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A MODIFIED OUTER APPROXIMATION ALGORITHM FOR CONVEX MINLP PROBLEMS

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Abstract. This paper proposes a modified outer approximation (OA) algorithm for convex mixed-integer nonlinear programming (MINLP) problems. Building upon the conventional OA framework, we introduce an enhancement in the construction of the mixed-integer linear programming (MILP) master problem. Specifically, when the nonlinear programming subproblem is feasible, a new class of linear cuts is generated by incorporating an adjustable parameter into the gradients of the constraint functions. The theoretical validity of these linear cuts is rigorously proved. These cuts are subsequently integrated into the MILP master problem, forming the core of our modified OA algorithm. The effectiveness of the proposed algorithm is demonstrated through two numerical examples, where its performance is compared with that of the conventional OA method. The results demonstrate that the modified OA algorithm generates master problems with either a smaller scale or tighter constraints while maintaining a comparable number of iterations. This leads to a reduction in overall solution time and a significant enhancement in computational efficiency.

Keywords. Convex MINLP; MILP; Master problem; Nonlinear subproblem; Outer approximation.
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1. INTRODUCTION

Mixed-Integer Nonlinear Programming (MINLP) is a powerful branch of mathematical optimization that integrates both continuous and discrete variables with nonlinear systems. It can characterize complex systems involving logical choices, fixed costs, discrete resource allocation, or nonlinear physical relationships, finding extensive applications across numerous fields such as chemical engineering ([17, 22, 33]), power generation and public utilities ([1, 30, 31]), communication and transportation networks ([8, 9]), facility planning ([5, 19]) and so on.

Based on the convexity of the continuous subproblems after relaxing the integer variables, MINLP problems are generally categorized into convex and non-convex classes. While solving MINLP problems is NP-hard ([20, 25]), the favorable mathematical structure of a convex

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MINLP where the continuous relaxation exhibits convexity enables the design of efficient global optimization algorithms. For convex MINLP problems, conventional solution methodologies include Branch and Bound ([6, 18]), Outer Approximation (OA) ([10, 13, 36]), Extended Cutting Plane (cf. [11, 37]), NLP/LP-based Branch and Bound ([24, 29]), and Generalized Benders Decomposition ([14, 15]). The reader is invited to consult bibliographies [3, 4, 12, 16, 27, 23] and references therein for more details on comprehensive review of MINLP methods.

The OA algorithm is widely regarded as one of the most effective for solving convex MINLP problems. Originally proposed by Duran and Grossmann [10] in 1986, the method was designed for MINLP problems where integer variables appear linearly and continuous variables are convex and differentiable, with its finite-step convergence proven under appropriate assumptions. Subsequently, the framework was generalized in 1994 by Fletcher and Leyffer [13] to cover cases where integer variables enter nonlinear terms. Addressing potential infeasibility of the NLP subproblem, they introduced feasibility subproblems, thereby enriching the theoretical foundation. In 2008, Bonami et al. [3] proposed a hybrid algorithmic framework based on outer approximation for convex and continuously differentiable MINLP problems. Through extensive computational experiments, they showed that such outer-approximation-based hybrid algorithms can efficiently solve large-scale convex MINLP problems. Then, in 2014, Hijazi et al. [21] focused on separable nonlinear structures that commonly arise in convex MINLPs and proposed three enhancements to the conventional outer approximation algorithm, namely a simple extended formulation, a refined outer approximation, and a heuristic inner approximation of the feasible region. These methods are implemented in the Bonmin solver and significantly improve the efficiency of outer approximation for solving convex MINLP problems with separable structure. In recent advancements for solving convex MINLPs, researchers have improved the OA algorithm from various perspectives. Su et al. [32] proposed a multi-generation cutting plane strategy that, after each master problem solution, solves multiple NLP subproblems corresponding to several integer solutions in parallel and simultaneously adds the resulting linear supporting cuts to the master problem. This approach of introducing multiple valid cuts at once provides a more comprehensive characterization of the nonlinear feasible region's boundary, thereby significantly reducing the number of iterations and enhancing solution efficiency while effectively approximating the original feasible set. In 2015, Wei and Ali [34] addressed nonsmooth convex MINLP problems by using subgradients instead of gradients to construct outer approximation linearizations based on KKT conditions, thereby extending the applicability of the OA algorithm from the differentiable case to the non-differentiable case. In 2019, Kronqvist et al. [23] introduced a center-cut algorithm that, instead of selecting boundary points of the outer approximation model as trial solutions, determines the next trial point by identifying the Chebyshev center of the current linear outer approximation region. This center-point selection method increases the likelihood of trial solutions satisfying the original nonlinear constraints, thus significantly accelerating the acquisition of feasible solutions and demonstrating the ability to converge to the optimum within finite iterations, offering a new way to obtain high-quality solutions quickly. In 2020, De Mauri et al. [7] combined outer approximation with the nonlinear feasibility pump for convex MINLP problems and proposed the Proximal Outer Approximation algorithm. By enhancing the generation of linear constraints and the selection of discrete assignments, the algorithm improves the solution efficiency. Muts et al. [26] proposed the Decomposition-based Outer Approximation algorithm in 2020, which adopts a

