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ON ε-SOLUTIONS FOR SECOND-ORDER CONE LINEAR FRACTIONAL OPTIMIZATION PROBLEMS

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Abstract. This paper concentrates on studying of approximate optimality conditions for approximate solutions to a second-order cone linear fractional optimization problem. In the absence of constraint qualifications, it provides approximate optimality conditions expressed by sequences. Our results yield approximate optimality conditions for the second-order cone linear fractional optimization problem under a suitable constraint qualification. Moreover, related dual models in both sequential approximate form and approximate form are formulated, and weak and strong duality assertions are proposed. Under the fulfillment of constraint qualification, converse duality assertion is also established. Finally, the relation between approximate solutions to a second-order cone linear fractional optimization problem and approximate solutions to a second-order cone linear optimization problem is investigated. Some examples are provided to support our main results.

Keywords. Constraint qualifications; Duality theorem; Linear fractional optimization; Optimality conditions; Second-order cone.
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1. INTRODUCTION

Linear fractional programming (LF) consists of minimizing a ratio of two linear functions subject to some constraints. In the past few decades, the interest in (LF) has grown considerably over a wide range of fields due to it occurs frequently in many real world problems, such as maritime transportation problem [3, p.63], product planning [3, p.64], blending problem [3, p.68], and location problem [3, p.70]. Due to the fact that (LF)s are neither linear nor convex, criteria which identify global minimizers of (LF) were developed. In order to obtain the equivalent nonfractional linear optimization problem from a linear fractional optimization problem, variable transformation considered in [6, 7] was typically employed. In particular, some recent contributions to (LF)s with semidefinite cone constraints related to optimality conditions and duality (nonfractional form) were investigated in [13, 15]. Nonetheless, in general, since the

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exact solutions are not always available, it might not always be able to identify the point of minimizers in an optimization issue. Additionally, it is computationally costly because numerical techniques typically produce only approximations of solutions. This fact gives rise to the idea of an approximate solution which becomes a significant and interesting area. Besides, though approximate optimality criteria are always considered, most heuristic or iterative algorithms do not experiment with constraint qualifications in any way. This fact leads one to modify approximate optimality conditions without constraint qualifications; see, e.g., [10, 11, 12, 17, 18, 20] and the references therein. Recently, the characterizations of approximate solutions to semidefinite linear fractional optimization problems were also obtained [16].

It is remarkable that linear optimization problem with semidefinite cones and linear optimization problem with second order cones [1, 19] have now come to be recognised as a valuable numerical modeling tool for many optimization problems because they can efficiently be solved. In light of the importance of the second order cones, examining characterizations of approximate solutions for second order cone linear fractional optimization problems is not only reasonable but also advantageous. Recently, Kim et al. [14] obtained characterizations of optimal solutions to second order cone linear fractional optimization problems.

In this paper, motivated by these points, we aim to study the approximate optimality conditions for approximate solutions of a linear fractional optimization problem admiring the following form:

$$\begin{aligned} &\text{Minimize} \quad \frac{c^T x+\alpha}{d^T x+\beta} \\ &\text{subject to} \quad x \in K,\ a_i^T x = b_i,\ i=1,\ldots,m, \end{aligned} \tag{SOCLF}$$

where $c,\ d,\ a_i \in \mathbb{R}^n$, are the column vectors, $\alpha,\ \beta,\ b_i$ are real numbers for $i=1,\ldots,m$ and

$$K := \left\{ x := (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \mid x_1 \geqq \sqrt{x_2^2 + x_3^2 + \cdots + x_n^2} \right\}.$$

Note that K is a second order cone in $\mathbb{R}^n$ and is self-dual cone, i.e.,

$$K = K^* := \left\{ v \in \mathbb{R}^n \mid v^T x \geqq 0,\ \forall x \in K \right\}.$$

We denote F as the feasible set of problem (SOCLF), given by

$$F := \left\{ x \in K \mid a_i^T x = b_i, i=1,\ldots,m \right\}.$$

Throughout this paper, we always suppose that $d^T x + \beta > 0$ for any $x \in F$. Our results are divided into two parts. In the first one, we deal with approximate solutions for (SOCLF). We establish necessary and sufficient optimality conditions for approximate solutions of (SOCLF) by using a closedness type constraint qualification introduced in the literature. In the second part, after the dual models (stated in both sequential approximate form and approximate form) being formulated, we propose the approximate weak, strong and converse duality theorems. Relation between approximate solutions to (SOCLF) and approximate solutions to second-order cone linear optimization problem is investigated.

The paper is organized as follows. In Section 2, we recall some notions and give some preliminary results. In Section 3, we obtain necessary and sufficient optimality conditions for approximate solutions of (SOCLF). In Section 4, we investigate approximate duality results

between the primal problem and its dual one. Finally, concluding remarks are given in Section 5.

2. PRELIMINARIES

We begin this section by defining notation and preliminary results which are used later in this paper. We denote by $\mathbb{R}^n$ the Euclidean space with dimension n whose Euclidean norm is denoted by $\|\cdot\|_2$, i.e., $\|x\|_2 = \sqrt{x^T x}$ for any column vector $x := (x_1, \ldots, x_n)^T$. For a nonempty set A in $\mathbb{R}^n$, by $\mathrm{int}(A)$ (resp. $\mathrm{cl}(A)$, $\mathrm{co}(A)$), we denote the interior (resp. closure, convex hull) of the set E. The indicator function $\delta_A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

For an extended real-valued function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the effective domain and the epigraph are respectively defined by $\mathrm{dom}\, h := \{x \in \mathbb{R}^n \mid h(x) < +\infty\}$ and $\mathrm{epi}\, h := \{(x,r) \in \mathbb{R}^n \times \mathbb{R} \mid h(x) \leqq r\}$. We say that h is proper if $h(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\mathrm{dom}(h) \neq \emptyset$. A function h is said to be convex if, for all $\mu \in [0,1]$,

$$h(\mu x + (1-\mu)y) \leqq \mu h(x) + (1-\mu)h(y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

The conjugate function of h, $h^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$h^*(v) := \sup\{v^T x - h(x) \mid x \in \mathbb{R}^n\}.$$

Moreover, if $\liminf_{y \to x} h(y) \geqq h(x)$ for all $x \in \mathbb{R}^n$, we say that h is lower semicontinuous.

The following Lemmas are well-known.

Lemma 2.1. [4, 8] *Let $h_1, h_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous, and convex functions. Then $\mathrm{epi}(h_1 + h_2)^* = \mathrm{cl}(\mathrm{epi}\, h_1^* + \mathrm{epi}\, h_2^*)$. If, in addition, one of h_1 and h_2 is continuous at some $x_0 \in \mathrm{dom}\, h_1 \cap \mathrm{dom}\, h_2$, then $\mathrm{epi}(h_1 + h_2)^* = \mathrm{epi}\, h_1^* + \mathrm{epi}\, h_2^*$.*

Lemma 2.2. [10] *Let I be an arbitrary index set and let $h_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, be proper, lower semicontinuous, and convex functions. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $\sup_{i \in I} h_i(x_0) < \infty$. Then $\mathrm{epi}\big(\sup_{i \in I} h_i\big)^* = \mathrm{clco} \bigcup_{i \in I} \mathrm{epi}\, h_i^*$, where $\sup_{i \in I} h_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $(\sup_{i \in I} h_i)(x) := \sup_{i \in I} h_i(x)$ for all $x \in \mathbb{R}^n$.*

For each $\varepsilon \geqq 0$, a feasible point $\bar{x} \in F$ is called an ε-solution of (SOCLF) if, for any $x \in F$,

$$\frac{c^T x + \alpha}{d^T x + \beta} \geqq \frac{c^T \bar{x} + \alpha}{d^T \bar{x} + \beta} - \varepsilon.$$

Particularly, if $\varepsilon = 0$, $\bar{x} \in F$ is called an optimal solution of (SOCLF).

By virtue of the parametric approach introduced in [2], we associate the problem (SOCLF) into the nonfractional linear optimization problem (SOCLNF$_{\bar{r}}$) with a parametric $\bar{r} \in \mathbb{R}_+$:

$$\begin{aligned} &\text{Minimize } \ c^T x + \alpha - \bar{r}(d^T x + \beta) \\ &\text{subject to } \ x \in K, \ a_i^T x = b_i, \ i = 1, \ldots, m. \end{aligned} \qquad \text{(SOCLNF}_{\bar{r}}\text{)}$$

We close the section by the following proposition that describes the relationship between ε-solution to the problem (SOCLF) and an ε-solution to the problem (SOCLNF$_{\bar{r}}$).

Proposition 2.1. *Let $\varepsilon \geqq 0$ and $\bar{x} \in F$. Then the following statements are equivalent:*

(i) $\bar{x}$ *is an* ε*-solution of* (SOCLF)*;*
(ii) $\bar{x}$ *is an* $\bar{\varepsilon}$*-solution of* (SOCLNF$_{\bar{r}}$)*, where* $\bar{\varepsilon} := \varepsilon(d^T\bar{x}+\beta)$ *and*

$$\bar{r} := \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon.$$

Proof. [(i) $\Rightarrow$ (ii)]: Suppose that $\bar{x}$ is an ε-solution of (SOCLF). Then by the definition

$$\frac{c^Tx+\alpha}{d^Tx+\beta} \geqq \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon, \ \forall x \in F,$$

from which it follows that, for any $x \in F$, $c^Tx+\alpha-\bar{r}(d^Tx+\beta) \geqq 0$, where $\bar{r} := \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon$ and $\bar{\varepsilon}$:=$\varepsilon(d^T\bar{x}+\beta)$. Since

$$c^T\bar{x}+\alpha-\bar{r}(d^T\bar{x}+\beta)-\varepsilon(d^T\bar{x}+\beta) = 0,$$

for any $x \in F$, one has

$$\begin{aligned} c^Tx+\alpha-\bar{r}(d^Tx+\beta) &\geqq c^T\bar{x}+\alpha-\bar{r}(d^T\bar{x}+\beta)-\varepsilon(d^T\bar{x}+\beta) \\ &= c^T\bar{x}+\alpha-\bar{r}(d^T\bar{x}+\beta)-\bar{\varepsilon}, \end{aligned}$$

showing that $\bar{x}$ is an $\bar{\varepsilon}$-solution of (SOCLNF$_{\bar{r}}$).

[(ii) $\Rightarrow$ (i)]: Assume that $\bar{x}$ is an $\bar{\varepsilon}$-optimal solution of (SOCLNF$_{\bar{r}}$). Then,

$$c^Tx+\alpha-\bar{r}(d^Tx+\beta) \geqq c^T\bar{x}+\alpha-\bar{r}(d^T\bar{x}+\beta)-\bar{\varepsilon}, \ \forall x \in F,$$

or, equivalently, $c^Tx+\alpha-\bar{r}(d^Tx+\beta) \geqq 0$, $\forall x \in F$ due to $c^T\bar{x}+\alpha-\bar{r}(d^T\bar{x}+\beta)-\bar{\varepsilon} = 0$. So, we obtain that, for any $x \in F$,

$$\frac{c^Tx+\alpha}{d^Tx+\beta} \geqq \bar{r} = \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon.$$

Consequently, $\bar{x}$ is an ε- solution of (SOCLF). Thus the desired result is obtained. □

3. ε-Optimality Theorems

In this section, we establish ε-optimality theorems for ε-solutions for second-order cone linear fractional optimization problem (SOCLF).

