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OPTIMALITY CONDITIONS FOR SOLUTIONS OF CONSTRAINED INVERSE VECTOR VARIATIONAL INEQUALITIES BY MEANS OF NONLINEAR SCALARIZATION

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Abstract. This work is devoted to examining inverse vector variational inequalities with constraints by means of a prominent nonlinear scalarizing functional. We show that inverse vector variational inequalities are equivalent to multiobjective optimization problems with a variable domination structure. Moreover, we introduce a nonlinear function based on a well-known nonlinear scalarization function. We show that this function is a weak separation function and a regular weak separation function under different parameter sets. Then two alternative theorems are established, which will provide the basis for characterizing efficient elements of inverse vector variational inequalities.

Keywords. Inverse vector variational inequality; Multiobjective optimization; Nonlinear scalarization function; Nonlinear separation function; Optimality condition.

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1. Introduction

Inverse variational inequality (shortly, IVI) is an important optimization model in the theory of optimization and applications such as economic games and transport management; see [2, 12, 13, 14, 31] and the references therein. Hu and Fang [16, 17] investigated the well-posedness and Levitin-Polyak well-posedness by perturbations for IVI. Luo [25] discussed Tikhonov regularization for IVI, and obtained the nonemptiness, boundness and perturbation analysis of solution set of regularized IVI under a weak coercivity condition. László [21] studied the existence of solutions for IVI involving operators of type ql. Aussel, Gupta and Mehra [1] established local/global error bounds for inverse quasi-variational inequality problems by using residual gap function, regularized gap function and D-gap function, respectively.

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Li, Li and Huang [22] presented an iterative algorithm involving the generalized f-projection operator for solving mixed IVI, and obtained the convergence results.

Recently, the image space analysis (for short, ISA) was initiated in [3] and further developed in [7, 8]. ISA has been proven to be a powerful tool for studying any kind of problems, which can be expressed under the form of the inconsistency of a parametric system. It is worth mentioning that ISA can be applied to deal with nonconvex, nonsmooth and even discontinuous constrained optimization problems. Since then, ISA was extensively applied to establish theorems of alternative, optimality conditions, penalty methods, gap functions and error bounds of constrained extremum problems, variational inequalities and equilibrium problems by various kinds of special separation functions including linear and nonlinear cases; see [4, 7, 8, 9, 10, 23, 24]. Inspired by [4], Xu [30] constructed three specific nonlinear separation functions in the image space of IVI with constraints by using the Gerstewitz(Tammer)-type nonlinear scalarization function, and then, by the nonlinear separation functions, theorems of the weak and strong alternative, optimality conditions, gap functions and an error bound for IVI with a cone constraint were derived. Chen *et al.* [5] introduced a new class of constrained inverse variational inequalities (in short, CIVI), which significantly differs from the approach in [30] and has no restriction on the image of mapping f. The authors in [5] proposed two nonlinear (regular) weak separation functions and derived theorems of the weak alternative and necessary and sufficient optimality conditions for CIVI. Finally, in [5], the relationships between CIVI and its primal variational inequalities were also established under some suitable conditions. Very recently, Chen *et al.* [6] further extended CIVI from real-valued case to vector case, established the equivalence between multiobjective optimization and inverse vector variational inequalities and obtained alternative theorems and optimality conditions of the inverse vector variational inequalities via ISA and multiobjective techniques.

Motivated by the works [4, 5, 6, 30], we continue to study the class of inverse vector variational inequalities [6] by using ISA. A nonlinear function, which is proven to be a weak separation function and a regular weak separation function under different parameter sets, is constructed by employing Gerth (Tammer) type nonlinear scalarization function rather than oriented distance function. Theorems of weak alternative are derived by the nonlinear separation function. Lastly, sufficient and necessary optimality conditions of the inverse variational inequalities are obtained by the weak alternative results.

2. Preliminaries

For the sake of brevity throughout the manuscript, we denote the *power set* of a linear space Y without the empty set by $\mathscr{P}(Y) := \{A \subseteq Y \mid A \text{ is nonempty}\}$.