two-phase solution strategy by solving subproblems in parallel to rapidly construct linear relaxations and iteratively refine the OA algorithm, effectively enhancing the solution efficiency for convex MINLP problems. In 2022, Bernal et al. [2] proposed the regularized outer approximation algorithm, which effectively enhances the solution efficiency for convex MINLP problems by introducing multiple regularization functions based on distance metrics and Lagrangian approximations to improve the selection of integer combinations. In 2024, Peng et al. [28] significantly enhanced the efficiency and robustness of the conventional OA algorithm for convex MINLPs by introducing convexification and domain reduction techniques during the initialization phase, which provide tighter initial approximations and thereby accelerate algorithmic convergence and boost performance. Inspired by these advances, this paper aims to propose a modified OA algorithm for convex MINLP. By introducing a new parameter to modify the nonlinear constraints in feasible NLP subproblems, the algorithm optimizes the construction of cutting planes. It is illustrated by numerical examples that this refinement is able to generate a more tighten polyhedron to approximate the original nonlinear feasible region, leading to faster convergence and enhanced solution efficiency of our OA algorithm.

The organization of this paper is as follows: Section 2 briefly reviews the computational steps of the conventional OA algorithm and presents its algorithmic framework for solving convex MINLP. In Section 3, we propose a novel modified OA algorithm for convex MINLP. By introducing a parameter, we generate a new of linear cuts that are proved to be valid for the original nonlinear region. Based on this, we construct a modified OA algorithm and prove the termination after a finite number of iterations. Section 4 selects two numerical examples to test the modified OA algorithm, demonstrating the computational results and verifying the effectiveness of the algorithm through a comparative analysis with the conventional OA method. The conclusions of this paper are presented in Section 5, the last section.

2. Background

In this section, we review the conventional OA algorithm for solving convex MINLP. The convex MINLP problem studied in this paper is formulated as follows:

$$(P)\begin{cases}\min\limits_{x,y} f(x,y)\\ \text{s.t. } g_i(x,y)\le 0, \quad i=1,\cdots,m,\\ \qquad x\in X, y\in Y\cap\mathbb{Z}^p,\end{cases}$$

where

- $f:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$ and $g_i:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$ $(i=1,\cdots,m)$ are convex and continuously differentiable functions;
- X is a nonempty bounded polyhedral set of $\mathbb{R}^n$;
- Y is a polyhedral set of $\mathbb{R}^p$.

For any fixed $y\in Y\cap\mathbb{Z}^p$, we consider the following nonlinear subproblem $NLP(y)$:

$$NLP(y)\begin{cases}\min\limits_{x} f(x,y)\\ \text{s.t. } g_i(x,y)\le 0, \quad i=1,\cdots,m,\\ \qquad x\in X.\end{cases}$$

We divide the integer variable set $Y \cap \mathbb{Z}^p$ into two disjoint subsets:

$$T := \{y \in Y \cap \mathbb{Z}^p : NLP(y) \text{ is feasible}\} \text{ and } S := \{y \in Y \cap \mathbb{Z}^p : NLP(y) \text{ is infeasible}\}.$$

Throughout this paper, we assume that the following Slater constraint qualification holds:
For any $y \in T$, there exist $\tilde{x} \in X$ such that $g_i(\tilde{x}, y) < 0$ for all $i = 1, \cdots, m$.