We begin with the following lemma.

Lemma 3.1. *Let* $\bar{x} \in F$, $\varepsilon \geqq 0$, *and* $\bar{r} := \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon$. *Then, the following statements are equivalent:*

(i) $F \subseteq \{x \in K \mid c^Tx+\alpha-\bar{r}(d^Tx+\beta) \geq 0\}$*;*
(ii) $(0,0) \in (c-\bar{r}d, -\alpha+\bar{r}\beta) + (\{0\}\times\mathbb{R}_+) + \mathrm{cl}\left(\bigcup_{\lambda_i\in\mathbb{R}} \{\sum_{i=1}^m \lambda_i(a_i,b_i)\} + (-K)\times\mathbb{R}_+\right)$.

Proof. First, let us verify that

$$\mathrm{epi}\,\delta_F^* = \mathrm{cl}\left(\bigcup_{\lambda_i\in\mathbb{R}} \left\{\sum_{i=1}^m \lambda_i(a_i,b_i)\right\} + (-K)\times\mathbb{R}_+\right).$$

Now, let $E := \{x \in \mathbb{R}^n \mid a_i^T x = b_i,\ i = 1,\ldots,m\}$. Then, $F = E \cap K$ and for any $x \in \mathbb{R}^n$,

$$\delta_F(x) = \delta_E(x) + \delta_K(x) \text{ and } \delta_E(x) = \sup_{\lambda_i \in \mathbb{R}} \sum_{i=1}^{m} \lambda_i \big(a_i^T x - b_i\big).$$

Thus, by Lemma 2.1 and Lemma 2.2 along with the fact that $K^* = K$, we have

$$\begin{aligned}
\operatorname{epi} \delta_F^* = \operatorname{epi}(\delta_E + \delta_K)^* &= \operatorname{cl}(\operatorname{epi}\delta_E^* + \operatorname{epi}\delta_K^*) \\
&= \operatorname{cl}\left(\operatorname{epi}\left(\sup_{\lambda_i \in \mathbb{R}} \left\{\sum_{i=1}^{m} \lambda_i\big(a_i^T(\cdot) - b_i\big)\right\}\right)^* + \operatorname{epi}\delta_K^*\right) \\
&= \operatorname{cl}\left(\operatorname{cl}\operatorname{co} \bigcup_{\lambda_i \in \mathbb{R}} \operatorname{epi}\left(\sum_{i=1}^{m} \lambda_i\big(a_i^T(\cdot) - b_i\big)\right)^* + \operatorname{epi}\delta_K^*\right) \\
&= \operatorname{cl}\left(\operatorname{cl} \bigcup_{\lambda_i \in \mathbb{R}} \operatorname{epi}\left(\sum_{i=1}^{m} \lambda_i\big(a_i^T(\cdot) - b_i\big)\right)^* + \operatorname{epi}\delta_K^*\right) \\
&= \operatorname{cl}\left(\operatorname{cl}\left[\bigcup_{\lambda_i \in \mathbb{R}} \left\{\sum_{i=1}^{m} \lambda_i(a_i, b_i)\right\} + \{0\} \times \mathbb{R}_+\right] + (-K^*) \times \mathbb{R}_+\right) \\
&= \operatorname{cl}\left(\bigcup_{\lambda_i \in \mathbb{R}} \left\{\sum_{i=1}^{m} \lambda_i(a_i, b_i)\right\} + (-K) \times \mathbb{R}_+\right). \qquad (3.1)
\end{aligned}$$

[(i)⇒(ii)]: Suppose that (i) holds. Then $c^T x + \alpha - \bar{r}(d^T x + \beta) \geq 0$, for any $x \in F$, or, equivalently, $c^T x + \alpha - \bar{r}(d^T x + \beta) + \delta_F(x) \geq 0$ for any $x \in \mathbb{R}^n$, that is,

$$0^T x - \{c^T x + \alpha - \bar{r}(d^T x + \beta) + \delta_F(x)\} \leq 0$$

for any $x \in \mathbb{R}^n$. Let $f(x) := c^T x + \alpha - \bar{r}(d^T x + \beta)$. It then follows that $0^T x - \{f(x) + \delta_F(x)\} \leq 0$ for any $x \in \mathbb{R}^n$, and consequently, $(0,0) \in \operatorname{epi}(f + \delta_F)^*$. Since f is continuous on $F = \operatorname{dom}(\delta_F)$, by Lemma 2.1, we obtain that

$$(0,0) \in \operatorname{epi}(f + \delta_F)^* = \operatorname{epi} f^* + \operatorname{epi}\delta_F^*. \qquad (3.2)$$

Note that

$$\begin{aligned}
f^*(v) &= \sup\{v^T x - f(x) \mid x \in \mathbb{R}^n\} \\
&= \sup\{v^T x - c^T x - \alpha + \bar{r}(d^T x + \beta) \mid x \in \mathbb{R}^n\} \\
&= \sup\{(v - c + \bar{r}d)^T x - \alpha + \bar{r}\beta \mid x \in \mathbb{R}^n\} \\
&= \sup\{(v - c + \bar{r}d)^T x \mid x \in \mathbb{R}^n\} - \alpha + \bar{r}\beta \\
&= \begin{cases} -\alpha + \bar{r}\beta, & \text{if} \quad v = c - \bar{r}d, \\ +\infty, & \text{if} \quad v \neq c - \bar{r}d, \end{cases}
\end{aligned}$$

which implies

$$\operatorname{epi} f^* = \{(c - \bar{r}d, t) \mid -\alpha + \bar{r}\beta \leq t\} = \{(c - \bar{r}d, -\alpha + \bar{r}\beta)\} + \{0\} \times \mathbb{R}_+. \qquad (3.3)$$

Hence, (ii) follow from (3.1), (3.2) and (3.3).

[(ii) ⇒ (i)]: Suppose that (ii) holds. Since (ii) implies $(0,0) \in \operatorname{epi}(f + \delta_F)^*$, equivalently $(f + \delta_F)^*(0) \leqq 0$, where $f(x) := c^T x + \alpha - \bar{r}(d^T x + \beta)$, one has $c^T x + \alpha - \bar{r}(d^T x + \beta) = f(x) \geqq 0$ for any $x \in F$. So, (i) holds and the proof is completed. □

Now, we are in a position to derive the approximate optimality condition for an ε-solution to the problem (SOCLF) without any the fulfilment of constraint qualification.

Theorem 3.1. *(Sequential ε-Optimality Theorem) Consider problem* (SOCLF) *and let $\bar{x} \in F$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an ε-solution of* (SOCLF)*;*

(ii) *There exist $\{\lambda_i^l\}_{l\in\mathbb{N}} \subset \mathbb{R}$, $i = 1,\ldots,m$, $\{v_l\}_{l\in\mathbb{N}} \subset K$ and $\{\theta_l\}_{l\in\mathbb{N}} \subset \mathbb{R}_+$ such that*

$$c - \left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon\right) d + \lim_{l\to\infty}\left[\sum_{i=1}^{m} \lambda_i^l a_i - v_l\right] = 0 \text{ and } 0 \leqq \lim_{l\to\infty}\left[v_l^T\bar{x} + \theta_l\right] \leqq \varepsilon(d^T\bar{x}+\beta).$$

Proof. [(i) $\Rightarrow$ (ii)]: Assume that $\bar{x}$ is an ε-solution of (SOCLF). It follows from Lemma 2.1 that $\bar{x}$ is an $\bar{\varepsilon}$-solution of (SOCLNF$_{\bar{r}}$), where $\bar{r} := \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon$ and $\bar{\varepsilon} := \varepsilon(d^T\bar{x}+\beta)$, that is,

$$c^Tx + \alpha - \bar{r}(d^Tx+\beta) \geqq c^T\bar{x} + \alpha - \bar{r}(d^T\bar{x}+\beta) - \bar{\varepsilon}$$

for any $x \in F$. As $c^T\bar{x} + \alpha - \bar{r}(d^T\bar{x}+\beta) - \bar{\varepsilon} = 0$, we have

$$F \subseteq \{x \in K \mid c^Tx + \alpha - \bar{r}(d^Tx+\beta) \geq 0\}.$$

In view of Lemma 3.1, we find that there exist $\theta \in \mathbb{R}_+$, $\{\lambda_i^l\}_{l\in\mathbb{N}} \subset \mathbb{R}$, $i = 1,\ldots,m$, $\{v_l\}_{l\in\mathbb{N}} \subset K$ and $\{\theta_l\}_{l\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$c - \bar{r}d + \lim_{l\to\infty}\left[\sum_{i=1}^{m} \lambda_i^l a_i - v_l\right] = 0 \tag{3.4}$$

and

$$-\alpha + \bar{r}\beta + \theta + \lim_{l\to\infty}\left[\sum_{i=1}^{m} \lambda_i^l b_i + \theta_l\right] = 0. \tag{3.5}$$

From (3.4), we see that

$$(c - \bar{r}d)^T\bar{x} + \lim_{l\to\infty}\left[\sum_{i=1}^{m} \lambda_i^l a_i^T\bar{x} - v_l^T\bar{x}\right] = 0.$$

Since $(c - \bar{r}d)^T\bar{x} = -\alpha + \bar{r}\beta + \bar{\varepsilon}$, we have

$$-\alpha + \bar{r}\beta + \bar{\varepsilon} + \lim_{l\to\infty}\left[\sum_{i=1}^{m} \lambda_i^l a_i^T\bar{x} - v_l^T\bar{x}\right] = 0. \tag{3.6}$$

As $\theta \in \mathbb{R}_+$, it follows from (3.5) and (3.6) that $\lim_{l\to\infty}(\theta_l + v_l^T\bar{x}) = \bar{\varepsilon} - \theta \leqq \bar{\varepsilon}$. Since $v_l \in K = K^*$ for all $l \in \mathbb{N}$ and $\bar{x} \in K$, we have $v_l^T\bar{x} \geqq 0$ for all $l \in \mathbb{N}$ and hence $0 \leqq \lim_{l\to\infty}(\theta_l + v_l^T\bar{x})$, which completes the proof the implication (i) $\Rightarrow$ (ii).