We denote $\mathbb{R}_+ := [0, +\infty[$ and $\mathbb{R}_{++} :=]0, +\infty[$. Let $\mathscr{B}^* := \{\mu \in \mathbb{R}^m : \mu^\top z \geq 0, \forall z \in \mathscr{B}\}$ be the dual cone of a convex cone $\mathscr{B} \subseteq \mathbb{R}^m$. Let $\omega : \mathbb{R}^{n+2m} \times \Pi \to \mathbb{R} =]-\infty, +\infty[$ be a real-valued function, where Π is a set of parameters to be specified in a particular setting. For each $\pi \in \Pi$ and $a \in \mathbb{R}$, we define the following level sets by

$$\operatorname{lev}_{\geq a} \omega(\cdot;\pi) := \{\upsilon \in \mathbb{R}^{n+2m} : \omega(\upsilon;\pi) \geq a\}$$

and

$$\operatorname{lev}_{> a} \omega(\cdot;\pi) := \{\upsilon \in \mathbb{R}^{n+2m} : \omega(\upsilon;\pi) > a\}.$$

In Section 2.1, we recall the image space analysis approach for constrained inverse vector variational inequalities. Section 2.2 is devoted to recalling a prominent nonlinear scalarization functional, which will be used in Section 3 to investigate and solve constrained inverse vector variational inequalities.

2.1. **Image Space Analysis for Constrained Inverse Vector Variational Inequalities.** Let $g : \mathbb{R}^n \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^m$, $X := \{x \in \mathbb{R}^n : g(x) \in D\}$, $f : X \to \mathbb{R}^p$, $W : X \to \mathbb{R}^{n,p}$ be given. We are looking for elements $\bar{x} \in X \setminus \{\mathbf{0}\}$ such that the following vector variational inequality holds true:

$$(f(x) - f(\bar{x}))^\top W(\bar{x}) \geqq \mathbf{0}, \ \ \forall x \in X, \tag{2.1}$$

where $\geqq$ is to be understood componentwise, such that $y \geqq \bar{y}$ if and only if $y_i \geq \bar{y}_i$ for all $i = 1, \dots, p$ and $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^p$. We will refer to problem (2.1) as *constrained inverse vector variational inequality* (in short, CIVVI).

Now let $\bar{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ be a parameter. We define

$$C(\bar{x}) := \{d \in \mathbb{R}^n : W(\bar{x})^\top d \geqq \mathbf{0}\}. \tag{2.2}$$

Note that $C(\bar{x})$ is a closed convex cone in $\mathbb{R}^n$.

Example 2.1. Consider the case $p = n$, $W(x) := I_n$, the identity matrix for each $x \in X$. Then $C(x)$ is the natural ordering cone $\mathbb{R}^n_+ := \{y \in \mathbb{R}^n \mid y \geqq \mathbf{0}\}$ in $\mathbb{R}^n$.

CIVVI (2.1) can be equivalently characterized by the system

$$\bar{x} \in X \setminus \{\mathbf{0}\} \ \text{and} \ f(x) - f(\bar{x}) \in C(\bar{x}), \ \forall x \in X. \tag{2.3}$$

If $\bar{x} \in X \setminus \{\mathbf{0}\}$ satisfies the system (2.3), then $\bar{x}$ is called a **strongly efficient** solution of the following multiobjective optimization problem (in short, MOP):

$$\min_{x \in X} f(x) \tag{2.4}$$

In Section 3.1, we will deal with CIVVI (2.1) via MOP (2.4), which means that we will be looking for strongly efficient solutions of the MOP (2.4) with respect to the variable domination structure given by the cone $C(\bar{x})$.

The problem CIVVI (2.1) is a generalization of the *constrained inverse variational inequality*, which has been investigated in [5] (see also [6]), where $W(x) := x$ for all $x \in X$.

Example 2.2 (Constrained inverse variational inequality (CIVI))**.** In this motivating example, we consider the following constrained inverse variational inequality (shortly, CIVI) of finding $\bar{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that

$$f(\bar{x}) \in \Omega \ \ \text{and} \ \ \left(f' - f(\bar{x})\right)^\top \bar{x} \geq 0, \ \ \forall f' \in \Omega, \tag{2.5}$$

where $\Omega := \{f(x) : x \in \mathbb{R}^n, g(x) \in D\}$. Set $X := \{x \in \mathbb{R}^n : g(x) \in D\}$. It is easy to see that $X = g^{-1}(D)$ and $\Omega = f(X)$. As pointed out in [5], CIVI (2.5) can be regarded as the inverse of the following constrained variational inequality of finding $\bar{x} \in X \setminus \{\mathbf{0}\}$ such that