It is well-recognized that to solve convex MINLP heavily depends on linear approximation algorithms. The central idea of such linear approximation algorithm is to approximate the MINLP problem (P) by an MILP problem, that is referred to as *master problem*. To obtain the master problem, we reformulate problem (P) as the following problem $(\bar{P})$ with the linear objective function by an auxiliary variable η:

$$(\bar{P}) \begin{cases} \min\limits_{x,y,\eta} \eta \\ \text{s.t. } f(x,y) \le \eta, \\ \quad g_i(x,y) \le 0, \quad i = 1, \cdots, m, \\ \quad x \in X,\ y \in Y \cap \mathbb{Z}^p, \eta \in \mathbb{R}. \end{cases}$$

Based on the convexity and differentiability of the nonlinear functions f and g_i, when performing a first-order Taylor series expansion at any point $(\hat{x}, \hat{y}) \in X \times (Y \cap \mathbb{Z}^p)$, we can obtain the following valid inequalities (2.1)–(2.2) for problem $(\bar{P})$,

$$f(\hat{x}, \hat{y}) + \nabla f(\hat{x}, \hat{y})^\top \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix} \le \eta, \tag{2.1}$$

$$g_i(\hat{x}, \hat{y}) + \nabla g_i(\hat{x}, \hat{y})^\top \begin{bmatrix} x - \hat{x} \\ y - \hat{y} \end{bmatrix} \le 0, \quad i = 1, \cdots, m, \tag{2.2}$$

where $\nabla f(\hat{x}, \hat{y})$ and $\nabla g_i(\hat{x}, \hat{y})$ represent the gradients of f and g_i at $(\hat{x}, \hat{y})$, respectively. It is noted that such $(\hat{x}, \hat{y})$ may not necessarily satisfy the constraints of the problem (P).

Following the above linearization process, let $L = \{(x^1, y^1), \ldots, (x^k, y^k)\}$ with k linearization points and we consider the following master problem:

$$(M^L) \begin{cases} \min\limits_{x,y,\eta} \quad \eta \\ \text{s.t.} \quad f(x^l, y^l) + \nabla f(x^l, y^l)^{\mathrm{T}} \begin{bmatrix} x - x^l \\ y - y^l \end{bmatrix} \le \eta, \quad \forall (x^l, y^l) \in L, \\ \qquad g_i(x^l, y^l) + \nabla g_i(x^l, y^l)^{\mathrm{T}} \begin{bmatrix} x - x^l \\ y - y^l \end{bmatrix} \le 0, \quad \forall (x^l, y^l) \in L,\ \forall i = 1, \cdots, m, \\ \qquad x \in X, y \in Y \cap \mathbb{Z}^p, \eta \in \mathbb{R}. \end{cases}$$

Clearly, (M^L) is an MILP problem. Solve this problem and obtain an optimal solution denoted by $(\hat{x}, \hat{y}, \hat{\eta})$. Then the value of $\hat{\eta}$ serves as a valid lower bound (LB) for the problem (P). If $(\hat{x}, \hat{y}, \hat{\eta})$ is feasible for problem $(\bar{P})$, then $\hat{\eta}$ is also an upper bound for $(\bar{P})$ and thus $(\hat{x}, \hat{y})$ will be an optimal solution for (P). Further, if $(\hat{x}, \hat{y}, \hat{\eta})$ is not feasible for problem $(\bar{P})$, it is necessary to add valid inequalities to (M^L) in order to cut off this solution. In general, it is to add a new point to the set L so that new linearizations could exclude $(\hat{x}, \hat{y}, \hat{\eta})$. This procedure is then iterative repeated until the difference between the curret best lower and upper bounds becomes sufficiently small. It is worth mentioning that the key factor that distinguishes a certain algorithm from the others is how to add a new linearization point to L.

It is well-known that outer approximation (OA) algorithm can be traced back to Duran and Grossmann [10] and it is consequently refined by Fletcher and Leyffer [13]. The main idea is still to adopt the master problem (M^L), and then add a new linearization point in L.

Suppose that $L = \{(x^1,y^1),\ldots,(x^k,y^k)\}$. Set $T^k := \{y^l : l \le k\} \cap T$ and $S^k := \{y^l : l \le k\} \cap S$. Consider the following master problem MP^k:

$$MP^k \begin{cases} \min\limits_{x,y,\eta} & \eta \\ \text{s.t.} & f(x^l,y^l) + \nabla f(x^l,y^l)^{\mathrm{T}} \begin{bmatrix} x-x^l \\ y-y^l \end{bmatrix} \le \eta, \quad \forall y^l \in T^k, \\ & g_i(x^l,y^l) + \nabla g_i(x^l,y^l)^{\mathrm{T}} \begin{bmatrix} x-x^l \\ y-y^l \end{bmatrix} \le 0, \quad \forall y^l \in T^k,\ \forall i = 1,\cdots,m, \\ & g_i(x^l,y^l) + \nabla g_i(x^l,y^l)^{\mathrm{T}} \begin{bmatrix} x-x^l \\ y-y^l \end{bmatrix} \le 0, \quad \forall y^l \in S^k,\ \forall i = 1,\cdots,m, \\ & x \in X, y \in Y \cap \mathbb{Z}^p, \eta \in \mathbb{R}. \end{cases}$$