[(ii) $\Rightarrow$ (i)]: Suppose that there exist $\{\lambda_i^l\}_{l\in\mathbb{N}} \subset \mathbb{R}$, $i = 1,\ldots,m$, $\{v_l\}_{l\in\mathbb{N}} \subset K$ and $\{\theta_l\}_{l\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$c - \left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon\right) d + \lim_{l\to\infty}\left[\sum_{i=1}^{m} \lambda_i^l a_i - v_l\right] = 0 \text{ and } 0 \leqq \lim_{l\to\infty}\left[v_l^T\bar{x} + \theta_l\right] \leqq \varepsilon(d^T\bar{x}+\beta).$$

Let $\bar{r} := \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon$ and $\bar{\varepsilon} := \varepsilon(d^T\bar{x}+\beta)$. Then, for any $x \in F$, one has

$$\begin{aligned} c^Tx+\alpha-\bar{r}(d^Tx+\beta)-c^T\bar{x}-\alpha+\bar{r}(d^T\bar{x}+\beta) &= c^Tx-\bar{r}d^Tx-c^T\bar{x}+\bar{r}d^T\bar{x} \\ &= (c-\bar{r}d)^T(x-\bar{x}) \\ &= -\lim_{l\to\infty}\left(\sum_{i=1}^m \lambda_i^l a_i - v_l\right)^T(x-\bar{x}) \\ &= -\lim_{l\to\infty}\left[\left(\sum_{i=1}^m \lambda_i^l(b_i-b_i)\right) - v_l^T(x-\bar{x})\right] \\ &= \lim_{l\to\infty} v_l^T(x-\bar{x}). \end{aligned}$$

Since $0 \leqq \lim_{l\to\infty}(\theta_l + v_l^T\bar{x}) \leqq \bar{\varepsilon}$, we have, for any $x \in F$ that

$$\begin{aligned} c^Tx+\alpha-\bar{r}(d^Tx+\beta)-c^T\bar{x}-\alpha+\bar{r}(d^T\bar{x}+\beta)+\bar{\varepsilon} &= \lim_{l\to\infty} v_l^T(x-\bar{x})+\bar{\varepsilon} \\ &\geqq \lim_{l\to\infty} v_l^T(x-\bar{x}) + \lim_{l\to\infty}(\theta_l+v_l^T\bar{x}) \\ &= \lim_{l\to\infty}(\theta_l+v_l^Tx) \\ &\geqq 0. \end{aligned}$$

This means that $\bar{x}$ is an $\bar{\varepsilon}$-solution of (SOCLNF$_{\bar{r}}$). According to Proposition 2.1, we conclude that $\bar{x}$ is an ε-solution of (SOCLF), thereby establishing the desired result. □

Regarding the proof of Theorem 3.1, it is worth mentioning here that the closedness of the set $\bigcup_{\lambda_i\in\mathbb{R}} \sum_{i=1}^m \lambda_i(a_i,b_i) + (-K)\times\mathbb{R}_+$ can be used as a constraint qualification for an ε- solution of (SOCLF) as in the following theorem shows.

Theorem 3.2. *(ε-Optimality Theorem) Consider the problem* (SOCLF) *and let $\bar{x} \in F$. Suppose that $\bigcup_{\lambda_i\in\mathbb{R}} \sum_{i=1}^m \lambda_i(a_i,b_i) + (-K)\times\mathbb{R}_+$ is closed in $\mathbb{R}^n\times\mathbb{R}$. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an ε-solution of* (SOCLF);

(ii) *There exist $\lambda_i \in \mathbb{R}$, $i = 1,\ldots,m$, and $v \in K$ such that*

$$c - \left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon\right)d + \sum_{i=1}^m \lambda_i a_i - v = 0 \text{ and } 0 \leqq v^T\bar{x} \leqq \varepsilon\left(d^T\bar{x}+\beta\right).$$

Proof. [(i) ⇒ (ii)]: Assume that $\bar{x}$ is an ε-solution of (SOCLF). Then, by Theorem 3.1, there exist $\{\lambda_i^l\}_{l\in\mathbb{N}} \subset \mathbb{R}$, $i = 1,\ldots,m$ and $\{v_l\}_{l\in\mathbb{N}} \subset K$ and $\{\theta_l\}_{l\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$c - \bar{r}d + \lim_{l\to\infty}\left[\sum_{i=1}^m \lambda_i^l a_i - v_l\right] = 0 \text{ and } 0 \leqq \lim_{l\to\infty}\left[v_l^T\bar{x}+\theta_l\right] \leqq \varepsilon(d^T\bar{x}+\beta),$$

where $\bar{r} := \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon$. Note that

$$\sum_{i=1}^m \lambda_i^l b_i + \theta_l = c^T\bar{x}+\alpha-\bar{r}(d^T\bar{x}+\beta)-\bar{\varepsilon}+\sum_{i=1}^m \lambda_i^l a_i^T\bar{x} - v_l^T\bar{x} + v_l^T\bar{x} + \theta_l$$

and $\lim_{l\to\infty}\left[\sum_{i=1}^{m}\lambda_i^l b_i+\theta_l\right]=\alpha-\bar{r}\beta-\bar{\varepsilon}+\omega$, where $\lim_{l\to\infty}[v_l^T\bar{x}+\theta_l]=\omega\in[0,\bar{\varepsilon}]$. As

$$\left\{\left(\sum_{i=1}^{m}\lambda_i^l a_i-v_l,\sum_{i=1}^{m}\lambda_i^l b_i+\theta_l\right)\right\}_{l\in\mathbb{N}}\subset\bigcup_{\lambda_i\in\mathbb{R}}\left\{\sum_{i=1}^{m}\lambda_i(a_i,b_i)\right\}+(-K)\times\mathbb{R}_+,$$

and $\bigcup\limits_{\lambda_i\in\mathbb{R}}\{\sum_{i=1}^{m}\lambda_i(a_i,b_i)\}+(-K)\times\mathbb{R}_+$ is closed, we see that

$$(-c+\bar{r}d,\alpha-\bar{r}\beta-\bar{\varepsilon}+\omega)\in\bigcup_{\lambda_i\in\mathbb{R}}\left\{\sum_{i=1}^{m}\lambda_i(a_i,b_i)\right\}+(-K)\times\mathbb{R}_+.$$

It follows that there exist $\lambda_i\in\mathbb{R}, i=1,\ldots,m$, $v\in K$ and $\theta\geqq 0$ such that

$$-c+\bar{r}d=\sum_{i=1}^{m}\lambda_i a_i-v,\quad \alpha-\bar{r}\beta-\bar{\varepsilon}+\omega=\sum_{i=1}^{m}\lambda_i b_i+\theta.$$

Therefore, $c-\bar{r}d+\sum_{i=1}^{m}\lambda_i a_i-v=0$ and

$$\begin{aligned}
0\leqq v^T\bar{x}&=(c-\bar{r}d)^T\bar{x}+\sum_{i=1}^{m}\lambda_i(a_i^T\bar{x})\\
&\leqq c^T\bar{x}-\bar{r}d^T\bar{x}+\sum_{i=1}^{m}\lambda_i(a_i^T\bar{x})+\alpha-\bar{r}\beta-\sum_{i=1}^{m}\lambda_i b_i\\
&=c^T\bar{x}+\alpha-\bar{r}(d^T\bar{x}+\beta)\\
&=c^T\bar{x}+\alpha-\left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta}-\varepsilon\right)(d^T\bar{x}+\beta)\\
&=\varepsilon(d^T\bar{x}+\beta).
\end{aligned}$$

The conclusion is obtained.

[(ii) $\Rightarrow$ (i)]: Suppose that there exist $\lambda_i\in\mathbb{R}, i=1,\ldots,m$ and $v\in K$ such that $c-\bar{r}d+\sum_{i=1}^{m}\lambda_i a_i-v=0$ and $0\leqq v^T\bar{x}\leqq\varepsilon(d^T\bar{x}+\beta)$, where $\bar{r}:=\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta}-\varepsilon$. By taking $\lambda_i^l:=\lambda_i$, $v_l:=v$ and $\theta_l:=0$ for all $l\in\mathbb{N}$, we have

$$c-\bar{r}d+\lim_{l\to\infty}\left[\sum_{i=1}^{m}\lambda_i^l a_i-v_l\right]=0 \text{ and } 0\leqq\lim_{l\to\infty}\left[v_l^T\bar{x}+\theta_l\right]\leqq\varepsilon(d^T\bar{x}+\beta),$$

Thanks to Theorem 3.1, we conclude that $\bar{x}$ is an ε-solution of (SOCLF), consequently, the proof is completed. □

Remark 3.1. It should be note here that if $a_1,\ldots,a_m$ are linearly independent in $\mathbb{R}^n$ and there exists $\widetilde{x}\in\operatorname{int}(K):=\{(x_1,x_2,\ldots,x_n)^T\mid x_1>\sqrt{x_2^2+x_3^2+\cdots+x_n^2}\}$ such that $a_i^T\widetilde{x}=b_i$, $i=1,\ldots,m$, then

$$\bigcup_{\lambda_i\in\mathbb{R}}\sum_{i=1}^{m}\lambda_i(a_i,b_i)+(-K)\times\mathbb{R}_+$$

is closed in $\mathbb{R}^n\times\mathbb{R}$. The reader is referred to [14, Proposition 2.2] and [9, Proposition 2.1] for more details.

Now, we give the following example to illustrate Theorem 3.2.

Example 3.1. Consider the following second order cone linear fractional optimization problem:

$$\begin{aligned} \text{Minimize} \quad & \frac{x_2 - x_3 + 2}{x_1 - x_2 - x_3 + 1} \\ \text{subject to} \quad & x := (x_1, x_2, x_3)^T \in K := \left\{ x \in \mathbb{R}^3 \mid x_1 \geqq \sqrt{x_2^2 + x_3^2} \right\} \\ & x_1 - x_2 + x_3 = 0. \end{aligned} \tag{SOCLF}$$

Let $c := (0,1,-1)^T$, $\alpha := 2$, $d := (1,-1,-1)^T$, $\beta := 1$, $a_1 := (1,-1,1)^T$, and $b_1 := 0$. Note that $\{a_1\}$ is actually linearly independent, $\widetilde{x} := (2,1,-1)^T \in \text{int}(K)$, and

$$a_1^T \widetilde{x} = (1,-1,1)(2,1,-1)^T = 2 - 1 - 1 = 0 = b_1,$$

we conclude by Remark 3.1 that

$$\bigcup_{\lambda \in \mathbb{R}} \lambda(1,-1,1,0) + (-K) \times \mathbb{R}_+$$

is closed in $\mathbb{R}^3 \times \mathbb{R}$. Note also that, for any $x \in F$,

$$x_1 \geqq \sqrt{(x_1 + x_3)^2 + x_3^2} \Leftrightarrow x_1^2 \geqq x_1^2 + 2x_1x_3 + 2x_3^2,\ x_1 \geqq 0 \Leftrightarrow 0 \geqq x_2x_3,\ x_1 \geqq 0,$$

and $x_2 = x_1 + x_3 \geqq x_3$. Therefore, $x_1,\ x_2 \geqq 0$ and $x_3 \leqq 0$. Observe that

$$\frac{x_2 - x_3 + 2}{x_1 - x_2 - x_3 + 1} = \frac{x_1 + 2}{-2x_3 + 1} \geqq \frac{1}{2},\ \forall x \in F$$

and $\inf(\text{SOCLF}) = \frac{1}{2}$. We can verify that this problem has no an optimal solution.

Now, let $\varepsilon := \frac{1}{2}$ be given. Clearly, $\bar{x} := (1,0,-1)^T$ is an ε-solution of (SOCLF). In what follows, we would verify the approximate optimality condition proposed in Theorem 3.2, that is, finding $\bar{\lambda}_1 \in \mathbb{R}$ and $\bar{v} \in K$ such that $c - \left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon\right) d + \bar{\lambda}_1 a_1 - \bar{v} = 0$ and $\bar{v}^T\bar{x} \leqq \varepsilon(d^T\bar{x} + \beta)$. So, by taking $\bar{\lambda}_1 := \frac{3}{2}$ and $\bar{v} := (1,0,1)^T \in K$, $(\bar{\lambda}_1, \bar{v})$ satisfy these conditions, i.e.,

$$\begin{aligned} & c - \left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon\right) d + \bar{\lambda}_1 a_1 - \bar{v} \\ &= (0,1,-1)^T - \left(1 - \frac{1}{2}\right)(1,-1,-1)^T + \frac{3}{2}(1,-1,1)^T - (1,0,1)^T \\ &= (0,0,0)^T \end{aligned}$$

and $\bar{v}^T\bar{x} = 0 \leqq \varepsilon(d^T\bar{x} + \beta)$.