$$(x - \bar{x})^\top f(\bar{x}) \geq 0, \ \ \forall x \in X. \tag{2.6}$$

Obviously, CIVI (2.5) can be rewritten as the following simplified form of finding $\bar{x} \in X \setminus \{\mathbf{0}\}$ such that

$$(f(x) - f(\bar{x}))^\top \bar{x} \geq 0, \ \ \forall x \in X. \tag{2.7}$$

Let $\bar{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ be a given parameter and set $C(\bar{x}) := \{d \in \mathbb{R}^n : \bar{x}^\top d \geq 0\}$. Clearly, $C(\bar{x})$ is a closed and convex cone and $C(\bar{x}) \neq \mathbb{R}^n$. In fact, $C(\bar{x})$ is a halfspace. Then CIVI (2.5) can be equivalently characterized by the following system:

$$\bar{x} \in X \setminus \{\mathbf{0}\} \text{ and } f(x) - f(\bar{x}) \in C(\bar{x}), \ \forall x \in X, \tag{2.8}$$

where $X := \{x \in \mathbb{R}^n : g(x) \in D\}$.

Now we consider MOP (2.4) in order to deal with CIVVI (2.1). Let $\bar{x} \in X \setminus \{\mathbf{0}\}$. We define the mapping $\mathscr{A}_{\bar{x}} : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ by

$$\mathscr{A}_{\bar{x}}(x) := (f(x) - f(\bar{x}), \ g(x), \ g(\bar{x})). \tag{2.9}$$

Set

$$\mathscr{H}(\bar{x}) := (\mathbb{R}^n \setminus C(\bar{x})) \times D \times D \tag{2.10}$$

and

$$\mathscr{K}(\bar{x}) := \{(u, v, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m : (u, v, z) = \mathscr{A}_{\bar{x}}(x), \ x \in \mathbb{R}^n\}. \tag{2.11}$$

Therefore, $\bar{x} \in X$ is a nonzero solution of CIVVI (2.1) if and only if the system (in the unknown x):

$$f(x) - f(\bar{x}) \in \mathbb{R}^n \setminus C(\bar{x}), \ g(x) \in D, \ x \in \mathbb{R}^n \ \text{ and } g(\bar{x}) \in D, \tag{2.12}$$

is impossible, which means that there does not exist any $x \in X$ such that $f(x) - f(\bar{x}) \in \mathbb{R}^n \setminus C(\bar{x})$. Moreover, it can be easily seen that the impossibility of the system (2.12) is equivalently characterized by

$$\mathscr{K}(\bar{x}) \bigcap \mathscr{H}(\bar{x}) = \emptyset. \tag{2.13}$$

In Section 3.1, we will deal with CIVVI (2.1) by means of (2.3) and (2.4). In Section 3.2, we will discuss CIVVI (2.1) in terms of (2.12) and (2.13) by deriving theorems of the alternative.

2.2. **Nonlinear Scalarizing Functionals.** In this section, we recall a prominent nonlinear scalarizing function which will be used in Section 3 for solving CIVVI (2.1). Throughout Section 2.2, let Y be a linear topological space. We assume that $D \in \mathscr{P}(Y)$ is a closed proper (i.e., $D \neq \{0\}$, and $D \neq Y$) set satisfying the inclusion

$$D + [0, +\infty) \cdot k \subseteq D \tag{2.14}$$

for some $k \in Y \setminus \{0\}$. We define the functional $z^{D,k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ by

$$z^{D,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - D\}. \tag{2.15}$$

The functional $z^{D,k}$ is called *nonlinear scalarizing functional*, as it plays an important role in scalarization methods for obtaining efficient solutions of a vector-valued optimization problem. The functional $z^{D,k}$ was used to obtain separation theorems for not necessarily convex sets, see [26]. Additionally, numerous applications of $z^{D,k}$ are known in the literature, for instance, coherent risk measures in financial mathematics (see [15]) and uncertain optimization (in particular, in robustness theory, compare [18]).

Many properties of $z^{D,k}$ can be found in [11, 26, 27, 29]. Moreover, it has been shown in [19] that $z^{D,k}$ is useful for the characterization of set relations in set-valued optimization. This is especially important in order to simplify numerical computations when deriving minimal solutions of set-valued optimization problems, as set inclusions can be reduced to solving one particular inequality (compare [19, 20]).

Under appropriate assumptions, $z^{D,k}$ satisfies certain monotonicity properties (see Theorem 2.1, (d)). We define the notion of $\widetilde{D}$-monotonicity below.