Solve problem MP^k and denote by $(\hat{x},\hat{y},\hat{\eta})$ the optimal solution for MP^k. Let $y^{k+1} := \hat{y}$. Consider the following two cases:

(i) If $y^{k+1} \in T$, we solve subproblem $NLP(y^{k+1})$ and denote by x^{k+1} the optimal solution for problem $NLP(y^{k+1})$. In this case, objective value $f(x^{k+1},y^{k+1})$ provides an upper bound (UB) for the problem (P) and also for $(\bar{P})$.

(ii) If $y^{k+1} \in S$, then subproblem $NLP(y^{k+1})$ is infeasible and it becomes necessary to solve the following feasibility problem $NLP^F(y^{k+1})$:

$$NLP^F(y^{k+1}) \begin{cases} \min & \sum\limits_{i=1}^{m} u_i \\ \text{s.t.} & g_i(x,y^{k+1}) \le u_i, \quad i = 1,\cdots,m, \\ & u_i \ge 0, \quad i = 1,\cdots,m, \\ & x \in X, u \in \mathbb{R}^m. \end{cases} \tag{2.3}$$

Solve problem $NLP^F(y^{k+1})$ and denote by x^{k+1} the optimal solution.

Update L by adding a new linearization point (x^{k+1},y^{k+1}) to L. The algorithm would initiate a new iteration. Due to the convexity of the nonlinear constraints and the inherent structure of the feasibility subproblem, subproblem (2.3) is guaranteed to be feasible and numerically solvable; given the solution (x^{k+1},y^{k+1}) obtained from this subproblem, corresponding linear cutting planes can be constructed as

$$g_i(x^{k+1},y^{k+1}) + \nabla g_i(x^{k+1},y^{k+1})^{\mathrm{T}} \begin{bmatrix} x-x^{k+1} \\ y-y^{k+1} \end{bmatrix} \le 0, \quad i = 1,\cdots,m.$$

As shown in [34, Theorem 3.3], these cutting planes, when added to the master problem, are able to exclude the current infeasible integer variable y^{k+1} in subsequent iterations, thereby progressively tightening the approximation of the original feasible region.

Further, as shown in [34, 35], this procedure could guarantee that the given integer variable would not be visited more than once by the algorithm and thus the algorithm would terminate

after a finite number of steps under the assumption that the number of integer variables is finite, and the optimal solution of problem (P) could also be attained in a finite number of iterations.

The OA algorithm for convex MINLP is presented as follows (see [13]).

Algorithm 1 (OA Algorithm for Convex MINLP)

Initialization: Given an initial point $(x^0, y^0) \in X \times (Y \cap \mathbb{Z}^p)$, $L \leftarrow (x^0, y^0)$, optimality gap $\varepsilon \geq 0$, $T^0 \leftarrow \emptyset$, $S^0 \leftarrow \emptyset$, $UB^1 \leftarrow +\infty$,
Solve master problem M^L to obtain $(\bar{x}, \bar{y}, \bar{\eta})$
$y^1 \leftarrow \bar{y}$, $LB^1 \leftarrow \bar{\eta}$, $k \leftarrow 1$
while $UB^k - LB^k > \varepsilon$ **do**
  if subproblem $NLP(y^k)$ is feasible **then**
    Solve $NLP(y^k)$ to obtain x^k and $f(x^k, y^k)$
    $T^{+1} \leftarrow T^{k-1} \cup \{y^k\}$, $S^k \leftarrow S^{k-1}$, $UB^{k+1} \leftarrow \min\{UB^k, f(x^k, y^k)\}$
    if $f(x^k, y^k) < UB^k$ **then**
      Update best solution: $(x^*, y^*) \leftarrow (x^k, y^k)$
    end if
  else
    Solve feasibility problem $NLP^F(y^k)$ to obtain x^k
    $T^k \leftarrow T^{k-1}$, $S^k \leftarrow S^{k-1} \cup \{y^k\}$, $UB^{k+1} \leftarrow UB^k$
  end if
  Solve master problem MP^k to obtain $(\hat{x}, \hat{y}, \hat{\eta})$
  $y^{k+1} \leftarrow \hat{y}$, $LB^{k+1} \leftarrow \hat{\eta}$
  $k \leftarrow k+1$
end while
return best solution (x^*, y^*)

3. The Modified OA Algorithm

Building upon the theoretical analysis of the conventional OA algorithm, this paper proposes a novel OA variant to further enhance computational efficiency and the tightening effect of the feasible region. Specifically, the proposed method introduces an optimized criterion for generating cutting planes at feasible points. This refinement aims to strengthen the approximation accuracy of the cuts relative to the original constraints, thereby accelerating algorithmic convergence. The detailed design and implementation are presented in the section.

