Step 5 **Stopping criterion.** Terminate the procedure if $\|u_k\| < \varepsilon$; otherwise, proceed to Step 6.

Step 6 **Compute a step length.** First estimate the parameter $s_k := -\frac{\Omega_e(\nabla f^{a^k_j}(z_k)^\top u_k)}{\|u_k\|^2}$. Then, find the smallest value of q such that ν^q estimates the step length t_k by

$$t_k := \max_{q\in\mathbb{N}\cup\{0\}} \{s_k,\ s_k\nu^q,\ s_k^2\nu^{2q}, \ldots\},$$

where the set $\{s_k,\ s_k\nu^q,\ s_k^2\nu^{2q}, \ldots\}$ satisfies the following relation given below:

$$f^{a^k_j}(z_k + \nu^q u_k) \preceq C^{a^k_j}(z_k) + \sigma_1 s_k \nu^q \nabla f^{a^k_j}(z_k)^\top u_k - \sigma_2 s_k^2 \nu^{2q}\|u_k\|^2 e \ \forall\ j \in [\omega_k].$$

Evaluate $z_{k+1} \leftarrow z_k + t_k u_k$.

Step 7 **Update the non-monotone parameter.** Evaluate the value of non-monotone parameter $C^i(z_{k+1})$. For $k = 0$, $C^i(z_0) := f^i(z_0)$. Next, we update $C^i(z_k)$ as follows:

$$C^i(z_{k+1}) := \theta_k C^i(z_k) + (1-\theta_k) f^i(z_{k+1}).$$

Step 8 **Update the iterate and approximation.**
Compute $s_k := z_{k+1} - z_k$ and $y^i_k := \nabla f^i(z_{k+1}) - \nabla f^i(z_k)$, $i \in [p]$.
Then, update the symmetric positive definite matrix by

$$B^i(z_{k+1}) := B^i(z_k) - \frac{B^i(z_k)s_k s_k^\top B^i(z_k)}{s_k^\top B^i(z_k) s_k} + \frac{y^i_k y^{i^\top}_k}{s_k^\top y^i_k} \text{ for all } i \in [p].$$

Update $k \leftarrow k+1$ and return to Step 3.

Assumption 4.1. (a) For all $i = 1, 2, \ldots, m$, each f^i is bounded from below on the level set $\mathscr{L}_0 := \{z \in \mathbb{R}^n : f^i(z) \preceq f^i(z_0)\}$, where z_0 is an initial point.

(b) For all $i = 1, 2, \ldots, m$, each function $\nabla f^i(z)$ of $f^i(z)$ is Lipschitz continuous in an open convex set $\mathscr{B}$ that contains $\mathscr{L}_0$, i.e., there exists a constant $L > 0$ such that

$$\|\nabla f^i(z) - \nabla f^i(y)\| \leq L\|z - y\|,\ z, y \in \mathscr{B}.$$

Lemma 4.5. *Let $\beta \in (0,1)$ be fixed, and consider the functionals $\xi_{\bar{z}}$ and Φ defined in (3.5) and (3.6), respectively. Suppose that $\{z_k\}$ is a sequence of non-stationary points of (SOP). Then, there exists a step-size $t_k > 0$ of Algorithm 3 that satisfies the non-monotone Armijo condition in the line search procedure, i.e., for all $t \in (0, t_k]$,*

$$j \in [\omega_k] : f^{a_j^k}(z_k + tu_k) \preceq C^{a_j^k}(z_k) + \sigma_1 t_k \nabla f^{a_j^k}(z_k)^\top u_k - \sigma_2 t_k^2 \|u_k\|^2 e, \tag{4.11}$$

where $C^{a_j^k}$ is defined in (4.8).

Proof. Assume contrarily that, for each $k \in \mathbb{N}$,

$$f^{a_j^k}(z_k + s_k\beta^i u_k) \not\preceq C^{a_j^k}(z_k) + \sigma_1 s_k \beta^i \nabla f^{a_j^k}(z_k)^\top u_k - \sigma_2 (s_k\beta^i)^2 \|u_k\|^2 e$$

$$\text{or, } \frac{f^{a_j^k}(z_k + s_k\beta^i u_k) - f^{a_j^k}(z_k)}{s_k\beta^i} \not\preceq \sigma_1 \nabla f^{a_j^k}(z_k)^\top u_k - \sigma_2 s_k \beta^i \|u_k\|^2 e \text{ from Lemma 4.4.}$$

Since each f^i, $i = 1, 2, \ldots, m$ is continuously differentiable, therefore, for any $\beta \in (0,1)$, by taking limit $k \to +\infty$, we obtain

$$\nabla f^{a_j^k}(z_k)^\top u_k \not\preceq \sigma_1 \nabla f^{a_j^k}(z_k)^\top u_k, \text{ i.e., } (1-\sigma_1)\nabla f^{a_j^k}(z_k)^\top u_k \not\preceq 0.$$

This implies that $\nabla f^{a_j^k}(z_k)^\top u_k \not\preceq 0$, as $\sigma_1 \in (0,1)$. This is a contradiction to the fact that z_k is a non-stationary point of (SOP). Therefore, result (4.11) holds. □

Lemma 4.6. *Let $\{z_k\}$ be a sequence of non-stationary points generated by Algorithm 3. Then, $z_k \in \mathscr{L}(z_0)$ for all k, and the sequence $\{C^{a_j^k}\}_k$ is non-increasing and convergent.*

Proof. Note that z_k is a non-stationary point of (SOP). Thus, in view of (3.1), there exists $u_k \in \mathbb{R}^n$ such that

$$\begin{aligned}
& \Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k) < 0 \\
\implies \quad & \Omega_e(f^{a_j^k}(z_k + t_k u_k) - C^{a_j^k}(z_k)) < 0 \quad \text{from Lemma 4.5} \\
\implies \quad & f^{a_j^k}(z_k + t_k u_k) - C^{a_j^k}(z_k) \in -\operatorname{int}(K) \quad \text{from Proposition 2.2(v)} \\
\implies \quad & f^{a_j^k}(z_{k+1}) \prec C^{a_j^k}(z_k).
\end{aligned} \tag{4.12}$$

In view of relation (4.8), we observe that

$$\begin{aligned}
C^{a_j^k}(z_{k+1}) &= \theta_{k+1} C^{a_j^k}(z_k) + (1-\theta_{k+1}) f^{a_j^k}(z_{k+1}) \\
&\preceq \theta_{k+1} C^{a_j^k}(z_k) + (1-\theta_{k+1}) C^{a_j^k}(z_k) \quad \text{from (4.12)} \\
&= C^{a_j^k}(z_k) \ \forall\ j \in [\omega_k].
\end{aligned} \tag{4.13}$$

Thus $\{C^{a_j^k}\}$ is non-increasing. Now, we prove the second part with the help of induction. For $k = 0$, $z_0 \in \mathscr{L}_0$ and the result is trivial. Now, assume that the result holds for some $z_k \in \mathscr{L}_0$ when $k \le l$. This means that, for all $k \le l$, $f^{a_j^k}(z_k) \preceq f^{a_j^k}(z_0)$. Then, in view of Lemma 4.4 and relation (4.13), we obtain

$$f^{a_j^k}(z_{l+1}) \preceq C^{a_j^k}(z_{l+1}) \preceq C^{a_j^k}(z_l) \preceq \cdots \preceq C^{a_j^k}(z_0) = f^{a_j^k}(z_0) \implies f^{a_j^k}(z_{l+1}) \preceq f^{a_j^k}(z_0).$$

Thus $z_{l+1} \in \mathscr{L}_0$ and the required result holds immediately. □

5. Convergence Analysis

5.1. **Convergence analysis of Algorithms 1 and 2.** In this section, we establish a lower bound for the step size produced by the non-monotone line search procedures in Algorithms 1 and 2.