It is worth noting that Theorem 3.2 is not valid if

$$\bigcup_{\lambda_i \in \mathbb{R}} \sum_{i=1}^{m} \lambda_i(a_i, b_i) + (-K) \times \mathbb{R}_+$$

is not closed in $\mathbb{R}^n \times \mathbb{R}$. On the other hand, Theorem 3.1 can apply for this case. The following example illustrates these facts modifying a known example (https://math.stackexchange.com/questions/712188/how-can-the-sum-of-two-closed-cones-be-not-closed).

Example 3.2. Consider the following second order cone linear fractional optimization problem:

$$\begin{aligned} \text{Minimize} \quad & \frac{-x_2+x_3+2}{x_1+x_2+x_3+1} \\ \text{subject to} \quad & x := (x_1,x_2,x_3)^T \in K := \left\{ x \in \mathbb{R}^3 \mid x_1 \geqq \sqrt{x_2^2+x_3^2} \right\} \\ & x_1 - x_3 = 0. \end{aligned} \tag{SOCLF}$$

Let $c := (0,-1,1)^T$, $\alpha := 2$, $d := (1,1,1)^T$, $\beta := 1$, $a_1 := (1,0,-1)^T$, and $b_1 := 0$. Note that, for any $x \in F$,

$$x_1 \geqq \sqrt{x_2^2+x_1^2} \Leftrightarrow x_1^2 \geqq x_2^2+x_1^2,\ x_1 \geqq 0 \Leftrightarrow x_2 = 0,\ x_1 \geqq 0.$$

So, $x_1 = x_3 \geqq 0$ and $x_2 = 0$. Note also that

$$\frac{-x_2+x_3+2}{x_1+x_2+x_3+1} = \frac{x_3+2}{2x_3+1} \geqq \frac{1}{2},\ \forall x \in F,$$

and $\inf(\text{SOCLF}) = \frac{1}{2}$. It is easily seen that the problem has no an optimal solution. Let $\bar{x} := (1,0,1)^T$ and let $\varepsilon := \frac{1}{2}$. We can see that $\bar{x}$ is ε-solution of (SOCLF). However, we can not find out $\lambda \in \mathbb{R}$ and $v \in K$ fulfilling

$$c - \left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} - \varepsilon \right) d + \lambda a_1 - v = 0$$

and $v^T\bar{x} \leqq \varepsilon(d^T\bar{x}+\beta)$. Otherwise, assume that there exist $\lambda \in \mathbb{R}$ and $v \in K$ such that the above constraints hold. We see that $(0,-1,1)^T - \frac{1}{2}(1,1,1)^T + \lambda(1,0,-1)^T \in K$ and hence $\left(\lambda - \frac{1}{2}, -\frac{3}{2}, \frac{1}{2} - \lambda\right)^T \in K$, i.e., $\lambda \geqq \frac{1}{2}$,

$$\lambda - \frac{1}{2} \geqq \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{1}{2}-\lambda\right)^2},$$

a contradiction. Thus, the conclusion of Theorem 3.2 does not hold. The reason is that $\bigcup_{\lambda\in\mathbb{R}} \lambda(a_1,b_1) + (-K) \times \mathbb{R}_+$ is not closed in $\mathbb{R}^3 \times \mathbb{R}$. In view of

$$\left((0,-1,0)^T,0\right) \notin \bigcup_{\lambda\in\mathbb{R}} \lambda(a_1,b_1) + (-K) \times \mathbb{R}_+,$$

one sees that

$$\begin{aligned} \left((0,-1,0)^T,0\right) &= \lim_{\lambda\to\infty} \left[\lambda \left((1,0,-1)^T,0\right) + \left(\left(-\sqrt{\lambda^2 + \left(1+\frac{1}{\lambda}\right)^2}, -1-\frac{1}{\lambda}, \lambda \right)^T, 0 \right) \right] \\ &\in \operatorname{cl}\left(\bigcup_{\lambda\in\mathbb{R}} \lambda(a_1,b_1) + (-K) \times \mathbb{R}_+ \right), \end{aligned}$$

which implies that the set $\bigcup_{\lambda\in\mathbb{R}} \lambda(a_1,b_1) + (-K) \times \mathbb{R}_+$ is not closed in $\mathbb{R}^3 \times \mathbb{R}$. On the other hand, taking $\lambda^l := l + \frac{1}{2}$, $v_l := \left(\sqrt{l^2 + \left(-\frac{3}{2}+\frac{1}{l}\right)^2}, -\frac{3}{2}+\frac{1}{l}, -l \right)^T$ and $\theta_l := 0$ for all $l \in \mathbb{N}$, one

has $\{\lambda^l\}_{l\in\mathbb{N}} \subset \mathbb{R}$, $\{v_l\}_{l\in\mathbb{N}} \subset K$, $\{\theta_l\}_{l\in\mathbb{N}} \subset \mathbb{R}_+$,

$$\begin{aligned}\lim_{l\to\infty}[\lambda^l a_1 - v_l] &= \lim_{l\to\infty}\left[\left(l+\frac{1}{2}\right)(1,0,-1)^T - \left(\sqrt{l^2+\left(-\frac{3}{2}+\frac{1}{l}\right)^2}, -\frac{3}{2}+\frac{1}{l}, -l\right)^T\right]\\ &= \lim_{l\to\infty}\left[\left(l+\frac{1}{2}-\sqrt{l^2+\left(-\frac{3}{2}+\frac{1}{l}\right)^2}, \frac{3}{2}-\frac{1}{l}, -\frac{1}{2}\right)^T\right]\\ &= \left(\frac{1}{2},\frac{3}{2},-\frac{1}{2}\right)^T\\ &= -c+\left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta}\right)d,\end{aligned}$$

and $\lim_{l\to\infty} v_l\bar{x} = 0 \leqq \varepsilon(d^T\bar{x}+\beta)$. By Theorem 3.1, we conclude that $\bar{x}$ is ε-solution of (SOCLF).

Remark 3.2. It should be note here that we can solve the following conditions via some iterative optimization algorithms

$$c-\left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta}-\varepsilon\right)d+\sum_{i=1}^{m}\lambda_i a_i - v = 0 \text{ and } 0 \leqq v^T\bar{x} \leqq \varepsilon\left(d^T\bar{x}+\beta\right),$$

when $\lambda_i \in \mathbb{R}$, $i=1,\ldots,m$, and $v \in K$. In this way, we get $\lambda_i \in \mathbb{R}$, $i=1,\ldots,m$, and $v \in K$ especially in terms of sequences such that $c-\left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta}-\varepsilon\right)d+\sum_{i=1}^{m}\lambda_i a_i - v \to 0$ and $c-\left(\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta}-\varepsilon\right)d+\sum_{i=1}^{m}\lambda_i a_i - v \neq 0$. This means that we cannot conclude that $\bar{x}$ is an ε-solution of (SOCLF) by Theorem 3.2. Thus, our sequential optimality conditions are suitable for practical situation whenever constraint qualification is violated.

4. ε-Duality Theorems

In this section, we formulate dual problems (stated in both sequential approximate form and approximate form) to the second-order cone linear fractional optimization problem and explore approximate duality relations between them.

Let $\varepsilon \geqq 0$ be given. In pace with optimality condition expressed in Theorem 3.1, we consider a dual problem (in approximate sequential form) connected with the primal problem (SOCLF) as follows:

$$\begin{aligned}&\text{Maximize } \gamma && (\text{SOCLFSD}_\varepsilon)\\ &\text{subject to } c-(\gamma-\varepsilon)d+\lim_{l\to\infty}\left(\sum_{i=1}^{m}\lambda_i^l a_i - v_l\right)=0,\\ &\qquad \alpha-\beta(\gamma-\varepsilon) \geqq \limsup_{l\to\infty}\sum_{i=1}^{m}\lambda_i^l b_i,\\ &\qquad \gamma\in\mathbb{R},\ \{\lambda^l\}_{l\in\mathbb{N}}\subset\mathbb{R}^m,\ \{v_l\}_{l\in\mathbb{N}}\subset K,\end{aligned}$$

where $\lambda^l := (\lambda_i^l)$ for $i=1,\ldots,m$. Let $F_{SD}^{\varepsilon} := \{(\gamma,\lambda^l,v_l) \in \mathbb{R}\times\mathbb{R}^m\times K \mid c-(\gamma-\varepsilon)d + \lim_{l\to\infty}\left(\sum_{i=1}^{m}\lambda_i^l a_i - v_l\right)=0,\ \alpha-\beta(\gamma-\varepsilon) \geqq \limsup_{l\to\infty}\sum_{i=1}^{m}\lambda_i^l b_i\}$ be a feasible solution set

of (SOCLFSD$_\varepsilon$) and sup(SOCLFSD$_\varepsilon$) be an optimal value of (SOCLFSD$_\varepsilon$). In what follows, a feasible solution $(\bar{\gamma}, \bar{\lambda}^l, \bar{v}_l) \in F_D^\varepsilon$ is called an ε-solution of (SOCLFSD$_\varepsilon$) if $\bar{\gamma} + \varepsilon \geqq \gamma$ for all $(\gamma, \lambda^l, v_l) \in F_{SD}^\varepsilon$.

The following theorem presents a weak duality relation between the primal problem (SOCLF) and the dual problem (SOCLFD$_\varepsilon$).

Theorem 4.1. *(ε-Weak Duality) Let $\varepsilon \geqq 0$ be given. For any $x \in F$ and any $(\gamma, \lambda^l, v_l) \in F_{SD}^\varepsilon$, it holds that*

$$\frac{c^T x + \alpha}{d^T x + \beta} \geqq \gamma - \varepsilon.$$

Proof. For each $\varepsilon \geqq 0$, let $x \in F$ and $(\gamma, \lambda^l, v_l) \in F_{SD}^\varepsilon$ be arbitrary. Suppose to the contrary that $\frac{c^T x+\alpha}{d^T x+\beta} < \gamma - \varepsilon$. As $d^T x + \beta > 0$, we see that $c^T x + \alpha - (\gamma - \varepsilon)(d^T x + \beta) < 0$, and so, $\alpha - (\gamma-\varepsilon)\beta + (c - (\gamma-\varepsilon)d)^T x < 0$. Hence $\alpha - (\gamma-\varepsilon)\beta - \lim_{l\to\infty}(\sum_{i=1}^m \lambda_i^l a_i^T x - v_l^T x) < 0$. Thus

$$\begin{aligned}
\alpha - (\gamma-\varepsilon)\beta &< \lim_{l\to\infty}\left(\sum_{i=1}^m \lambda_i^l a_i^T x - v_l^T x\right) \\
&\leqq \limsup_{l\to\infty} \sum_{i=1}^m \lambda_i^l a_i^T x + \limsup_{l\to\infty}(-v_l^T x) \\
&\leqq \limsup_{l\to\infty} \sum_{i=1}^m \lambda_i^l a_i^T x \\
&= \limsup_{l\to\infty} \sum_{i=1}^m \lambda_i^l b_i,
\end{aligned}$$

which contradicts to the fact that $(\gamma, \lambda^l, v_l) \in F_{SD}^\varepsilon$. Consequently, $\frac{c^T x+\alpha}{d^T x+\beta} \geqq \gamma - \varepsilon$. □

The forthcoming theorem declares strong duality relation between (SOCLF) and the dual problem (SOCLFSD$_\varepsilon$).