In this section, within the framework of the conventional OA algorithm, the nonlinear constraints in the feasible subproblems are modified by introducing newly defined parameters, ensuring that the construction of linear cutting planes no longer relies solely on conventional first-order Taylor expansion.

Suppose that nonlinear subproblem $NLP(y^l)$ is feasible and let x^l be an optimal solution of $NLP(y^l)$. We denote by $I(x^l)$ and $J(x^l)$ the *active* and *inactive* index sets, respectively and they are defined by

$$I(x^l) := \{i \in \{1, \cdots, m\} : g_i(x^l, y^l) = 0\} \text{ and } J(x^l) := \{i \in \{1, \cdots, m\} : g_i(x^l, y^l) < 0\}.$$

We consider the case that $J(x^l) \neq \emptyset$. Let Π_l be given by

$$\Pi_l := \max_{i \in J(y^l)} \max_{(x,y) \in X \times (Y \cap \mathbb{Z}^p)} \left\{ \nabla g_i(x^l, y^l)^\top \begin{pmatrix} x - x^l \\ y - y^l \end{pmatrix} \right\}. \tag{3.1}$$

The following proposition is immdiate from the definition of Π_l.

Proposition 3.1. *Let Π_l be given as (3.1). Then $\Pi_l \geq 0$. Further, if $\Pi_l = 0$, the following linearization inequalities are redundant:*

$$g_i(x^l, y^l) + \nabla g_i(x^l, y^l)^\top \begin{pmatrix} x - x^l \\ y - y^k \end{pmatrix} \leq 0, \quad i \in J(x^l).$$

Remark 3.1. It is known from [13] and [2] that when using OA method to construct a linear approximation (or relaxation) of the feasible set of the nonlinear constraints, the index set contains the indices of the nonlinear constraints active at the solution (x^l, y^l). Theoretically, Proposition 3.1 is able to provide a mathematical explanation for the case of $\Pi_l = 0$.

Now, we are in a position to define the coefficient β_l as follows:

$$\beta_l := \begin{cases} \dfrac{-\max\limits_{i \in J(x^l)} g_i(x^l, y^l)}{\Pi_l}, & \text{if } J(x^l) \neq \emptyset \text{ and } \Pi_l > 0, \\ 1, & \text{otherwise.} \end{cases} \tag{3.2}$$

The theorem is presented below, which illustrates the validity of cut, and its corresponding proof is provided.

Theorem 3.1. *For $y^l \in T^k$, let x^l be an optimal solution of the feasible subproblem* NLP(y^l). *Then the following linearization inequalities*

$$g_i(x^l, y^l) + \beta_l \nabla g_i(x^l, y^l)^\top \begin{pmatrix} x - x^l \\ y - y^l \end{pmatrix} \leq 0, \quad i = 1, \cdots, m \tag{3.3}$$

are valid for MINLP problem (P).

Proof. Take any feasible point $(\bar{x}, \bar{y})$ of the original problem (P) and let $i \in \{1, \cdots, m\}$. If $i \in I(x^l)$, then $g_i(x^l, y^l) = 0$. Note that g_i is convex. Thus

$$g_i(x^l, y^l) + \nabla g_i(x^l, y^l)^\top \begin{pmatrix} \bar{x} - x^l \\ \bar{y} - y^l \end{pmatrix} \leq g_i(\bar{x}, \bar{y}).$$

This and the feasibility of $(\bar{x}, \bar{y})$ give $\nabla g_i(x^l, y^l)^\top \begin{pmatrix} \bar{x} - x^l \\ \bar{y} - y^l \end{pmatrix} \leq 0$. It follows that

$$g_i(x^l, y^l) + \beta_l \nabla g_i(x^l, y^l)^\top \begin{pmatrix} \bar{x} - x^l \\ \bar{y} - y^l \end{pmatrix} \leq 0.$$

If $i \in J(x^l)$, then $g_i(x^l, y^l) < 0$. Let $M := -\max_{i \in J(y^l)} g_i(x^l, y^l) > 0$. Then

$$g_i(x^l, y^l) + \beta_l \nabla g_i(x^l, y^l)^\top \begin{pmatrix} \bar{x} - x^l \\ \bar{y} - y^l \end{pmatrix} \leq g_i(x^l, y^l) + M \leq 0.$$