Lemma 5.1. *Let $\beta, \nu \in (0,1)$ and z_k be a sequence of non-stationary points of (SOP). Let, for all $a \in P_z$ and $j \in [\omega(z_k)]$, the functions $\nabla f^{a_j^k}$'s be Lipschitz continuous with Lipschitz constant $\tilde{L}$, i.e., $\|\nabla f^{a_j^k}(z) - \nabla f^{a_j^k}(z_k)\| \leq \tilde{L}\|z - z_k\|$ for all z on the segment connecting z_k and $z_k + \frac{t_k}{\nu}u_k$, where $t_k \leq \nu$. Assume that there exists $t_k > 0$ of Algorithms 1 and 2 such that the non-monotone Armijo's conditions are satisfied. Then,*

$$t_k \geq \min\left\{\nu, \frac{2\nu(1-\beta)\|\nabla f^{a_j^k}(z_k)^\top u_k\|}{\tilde{L}\|u_k\|^2}\right\}. \tag{5.1}$$

Proof. If $t_k \geq \nu$, then (5.1) trivially holds. If $t_k < \nu$, then there exists a least non-negative integer q_k such that $t_k = \nu^{q_k}$ which satisfies (4.1). Thus

$$\begin{aligned} f^{a_j^k}(z_k + \tfrac{t_k}{\nu}u_k) &\succ C^{a_j^k} + \beta\tfrac{t_k}{\nu}\nabla f^{a_j^k}(z_k)^\top u_k \\ &\succeq f^{a_j^k}(z_k) + \beta\tfrac{t_k}{\nu}\nabla f^{a_j^k}(z_k)^\top u_k, \text{ from Lemma 4.2, for all } j \in [\omega(z_k)]. \end{aligned} \tag{5.2}$$

Since $\nabla f^{a_j^k}$'s are Lipschitz continuous, for all $j \in [\omega(z_k)]$, therefore we have

$$\begin{aligned} f^{a_j^k}(z_k + \tfrac{t_k}{\nu}u_k) - f^{a_j^k}(z_k) &= \tfrac{t_k}{\nu}\nabla f^{a_j^k}(z_k)^\top u_k + \int_0^{\frac{t_k}{\nu}} [\nabla f^{a_j^k}(z_k + tu_k) - \nabla f^{a_j^k}(z_k)]^\top u_k \, dt \\ &\preceq \tfrac{t_k}{\nu}\nabla f^{a_j^k}(z_k)^\top u_k + \int_0^{\frac{t_k}{\nu}} t\tilde{L}\|u_k\|^2 dt \\ &= \tfrac{t_k}{\nu}\nabla f^{a_j^k}(z_k)^\top u_k + \tfrac{1}{2}\tilde{L}\tfrac{t_k^2}{\nu^2}\|u_k\|^2. \end{aligned} \tag{5.3}$$

In view of (5.2) and (5.3), we obtain

$$\begin{aligned} & \beta\tfrac{t_k}{\nu}\nabla f^{a_j^k}(z_k)^\top u_k \preceq \tfrac{t_k}{\nu}\nabla f^{a_j^k}(z_k)^\top u_k + \tfrac{1}{2}\tilde{L}\tfrac{t_k^2}{\nu^2}\|u_k\|^2 \\ \implies & (\beta-1)\tfrac{t_k}{\nu}\nabla f^{a_j^k}(z_k)^\top u_k \preceq \tfrac{1}{2}\tilde{L}\tfrac{t_k^2}{\nu^2}\|u_k\|^2 \\ \implies & t_k \succeq \frac{2\nu(\beta-1)\nabla f^{a_j^k}(z_k)^\top u_k}{\tilde{L}\|u_k\|^2}. \end{aligned}$$

Note that in the above relation $(\beta-1)\nabla f^{a_j^k}(z_k)^\top u_k \succ 0$. Thus, we conclude that

$$t_k \geq \min\left\{\nu, \frac{2\nu(1-\beta)\|\nabla f^{a_j^k}(z_k)^\top u_k\|}{\tilde{L}\|u_k\|^2}\right\}.$$

□

We now turn to establishing the global convergence of Algorithms 1 and 2.

Theorem 5.1. *Let $U \subseteq \mathbb{R}^n$ be a non-empty set, and z_k be a sequence of non-stationary points of (SOP). For any $z \in U$ and $a \in P_z$, suppose that there exist constants $\mathsf{a}, \mathsf{b} \in \mathbb{R}$ with $0 < \mathsf{a} \leq \mathsf{b}$*

such that $\mathsf{a}I \leq B^{a_j}(z) \leq \mathsf{b}I$ *for all* $j \in [\omega(z)]$. *Assume further that each function* $f^{a_j^k}$ *is bounded from below,* $\eta_{\max} < 1$, *and there exists a constant* $c_1 > 0$ *satisfying*

$$\|\nabla f^{a_j^k}(z_k)^\top u_k\| > c_1\|u_k\|^2, \tag{5.4}$$

for all $j \in [\omega(z_k)]$. *In addition, suppose that* $\nabla f^{a_j^k}$*'s are Lipschitz continuous with Lipschitz constant* $\tilde{L}$, *and* Lemma 5.1 *holds. Then, every limit point of the sequence* $\{z_k\}$ *is a stationary point of* (SOP).

Proof. We first show that

$$f^{a_j^k}(z_{k+1}) \leq C^{a_j^k} - \gamma\|\nabla f^{a_j^k}(z_k)^\top u_k\|e \quad \text{for all } j \in [\omega(z_k)] \text{ and } e \in \operatorname{int}(K), \tag{5.5}$$

where $\gamma := \min\left\{\beta\nu, \frac{2\beta\nu(1-\beta)c_1}{\tilde{L}}\right\}$. We observe that there are following two cases:

(1) If the non-monotone Armijo's condition is employed and $t_k \geq \nu$, then, for every $j \in [\omega(z_k)]$,

$$\begin{aligned} f^{a_j^k}(z_{k+1}) &\preceq C^{a_j^k} + \beta t_k \nabla f^{a_j^k}(z_k)^\top u_k \\ &= C^{a_j^k} - \beta t_k \|\nabla f^{a_j^k}(z_k)^\top u_k\| e \\ &\preceq C^{a_j^k} - \beta\nu\|\nabla f^{a_j^k}(z_k)^\top u_k\| e \quad \text{since } t_k \geq \nu. \end{aligned}$$

(2) If the non-monotone Armijo's condition is employed and $t_k < \nu$, then we find from Lemma 5.1 for all $j \in [\omega(z_k)]$ that

$$\begin{aligned} f^{a_j^k}(z_{k+1}) &\preceq C^{a_j^k} + \beta t_k \nabla f^{a_j^k}(z_k)^\top u_k \qquad (5.6) \\ &\preceq C^{a_j^k} - \frac{2\beta\nu(1-\beta)}{\tilde{L}} \frac{\|\nabla f^{a_j^k}(z_k)^\top u_k\|^2}{\|u_k\|^2} e \quad \text{from Lemma 5.1} \\ &\preceq C^{a_j^k} - \frac{2\beta\nu(1-\beta)c_1}{\tilde{L}} \|\nabla f^{a_j^k}(z_k)^\top u_k\| e \quad \text{from (5.4)} \\ &\preceq C^{a_j^k} - \gamma\|\nabla f^{a_j^k}(z_k)^\top u_k\| e. \qquad (5.7) \end{aligned}$$

Therefore, in view of Step 7 of Algorithm 2 and the above relation, we have

$$\begin{aligned} C^{a_j^{k+1}} &= \frac{\eta_k q_k C^{a_j^k} + f^{a_j^k}(z_{k+1})}{q_{k+1}} \\ &\preceq \frac{\eta_k q_k C^{a_j^k} + C^{a_j^k} - \gamma\|\nabla f^{a_j^k}(z_k)^\top u_k\| e}{q_{k+1}} \\ &= C^{a_j^k} - \frac{\gamma\|\nabla f^{a_j^k}(z_k)^\top u_k\| e}{q_{k+1}}. \qquad (5.8) \end{aligned}$$

Since $f^{a_j^k}$'s are bounded from below, we conclude from Lemma 5.1 that $C^{a_j^k}$'s are bounded from below. Thus, from (5.8), we obtain

$$\sum_{k=0}^{+\infty} \frac{\|\nabla f^{a_j^k}(z_k)^\top u_k\|}{q_{k+1}} < +\infty. \tag{5.9}$$

In light of the above relation, for all $a \in P_z$ and $u \in \mathbb{R}^n$, we conclude that

$$\begin{aligned}
\Phi(z_k) &= \min_{(a,u)\in P_{z_k}\times\mathbb{R}^n} \xi_{z_k}(a,u) \\
&\leq \xi_{z_k}(a^k,u_k) \\
&= \max_{j\in[\omega(z_k)]} \left\{\Omega_e(\nabla f^{a^k_j}(z_k)^\top u_k + \tfrac{1}{2}u_k{}^\top B^{a^k_j}(z_k)u_k)\right\} \\
&\leq \max_{j\in[\omega(z_k)]} \left\{\hat{L}(\|\nabla f^{a^k_j}(z_k)^\top u_k + \tfrac{1}{2}u_k^\top B^{a^k_j}(z_k)u_k\|)\right\} \quad \text{from Proposition 2.2 (iii)} \\
&\leq \hat{L} \max_{j\in[\omega(z_k)]} \left\{\|\nabla f^{a^k_j}(z_k)^\top u_k\| + \tfrac{\mathsf{b}}{2}\|u_k\|^2\right\} \quad \text{since } B^{a^k_j}(z_k) \leq \mathsf{b}I \\
&\leq \hat{L} \max_{j\in[\omega(z_k)]} \left\{\|\nabla f^{a^k_j}(z_k)^\top u_k\| + \tfrac{\mathsf{b}}{2c_1}\|\nabla f^{a^k_j}(z_k)^\top u_k\|\right\} \quad \text{from (5.4)} \\
&\leq \hat{L} \max_{j\in[\omega(z_k)]} \left(1+\tfrac{\mathsf{b}}{2c_1}\right)\|\nabla f^{a^k_j}(z_k)^\top u_k\|.
\end{aligned} \tag{5.10}$$