Theorem 4.2. *(ε-Strong Duality) Let $\varepsilon \geqq 0$ and $\bar{x} \in F$ be an ε-solution of (SOCLF). Then, there exists $\{\bar{\lambda}^l\}_{l\in\mathbb{N}} \subset \mathbb{R}^m$, $\{\bar{v}_l\}_{l\in\mathbb{N}} \subset K$ such that $(\bar{\gamma}, \bar{\lambda}^l, \bar{v}_l) \in F_{SD}^\varepsilon$, $\sup(\text{SOCLFSD}_\varepsilon) \geqq \bar{\gamma} \geqq \sup(\text{SOCLFSD}_\varepsilon) - \varepsilon$, and moreover, $(\bar{\gamma}, \bar{\lambda}^l, \bar{v}_l)$ is an ε-solution of (SOCLFSD$_\varepsilon$), where $\bar{\gamma} := \frac{c^T \bar{x}+\alpha}{d^T \bar{x}+\beta}$.*

Proof. Let $\bar{x}$ be an ε-solution of (SOCLF). By Theorem 3.1, there exist $\{\bar{\lambda}^l\}_{l\in\mathbb{N}} \subset \mathbb{R}^m$ and $\{\bar{v}_l\}_{l\in\mathbb{N}} \subset K$ and $\{\bar{\theta}_l\}_{l\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$c - \left(\frac{c^T \bar{x}+\alpha}{d^T \bar{x}+\beta} - \varepsilon\right) d + \lim_{l\to\infty}\left(\sum_{i=1}^m \bar{\lambda}_i^l a_i - \bar{v}_l\right) = 0 \text{ and } 0 \leqq \lim_{l\to\infty}(\bar{v}_l^T \bar{x} + \bar{\theta}_l) \leqq \bar{\varepsilon}.$$

Now, for each $l \in \mathbb{N}$,

$$c^T \bar{x} - (\bar{\gamma}-\varepsilon)d^T \bar{x} + \left(\sum_{i=1}^m \bar{\lambda}_i^l b_i + \bar{\theta}_l\right) = c^T \bar{x} - (\bar{\gamma}-\varepsilon)d^T \bar{x} + \left(\sum_{i=1}^m \bar{\lambda}_i^l a^T \bar{x} - \bar{v}_l^T \bar{x} + \bar{v}_l^T \bar{x} + \bar{\theta}_l\right).$$

By passing the limit $l \to \infty$, we have

$$c^T\bar{x} - (\bar{\gamma}-\varepsilon)d^T\bar{x} + \lim_{l\to\infty}\left(\sum_{i=1}^{m}\bar{\lambda}_i^l b_i + \bar{\theta}_l\right) = 0 + \lim_{l\to\infty}(\bar{v}_l^T\bar{x} + \bar{\theta}_l) \leqq \varepsilon(d^T\bar{x}+\beta). \tag{4.1}$$

Since $\bar{\gamma} = \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta}$, one has $c^T\bar{x} - \bar{\gamma}d^T\bar{x} + \alpha - \bar{\gamma}\beta = 0$, which together with (4.1) yields

$$-\alpha + \bar{\gamma}\beta + \varepsilon d^T\bar{x} + \limsup_{l\to\infty}\sum_{i=1}^{m}\bar{\lambda}_i^l b_i \leqq \varepsilon(d^T\bar{x}+\beta),$$

or equivalently, $\limsup_{l\to\infty}\sum_{i=1}^{m}\bar{\lambda}_i^l b_i \leqq \alpha - \beta(\bar{\gamma}-\varepsilon)$. Thus $(\bar{\gamma},\bar{\lambda}^l,\bar{v}_l) \in F_{SD}^{\varepsilon}$. Moreover, it follows from Theorem 4.1 that $\sup(\text{SOCLFSD}_\varepsilon) \geqq \bar{\gamma} = \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} \geqq \sup(\text{SOCLFSD}_\varepsilon) - \varepsilon$. In other words, $(\bar{\gamma},\bar{\lambda}^l,\bar{v}_l)$ is an ε-solution to ($\text{SOCLFSD}_\varepsilon$). □

Let us give an example to illustrate our duality theorems 4.2.

Example 4.1. Let c,α,d,β,a_1, and b_1 be defined as in Example 3.2. Let $\bar{x} := (1,0,1)^T$ and let $\varepsilon := \frac{1}{2}$. In view of Example 3.2, we see that $(\bar{\gamma},\bar{\lambda}^l,\bar{v}_l) \in F_{SD}^{\varepsilon}$, where $\bar{\gamma} = \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta} = 1$, $\bar{\lambda}^l := l + \frac{1}{2}$, $\bar{v}_l := \left(\sqrt{l^2 + \left(-\frac{3}{2}+\frac{1}{l}\right)^2}, -\frac{3}{2}+\frac{1}{l}, -l\right)^T$ for all $l \in \mathbb{N}$. So, $F_{SD}^{\varepsilon} \neq \emptyset$. Note by Theorem 4.1 that $\frac{1}{2} = \inf(\text{SOCLF}) \geqq \gamma - \varepsilon = \gamma - \frac{1}{2}$ for all $(\gamma,\lambda^l,v_l) \in F_{SD}^{\varepsilon}$, which implies that $\sup(\text{SOCLFSD}_\varepsilon) = 1$. Thus, the ε-strong duality of ($\text{SOCLFD}_\varepsilon$) with respect to $\bar{x} := (1,0,1)^T$ holds, i.e, $\sup(\text{SOCLFSD}_\varepsilon) \geqq \bar{\gamma} \geqq \sup(\text{SOCLFSD}_\varepsilon) - \varepsilon$, and moreover, $(\bar{\gamma},\bar{\lambda}^l,\bar{v}_l)$ is an ε-solution of ($\text{SOCLFSD}_\varepsilon$).

In the similar argument, concerning on Theorem 3.2, let us now consider a dual problem (in approximate form) connected with the primal problem (SOCLF) as follows:

$$\begin{aligned} &\text{Maximize } \gamma && (\text{SOCLFD}_\varepsilon)\\ &\text{subject to } c - (\gamma-\varepsilon)d + \sum_{i=1}^{m}\lambda_i a_i \in K,\\ &\qquad \alpha - \beta(\gamma-\varepsilon) \geqq b^T\lambda,\\ &\qquad \gamma \in \mathbb{R},\ \lambda := (\lambda_i) \in \mathbb{R}^m. \end{aligned}$$

Let $F_D^{\varepsilon} := \{(\gamma,\lambda) \in \mathbb{R}\times\mathbb{R}^m \mid c - (\gamma-\varepsilon)d + \sum_{i=1}^{m}\lambda_i a_i \in K,\ \alpha - \beta(\gamma-\varepsilon) \geqq b^T\lambda\}$ be a feasible solution set of ($\text{SOCLFD}_\varepsilon$) and $\sup(\text{SOCLFD}_\varepsilon)$ be an optimal value of ($\text{SOCLFD}_\varepsilon$). In what follows, a feasible solution $(\bar{\gamma},\bar{\lambda}) \in F_D^{\varepsilon}$ is called an ε-solution of ($\text{SOCLFD}_\varepsilon$) if $\bar{\gamma}+\varepsilon \geqq \gamma$ for all $(\gamma,\lambda) \in F_D^{\varepsilon}$.

The following theorem presents a weak duality relation between the primal problem (SOCLF) and the dual problem ($\text{SOCLFD}_\varepsilon$).

Theorem 4.3. *(ε-Weak Duality) Let $\varepsilon \geqq 0$ be given. For any $x \in F$ and any $(\gamma,\lambda) \in F_D^{\varepsilon}$, it holds that*

$$\frac{c^Tx+\alpha}{d^Tx+\beta} \geqq \gamma - \varepsilon.$$

Proof. For each $\varepsilon \geqq 0$, let $x \in F$ and $(\gamma, \lambda) \in F_D^\varepsilon$ be arbitrary. So, $(\gamma, \lambda^l, v_l) \in F_{SD}^\varepsilon$, where $\lambda^l := \lambda$ and $v_l := c - (\gamma - \varepsilon)d + \sum_{i=1}^m \lambda_i a_i$ for all $l \in \mathbb{N}$. Therefore, the conclusion follows Theorem 4.1 immediately. □

In the following result, we will give a strong duality relation between the primal problem (SOCLF) and the dual problem (SOCLFD$_\varepsilon$).

Theorem 4.4. (ε-Strong Duality) *Let $\varepsilon \geqq 0$ and $\bar{x} \in F$ be an ε-solution of (SOCLF). If $\bigcup_{\lambda_i \in \mathbb{R}} \sum_{i=1}^m \lambda_i (a_i, b_i) + (-K) \times \mathbb{R}_+$ is closed in $\mathbb{R}^n \times \mathbb{R}$, then there exists $\bar{\lambda} := (\bar{\lambda}_i) \in \mathbb{R}^m$ such that $(\bar{\gamma}, \bar{\lambda}) \in F_D^\varepsilon$, $\sup(\text{SOCLFD}_\varepsilon) \geqq \bar{\gamma} \geqq \sup(\text{SOCLFD}_\varepsilon) - \varepsilon$, and moreover, $(\bar{\gamma}, \bar{\lambda})$ is an ε-solution of (SOCLFD$_\varepsilon$), where $\bar{\gamma} := \frac{c^T \bar{x} + \alpha}{d^T \bar{x} + \beta}$.*

Proof. Invoking Theorem 3.2, there exist $\bar{\lambda} := (\bar{\lambda}_i) \in \mathbb{R}^m$ and $\bar{v} \in K$ such that

$$c - \left(\frac{c^T \bar{x} + \alpha}{d^T \bar{x} + \beta} - \varepsilon \right) d + \left(\sum_{i=1}^m \bar{\lambda}_i a_i - \bar{v} \right) = 0$$

$$\text{and} \quad 0 \leqq \bar{v}^T \bar{x} \leqq \bar{\varepsilon}.$$

These imply that $c - (\bar{\gamma} - \varepsilon)d + \sum_{i=1}^m \bar{\lambda}_i a_i \in K$ and

$$\begin{aligned}
\bar{\varepsilon} &\geqq \bar{v}^T \bar{x} \\
&= c^T \bar{x} - \bar{\gamma} d^T \bar{x} + \varepsilon d^T \bar{x} + \sum_{i=1}^m \bar{\lambda}_i (a_i^T \bar{x}) \\
&= c^T \bar{x} + \alpha - \bar{\gamma}(d^T \bar{x} + \beta) + \varepsilon (d^T \bar{x} + \beta) + \sum_{i=1}^m \bar{\lambda}_i b_i - \alpha + \bar{\gamma}\beta - \varepsilon\beta \\
&\geqq \bar{\varepsilon} + b^T \bar{\lambda} - \alpha + \bar{\gamma}\beta - \varepsilon\beta,
\end{aligned}$$

which is equivalent to $\alpha - \beta(\bar{\gamma} - \varepsilon) - b^T \bar{\lambda} \geqq 0$. Thus $(\bar{\gamma}, \bar{\lambda}) \in F_D^\varepsilon$. Moreover, it follows from Theorem 4.3 that $\sup(\text{SOCLFD}_\varepsilon) \geqq \bar{\gamma} = \frac{c^T \bar{x} + \alpha}{d^T \bar{x} + \beta} \geqq \sup(\text{SOCLFD}_\varepsilon) - \varepsilon$. In other words, $(\bar{\gamma}, \bar{\lambda})$ is an ε-solution to (SOCLFD$_\varepsilon$). □

The following example demonstrates that the closedness of $\bigcup_{\lambda_i \in \mathbb{R}} \sum_{i=1}^m \lambda_i (a_i, b_i) + (-K) \times \mathbb{R}_+$ cannot be dropped.