Hence, (3.3) holds for $(\bar{x}, \bar{y})$ and this means that (3.3) are valid for MINLP problem (P). □

When considering the coefficient β_l, the original OA relaxation problem (2) can be reformulated as follows:

$$\widetilde{MP}^k \begin{cases} \min\limits_{x,y,\eta} & \eta \\ \text{s.t.} & f(x^l,y^l)+\nabla f(x^l,y^l)^{\mathrm{T}}\begin{bmatrix} x-x^l \\ y-y^l \end{bmatrix} \le \eta, \quad \forall y^l \in T^k, \\ & g_i(x^l,y^l)+\beta_l \nabla g_i(x^l,y^l)^{\mathrm{T}}\begin{bmatrix} x-x^l \\ y-y^l \end{bmatrix} \le 0, \quad \forall y^l \in T^k,\ \forall i=1,\cdots,m, \\ & g_i(x^l,y^l)+\nabla g_i(x^l,y^l)^{\mathrm{T}}\begin{bmatrix} x-x^l \\ y-y^l \end{bmatrix} \le 0, \quad \forall y^l \in S^k,\ \forall i=1,\cdots,m, \\ & x \in X, y \in Y \cap \mathbb{Z}^p, \eta \in \mathbb{R}. \end{cases} \tag{3.4}$$

Based on the above improvement strategies, Algorithm 2 presents the complete computational steps of the modified OA algorithm in the form of pseudocode, elaborating in detail its implementation procedure for solving the original problem (P).

Algorithm 2 (Modified OA Algorithm for Convex MINLP)

Initialization: Given an initial point $(x^0,y^0) \in X \times (Y \cap \mathbb{Z}^p)$, $L \leftarrow (x^0,y^0)$, optimality gap $\varepsilon \ge 0$, $T^0 \leftarrow \emptyset$, $S^0 \leftarrow \emptyset$, $UB^1 \leftarrow +\infty$,
Solve master problem M^L to obtain $(\bar{x},\bar{y},\bar{\eta})$
$y^1 \leftarrow \bar{y}$, $LB^1 \leftarrow \bar{\eta}$, $k \leftarrow 1$
while $UB^k - LB^k > \varepsilon$ **do**
  if subproblem $NLP(y^k)$ is feasible **then**
    Solve $NLP(y^k)$ to obtain x^k and $f(x^k,y^k)$
    Compute coefficients β_k according to (3.2)
    $T^k \leftarrow T^{k-1} \cup \{y^k\}$, $S^k \leftarrow S^{k-1}$, $UB^{k+1} \leftarrow \min\{UB^k, f(x^k,y^k)\}$
    if $f(x^k,y^k) < UB^k$ **then**
      Update best solution: $(x^*,y^*) \leftarrow (x^k,y^k)$
    end if
  else
    Solve feasibility problem $NLP^F(y^k)$ to obtain x^k
    $T^k \leftarrow T^{k-1}$, $S^k \leftarrow S^{k-1} \cup \{y^k\}$, $UB^{k+1} \leftarrow UB^k$
  end if
  Solve master problem $\widetilde{MP}^k$ to obtain $(\hat{x},\hat{y},\hat{\eta})$ according to (3.4)
  $y^{k+1} \leftarrow \hat{y}$, $LB^{k+1} \leftarrow \hat{\eta}$
  $k \leftarrow k+1$
end while
return best solution (x^*,y^*)

We next study the convergence of Algorithm 2. The following theorem shows that under the assumption of finite cardinality of integer variables, Algorithm 2 could terminate after a finite number of iterations.

Theorem 3.2. *Suppose that the cardinality of $Y \cap \mathbb{Z}^p$ is finite. Then, for any $\varepsilon > 0$, Algorithm 2 terminates in a finite number steps at an optimal value of problem (P).*

Proof. As shown in [35], the given integer variable y^k would not be generated twice by the algorithm and thus Algorithm 2 would terminate after a finite number of steps since the cardinality of $Y \cap \mathbb{Z}^p$ is finite.