From (5.9) and (5.10), we obtain

$$\sum_{k=0}^{+\infty} \frac{|\Phi(z_k)|}{q_{k+1}} < +\infty. \tag{5.11}$$

Now, let $\bar{z}$ be a limit point of the sequence $\{z_k\}$. Then, there exists a subsequence $\{z_k\}_{k\in Q}$ which converges to $\bar{z}$. We aim to show that $\Phi(\bar{z}) = 0$. Suppose, for the sake of contradiction, that $\Phi(\bar{z}) < 0$. Then, there exist constants $\varepsilon > 0$ and $\delta_0 > 0$ such that, for every $0 < \delta \leq \delta_0$ and for all $k \in Q$ satisfying $\|z_k - \bar{z}\| \leq \delta$,

$$|\Phi(z_k)| \geq \varepsilon > 0 \text{ and } \sum_{k=0}^{+\infty} \frac{|\Phi(z_k)|}{q_{k+1}} \geq \sum_{k\in\{k\in Q\,:\,\|z_k-\bar{z}\|\leq\delta\}} \frac{\varepsilon}{q_{k+1}}. \tag{5.12}$$

Therefore, in view of (4.3) and $\eta_{\max} < 1$, we have

$$q_{k+1} = 1 + \sum_{i=0}^{k}\prod_{l=0}^{i}\eta_{k-l} \leq 1 + \sum_{i=0}^{k}\eta_{\max}^{i+1} \leq \sum_{i=0}^{+\infty}\eta_{\max}^{i} = \frac{1}{1-\eta_{\max}}.$$

Consequently, applying the above relation in (5.12), we have

$$\begin{aligned}
\sum_{k=0}^{+\infty} \frac{|\Phi(z_k)|}{q_{k+1}} &\geq \sum_{k\in\{k\in Q\,:\,\|z_k-\bar{z}\|\leq\delta\}} \frac{\varepsilon}{q_{k+1}} \\
&\geq \sum_{k\in\{k\in Q\,:\,\|z_k-\bar{z}\|\leq\delta\}} (1-\eta_{\max})\varepsilon = +\infty,
\end{aligned}$$

which is a contradiction to (5.11). Hence, $\Phi(\bar{z}) = 0$ and $\bar{z}$ is a stationary point of (SOP). □

Lemma 5.2. *Let U be a non-empty subset of $\mathbb{R}^n$. Assume that there exist $\mathsf{a},\mathsf{b} \in \mathbb{R}$ such that $0 < \mathsf{a} \leq \mathsf{b}$ and for any $z \in U$ and $a \in P_z$, $B^{a^k}(z) \leq \mathsf{b}I$ for all $j \in [\omega(z)]$. Then, for any $a \in P_x$, there exist $\lambda_j \geq 0$, $j \in [\omega(x)]$ with $\sum_{j=1}^{\omega(x)} \lambda_j = 1$ such that $|\Phi(z)| \leq \frac{3\hat{L}}{2q}\left\|\sum_{j=1}^{\omega(z)} \lambda_j \nabla f^{a^k}(z)\right\|^2$, where $\hat{L}$ is a Lipschitz constant for Ω_e.*

Proof. The proof is similar to Proposition 4.2 in [15]. □

Proposition 5.1. *Let U be a non-empty subset of $\mathbb{R}^n$ and z_k be a bounded sequence of non-stationary points of (SOP). For any $z \in U$ and $a \in P_z$, let $\mathsf{a}, \mathsf{b}, \delta, \varepsilon > 0$ and $\mathsf{a}I \leq B^{a_j}(z) \leq \mathsf{b}I$ for all j in $[\omega(z)]$. Then, $\lim_{k\to+\infty} \Phi(z_k) = 0$ and $\lim_{k\to+\infty} \|u_k(z_k)\| = 0$.*

Proof. Let us assume contrarily that, for $\varepsilon > 0$, there exist a subsequence $\{z_k\}_{k\in Q}$ such that $|\Phi(z_k)| \geq \varepsilon$ for all $k \in Q$. Since $\{z_k\}_{k\in Q}$ is a bounded sequence, it has at least one limit point $\bar{z}$, and we must have $|\Phi(\bar{z})| \geq \varepsilon$, which is a contradiction to the hypothesis that $\bar{z}$ must be a non-stationary point of (SOP). Therefore, $\Phi(z_k) \to 0$ as $k \to +\infty$. Now, in view of Lemma 5.2, we obtain the desired conclusion immediately. □

5.2. **Convergence analysis of Algorithm 3.** In this section, we give convergence results for the method with a modified non-monotone line search defined in (4.8). First, we assume that, at any given point z_k, search direction u_k satisfies Assumption 4.1.

Lemma 5.3. *Assume that Assumption 4.1 holds. Then, there exists a constant $\eta > 0$ such that*

$$C^{a_j^k}(z_k) - f^{a_j^k}(z_{k+1}) \succeq \eta \left(\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|} \right)^2 e \quad \text{for all } j \in [\omega_k].$$

Proof. Let us define the following sets:

$$P_1 := \{k : t_k = s_k\} \text{ and } P_2 := \{k : t_k < s_k\}.$$

First, we assume that $k \in P_1$. Then, in view of Step 6 of Algorithm 3 and $s_k = -\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|^2}$, we have

$$C^{a_j^k}(z_k) - f^{a_j^k}(z_{k+1}) \succeq -\sigma_1 s_k \nu^q \nabla f^{a_j^k}(z_k)^\top u_k + \sigma_2 s_k^2 \nu^{2q} \|u_k\|^2 e \; \forall j \in [\omega_k]. \tag{5.13}$$

Then, from Definition 2.2, we deduce $\nabla f^{a_j^k}(z_k)^\top u_k \preceq \Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)e$. Therefore, in view of the above relation, (5.13) reduces to

$$\begin{aligned} C^{a_j^k}(z_k) - f^{a_j^k}(z_{k+1}) &\succeq \sigma_1 \left(\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|} \right)^2 e + \sigma_2 \left(\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|} \right)^2 e \quad \forall j \in [\omega_k] \\ &= (\sigma_1 + \sigma_2) \left(\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|} \right)^2 e \quad \forall j \in [\omega_k]. \end{aligned} \tag{5.14}$$

Next, we assume that $k \in P_2$ such that $t_k < s_k$ and let $t = \frac{t_k}{\sigma}$ for some $\sigma \in (0,1)$. In view of Step 6 of Algorithm 3, we obtain by Lemma 4.4 that

$$\begin{aligned} & f^{a_j^k}(z_k + t u_k) \not\preceq C^{a_j^k}(z_k) + \sigma_1 t_k \nabla f^{a_j^k}(z_k)^\top u_k - \sigma_2 t_k^2 \|u_k\|^2 e \quad \forall j \in [\omega_k] \\ \text{i.e.,} \quad & f^{a_j^k}(z_k + t u_k) \not\preceq f^{a_j^k}(z_k) + \sigma_1 t_k \nabla f^{a_j^k}(z_k)^\top u_k - \sigma_2 t_k^2 \|u_k\|^2 e \quad \forall j \in [\omega_k]. \end{aligned} \tag{5.15}$$

Now, using the mean-value theorem on the above relation, we deduce that there exists a $\lambda \in [0,1]$ such that

$$\begin{aligned} & t \nabla f^{a_j^k}(z_k + t\lambda u_k)^\top u_k \not\preceq \sigma_1 t_k \nabla f^{a_j^k}(z_k)^\top u_k - \sigma_2 t_k^2 \|u_k\|^2 e \quad \forall j \in [\omega_k] \\ \text{i.e.,} \quad & \nabla f^{a_j^k}(z_k + t\lambda u_k)^\top u_k \not\preceq \sigma_1 \nabla f^{a_j^k}(z_k)^\top u_k - \sigma_2 t_k \|u_k\|^2 e \quad \forall j \in [\omega_k]. \end{aligned}$$

From the Lipschitz continuity of $\nabla f^{a_j^k}$, we have $\|\nabla f^{a_j^k}(z_k+t\lambda u_k)-\nabla f^{a_j^k}(z_k)\| \le Lt\|u_k\|$, where $\lambda \in [0,1]$. Applying the Cauchy–Schwarz inequality, we obtain

$$(\nabla f^{a_j^k}(z_k+t\lambda u_k)-\nabla f^{a_j^k}(z_k))^\top u_k \le \|\nabla f^{a_j^k}(z_k+t\lambda u_k)-\nabla f^{a_j^k}(z_k)\|\|u_k\| \le Lt\|u_k\|^2,$$

which together with inequality (4.4) yields

$$Lt\|u_k\|^2 \ge -(1-\sigma_1)\nabla f^{a_j^k}(z_k)^\top u_k - \sigma_2 t\|u_k\|^2. \tag{5.16}$$