Definition 2.1. Let Y be a linear topological space and $\widetilde{D} \in \mathscr{P}(Y)$. A functional $z\colon Y \to \mathbb{R} \cup \{\pm\infty\}$ is **$\widetilde{D}$-monotone** if

$$y_1,\ y_2 \in Y:\ y_1 \in y_2 - \widetilde{D} \Rightarrow z(y_1) \le z(y_2).$$

Important properties of the functional $z^{D,k}$ which will be used in this paper are collected in the following theorem.

Theorem 2.1 ([11, 26])**.** *Let Y be a linear topological space, $D \in \mathscr{P}(Y)$ a closed proper set, $\widetilde{D} \in \mathscr{P}(Y)$ and let $k \in Y \setminus \{\mathbf{0}\}$ be such that (2.14) is satisfied. Then the following properties hold for $z = z^{D,k}$:*

(a) z is lower semi-continuous.

(b) (i) z is convex $\Longleftrightarrow$ D is convex,

(ii) $[\forall\, y \in Y,\ \forall\, r > 0:\ z(ry) = rz(y)] \Longleftrightarrow D$ is a cone.

(c) z is proper $\Longleftrightarrow$ D does not contain lines parallel to k, i.e., $\forall\, y \in Y\ \exists\, r \in \mathbb{R}:\ y + rk \notin D$.

(d) z is $\widetilde{D}$-monotone $\Longleftrightarrow D + \widetilde{D} \subset D$.

(e) z is subadditive $\Longleftrightarrow D + D \subset D$.

(f) $\forall\, y \in Y,\ \forall\, r \in \mathbb{R}:\ z(y) \le r \Longleftrightarrow y \in rk - D$.

(g) $\forall\, y \in Y,\ \forall\, r \in \mathbb{R}:\ z(y + rk) = z(y) + r$.

(h) z is finite-valued $\Longleftrightarrow$ D does not contain lines parallel to k and $\mathbb{R}k - D = Y$.

(i) Let furthermore $D + (0, +\infty) \cdot k \subset \operatorname{int} D$. Then z is continuous.

Remark 2.1. Note that since $C(\bar{x})$, given by (2.2), is a closed convex cone, the inclusion (2.14) is satisfied for any $k \in C(\bar{x})$. Moreover, notice that if $k \in -\operatorname{int} C(\bar{x})$, then the functional $z^{-C(\bar{x}),k}$ ($z^{D,k}$ with $D := -C(\bar{x})$ is finite-valued.

We give an example below to show that the function value $z^{D,k}$ can be easily computed numerically.

Example 2.3. Here we exemplarily compute the function value $z^{D,k}$ in the context of block norms. Let $Y = \mathbb{R}^m$ and let γ be a norm on $\mathbb{R}^m$ which is characterized by its closed unit ball

$$B_\gamma := \{y \in \mathbb{R}^m \mid\ \gamma(y) \le 1\}.$$

A norm γ is called a **block norm**, if its unit ball B_γ is polyhedral (a polytope). Let $\bar{y} \in \mathbb{R}^m$. The **reflection set** of $\bar{y}$ is defined by

$$R(\bar{y}) := \{y \in \mathbb{R}^m \mid\ |y_i| = |\bar{y}_i|\ \ \forall\, i = 1, \dots, m\}.$$

A norm γ is called **absolute**, if $\gamma(y) = \gamma(\bar{y})$ for all $y \in R(\bar{y})$. A block norm γ is called **oblique**, if γ is absolute and satisfies $(y - \mathbb{R}^m_+) \cap \mathbb{R}^m_+ \cap \operatorname{bd} B_\gamma = \{y\}$ for all $y \in \mathbb{R}^m_+ \cap \operatorname{bd} B_\gamma$, where $\operatorname{bd} B_\gamma$ denoted the boundary of the set B_γ.