Now, we suppose that Algorithm 2 terminates at k_0-th step for some $k_0 \in \mathbb{N}$. Then the relaxed master problem $\widetilde{MP}^{k_0}$ is infeasible. Let $(\bar{x},\bar{y})$ be an optimal solution to problem (P) with the optimal value $f(\bar{x},\bar{y})$. We consider the case that $f(x_j,y_j) > f(\bar{x},\bar{y})$ for all $y_j \in T^{k_0-1}$. Then $UB^{k_0-1} > f(\bar{x},\bar{y})$. We next prove that

$$y^{k_0} \in T^{k_0} \text{ and } f(x^{k_0},y^{k_0}) = f(\bar{x},\bar{y}). \tag{3.5}$$

Indeed, $y^{k_0} \in \mathrm{T}^{k_0}$ (if $y^{k_0} \in S^{k_0}$, then $UB^{k_0} = UB^{k_0-1}$ by Algorithm 2 and thus one can verify that $(\bar{x},\bar{y},f(\bar{x},\bar{y}))$ is feasible to $\widetilde{MP}^{k_0}$, a contradiction as $\widetilde{MP}^{k_0}$ is infeasible). Suppose on the contrary that $f(x^{k_0},y^{k_0}) > f(\bar{x},\bar{y})$. If $f(x^{k_0},y^{k_0}) \le UB^{k_0-1}$, then $f(\bar{x},\bar{y}) < f(x^{k_0},y^{k_0}) = UB^{k_0}$. Thus $(\bar{x},\bar{y},f(\bar{x},\bar{y}))$ is feasible for $\widetilde{MP}^{k_0}$, a contradiction as $\widetilde{MP}^{k_0}$ is infeasible. If $f(x^{k_0},y^{k_0}) > UB^{k_0-1}$, then $f(\bar{x},\bar{y}) < UB^{k_0-1} = UB^{k_0}$. Thus $(\bar{x},\bar{y},f(\bar{x},\bar{y}))$ is also feasible to $\widetilde{MP}^{k_0}$, a contradiction as $\widetilde{MP}^{k_0}$ is infeasible. This means $f(x^{k_0},y^{k_0}) = f(\bar{x},\bar{y})$ and (3.5) holds immediately. □

4. Numerical Examples

In this section, we show a detailed step-by-step implementation of the modified OA algorithm through two numerical examples. By comparing the performance of Algorithm 2 with Algorithm 1 in terms of the number of iterations and total computation time, the improvement achieved by the modified version is demonstrated. The specific implementation steps are outlined as follows.

Example 4.1. The following MINLP problem is considered:

$$\begin{cases} \min\limits_{x,y} f(x,y) = \dfrac{x^2}{10} - \dfrac{y}{4.5} + 2 \\ \text{s.t. } g_1(x,y) = \dfrac{x^2}{40} + y \le 20, \\ \qquad g_2(x,y) = \dfrac{(x-1)^2}{80} - y \le -4, \\ \qquad g_3(x,y) = 0.275y^{1.5} - 10(x+0.1)^{0.5} \le 0, \\ \qquad x \in [0,20],\ y \in [0,20] \cap \mathbb{Z}. \end{cases}$$

In Algorithm 2, a parameter β_l is computed at each iteration to adjust the tightness of the linearized constraints, thereby accelerating convergence. The algorithm is run with a tolerance $\varepsilon = 0.001$ and starts from the initial integer point $y^1 = 4$. In the first iteration, $\Pi_1 = 17.9650$ is computed, yielding $\beta_1 = 0.4611$. Solving $\widetilde{MP}^1$ gives $(\hat{x},\hat{y},\hat{\theta}) = (0,20,-2.5443)$. In the second iteration with $y^2 = 20$, the subproblem is infeasible; therefore, the feasibility subproblem $NLP^F(y^2)$ is solved, obtaining $x^2 = 5.9500$. After adding feasibility cuts and solving $\widetilde{MP}^2$, the result is $(\hat{x},\hat{y},\hat{\theta}) = (5.0425,19,-1.3144)$. For the third iteration with $y^3 = 19$, $\Pi_3 = 20.5238$ and $\beta_3 = 0.0172$ are computed. Solving $\widetilde{MP}^3$ yields $(\hat{x},\hat{y},\hat{\theta}) = (2.6301,16,-1.1298)$. In the fourth iteration with $y^4 = 16$, $\Pi_4 = 16.8491$ and $\beta_4 = 0.2241$ are computed. Solving $\widetilde{MP}^4$ gives