Example 4.2. Let c, α, d, β, a_1, and b_1 be defined as in Example 3.2. Let $\bar{x} := (1, 0, 1)^T$ and let $\varepsilon := \frac{1}{2}$. We see that $\bar{\gamma} = 1$ and (SOCLFD$_\varepsilon$) associated with (SOCLF) becomes

$$\begin{aligned}
&\text{Maximize} \quad \gamma \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{SOCLFD}_\varepsilon) \\
&\text{subject to} \quad \left(-\gamma + \frac{1}{2} + \lambda, -1 - \gamma + \frac{1}{2}, 1 - \gamma + \frac{1}{2} - \lambda \right)^T \in K, \\
&\qquad\qquad 2 - (1)(\gamma - \frac{1}{2}) \geqq (0)\lambda, \ \gamma \in \mathbb{R}, \ \lambda \in \mathbb{R}.
\end{aligned}$$

It then follows that

$$F_D^\varepsilon = \{(\gamma, \lambda) \in \mathbb{R}^2 \mid \left(\lambda - \gamma + \frac{1}{2} \right)^2 \geqq \left(\gamma + \frac{1}{2} \right)^2 + \left(\lambda + \gamma - \frac{3}{2} \right)^2, \ \lambda - \gamma + \frac{1}{2} \geqq 0, \ \gamma \leqq \frac{5}{2} \}.$$

Note that

$$\begin{aligned}
&\left(\lambda-\gamma+\frac{1}{2}\right)^2 \geqq \left(\gamma+\frac{1}{2}\right)^2+\left(\lambda+\gamma-\frac{3}{2}\right)^2\\
\Leftrightarrow\ &(\lambda-\gamma)^2+2(\lambda-\gamma)(\frac{1}{2})+\frac{1}{4} \geqq \gamma^2+2\gamma(\frac{1}{2})+\frac{1}{4}+(\lambda+\gamma)^2-2(\lambda+\gamma)(\frac{3}{2})+\frac{9}{4}\\
\Leftrightarrow\ &-4\lambda\gamma+\lambda-\gamma \geqq \gamma^2+\gamma-3\lambda-3\gamma+\frac{9}{4}\\
\Leftrightarrow\ &\lambda(1-\gamma) \geq \frac{1}{4}(1-\gamma)^2+\frac{5}{16}.
\end{aligned}$$

Thus $F_D^\varepsilon = \{(\gamma,\lambda)\in\mathbb{R}^2 \mid \lambda(1-\gamma) \geqq \frac{1}{4}(1-\gamma)^2+\frac{5}{16},\ \lambda \geqq \gamma-\frac{1}{2},\ \gamma \leqq \frac{5}{2}\}$. Now, consider the following three cases:

Case 1. If $\gamma=1$, then $\lambda\in\mathbb{R}$ does not exist.

Case 2. If $1<\gamma\leqq\frac{5}{2}$, then $\lambda \leqq \frac{1}{4}(1-\gamma)+\frac{5}{16}(\frac{1}{1-\gamma})$. Assume that there exists $\widetilde{\lambda}\in\mathbb{R}$ such that $\widetilde{\lambda} \leqq \frac{1}{4}(1-\gamma)+\frac{5}{16}(\frac{1}{1-\gamma})$ and $\widetilde{\lambda} \geqq \gamma-\frac{1}{2}$. Then,

$$\frac{1}{4}(1-\gamma)+\frac{5}{16}(\frac{1}{1-\gamma}) \geqq \gamma-\frac{1}{2} \Leftrightarrow 0 \geqq 5\gamma^2-8\gamma+\frac{17}{4}=5(\gamma-\frac{4}{5})^2+\frac{21}{4}>0,$$

which is a contradiction. Thus $\lambda\in\mathbb{R}$ does not exist.

Case 3. If $\gamma<1$, then $\lambda \geqq \frac{1}{4}(1-\gamma)+\frac{5}{16}(\frac{1}{1-\gamma})$. Letting $\widetilde{\lambda} := \max\{\gamma-\frac{1}{2}, \frac{1}{4}(1-\gamma)+\frac{5}{16}(\frac{1}{1-\gamma})\}$, we see that $\widetilde{\lambda} \geqq \frac{1}{4}(1-\gamma)+\frac{5}{16}(\frac{1}{1-\gamma})$ and $\widetilde{\lambda} \geqq \gamma-\frac{1}{2}$.

Note also that $\sup(\text{SOCLFD}_\varepsilon)=1$ and there exists no $\bar{\lambda}\in\mathbb{R}$ such that $(\bar{\gamma},\bar{\lambda})\in F_D^\varepsilon$. The reason is that $\bigcup\limits_{\lambda\in\mathbb{R}} \lambda(a_1,b_1)+(-K)\times\mathbb{R}_+$ is not closed in $\mathbb{R}^3\times\mathbb{R}$.

We now present the converse duality relation for an ε-solution between the primal problem (SOCLF) and the dual problem ($\text{SOCLFD}_\varepsilon$).

Theorem 4.5. **(ε-Converse Duality)** *Let $\varepsilon \geqq 0$ be given. Assume that*

$$\begin{aligned}
C := \quad & \bigcup_{v\in K}\left\{\left(v^Td, (-v^Ta_1,\ldots,-v^Ta_m)^T, v^T(c+\varepsilon d)\right)\right\}\\
&+\bigcup_{\eta\geqq 0}\left\{\eta\left(\beta,(b_1,\ldots,b_m)^T,\alpha+\varepsilon\beta\right)\right\}+(\{(0,0)\}\times\mathbb{R}_+)
\end{aligned}$$

is closed in $\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}$. If $(\bar{\gamma},\bar{\lambda})\in F_D^\varepsilon$ is an ε-solution of ($\text{SOCLFD}_\varepsilon$), then there exist $\bar{v}\in K$ and $\bar{\eta}\geqq 0$ such that

$$\begin{aligned}
&\bar{v}^Td+\bar{\eta}\beta-1=0;\\
&\bar{v}^Ta_i-\bar{\eta}b_i=0,\ i=1,\ldots,m;\\
&\bar{v}^Tc+\bar{\eta}\alpha-\bar{\gamma}\leqq 0.
\end{aligned}$$

Additionally, if F is a bounded set, then $\bar{\eta}>0$, $\frac{c^T\left(\frac{1}{\bar{\eta}}\bar{v}\right)+\alpha}{d^T\left(\frac{1}{\bar{\eta}}\bar{v}\right)+\beta} \leqq \bar{\gamma} \leqq \frac{c^T\left(\frac{1}{\bar{\eta}}\bar{v}\right)+\alpha}{d^T\left(\frac{1}{\bar{\eta}}\bar{v}\right)+\beta}+\varepsilon$, and $\frac{1}{\bar{\eta}}\bar{v}$ is an ε-solution of (SOCLF).

Proof. Let $f(\gamma,\lambda) := -\gamma$ and $g(\gamma,\lambda) := -c + (\gamma-\varepsilon)d - \sum_{i=1}^{m}\lambda_i a_i$ and $h(\gamma,\lambda) := b^T\lambda + \beta(\gamma-\varepsilon) - \alpha$. Then, the (SOCLFD$_\varepsilon$) becomes:

$$\begin{aligned} &\text{Minimize} \quad f(\gamma,\lambda) \\ &\text{subject to} \quad (g(\gamma,\lambda),h(\gamma,\lambda)) \in (-K)\times(-\mathbb{R}_+) \\ &\qquad\qquad \gamma\in\mathbb{R},\ \lambda := (\lambda_i)\in\mathbb{R}^m. \end{aligned}$$

Let $G := \{(\gamma,\lambda)\in\mathbb{R}\times\mathbb{R}^m \mid g(\gamma,\lambda)\in -K\}$ and $H := \{(\gamma,\lambda)\in\mathbb{R}\times\mathbb{R}^m \mid h(\gamma,\lambda)\in -\mathbb{R}_+\}$. Let $(\bar\gamma,\bar\lambda)$ be an ε-solution of (SOCLFD$_\varepsilon$). Then, $f(\gamma,\lambda)+\delta_{G\cap H}(\gamma,\lambda) \geqq f(\bar\gamma,\bar\lambda)+\delta_{G\cap H}(\bar\gamma,\bar\lambda)-\varepsilon$, or equivalently, $(f+\delta_{G\cap H})^*(0,0) \leqq -f(\bar\gamma,\bar\lambda)-\delta_{G\cap H}(\bar\gamma,\bar\lambda)+\varepsilon = -f(\bar\gamma,\bar\lambda)+\varepsilon$. It follows from Lemma 2.1 that

$$\left(0,0,-f(\bar\gamma,\bar\lambda)+\varepsilon\right) \in \operatorname{epi}(f+\delta_{G\cap H})^* = \operatorname{epi} f^* + \operatorname{epi}\delta^*_{G\cap H} = \operatorname{epi} f^* + \operatorname{cl}\left(\operatorname{epi}\delta^*_G + \operatorname{epi}\delta^*_H\right). \tag{4.2}$$

Note that

$$\begin{aligned} f^*(s,u) = \sup_{(\gamma,\lambda)\in\mathbb{R}\times\mathbb{R}^m}[(s,u)^T(\gamma,\lambda)-f(\gamma,\lambda)] &= \sup_{(\gamma,\lambda)\in\mathbb{R}\times\mathbb{R}^m}[s\gamma+u^T\lambda+\gamma] \\ &= \begin{cases} 0 & \text{if } (s,u)=(-1,0); \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Thus $\operatorname{epi} f^* = \{((s,u),t)\in\mathbb{R}\times\mathbb{R}^m\times\mathbb{R} \mid f^*(s,u)\leqq t\} = \{(-1,0)\}\times\mathbb{R}_+$. Now, as $G=\{(\gamma,\lambda)\in\mathbb{R}\times\mathbb{R}^m \mid v^Tg(\gamma,\lambda)\leqq 0,\ \forall v\in K\}$, one has $\delta_G(\gamma,\lambda) = \sup_{v\in K} v^Tg(\gamma,\lambda)$. Along with Lemma 2.2, we get $\operatorname{epi}\delta^*_G = \operatorname{epi}\left(\sup_{v\in K} v^Tg(\cdot,\cdot)\right)^* = \operatorname{cl}\bigcup_{v\in K}\operatorname{epi}\left(v^Tg(\cdot,\cdot)\right)^*$. Since

$$\begin{aligned} \left(v^Tg(\cdot,\cdot)\right)^*(s,u) &= \sup_{(\gamma,\lambda)\in\mathbb{R}\times\mathbb{R}^m}[(s,u)^T(\gamma,\lambda)-v^Tg(\gamma,\lambda)] \\ &= \sup_{(\gamma,\lambda)\in\mathbb{R}\times\mathbb{R}^m}\left[s\gamma+u^T\lambda+v^Tc-(\gamma-\varepsilon)v^Td+\sum_{i=1}^{m}\lambda_i v^Ta_i\right] \\ &= \begin{cases} v^T(c+\varepsilon d) & \text{if } (s,u)=\left(v^Td,(-v^Ta_1,\ldots,-v^Ta_m)^T\right); \\ +\infty & \text{otherwise,} \end{cases} \end{aligned}$$

we have

$$\begin{aligned} \operatorname{epi}\left(v^Tg(\cdot,\cdot)\right)^* &= \{((s,u),t)\in\mathbb{R}\times\mathbb{R}^m\times\mathbb{R} \mid \left(v^Tg(\cdot,\cdot)\right)^*(s,u)\leqq t\} \\ &= \left\{\left(v^Td,(-v^Ta_1,\ldots,-v^Ta_m)^T,v^T(c+\varepsilon d)\right)\right\}+\{(0,0)\}\times\mathbb{R}_+. \end{aligned}$$