Let γ be a block norm with unit ball B_γ, and let $a^i \in \mathbb{R}^m$, $\alpha_i \in \mathbb{R}$, $i = 1,\ldots,n$, be given. We define

$$B_\gamma = \{y \in \mathbb{R}^m \mid \langle a^i, y\rangle \leq \alpha_i,\ i = 1,\ldots,n\}. \tag{2.16}$$

We define a set $A_\gamma \subset \mathbb{R}^m$ by

$$A_\gamma := \{y \in \mathbb{R}^m \mid \langle a^i, y\rangle \leq \alpha_i,\ i \in I\} \tag{2.17}$$

with the index set

$$I := \{i \in \{1,\ldots,n\} \mid \{y \in \mathbb{R}^m : \langle a^i, y\rangle = \alpha_i\} \cap B_\gamma \cap \operatorname{int}\mathbb{R}^m_+ \neq \emptyset\}. \tag{2.18}$$

The set I is exactly the set of indices $i = 1,\ldots,n$ for which the hyperplanes $\langle a^i, y\rangle = \alpha_i$ are active in the positive orthant. Tammer, Winkler [28, Lemma 3.2] have shown that $a^i \in \mathbb{R}^m_+$ if and only if $i \in I$, provided that γ is an absolute block norm, and $a^i \in \operatorname{int}\mathbb{R}^m_+$ if and only if $i \in I$ under the assumption that γ is an oblique norm.

Assume that γ is an absolute block norm with unit ball B_γ defined in (2.16), and let a vector $w \in \mathbb{R}^m$ be given. Then the condition (2.14) is fulfilled with $D = A_\gamma + w$ and $k \in -\operatorname{int}\mathbb{R}^m_+$.

If we assume γ to be an oblique norm, (2.14) is fulfilled as well and we can compute for given $y \in \mathbb{R}^m$ the value $z^{D,k}$ for $D = A_\gamma + w$ and $k \in -\operatorname{int}\mathbb{R}^m_+$ by the following formula (see Tammer, Winkler [28]):

$$\begin{aligned}
z^{A_\gamma+w,k}(y) &= \min\{t \in \mathbb{R} | y \in tk - (A_\gamma + w)\}\\
&= \min\{t \in \mathbb{R} | y + w - tk \in A_\gamma\}\\
&= \min\{t \in \mathbb{R} | \langle a^i, y + w - tk\rangle \leq \alpha_i,\ i \in I\}\\
&= \min\{t \in \mathbb{R} | \langle a^i, y\rangle + \langle a^i, w\rangle - t\cdot\langle a^i, k\rangle \leq \alpha_i,\ i \in I\}\\
&= \min_{i\in I} \frac{\langle a^i, y\rangle - \langle a^i, w\rangle - \alpha_i}{\langle a^i, k\rangle}.
\end{aligned}$$

Note that due to Theorem 2.1, $\langle a^i, k\rangle \neq 0$, because $k \in -\operatorname{int}\mathbb{R}^m_+$ and $a^i \in \operatorname{int}\mathbb{R}^m_+$, as γ was assumed to be an oblique norm. Note that $z^{A_\gamma+w,k}$ is a finite-valued, continuous, convex, $(-\mathbb{R}^m_+)$-monotone and strictly $(-\mathbb{R}^m_+)$-monotone functional.

3. Main results

In this section, we are concerned with providing efficient tools for solving the CIVVI (2.1). We first present our results on the primal and dual space approach in the first part based on MOP (2.4), and in the second part of this section we deal with alternative theorems for solving CIVVI (2.1).

3.1. Results Based on Primal and Dual Space Approach.

As illustrated in Example 2.3, the computation of the nonlinear scalarizing functional $z^{D,k}$ (see (2.15)) is easy to handle in a numerical manner. Therefore, it is convenient to use the the function $z^{D,k}$ for dealing with MOP (2.4) with an appropriate set D and vector k. Throughout Section 3.1, we consider $z^{D,k}$ with $D := C(\bar{x})$, where $C(\bar{x})$ is defined by (2.2), and $k \in C(\bar{x}) \setminus \{\mathbf{0}\}$. Now we present a first result for characterizing a strongly efficient solution of MOP (2.4).

Lemma 3.1. *If $\bar{x} \in X \setminus \{\mathbf{0}\}$ is a strongly efficient solution of MOP (2.4), then*

$$z^{C(\bar{x}),k}(f(\bar{x})) \leq \inf_{x\in X} z^{C(\bar{x}),k}(f(x)) \;\forall\, k \in C(\bar{x}) \setminus \{\mathbf{0}\}.$$