Note that z_k is a non-stationary point of (SOP). Then, by Theorem 3.2, there exists $a^k \in P_{z_k}$ such that

$$\begin{aligned}
&\max_{j\in[\omega(z_k)]}\left\{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k + \tfrac{1}{2}u_k^\top B^{a_j^k}(z_k)u_k)\right\} < 0\\
\Longrightarrow\quad & \Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k + \tfrac{1}{2}u_k^\top B^{a_j^k}(z_k)u_k) < 0 \ \forall j \in [\omega(z_k)]\\
\overset{(3.4)}{\Longrightarrow}\quad & \Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k) < -\tfrac{1}{2}\rho_{\min}\|u_k\|^2 \ \forall j \in [\omega(z_k)]\\
\Longrightarrow\quad & \nabla f^{a_j^k}(z_k)^\top u_k \prec -\tfrac{1}{2}\rho_{\min}\|u_k\|^2 e \ \forall j \in [\omega(z_k)] \text{ by Proposition 2.2}(iv)\&(v). \qquad (5.17)
\end{aligned}$$

In view of Step 6, (5.16), and (5.17), we have

$$\begin{aligned}
C^{a_j^k}(z_k) - f^{a_j^k}(z_{k+1}) &\succeq -\sigma_1 t_k \nabla f^{a_j^k}(z_k)u_k + \sigma_2 t_k^2\|u_k\|^2 e\\
&\succeq \tfrac{\sigma_1(1-\sigma_1)}{L+\sigma_2}\left(\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|^2}\right)^2 e + \tfrac{\sigma_2(1-\sigma_1)^2}{(L+\sigma_2)^2}\left(\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|^2}\right)^2 e\\
&= \left(\tfrac{\sigma_1(1-\sigma_1)}{L+\sigma_2} + \tfrac{\sigma_2(1-\sigma_1)^2}{(L+\sigma_2)^2}\right)\left(\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|^2}\right)^2 e.
\end{aligned}$$

Let $\eta := \min\{\sigma_1+\sigma_2, \frac{\sigma_1(1-\sigma_1)}{(L+\sigma_2)^2} + \frac{\sigma_2(1-\sigma_1)^2}{L+\sigma_2}\}$. Therefore, from (5.13) and the relation above, we have the required result immediately. □

Theorem 5.2. *Assume that Assumption 4.1 holds and suppose that there exists constants* $\mathsf{a},\mathsf{b} \in \mathbb{R}$ *with* $0 < \mathsf{a} \le \mathsf{b}$ *such that* $\mathsf{a}I \le B^{a_j}(z) \le \mathsf{b}I$ *for all* $j \in [\omega(z)]$. *Let* $\{z_k\}$ *be a sequence of non-stationary points generated by Algorithm 3. Then* $\lim_{k\to+\infty}|\Phi(z_k)| = 0$.

Proof. It can be noted from Lemma 4.6 that $\{C^{a_j^k}\}_k$ is non-increasing and admits a limit when $k \to \infty$. Then, in view of relation (4.8), for all $j \in [\omega_k]$, we observe that

$$\begin{aligned}
&C^{a_j^k}(z_{k+1}) = \theta_{k+1}C^{a_j^k}(z_k) + (1-\theta_{k+1})f^{a_j^k}(z_{k+1})\\
&\Longrightarrow C^{a_j^k}(z_{k+1}) \preceq \theta_{k+1}C^{a_j^k}(z_k) + (1-\theta_{k+1})\left(C^{a_j^k}(z_k) - \eta\left(\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|}\right)^2 e\right) \text{ Lemma 5.3}\\
&\Longrightarrow (1-\theta_{k+1})\eta\left(\frac{\Omega_e(\nabla f^{a_j^k}(z_k)^\top u_k)}{\|u_k\|}\right)^2 e \preceq C^{a_j^k}(z_k) - C^{a_j^k}(z_{k+1}) \preceq 0 \text{ Lemma 4.6.}
\end{aligned} \tag{5.18}$$

Note that, for every $a^k \in P_k$, $j \in [\omega(z_k)]$, relation (3.4) holds

$$\max_{j\in[\omega(z_k)]} \{\Omega_e(\nabla f^{a^k_j}(z_k)^\top u_k)\} \le \max_{j\in[\omega(z_k)]} \{\Omega_e(\nabla f^{a^k_j}(z_k)^\top u_k + \tfrac{1}{2}u_k^\top B^{a^k_j}(z_k)^\top u_k)\} \tag{5.19}$$

and $u_k \in \mathbb{R}^n$. Therefore, in view of Proposition 2.2(iv)&(v), we deduce that

$$\begin{aligned}
&\max_{j\in[\omega(z_k)]} \{\Omega_e(\nabla f^{a^k_j}(z_k)^\top u_k)\} \\
&\le \min_{(a^k,u_k)\in P_{z_k}\times\mathbb{R}^n} \left\{ \max_{j\in[\omega(z_k)]} \Omega_e\left(\nabla f^{a^k_j}(z_k)^\top u_k + \tfrac{1}{2}u_k^\top B^{a^k_j}(z_k)u_k\right)\right\} \text{ from (3.5)} \qquad (5.20)\\
&= -|\Phi(z_k)|. \qquad (5.21)
\end{aligned}$$

Since $u_k^\top B^{a^k_{j^*}}(z_k)u_k \ge \mathsf{a}\|u_k\|^2 =: \rho_{\min}\|u_k\|^2$, we have $\Omega_e(\nabla f^{a^k_{j^*}}(z_k)^\top u_k) \le \Phi(z_k) - \frac{1}{2}\rho_{\min}\|u_k\|^2 < 0$, which yields $\nabla f^{a^k_j}(z_k)^\top u_k \prec -\frac{1}{2}\rho_{\min}\|u_k\|^2 e$ for all j in $[\omega_k]$. Since a^k and u_k are chosen in Algorithm 3 to achieve $\Phi(z_k) = \xi_{z_k}(a^k,u_k)$, let $j^* \in [\omega(z_k)]$ be the index attaining the maximum in $\xi_{z_k}(a^k,u_k) = \Phi(z_k) < 0$. Then, $\Omega_e(\nabla f^{a^k_{j^*}}(z_k)^\top u_k) \le \Phi(z_k) - \frac{1}{2}\rho_{\min}\|u_k\|^2 < 0$. It follows that

$$\left|\Omega_e(\nabla f^{a^k_{j^*}}(z_k)^\top u_k)\right| \ge |\Phi(z_k)| + \frac{1}{2}\rho_{\min}\|u_k\|^2,$$

and by boundedness of $\{u_k\}$ from Theorem 4.1 under Assumption 4.1,

$$\begin{aligned}
\left(\frac{\Omega_e(\nabla f^{a^k_{j^*}}(z_k)^\top u_k)}{\|u_k\|}\right)^2 &\ge \frac{\left(|\Phi(z_k)| + \frac{1}{2}\rho_{\min}\|u_k\|^2\right)^2}{\|u_k\|^2}\\
&\ge \frac{|\Phi(z_k)|^2}{\|u_k\|^2} + \rho_{\min}^2\|u_k\|^2\\
&\ge \tilde{c}\left(\frac{|\Phi(z_k)|^2}{\|u_k\|^2} + |\Phi(z_k)|\right),
\end{aligned}$$

for some $\tilde{c} > 0$, using $\|u_k\|^2 \ge \gamma|\Phi(z_k)|$ near zero by strong convexity. Substituting into (5.18) for $j = j^*$ yields

$$C^{a^k_j}(z_k) - C^{a^k_j}(z_{k+1}) \succeq (1-\theta_{k+1})\eta_k \left(\frac{\Omega_e(\nabla f^{a^k_{j^*}}(z_k)^\top u_k)}{\|u_k\|}\right)^2 e \succeq \delta_k|\Phi(z_k)|e,$$

where $\delta_k := \min\{(1-\theta_{k+1})\eta_k, \tilde{c}\} > 0$ is bounded away from zero under Assumption 4.1 (e.g., $\inf\eta_k > 0$ from Lemma 5.3, $\inf(1-\theta_k) > 0$). Summing over $k = 0,1,\ldots,N-1$ and telescoping gives

$$\sum_{k=0}^{N-1}\left(C^{a^k_j}(z_k) - C^{a^k_j}(z_{k+1})\right) \succeq \left(\sum_{k=0}^{N-1}\delta_k|\Phi(z_k)|\right)e.$$