Hence,

$$\operatorname{epi}\delta^*_G = \operatorname{cl}\left[\bigcup_{v\in K}\left\{\left(v^Td,(-v^Ta_1,\ldots,-v^Ta_m)^T,v^T(c+\varepsilon d)\right)\right\}+\{(0,0)\}\times\mathbb{R}_+\right].$$

On the other hand, as $H=\{(\gamma,\lambda)\in\mathbb{R}\times\mathbb{R}^m \mid \eta h(\gamma,\lambda)\leqq 0,\ \forall \eta\geqq 0\}$, one has $\delta_H(\gamma,\lambda)=\sup_{\eta\geqq 0}\eta h(\gamma,\lambda)$. Notice that, for each $\eta\geqq 0$,

$$\begin{aligned}(\eta h(\cdot,\cdot))^*(s,u) &= \sup_{(\gamma,\lambda)\in\mathbb{R}\times\mathbb{R}^m}[(s,u)^T(\gamma,\lambda)-\eta h(\gamma,\lambda)]\\ &= \sup_{(\gamma,\lambda)\in\mathbb{R}\times\mathbb{R}^m}\left[s\gamma+u^T\lambda-\eta b^T\lambda-\eta\beta(\gamma-\varepsilon)+\eta\alpha\right]\\ &= \begin{cases}\eta(\alpha+\varepsilon\beta) & \text{if } (s,u)=(\eta\beta,\eta b);\\ +\infty & \text{otherwise,}\end{cases}\end{aligned}$$

which yields

$$\begin{aligned}\operatorname{epi}(\eta h(\cdot,\cdot))^* &= \{((s,u),t)\in\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}\mid (\eta h(\cdot,\cdot))^*(s,u)\leqq t\}\\ &= \{(\eta\beta,\eta b,\eta(\alpha+\varepsilon\beta))\}+\{(0,0)\}\times\mathbb{R}_+.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{epi}\delta_H^* = \operatorname{epi}\left(\sup_{\eta\geqq 0}\eta h(\cdot,\cdot)\right)^* &= \operatorname{cl}\bigcup_{\eta\geqq 0}\operatorname{epi}(\eta h(\cdot,\cdot))^*\\ &= \operatorname{cl}\bigcup_{\eta\geqq 0}[\{(\eta\beta,\eta b,\eta(\alpha+\varepsilon\beta))\}+\{(0,0)\}\times\mathbb{R}_+].\end{aligned}$$

Therefore, in view of (4.2), it entails that

$$\begin{aligned}&(0,0,\bar\gamma+\varepsilon)\\ &\in \operatorname{epi}f^*+\operatorname{cl}(\operatorname{epi}\delta_G^*+\operatorname{epi}\delta_H^*)\\ &= (\{(-1,0)\}\times\mathbb{R}_+)\\ &\quad+\operatorname{cl}\Big(\operatorname{cl}\Big[\bigcup_{v\in K}\left\{\left(v^Td,(-v^Ta_1,\ldots,-v^Ta_m)^T,v^T(c+\varepsilon d)\right)\right\}+\{(0,0)\}\times\mathbb{R}_+\Big]\\ &\quad+\operatorname{cl}\bigcup_{\eta\geqq 0}[\{(\eta\beta,\eta b,\eta(\alpha+\varepsilon\beta))\}+\{(0,0)\}\times\mathbb{R}_+]\Big)\\ &\subseteq (\{(-1,0)\}\times\mathbb{R}_+)+\operatorname{cl}\Big(\operatorname{cl}\Big[\bigcup_{v\in K}\left\{\left(v^Td,(-v^Ta_1,\ldots,-v^Ta_m)^T,v^T(c+\varepsilon d)\right)\right\}\\ &\quad+\bigcup_{\eta\geqq 0}\{(\eta\beta,\eta b,\eta(\alpha+\varepsilon\beta))\}+\{(0,0)\}\times\mathbb{R}_+\Big]\Big)\\ &= (\{(-1,0)\}\times\mathbb{R}_+)+\operatorname{cl}\Big[\bigcup_{v\in K}\left\{\left(v^Td,(-v^Ta_1,\ldots,-v^Ta_m)^T,v^T(c+\varepsilon d)\right)\right\}\\ &\quad+\bigcup_{\eta\geqq 0}\{(\eta\beta,\eta b,\eta(\alpha+\varepsilon\beta))\}+\{(0,0)\}\times\mathbb{R}_+\Big].\end{aligned}$$

It then follows that there exist $\bar v\in K$, $\bar\eta\geqq 0$ and $\bar\theta\geqq 0$ such that

$$\begin{aligned}(0,0,\bar\gamma+\varepsilon) = &\left(-1,0,\bar\theta\right)+\left(\bar v^Td,(-\bar v^Ta_1,\ldots,-\bar v^Ta_m)^T,\bar v^T(c+\varepsilon d)\right)\\ &+(\bar\eta\beta,\bar\eta b,\bar\eta(\alpha+\varepsilon\beta)),\end{aligned}$$

that is,

$$0 = -1 + \bar{v}^T d + \bar{\eta}\beta,$$
$$0 = -\bar{v}^T a_i + \bar{\eta} b_i,\ i = 1,\dots,m,$$
$$\bar{\gamma} + \varepsilon = \bar{\theta} + \bar{v}^T c + \bar{\eta}\alpha + \varepsilon(\bar{v}^T d + \bar{\eta}\beta).$$

This means that there exist $\bar{v} \in K$ and $\bar{\eta} \geqq 0$ such that

$$\bar{v}^T d + \bar{\eta}\beta - 1 = 0, \tag{4.3}$$
$$\bar{v}^T a_i - \bar{\eta} b_i = 0,\ i = 1,\dots,m, \tag{4.4}$$
$$\bar{v}^T c + \bar{\eta}\alpha - \bar{\gamma} \leqq 0. \tag{4.5}$$

Now, we claim that $\bar{\eta} > 0$. Otherwise, assume that $\bar{\eta} = 0$. It follows from (4.3) that $\bar{v}^T d = 1$ and hence $\bar{v} \neq 0$. For any $x \in F$ and any $\eta \geqq 0$, $x + \eta\bar{v} \in K$ and $a_i^T(x + \eta\bar{v}) = a_i^T x + \eta a_i^T \bar{v} = a_i^T x = b_i$, $i = 1,\dots,m$. So, $x + \eta\bar{v} \in F$, which contradicts the boundedness of F. Thus $\bar{\eta} > 0$. From (4.4), we conclude that $\frac{1}{\bar{\eta}}\bar{v} \in F$. From (4.3), one has

$$\bar{\gamma}\bar{v}^T d + \bar{\gamma}\bar{\eta}\beta - \bar{\gamma} = 0. \tag{4.6}$$

In view of (4.5) together with (4.6), we have

$$\bar{\gamma}\bar{v}^T d + \bar{\gamma}\bar{\eta}\beta - \bar{v}^T c - \bar{\eta}\alpha \geqq \bar{\gamma} - \bar{\gamma} = 0. \tag{4.7}$$

Since $(\bar{\gamma}, \bar{\lambda}) \in F_D^{\varepsilon}$, we have $\bar{v}^T c - (\bar{\gamma} - \varepsilon)\bar{v}^T d + \sum_{i=1}^m \bar{\lambda}_i \bar{v}^T a_i \geqq 0$ and $\bar{\eta}(\alpha - \beta(\bar{\gamma} - \varepsilon) - b^T\bar{\lambda}) \geqq 0$. Thus,

$$\bar{\gamma}\bar{v}^T d + \bar{\gamma}\bar{\eta}\beta - \bar{v}^T c - \bar{\eta}\alpha \leqq \bar{v}^T c + \varepsilon\bar{v}^T d + \sum_{i=1}^m \bar{\lambda}_i \bar{v}^T a_i + \varepsilon\bar{\eta}\beta - \bar{\eta} b^T\bar{\lambda} - \bar{v}^T c = \varepsilon. \tag{4.8}$$

It follows from (4.3), (4.7), and (4.8) that

$$\begin{aligned}
&0 \leqq \bar{\gamma}\bar{v}^T d + \bar{\gamma}\bar{\eta}\beta - \bar{v}^T c - \bar{\eta}\alpha \leqq \varepsilon \\
\Leftrightarrow\ &0 \leqq \bar{\gamma}\left(d^T\left(\frac{1}{\bar{\eta}}\bar{v}\right) + \beta\right) - c^T\left(\frac{1}{\bar{\eta}}\bar{v}\right) - \alpha \leqq \frac{\varepsilon}{\bar{\eta}} \\
\Leftrightarrow\ &0 \leqq \bar{\gamma} - \frac{\left(c^T\left(\frac{1}{\bar{\eta}}\bar{v}\right) + \alpha\right)}{\frac{1}{\bar{\eta}}} \leqq \varepsilon \\
\Leftrightarrow\ &\frac{c^T\left(\frac{1}{\bar{\eta}}\bar{v}\right) + \alpha}{d^T\left(\frac{1}{\bar{\eta}}\bar{v}\right) + \beta} \leqq \bar{\gamma} \leqq \frac{c^T\left(\frac{1}{\bar{\eta}}\bar{v}\right) + \alpha}{d^T\left(\frac{1}{\bar{\eta}}\bar{v}\right) + \beta} + \varepsilon.
\end{aligned}$$

Moreover, by Theorem 4.3, we have that for any $x \in F$, $\frac{c^T x + \alpha}{d^T x + \beta} \geqq \bar{\gamma} - \varepsilon \geqq \frac{c^T\left(\frac{1}{\bar{\eta}}\bar{v}\right) + \alpha}{d^T\left(\frac{1}{\bar{\eta}}\bar{v}\right) + \beta} - \varepsilon$, showing that $\frac{1}{\bar{\eta}}\bar{v}$ is an ε-solution of (SOCLF). □

Remark 4.1. An analogous manner as in Remark 3.1, if there exist $\widetilde{\gamma} \in \mathbb{R}$ and $\widetilde{\lambda} := (\widetilde{\lambda}_i) \in \mathbb{R}^m$ such that $c - (\widetilde{\gamma} - \varepsilon)d + \sum_{i=1}^m \widetilde{\lambda}_i a_i \in \text{int}K$ and $\alpha - \beta(\widetilde{\gamma} - \varepsilon) > b^T\widetilde{\lambda}$, then C is closed in $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$.

Below, we give a simple example to illustrate Theorem 4.5.