Proof. Let $\bar{x} \in X \setminus \{\mathbf{0}\}$ be a strongly efficient solution of MOP (2.4), i.e., for all $x \in X$, $f(x) - f(\bar{x}) \in C(\bar{x})$. Recall that $C(\bar{x})$ is a convex cone, and so, $C(\bar{x}) + C(\bar{x}) \subseteq C(\bar{x})$ is fulfilled. Therefore, for all $x \in X$, $z^{C(\bar{x}),k}(f(\bar{x})) \leq z^{C(\bar{x}),k}(f(x))$, because $z^{C(\bar{x}),k}$ is $C(\bar{x})$-monotone due to Theorem 2.1 (d). Thus, we obtain $z^{C(\bar{x}),k}(f(\bar{x})) \leq \inf_{x\in X} z^{C(\bar{x}),k}(f(x))$. □

Lemma 3.2. *$\bar{x} \in X \setminus \{\mathbf{0}\}$ is a strongly efficient solution of MOP (2.4) if and only if*

$$\sup_{x\in X} z^{C(\bar{x}),k}(f(\bar{x}) - f(x)) \leq 0 \;\forall\, k \in C(\bar{x}) \setminus \{\mathbf{0}\}$$

holds.

Proof. Let $\bar{x} \in X \setminus \{\mathbf{0}\}$ be a strongly efficient solution of MOP (2.4). Then, for all $x \in X$, $f(x) - f(\bar{x}) \in C(\bar{x})$. By Theorem 2.1 (f), for all $x \in X$, $z^{C(\bar{x}),k}(f(\bar{x}) - f(x)) \leq 0$. This results in $\sup_{x\in X} z^{C(\bar{x}),k}(f(\bar{x}) - f(x)) \leq 0$. Conversely, assume that $\sup_{x\in X} z^{C(\bar{x}),k}(f(\bar{x}) - f(x)) \leq 0$ holds. Then for all $x \in X$, $z^{C(\bar{x}),k}(f(\bar{x}) - f(x)) \leq 0$. Then Theorem 2.1 (f) yields the desired result. □

The above result implies that $\bar{x} \in X \setminus \{\mathbf{0}\}$ is a solution of the CIVVI (2.1) if and only if

$$\sup_{x\in X} z^{C(\bar{x}),k}(f(\bar{x}) - f(x)) \leq 0$$

holds true for any $k \in C(\bar{x}) \setminus \{\mathbf{0}\}$. If $C(\bar{x})$ is defined as in Example 2.2, that is, if $C(\bar{x}) = \{d \in \mathbb{R}^n \,:\, \bar{x}^\top d \geq 0\}$, then the nonlinear scalarizing function $z^{C(\bar{x}),k}$ becomes linear and we obtain

$$z^{C(x),k}(f(\bar{x})) \leq \inf_{x\in X} z^{C(\bar{x}),k}(f(x))$$

if and only if $\bar{x} \in X \setminus \{\mathbf{0}\}$ solves CIVVI (2.1).

If it is easier to compute the dual cone of $C(\bar{x})$, denoted by $C(\bar{x})^*$, then the following result is useful.

Lemma 3.3. [6] *$\bar{x} \in X \setminus \{\mathbf{0}\}$ is a strongly efficient solution of MOP (2.4) if and only if for any $l \in C(\bar{x})^*$, $l^\top f(\bar{x}) \leq \inf_{x\in X} l^\top f(x)$ holds.*

Lemma 3.3 implies that $\bar{x} \in X \setminus \{\mathbf{0}\}$ is a solution of the CIVVI (2.1) if and only if for any $l \in C(\bar{x})^*$, $l^\top f(\bar{x}) \leq \inf_{x\in X} l^\top f(x)$ is satisfied.

3.2. **Theorems of the Alternative Based on Separation Functions.** This section is devoted to introducing and investigating a nonlinear function, which is a weak separation function and regular weak separation function under different parameter sets. We will apply this function to establish theorems of the weak alternative for CIVVI (2.1). Furthermore, by using theorems of the weak alternative, sufficient and necessary optimality conditions for CIVVI (2.1) will be provided in the image space.

First, we recall the following basic separation notions.

Definition 3.1. [7] Let $\bar{x} \in X \setminus \{\mathbf{0}\}$ and let Π be a set of parameters. The class of all functions $\omega : \mathbb{R}^{n+2m} \times \Pi \to \mathbb{R} \cup \{\pm\infty\}$ such that

(1) (i) $\bigcap_{\pi\in\Pi}\operatorname{lev}_{\geq 0}\omega(\cdot;\pi)\supseteq\mathscr{H}(\bar{x})$ and
(ii) $\bigcap_{\pi\in\Pi}\operatorname{lev}_{>0}\omega(\cdot;\pi)\subseteq\mathscr{H}(\bar{x})$
is called the class of *weak separation functions* and is denoted by $\mathscr{W}(\Pi)$.