The left side is $C^{a^0_j}(z_0) - C^{a^N_j}(z_N) \preceq C^{a^0_j}(z_0) - L < +\infty$. Thus $\sum_{k=0}^{\infty}\delta_k|\Phi(z_k)| < +\infty$. Assumption 4.1 implies $\sum|\Phi(z_k)| < +\infty$. So, $|\Phi(z_k)| \to 0$ as $k \to \infty$. In view of (5.20) and the non-increasing $\{C^k\}$, the relations (5.18)–(5.20) ensure consistency, which completes the proof. □

6. Numerical Demonstrations and Execution of Results

In this section, we compare the performance of the proposed non-monotone quasi-Newton methods with max-type line search, average-type, and modified-hybrid-type line search given in Algorithms 1, 2, and 3 on 20 numerical set-valued optimization problems. These problems are defined in Appendix A. In order to compare the methods, we follow the performance profile proposed by Dolan and Moré [6] for a set of solvers S (Max, Avg, Modified) on a set of Problems P (20 problems in this paper). The performance ratio $r_{p,s}$ of algorithm $s \in S$ for solving the problem $p \in P$ is defined as

$$r_{p,s} := \frac{t_{p,s}}{\min\{t_{p,s} : s \in S\}},$$

where $t_{p,s}$ is a metric of interest (e.g., number of iterations, CPU time, etc.) obtained by running algorithm s on problem p. For a problem p, the best performing algorithm s will have $r_{p,s} = 1$, and the rest will have $r_{p,s} > 1$. Using $r_{p,s}$, we can obtain the performance profile ρ_s for algorithm s by computing the cumulative distribution function of $r_{p,s}$, i.e.,

$$\rho_s(\tau) := \frac{1}{|P|}|\{p \in P : r_{p,s} \leq \tau\}|.$$

The term $\rho_s(\tau)$ is the probability that the performance ratio $r_{p,s}$ of algorithm s lies in $[1,\tau]$. At $\tau = 1$, the value $\rho_s(1)$ gives the probability of algorithm s being the best performing algorithm, i.e., for how many problems, out of total $|P|$ problems, the algorithm s was the winner. For instance, $\rho_s(1) = 0.8$ means algorithm s is the best performing for 80 percent of the problem. However, for better judgment about the algorithms, we should also look at the larger values of τ where better-performing algorithms should have higher values of $\rho_s(\tau)$.

For analyzing our experiments, we have used MATLAB R2023b software. This MATLAB software was installed in an ios machine equipped with a 12-core CPU and 8 GB RAM. We compare the Max (Algorithm 1), Avg (Algorithm 2), and Modified (Algorithm 3) with the quasi-Newton-type method [15] for set-valued optimization. For the numerical experiments, the algorithms were implemented using the following parameter settings.

- We explore all the test problems with respect to the standard ordering cone $K = \mathbb{R}^2_+$ except for Example 6.2. The parameter e for the scalarizing function Ω_e is chosen to be $(1,1,\ldots,1)^\top \in \text{int}(K)$.
- In Step 6 of Algorithms 1, 2, and 3, the line search parameters were fixed as $\beta = 0.5$ and $\nu = 0.6$
- The stopping condition for each method is employed as $\|u_k\| < 10^{-4}$.
- To figure out the set Min $(\mathscr{F}(z_k), K)$ at the k-th iteration in Step 2 of Algorithms 1, 2, and 3, we adopted the common method of comparing the elements in $\mathscr{F}(z_k)$ pair-wise.
- In each test problem cases, we generated 100 randomly chosen initial points within the box $\text{lb} \leq x \leq \text{ub}$. In each experiment, we computed the minimum (min), median, and maximum (max) values for the following instances:
  - **CPU Time:** The duration (in seconds) required by the algorithm to satisfy the stopping condition for each initial point.
  - **Iterations:** The total number of iterations taken by the algorithm to reach the stopping condition for each initial point.

We modified some test problems on vector optimization present in the literature into SOPs, along with some freshly introduced SOPs; see Appendix A. The comparison performance of the proposed methods—Algorithms 1, 2, and 3—with quasi-Newton method for SOPs [15] listed in Table 1 is given in Table 2. The results in Table 2 indicate that the proposed modified-hybrid-type method (Algorithm 3) outperforms the max-type 1, average-type 2, quasi-Newton-type [15] methods in various cases.

Figure 1 presents the performance profiles obtained from the above data in terms of CPU time and iteration count. It can be clearly observed that the modified-hybrid-type non-monotone quasi-Newton method consistently demonstrates superior performance compared to the average-type, max-type, and monotone quasi-Newton methods.

We discussed the geometrical illustrations of proposed methods for a few test problems in the literature. These are given in Examples 6.1 and 6.2. For each problem examined, the values of F at the initial and final points are represented by black *bullet* • and red *bullet* •. We have used blue *bullets* •, green *bullets* •, and cyan *bullets* • to represent the location of the intermediate iterate values of $\mathscr{F}$ generated by Algorithms 1, 2, and 3, respectively.

Test Problem	Reference	m	n	lb	ub
MOP1	[22]	2	1	-10^5	10^5
MOP7	[22]	3	2	$(-400,-400)$	$(400,400)$
BK1	[3]	2	2	$(-5,-5)$	$(10,10)$
DGO2	[22]	2	1	-9	9
DGO1	[22]	2	1	-10	13
VFM1	[40]	3	2	-2	2
ZLT1	[43]	2	3	-1000	1000
JOS1	[27]	2	4	$(-5,-5)$	$(5,5)$
QV1	[39]	2	2	-5.12	5.12
FES2	[11]	3	2	0	1
FES1	[11]	2	2	0	1
LRS1	[35]	2	4	-50	50
Problem 1	New	3	2	-10	10
Problem 2	New	2	3	-1.8	1.8
Problem 3	New	2	1	-1.5	1.5
Problem 4	New	3	3	-4	4
Problem 5	New	2	2	-3.5	3.5
Problem 6	New	2	2	-2.5	2.5
Problem 7	New	3	1	-2.5	2.5
Problem 8	New	2	2	-4	4

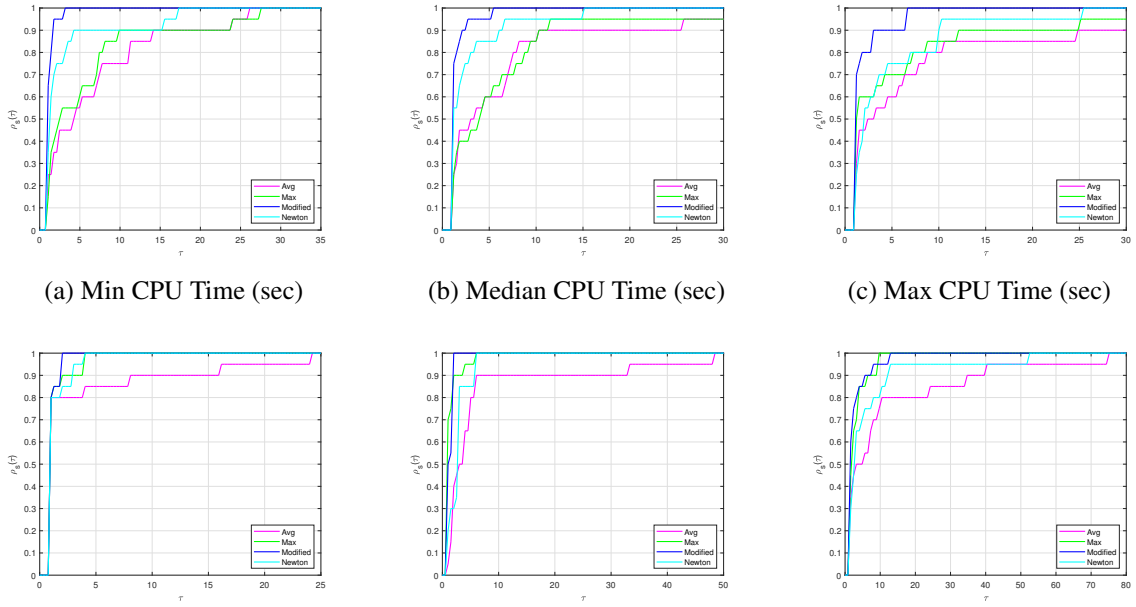
TABLE 1. Test problems for comparing the considered Algorithms

Further, we have discussed the geometrical illustrations of proposed methods for a few test problems in the literature. These are given in Examples 6.1 and 6.2.

Now, to visualize the performances of the proposed methods, we consider the following problems.