Example 4.3. Consider the following second order cone linear fractional optimization problem:

$$\begin{aligned} \text{Minimize} \quad & \frac{-x_2+1}{x_1-x_2+1} \\ \text{subject to} \quad & x:=(x_1,x_2)^T \in K:=\left\{x\in\mathbb{R}^2 \mid x_1 \geqq \sqrt{x_2^2}\right\}, \\ & x_1=1. \end{aligned} \tag{SOCLF}$$

Let $c:=(0,-1)^T$, $\alpha:=1$, $d:=(1,-1)^T$, $\beta:=1$, $a_1:=(1,0)^T$, and $b_1:=1$. Note that $F=\{0\}\times[-1,1]$ is bounded. Letting $\bar{x}:=(1,0,1)^T$ and $\varepsilon:=1$ and $(\widetilde{\gamma},\widetilde{\lambda}):=(-1,2)$, we see that $F_D^{\varepsilon}=\{(\gamma,\lambda)\in\mathbb{R}^2 \mid -\gamma+1+\lambda\geqq|-2+\gamma|,\ 2-\gamma\geqq\lambda\}$. So, $c-(\widetilde{\gamma}-\varepsilon)d+\widetilde{\lambda}_1 a_1\in \text{int}K$ and $\alpha-\beta(\widetilde{\gamma}-\varepsilon)>b_1\widetilde{\lambda}$, and hence C is closed in $\mathbb{R}^3$. In view of $-\gamma+1+\lambda\geqq|-2+\gamma|\geqq 2-\gamma\geqq\lambda$, we get $\gamma\leqq 1$. Thus, $(\bar{\gamma},\bar{\lambda}):=(1,1)$ is an ε-optimal solution of (SOCLFD$_\varepsilon$). In what follows, we would verify the inequality proposed in Theorem 4.5, that is, finding $\bar{v}\in K$ and $\bar{\eta}>0$ such that

$$\begin{cases} \bar{v}^T(1,-1)^T+\bar{\eta}=1, \\ \bar{v}^T(1,0)^T-\bar{\eta}=0, \\ \bar{v}^T(0,-1)^T+\bar{\eta}\leqq 1 \end{cases}$$

and $\frac{c^T\left(\frac{1}{\bar{\eta}}\bar{v}\right)+\alpha}{d^T\left(\frac{1}{\bar{\eta}}\bar{v}\right)+\beta}\leqq\bar{\gamma}\leqq\frac{c^T\left(\frac{1}{\bar{\eta}}\bar{v}\right)+\alpha}{d^T\left(\frac{1}{\bar{\eta}}\bar{v}\right)+\beta}+\varepsilon$. By taking $\bar{v}=(\bar{v}_1,\bar{v}_2)^T:=(1,1)^T\in K$ and $\bar{\eta}:=1>0$, we can verify that $(\bar{v},\bar{\eta})$ satisfy those conditions.

The next example demonstrates that the conclusion of Theorem 4.5 may fail under the violation of boundedness of F.

Example 4.4. Let c,α,d,β,a_1, and b_1 be defined as in Example 3.2. Letting $(\widetilde{\gamma},\widetilde{\lambda}):=(\frac{1}{2},\frac{3}{4})$, we see that $c-(\widetilde{\gamma}-\varepsilon)d+\widetilde{\lambda}_1 a_1\in\text{int}K$ and $\alpha-\beta(\widetilde{\gamma}-\varepsilon)>b_1\widetilde{\lambda}$, and so C is closed in $\mathbb{R}^3$. Now, it can be verify that $(\bar{\gamma},\bar{\lambda}):=(\frac{1}{2},\frac{3}{4})\in F_D^{\varepsilon}$ is an ε-solution of (SOCLFD$_\varepsilon$). According to Theorem 4.5, there exist $\bar{v}:=(\bar{v}_1,\bar{v}_2,\bar{v}_3)^T\in K$ and $\bar{\eta}\geqq 0$ such that

$$\begin{cases} \bar{v}^T(1,1,1)^T+\bar{\eta}=1, \\ \bar{v}^T(1,0,-1)^T=0, \\ \bar{v}^T(0,-1,1)^T+2\bar{\eta}\leqq\frac{1}{2} \end{cases} \tag{4.9}$$

As $\bar{v}_1-\bar{v}_3=0$ and $\bar{v}\in K$, we see that $\bar{v}_2=0$. So, we conclude that $\bar{\eta}=1-2\bar{v}_3$, $\bar{v}_1=\bar{v}_3$, and $\bar{v}_3\leqq\frac{1}{2}$. As $\bar{\eta}\geqq 0$, we have $\bar{v}_3=\frac{1}{2}$. Clearly, we cannot find $\bar{v}:=(\bar{v}_1,\bar{v}_2,\bar{v}_3)^T\in K$ and $\bar{\eta}>0$ satisfying (4.9). Hence, Theorem 4.5 goes awry. The reason is that $F=\{(x_1,x_2,x_3)^T\in\mathbb{R}^3 \mid x_1=x_3\geqq 0,\ x_2=0\}$ is not bounded.

Before closing this section, we point out that in the study of linear fractional optimization problems, the authors in [7] formulated the dual problem by using the equivalent nonfractional linear optimization problem. This dual problem is a nonfractional dual one and can be used to obtain an opitimal solution of linear fractional optimization problems. Note also that the dual problem differs from the dual problem obtained from the linear fractional optimization problem. In what follows, regarding the ε-converse duality, let us consider the second-order cone linear

programming problem:

$$\begin{aligned} \text{Minimize} \quad & v^T c + \eta\alpha \\ \text{subject to} \quad & v^T d + \eta\beta = 1, \\ & v^T a_i - \eta b_i = 0,\ i = 1,\ldots,m, \\ & v \in K,\ \eta \geqq 0. \end{aligned} \tag{SOCL}$$

We see now how we can obtain an ε-solution of (SOCLF) via (SOCL).

Theorem 4.6. *Let $\varepsilon \geqq 0$. Assume that*

$$\bigcup_{\lambda,\lambda_i\in\mathbb{R}} \left\{ \lambda(d,\beta,1) + \sum_{i=1}^{m} \lambda_i(a_i,-b_i,0) \right\} + (-K)\times(-\mathbb{R}_+)\times\mathbb{R}_+$$

is closed in $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$. If $(\bar{v},\bar{\eta})^T$ is an ε-solution of (SOCL) with $\bar{\eta}>0$, then $\frac{1}{\bar{\eta}}\bar{v}$ is an ε-solution of (SOCLF).

Proof. Suppose that $\bar{x} := (\bar{v},\bar{\eta})^T$ is an ε-solution of (SOCL) with $\bar{\eta}>0$. It then follows that $\bar{x}$ is an ε-solution of the following second-order cone linear fractional programming problem:

$$\begin{aligned} \text{Minimize} \quad & \frac{\widetilde{c}^T y + \widetilde{\alpha}}{\widetilde{d}^T y + \widetilde{\beta}} \\ \text{subject to} \quad & y \in \widetilde{K},\ \widetilde{a}_i^T y = \widetilde{b}_i,\ i = 0,1,\ldots,m, \end{aligned}$$

where $\widetilde{c} := (c,\alpha)^T$, $\widetilde{\alpha} := 0$, $\widetilde{d} := 0$, $\widetilde{\beta} := 1$, $y := (v,\eta)^T$, $\widetilde{a}_0 := (d,\beta)^T$, $\widetilde{b}_0 := 1$, $\widetilde{a}_i := (a_i,-b_i)^T$, $\widetilde{b}_i := 0$ for $i = 1,\ldots,m$ and $\widetilde{K} := K\times\mathbb{R}_+$. Let $\bar{\gamma} := \widetilde{c}^T\bar{x}$. Note that $\bigcup_{\lambda_i\in\mathbb{R}} \sum_{i=0}^{m} \lambda_i(\widetilde{a}_i,\widetilde{b}_i) + (-K)\times\mathbb{R}_+$ is closed in $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$. Invoking Theorem 4.4, there exist $\bar{\lambda}_0\in\mathbb{R}$ and $\bar{\lambda} := (\bar{\lambda}_i)\in\mathbb{R}^m$ such that $\widetilde{c} - (\bar{\gamma}-\varepsilon)\widetilde{d} + \sum_{i=0}^{m}\bar{\lambda}_i\widetilde{a}_i \in K\times\mathbb{R}_+$ and $\widetilde{\alpha} - \widetilde{\beta}(\bar{\gamma}-\varepsilon) \geqq \sum_{i=0}^{m}\bar{\lambda}_i\widetilde{b}_i$. Hence,

$$\begin{aligned} & c + \bar{\lambda}_0 d + \sum_{i=1}^{m}\bar{\lambda}_i a_i \in K, \\ & \alpha + \bar{\lambda}_0\beta - \sum_{i=1}^{m}\bar{\lambda}_i b_i \geqq 0, \\ & -(\bar{\gamma}-\varepsilon) \geqq \bar{\lambda}_0. \end{aligned}$$

Observe that $(-\bar{\lambda}_0+\varepsilon,\bar{\lambda}) \in F_D^{\varepsilon}$. Thus, $\sup(\text{SOCLFD}_{\varepsilon}) \geqq -\bar{\lambda}_0+\varepsilon \geqq \bar{\gamma}$. By Theorem 4.3, we see that $\inf(\text{SOCLF}) \geqq \bar{\gamma}-\varepsilon$. Since $\bar{\gamma} = \frac{c^T\left(\frac{1}{\bar{\eta}}\bar{v}\right)+\alpha}{d^T\left(\frac{1}{\bar{\eta}}\bar{v}\right)+\beta}$, we obtain that $\frac{1}{\bar{\eta}}\bar{v}$ is an ε-solution of (SOCLF). □

Remark 4.2. An analogous manner as in Remark 3.1, if $(d,\beta)^T$ and $(a_i,-b_i)^T$, $i=1,\ldots,m$, are linearly independent in $\mathbb{R}^n\times\mathbb{R}$ and there exists $(\widetilde{v},\widetilde{\eta})\in\text{int}(K)\times(0,+\infty)$ such that $\widetilde{v}^T d + \widetilde{\eta}\beta = 1$ and $\widetilde{v}^T a_i - \widetilde{\eta} b_i = 0$ for all $i = 1,\ldots,m$, then

$$\bigcup_{\lambda,\lambda_i\in\mathbb{R}} \left\{ \lambda(d,\beta,1) + \sum_{i=1}^{m} \lambda_i(a_i,-b_i,0) \right\} + (-K)\times(-\mathbb{R}_+)\times\mathbb{R}_+$$

is closed in $\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$.

Remark 4.3. Let us notice that we can solve (SOCLFD$_\varepsilon$) by using primal-dual interior point algorithms in [21] and then, we obtain ε-solutions of (SOCLFD$_\varepsilon$). Moreover, in view of our ε-converse duality theorem (Theorem 4.5), we can obtain an ε-solution of (SOCL). Thus, by using the primal-dual interior point algorithms, and the ε-converse duality result, we can design an efficient algorithm to seek the ε-solution for (SOCLF).

Remark 4.4. Note also that it is possible to obtain converse duality result like Theorem 4.5 by transformation techniques in [5, 6, 7], which differ from our dual approach.

5. Conclusion

In this paper, we mainly studied the approximate optimality theorem for an approximate solution to the second order cone linear fractional optimization problem (SOCLF) under the fulfillment of the closedness assumption. Finally, we formulated dual models in approximate form and studied approximate weak, strong and converse duality theorems.

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