$(\hat{x},\hat{y},\hat{\theta}) = (1.8360, 14, -0.8441)$. The fifth iteration uses $y^5 = 14$, computing $\Pi_5 = 14.4394$ and $\beta_5 = 0.4088$. Solving $\widetilde{MP}^5$ results in $(\hat{x},\hat{y},\hat{\theta}) = (2.4198, 15, -0.7676)$. Finally, in the sixth iteration with $y^6 = 15$, $\Pi_6 = 15.6371$ and $\beta_6 = 0.3101$ are computed. Solving $\widetilde{MP}^6$ produces $(\hat{x},\hat{y},\hat{\theta}) = (2.4523, 15, -0.7319)$. At this point, the algorithm terminates as the gap tolerance $\varepsilon = 0.001$ is satisfied. The final solution from this modified OA algorithm is $(x^*, y^*) = (2.4523, 15)$ with the optimal objective value $f(x^*, y^*) = -0.7319$. In terms of computational time, the modified OA algorithm takes 1.36 seconds for solution, while the conventional OA algorithm takes 5.08 seconds.

TABLE 1. Performance Comparison of Outer Approximation Algorithms

Metric	**OA**	**modified OA**
Total iterations	5	6
Feasible subproblems	5	5
Infeasible subproblems	0	1
Optimal solution (x^*, y^*)	(2.4523, 15)	(2.4523, 15)
Final objective value	-0.7319	-0.7319
Computational Time (s)	5.42	1.36

Example 4.2. The following MINLP problem is considered:

$$\begin{cases} \min\limits_{x,y} & f(x,y) = -3x - y \\ \text{s.t.} & g_1(x,y) = x^2 + y^2 \le 25, \\ & g_2(x,y) = x^2 + (5-y)^2 \le 36, \\ & g_3(x,y) = (6-x)^2 + y^2 \le 36, \\ & x \in [0,10], y \in [0,10] \cap \mathbb{Z} \end{cases}$$

Given a tolerance $\varepsilon = 0.001$ and Algorithm 2 starts from the initial integer point $y^1 = 3$. In the first iteration, $\Pi_1 = 60.0000$ is computed, yielding $\beta_1 = 0.2667$. Solving $\widetilde{MP}^1$ gives $(\hat{x},\hat{y},\hat{\theta}) = (6.2500, 0, -18.7500)$. In the second iteration with $y^2 = 0$, the subproblem is feasible; therefore, $\Pi_2 = 44.3325$ and $\beta_2 = 0.3158$ are computed. Solving $\widetilde{MP}^2$ gives $(\hat{x},\hat{y},\hat{\theta}) = (4.7500, 2, -16.2500)$. For the third iteration with $y^3 = 2$, $\Pi_3 = 61.6515$ and $\beta_3 = 0.0973$ are computed. Solving $\widetilde{MP}^3$ yields $(\hat{x},\hat{y},\hat{\theta}) = (4.5826, 2, -15.7477)$. At this point, the algorithm terminates as the gap tolerance $\varepsilon = 0.001$ is satisfied. The final solution from this modified OA algorithm is $(x^*, y^*) = (4.5826, 2)$ with the optimal objective value $f(x^*, y^*) = -15.7477$.The computational time of the modified OA algorithm is 2.03 seconds, whereas that of the original OA algorithm is 4.64 seconds.

The results of the two numerical experiments demonstrate that the modified OA algorithm achieves faster solution times and higher computational efficiency. The integration of the newly designed parameter is key to this acceleration, significantly improving the algorithm's efficiency in locating the optimal value.

TABLE 2. Performance Comparison of Outer Approximation Algorithms

Metric	**OA**	**modified OA**
Total iterations	2	3
Feasible subproblems	2	3
Infeasible subproblems	0	0
Optimal solution (x^*, y^*)	(4.5826, 2)	(4.5826, 2)
Final objective value	-15.7477	-15.7477
Computational Time (s)	4.64	2.03

5. CONCLUTIONS

This paper proposes a modified OA algorithm for solving convex MINLP problems. Building upon the conventional OA framework, the key innovation lies in the construction of linear cuts. Specifically, when the nonlinear subproblem is feasible, a parameter is introduced to formulate the linear cuts for the constraint functions. This modification preserves the fundamental structure of the OA method while offering an alternative approach for constructing the *master problem*. To validate the proposed algorithm, we provide a theoretical proof confirming the validity of the constructed linear cuts and design two numerical examples for empirical testing. The experimental results show that, while maintaining a comparable number of iterations, the modified OA algorithm achieves a significant advantage in computational time over the conventionalOA method. These findings confirm both the feasibility and the enhanced computational efficiency of the proposed algorithm. Future research will focus on controlling the scale of the master problem to mitigate the computational burden associated with an excessive number of linear cuts. This work is expected to further improve the overall solution efficiency of the algorithm.

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