Type of SOP	Min time				Median time				Max time				Min iteration				Median iteration				Max iteration			
	Avg	Max	Modified	Quasi-Newton	Avg	Max	Modified	Quasi-Newton	Avg	Max	Modified	Quasi-Newton	Avg	Max	Modified	Quasi-Newton	Avg	Max	Modified	Quasi-Newton	Avg	Max	Modified	Quasi-Newton
MOP1	0.0349	0.0201	0.0294	0.0660	0.9367	0.4300	0.2764	0.9342	4.2339	3.5725	0.4943	4.9260	1	1	1	1	1.5000	1	0.5000	1.5000	10	9	1	10
MOP7	36.9425	38.9235	1.4217	51.5215	74.5942	76.1315	1.5443	81.5396	181.8557	183.3668	55.9818	183.6115	2	1	2	3	4	1	2	3	19	3	17	38
BK1	1.3128	1.5111	0.1984	1.1941	1.3397	1.5435	0.2029	1.1980	2.5393	2.8097	0.3297	3.3390	1	1	1	1	6	1	2	3	7	1	2	13
DGO2	0.1762	0.1205	0.0247	0.4275	0.7641	0.6246	0.4814	0.9257	3.9754	3.5675	3.1011	5.9263	1	1	1	1	4	4	2	6	12	14	8	16
DGO1	0.0333	0.0156	0.0261	0.2256	0.1568	0.1082	0.1825	0.7030	1.6529	0.2895	1.8982	1.9972	1	1	1	1	5	3	5	17	44	8	13	46
VFM1	6.0035	7.4514	1.4294	7.4238	6.5199	8.1203	1.5019	8.4969	286.7603	293.0219	11.6753	295.0933	4	2	1	5	4	1	2	4	19	22	23	24
ZLT1	2.2666	3.0434	3.2025	3.5681	3.5017	4.1714	5.0647	6.1740	232.4231	242.9863	245.254	481.114	1	1	1	1	5	1	1	1	260	41	11	575
Problem 1	2.4734	2.4734	0.1827	3.1039	2.7473	3.1546	0.3524	3.3403	50.7459	34.3491	57.8647	63.7912	1	1	1	1	2	1	2	3	90	209	210	211
LRS1	12.2444	15.6571	16.2339	16.7474	29.5046	33.5044	34.0673	34.6679	77.0676	81.1120	81.9955	82.8687	6	4	1	1	3	6	1	1	7	9	1	12
Problem 2	0.4117	0.8212	1.2354	1.6471	7.7591	34.9662	41.2315	46.4260	154.9122	183.5796	187.7915	192.8001	1	1	1	1	4	1	1	1	43	36	32	49
Problem 3	0.0583	0.0534	0.0350	1.10191	0.0697	0.0609	0.0448	1.1316	0.7319	0.1662	0.1908	1.6153	1	1	1	1	2	2	1	3	3	2	1	3
Problem 4	2.5525	2.8007	3.0895	3.1515	8.3736	8.7950	9.1012	17.8408	120.6466	121.0045	121.4202	121.8426	18	19	20	20	57	58	59	59.5	652	653	654	655
FES1	2.0339	2.6940	0.2731	0.2794	5.4539	7.3100	1.391	7.7213	17.7028	17.7705	14.9621	26.5600	1	1	1	2	16	4	8	12	180	513	523	531
Problem 5	1.0750	1.0387	1.0839	1.1091	7.1939	9.6456	2.3950	9.6884	27.2489	114.8478	75.0957	115.8231	1	1	2	4	6	1	2	6	22	78	79	104
JOS1	19.5247	2.2314	1.7295	2.2620	63.1415	9.3315	2.4634	2.6657	858.4302	24.7893	17.4331	58.1446	8	4	1	1	4	2	2	6	12	10	7	17
QV1	1.1074	0.2244	0.0992	0.0982	0.3100	0.5176	0.1786	0.1748	95.1435	97.2829	0.8590	0.3040	1	1	2	3	1.5000	1	2	3	412	12	13	14
FES2	3.892	2.0545	0.2776	4.2759	5.9918	3.7204	0.5860	8.810	37.9866	25.7456	17.0689	44.4271	1	1	1	1	4.5000	2	4	5	70	59	46	86
Problem 6	0.3299	0.4888	0.0748	0.0723	0.5725	0.6786	0.1943	0.0798	1.9367	2.0199	2.0282	0.3080	1	1	1	1	5	1	1	1	14	7	11	2
Problem 7	6.8265	7.9723	3.0814	5.9276	3.8640	4.001	3.8025	6.001	26.0186	23.0172	19.8652	28.3592	1	1	1	1	3	2	2	3	17	15	13	18
Problem 8	0.4711	0.6707	0.0943	0.0911	0.6808	1.1784	0.1040	0.1035	3.7291	4.2608	0.3581	0.3569	1	1	1	1	5	4	1	6	23	19	3	24

TABLE 2. Performance comparison metric of avg-type (denoted as Avg), max-type (denoted as Max), and modified-hybrid-type (denoted as Modified) non-monotone quasi-Newton methods for 20 set-valued optimization problems



(a) Min CPU Time (sec) (b) Median CPU Time (sec) (c) Max CPU Time (sec)

(d) Min Iterations (e) Median Iterations (f) Max Iterations

FIGURE 1. Performance profile of avg-type, max-type, modified-hybrid-type non-monotone quasi-Newton methods with monotone quasi-Newton method for set optimization problems.

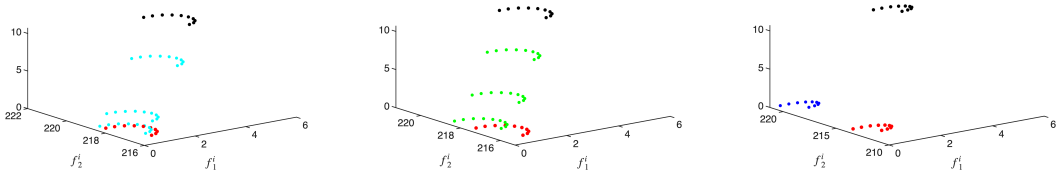
Example 6.1. Let $\mathscr{F} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^3$ be defined as $\mathscr{F}(z) = \{f^1(z), f^2(z), \ldots, f^{20}(z)\}$, where, for each $i \in [10]$, $f^i : \mathbb{R} \to \mathbb{R}^2$ is given by

$$f^i(z) = \begin{pmatrix} z_1^2 + \sin\left(\frac{2\pi(i-1)}{20}\right) \\ (z_1-10)^2 + (z_2-10)^2 + \cos\left(\frac{2\pi(i-1)}{20}\right) \\ z_1^2 + z_2^2 + \frac{(i-1)}{20}. \end{pmatrix}.$$

Figures 2(a), 2(b), and 2(c) depict the sequence $\{F(z_k)\}$ generated by Algorithms 1, 2, and 3 for the starting point $z_0 = (2.0000, -2.500)^\top$. The performance of Algorithms 1, 2, and 3, and quasi-Newton method for Example 6.1 is shown in Table 3. We observe that the modified-hybrid-type line search outperforms the average-type, max-type, and monotone line search.

Number of initial points	Algorithm	Iterations (Min, Max, Mean, Median)	CPU time (Min, Max, Mean, Median)
100	Avg line search	(1, 5, 1.9400, 1)	(0.7529, 6.1944, 2.3752, 1.0240)
	Max line search	(1, 3, 1, 1)	(0.36420, 7.1597, 3.3879, 2.0236)
	Modified line search	(1, 1, 1, 1)	(0.0163, 8.1244, 4.3971, 3.0248)
	Monotone line search	(1, 7, 4.0752, 4)	(0.9599, 8.1759, 6.7430, 6.7989)

TABLE 3. Performance of the considered Algorithms on Example 6.1



(a) The value of F across iterations generated by Algorithm 1 with average line search for initial point $z_0 = (2.0000, -2.5000)^\top$ for Example 6.1

(b) The value of F across iterations generated by Algorithm 2 with max line search for initial point $z_0 = (2.0000, -2.5000)^\top$ for Example 6.1

(c) The value of F across iterations generated by Algorithm 3 with modified line search for initial point $z_0 = (2.0000, -2.5000)^\top$ for Example 6.1

FIGURE 2. Output of Algorithms 1, 2, and 3 for Example 6.1

In Example 6.2, we consider the ordering cone K different from $\mathbb{R}^m_+$ and observe the performance of the proposed Algorithm 1, 2, and 3. The cone to be considered is given by $K = \{(y_1, y_2)^\top \in \mathbb{R}^2 : 5y_1 - 5y_2 \geq 0, -6y_1 + 8y_2 \geq 0\}$.

Example 6.2. Consider the set-valued map $\mathscr{F} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ defined by

$$\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{20}(z)\},$$

where, for each $i \in [20]$, the mapping $f^i : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} z_1^2 + e^{(z_1+z_2)} + \cos\left(\frac{2\pi(i-1)}{40}\right)\sin^2\left(\frac{2\pi(i-1)}{40}\right) \\ z_2^2 + e^{(z_1+z_2)} + \cos^2\left(\frac{2\pi(i-1)}{40}\right)\sin\left(\frac{2\pi(i-1)}{40}\right) \end{pmatrix}.$$

The first and last few mappings are

$$\begin{aligned} f^1(z) &= \begin{pmatrix} z_1^2 + e^{(z_1+z_2)} \\ z_2^2 + e^{(z_1+z_2)} \end{pmatrix}, \\ f^2(z) &= \begin{pmatrix} z_1^2 + e^{(z_1+z_2)} + \cos\left(\frac{\pi}{20}\right)\sin^2\left(\frac{\pi}{20}\right) \\ z_2^2 + e^{(z_1+z_2)} + \cos^2\left(\frac{\pi}{20}\right)\sin\left(\frac{\pi}{20}\right) \end{pmatrix}, \\ &\vdots \\ f^{20}(z) &= \begin{pmatrix} z_1^2 + e^{(z_1+z_2)} + \cos\left(\frac{19\pi}{20}\right)\sin^2\left(\frac{19\pi}{20}\right) \\ z_2^2 + e^{(z_1+z_2)} + \cos^2\left(\frac{19\pi}{20}\right)\sin\left(\frac{19\pi}{20}\right) \end{pmatrix}. \end{aligned}$$

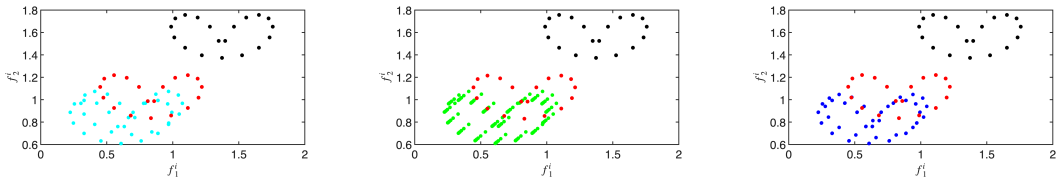
Let the initial point be $z_0 = (0.15, 0.15)^\top$. The iterative procedure generated by Algorithms 1, 2, and 3 is discussed below.

Step 1 (Initialization). Set the parameters as follows:

$$\beta = 0.5, \quad \nu = 0.6, \quad \eta = 0.85, \quad q_0 = 1, \quad B^1 = I_2, \quad B^2 = I_2,$$

where I_2 is a 2×2 identity matrix. Compute the initial function values $C^i = f^i(z_0)$ $i = 1, 2, \ldots, 20$. For instance,

$$C^1 = \begin{pmatrix} 1.3724 \\ 1.3724 \end{pmatrix}, \quad C^2 = \begin{pmatrix} 1.3965 \\ 1.5250 \end{pmatrix}, \quad \ldots, \quad C^{20} = \begin{pmatrix} 1.3482 \\ 1.5250 \end{pmatrix}.$$



(a) The value of F at each iteration generated by Algorithm 1 with average line search for the initial point $z_0 = (1.0000, -1.5000)^\top$ for Example 6.2

(b) The value of F at each iteration generated by Algorithm 2 with max line search for the initial point $z_0 = (1.0000, -1.5000)^\top$ for Example 6.2

(c) The value of F at each iteration generated by Algorithm 3 with modified line search for the initial point $z_0 = (1.0000, -1.5000)^\top$ for Example 6.2

FIGURE 3. Output of Algorithms 1 and 2 for Example 6.2

Step 2 (First index set selection and computation of direction). At $k = 0$, the initial index set is obtained as $M_0 = \{f^4, f^5, f^6, f^7, f^8, f^9, f^{15}, f^{16}, f^{17}, f^{18}, f^{19}\}$, $\omega_0 = 11$. Next, the search direction is computed as $u_0 = (-1.4997, -1.4997)^\top$, with norm $\|u_0\| = 2.1209 > \varepsilon$. Hence, the stopping criterion is not satisfied.

Step 3 (Step-size update and next iterative point z_1). For the average-type and max-type line search strategy, we obtain $\nu = 0.1250$, $z_1 = (-0.0375, -0.0375)^\top$, while, for the modified-type line search, $\nu = 0.5$, $z_1 = (-0.5999, -0.5999)^\top$.

Step 4 (For this new point z_1, the update of index sets is as follows).

(i) Average-type case: $M_1 = \{f^1, f^4, f^5, f^6, f^7, f^8, f^9, f^{11}, f^{13}, f^{14}, f^{15}, f^{16}, f^{17}, f^{18}, f^{19}\}$, $\omega_1 = 15$.

(ii) Maximum-type case: Similar index set and non-monotone parameters are computed.

(iii) Modified case: $M_1 = \{f^4, f^5, f^6, f^7, f^8, f^9, f^{11}, f^{13}, f^{15}, f^{16}, f^{17}, f^{18}\}$, $\omega_1 = 13$.

Step 5 (Next direction and iterate z_2). The updated index sets and the new directions are:

- **Average-type:**

$$u_1 = (-0.8902, 0.8902)^\top, \quad \nu = 0.5, \quad z_2 = (-0.2600, -0.2600)^\top.$$

- **Maximum-type:**

$$u_1 = (-0.8902, 0.8902)^\top, \quad \nu = 0.25, \quad z_2 = (-0.4826, -0.4826)^\top.$$

- **Modified-type:**

$$u_1 = (0.2985, 0.2985)^\top, \quad \nu = 0.5, \quad z_2 = (-0.4506, -0.4506)^\top.$$

Step 6 (Final iterate). After repeating the above steps, the iterates converge to the following solutions:

- **Average-type:** $z^* = (-0.4261, -0.4261)^\top$.
- **Maximum-type:** $z^* = (-0.4267, -0.4267)^\top$.
- **Modified-type:** $z^* = (-0.4283, -0.4283)^\top$.

The numerical behavior of Algorithms 1, 2, and 3 is illustrated in Figures 3(a), 3(b), and 3(c), respectively.

7. Conclusion

In this paper, we analyzed a solution concept for set optimization problems with respect to the lower set less relation. The set-valued objective function was given by finitely many twice continuously differentiable and strongly convex functions. We proposed a non-monotone quasi-Newton method with max-type (Algorithm 1), average-type (Algorithm 2), and modified-hybrid-type (Algorithm 3) line searches to find the weakly minimal solution to the considered problem. At each iteration k, for a carefully chosen initial point, we evaluated the partition set P_{z_k} for the current iterate z_k. After that, for a selected element a^k from the partition set P_{z_k}, we figured out the quasi-Newton direction u_k of the vector optimization problem (VOP) of the current iterate z_k, with the help of the concepts in [42, 17]. This decent direction u_k has been used to find the next iterate z_{k+1}. Each of the Algorithms 1, 2, and 3 kept generating iterates until the stopping condition was met. The employed stopping condition is a necessary optimality condition (Proposition 3.1) of weakly minimal or stationary points of problem (SOP). Some results for the non-monotone quasi-Newton method for set-valued optimization problems are analogous to those for the monotone ones. Thus, we did not provide their proof in this paper. These results are

(i) an equivalent condition of non-stationarity of a point (Proposition 3.2),
(ii) boundedness of the sequence of generated quasi-Newton direction (Proposition 4.1).

The well-definedness of Algorithms 1, 2, and 3 for the derived nonmonotone quasi-Newton method was discussed in Subsections 4.1 4.2, 4.3. To the well-definedness, we ensured the existence of (a^k, u_k) in Step 4 and the existence of a step length t_k in Step 6 (Propositions 4.3 and 4.5). The convergence analysis of Algorithms 1, 2, and 3 was discussed in Section 5 with the help of Lemmas 4.6, 5.1, 5.2, and 5.3. Finally, we tested the performance of the proposed non-monotone quasi-Newton method on some existing and freshly introduced numerical test problems in Section 6. It was found that the proposed non-monotone quasi-Newton method outperforms the monotone one.

A potential direction for future work is to analyze the proposed work with other set relations discussed in [24]. Further, one can study the problem with other scalarizing functionals, such as separating functionals [13] with uniform level sets, Hiriart-Urruty functional [21], and Drummond-Svaiter functional [8] instead of Gerstewitz scalarizing function. Moreover, the theory can be further analyzed to different line searches such as adaptive line search, Wolfe line search, etc.

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APPENDIX A. MODIFIED SET-VALUED TEST PROBLEMS

MOP1: The set-valued map $\mathscr{F} : \mathbb{R} \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{50}(z)\}$, where, for each $i \in [50]$, $f^i : \mathbb{R} \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} z^2 + \frac{(i-1)}{2}z \\ (z-2)^2 + \frac{(i-1)}{2}z \end{pmatrix}.$$

MOP7: The set-valued map $\mathscr{F} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^3$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{25}(z)\}$, where, for each $i \in [25]$, $f^i : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$f^i(z_1, z_2) := \begin{pmatrix} \frac{(z_1-2)^2}{2} + \frac{(z_2+1)^2}{13} + \frac{(i-1)}{2} + 3 \\ \frac{(z_1-z_2-3)^2}{36} + \frac{(-z_1+z_2+2)^2}{8} - 17 \\ \frac{(z_1+2z_2-1)^2}{175} + \frac{(-z_1+2z_2)^2}{17} + \frac{(i-1)}{2} - 13 \end{pmatrix}.$$

BK1: The set-valued map $\mathscr{F}:\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{50}(z)\}$, where, for each $i \in [50]$, $f^i:\mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} z_1^2 + z_2^2 + \frac{(i-1)}{2} \\ (z_1-5)^2 + (z_2-5)^2 + \frac{(i-1)}{2} \end{pmatrix}.$$

DGO2: The set-valued map $\mathscr{F}:\mathbb{R} \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{20}(z)\}$, where, for each $i \in [20]$, $f^i:\mathbb{R} \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} z^2 + \frac{(i-1)}{2} z \\ 9 - \sqrt{81 - z^2} + \frac{(i-1)}{2}. \end{pmatrix}$$

DGO1: The set-valued map $\mathscr{F}:\mathbb{R} \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{15}(z)\}$, where, for each $i \in [15]$, $f^i:\mathbb{R} \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} z^2 + \sin\left(\frac{2\pi(i-1)z}{2}\right) \\ 2z^2 + \sin\left(\frac{2\pi(i-1)z}{2}\right) \end{pmatrix}.$$

VFM1: The set-valued map $\mathscr{F}:\mathbb{R}^2 \rightrightarrows \mathbb{R}^3$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{25}(z)\}$, where, for each $i \in [25]$, $f^i:\mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} z_1^2 + (z_2-1)^2 + \frac{(i-1)}{2} \\ z_1^2 + (z_2+1)^2 + \frac{(i-1)}{2} \\ (z_1-1)^2 + z_2^2 + 2 + + \frac{(i-1)}{2} \end{pmatrix}.$$

ZLT1: The set-valued map $\mathscr{F}:\mathbb{R}^3 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{30}(z)\}$, where, for each $i \in [30]$, $f^i:\mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} 10 + (z_1-1)^2 + z_2^2 + z_3^2 + \frac{(i-1)}{2} \\ 20 + z_1^2 + (z_2-1)^2 + (z_3-1)^2 + \frac{(i-1)}{2} \end{pmatrix}.$$

LRS1: The set-valued map $\mathscr{F}:\mathbb{R}^4 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{30}(z)\}$, where, for each $i \in [30]$, $f^i:\mathbb{R}^4 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} 10 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + \frac{(i-1)}{2} \\ 20 + (z_1+10)^2 + z_2^2 + z_3^2 + z_4^2 + \frac{(i-1)}{2} \end{pmatrix}.$$

FES1: The set-valued map $\mathscr{F}:\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{50}(z)\}$, where, for each $i \in [50]$, $f^i:\mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} z_1^2 + \frac{e^{2k}}{9} - \frac{2}{3} z_1 e^k + z_2^2 + \frac{e^{2k}}{9} - \frac{2}{3} z_2 e^k \\ z_1^2 + \cos^2(5\pi i) - z_1 \cos(5\pi i) + z_2^2 + \cos^2(5\pi i) - z_2 \cos(5\pi i) \end{pmatrix}.$$

JOS1: The set-valued map $\mathscr{F}:\mathbb{R}^4 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{30}(z)\}$, where, for each $i \in [30]$, $f^i:\mathbb{R}^4 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} z_1^2 + z_2^2 + z_3^2 + z_4^2 + \frac{(i-1)}{2} z_1^2 \\ (z_1-2)^2 + \frac{(z_2-2)^2}{2} + \frac{(z_3-1)^2}{3} + \frac{(z_4-2)^2}{4} + z_3^2 \frac{(i-1)}{2} \end{pmatrix}.$$

QV1: The set-valued map $\mathscr{F}:\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{25}(z)\}$, where, for each $i \in [25]$, $f^i : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} \frac{z_1^2 - 10\cos(2\pi i z_1)+10}{2} + \frac{z_2^2 - 10\cos(2\pi i z_2)+10}{2} \\ \frac{(z_1-1.5)^2 - 10\cos(2\pi i(z_1-1.5))+10}{2} + \frac{(z_2-1.5)^2 - 10\cos(2\pi i(z_2-1.5))+10}{2} \end{pmatrix}.$$

FES2: The set-valued map $\mathscr{F}:\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{25}(z)\}$, where, for each $i \in [25]$, $f^i : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$f^i(z) := \begin{pmatrix} (z_1 - 0.5 - 0.5\cos(20\pi i))^2 + (z_2 - 0.5 - 0.5\cos(20\pi i))^2 \\ z_1^2 + (z_2 - \sin^2(i))\cos^2(i))^2 \\ (z_1 + 0.125)^2 + (z_2 - 0.25\cos(i)\cos(2i) - 0.5)^2 \end{pmatrix}.$$

Problem 1: The set-valued map $\mathscr{F}:\mathbb{R}^2 \rightrightarrows \mathbb{R}^3$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{50}(z)\}$, where, for each $i \in [30]$, $f^i : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} 2e^{z_2}(z_1 + z_2)^2 + 0.7\sin\left(\frac{2\pi(i-1)}{50} z_1\right) \\ 2z_1^2 + 2z_2^2 + 0.7\cos\left(\frac{2\pi(i-1)}{50} z_2\right) \\ 5z_2^2 e^{z_1 z_2} + \left(\frac{2\pi(i-1)}{50} z_1 z_2\right) \end{pmatrix}.$$

Problem 2: The set-valued map $\mathscr{F}:\mathbb{R}^3 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{10}(z)\}$, where, for each $i \in [10]$, $f^i : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} 10 - \cos\left(\frac{2\pi(i-1)}{2} z_1\right) \\ 20 + z_3^2 + \sin\left(\frac{2\pi(i-1)}{2} z_2\right) \end{pmatrix}.$$

Problem 3: The set-valued map $\mathscr{F}:\mathbb{R} \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{20}(z)\}$, where, for each $i \in [20]$, $f^i : \mathbb{R} \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} 10 + z^2 - 10\cos\left(\frac{2\pi(i-1)}{2} z\right) \\ (z-4)^2 + \frac{(i-1)}{2} z \end{pmatrix}.$$

Problem 4: The set-valued map $\mathscr{F}:\mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{10}(z)\}$, where, for each $i \in [10]$, $f^i : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$f^i(z) := \begin{pmatrix} \frac{1}{9}(z_1 - 1)^4 + 2(z_2 - 2)^4 + 3(z_3 - 3)^4 + \frac{2\pi(i-1)}{2} \\ e^{\left(z_1 + \frac{z_2}{2} + \frac{z_3}{3}\right)} + \frac{(i-1)}{2} z \\ \frac{3}{12}(3e^{-z_1} + 4e^{-z_2} + 3e^{-z_3} + \frac{(i-1)}{2}; \end{pmatrix}.$$

Problem 5: The set-valued map $\mathscr{F}:\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{50}(z)\}$, where, for each $i \in [50]$, $f^i : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} e^{2k+1} z_1^2 + (z_2 + 1)^2 + (z_1 - 1)^2 \\ e^{2k+1} z_2^2 + (z_1 + 1)^2 + (z_2 - 1)^2 \end{pmatrix}.$$

Problem 6: The set-valued map $\mathscr{F} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{10}(z)\}$, where, for each $i \in [10]$, $f^i : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} 10 + z_1^2 + e^{(z_1+z_2)} + 0.5\sin\left(\frac{2\pi(i-1)}{100}\right) \\ 11 + z_2^2 + \cos(z_1+z_2) + 0.5\sin\left(\frac{2\pi(i-1)}{100}\right) \end{pmatrix}.$$

Problem 7: The set-valued map $\mathscr{F} : \mathbb{R} \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{30}(z)\}$, where, for each $i \in [30]$, $f^i : \mathbb{R} \to \mathbb{R}^3$ is given by

$$f^i(z) := \begin{pmatrix} \frac{2\pi}{50}\sin\left(\frac{2\pi(i-1)}{50}\right)\cos\left(\frac{2\pi(i-1)}{100}\right) + z^2 \\ \frac{2\pi}{50}\cos\left(\frac{2\pi(i-1)}{50}\right) + z^2 + \cos(2z) + e^{(1+z)} \\ \frac{2\pi}{50}\sin\left(\frac{2\pi(i-1)}{50}\right) + z^2 \end{pmatrix}.$$

Problem 8: The set-valued map $\mathscr{F} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined as $\mathscr{F}(z) := \{f^1(z), f^2(z), \ldots, f^{10}(z)\}$, where, for each $i \in [10]$, $f^i : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$f^i(z) := \begin{pmatrix} 10z_1^2 + e^{(z_1+z_2)} + 0.5\sin\left(\frac{2\pi(i-1)}{100}\right) \\ 11z_2^2 + \cos(z_1+z_2) + 0.5\sin\left(\frac{2\pi(i-1)}{100}\right) \end{pmatrix}.$$