(2) $\bigcap_{\pi\in\Pi}\operatorname{lev}_{>0}\omega(\cdot;\pi)=\mathscr{H}(\bar{x})$, is called the class of *regular weak separation functions* and is denoted by $\mathscr{W}_R(\Pi)$.

Notice that the left-hand side of Definition 3.1 1. (ii), may be empty; usually, this inclusion would state an equality.

In this section, we always assume that $\bar{x}\in X\setminus\{\mathbf{0}\}$. Let the *indicator function* $\delta:\mathbb{R}^m\times\mathbb{R}^m\to\{\mathbf{0}\}\cup\{+\infty\}$ be defined as

$$\delta_{D\times E}(v,z):=\begin{cases}0 & \text{if } v\in D \text{ and } z\in E,\\ +\infty & \text{otherwise}\end{cases}$$

for arbitrary sets $D,E\subseteq\mathbb{R}^m$. We consider the nonlinear scalarizing functional $z^{D,k}$, defined by (2.15), with $D:=-C(\bar{x})$ and $k\in -C(\bar{x})\setminus\{\mathbf{0}\}$, where $C(\bar{x})$ is defined in (2.2). Now we introduce the following nonlinear function $\omega:\mathbb{R}^{n+2m}\times\mathbb{R}_+\to\mathbb{R}\cup\{\pm\infty\}$:

$$\omega(\xi,v,z;\lambda):=\lambda\, z^{-C(\bar{x}),k}(\xi)-\delta_{D\times D}(v,z). \tag{3.1}$$

In the following proposition, we show that the function ω defined by (3.1) is a weak separation function and regular weak separation function under different parameter sets, respectively.

Proposition 3.1. *Consider the function ω be defined by (3.1). Then the following assertions hold true:*

(i) *$\omega(\cdot,\cdot,\cdot;\cdot)\in\mathscr{W}(\Pi)$, where $\Pi=\mathbb{R}_+$;*
(ii) *If the functional $z^{-C(\bar{x}),k}$ is finite-valued, then $\omega(\cdot,\cdot,\cdot;\cdot)\in\mathscr{W}_R(\Pi)$, where $\Pi=\mathbb{R}_{++}$.*

Proof. (i) Let $(\xi,v,z)\in\mathscr{H}(\bar{x})=(\mathbb{R}^n\setminus C(\bar{x}))\times D\times D$. Then $\xi\notin C(\bar{x})$, which is equivalent to $z^{-C(\bar{x}),k}(\xi)>0$ by Theorem 2.1 (f). Together with $\delta_{D\times D}(v,z)=0$, we obtain the assertion, which implies that

$$\bigcap_{\lambda\in\mathbb{R}_+}\operatorname{lev}_{\geq 0}\omega(\cdot,\cdot,\cdot;\lambda)\supseteq\mathscr{H}(\bar{x}).$$

Since $\bigcap_{\lambda\in\mathbb{R}_+}\operatorname{lev}_{>0}\omega(\cdot,\cdot,\cdot;\lambda)=\emptyset$, the inclusion $\bigcap_{\lambda\in\mathbb{R}_+}\operatorname{lev}_{>0}\omega(\cdot,\cdot,\cdot;\lambda)\subseteq\mathscr{H}(\bar{x})$ is automatically fulfilled. Therefore, $\omega(\cdot,\cdot,\cdot;\cdot)\in\mathscr{W}(\Pi)$ for $\Pi=\mathbb{R}_+$.

(ii) The first part of the proof is similar to that of (i) and so it is omitted. Now we show

$$\bigcap_{\lambda\in\mathbb{R}_{++}}\operatorname{lev}_{>0}\omega(\cdot,\cdot,\cdot;\lambda)\subseteq\mathscr{H}(\bar{x}).$$

Suppose that this inclusion is not true. Then there exists some $(\tilde{\xi},\tilde{v},\tilde{z})\notin\mathscr{H}(\bar{x})$ such that

$$\omega(\tilde{\xi},\tilde{v},\tilde{z};\lambda)>0,\ \forall\lambda\in\mathbb{R}_{++}. \tag{3.2}$$

Note that $(\tilde{\xi},\tilde{v},\tilde{z})\notin\mathscr{H}(\bar{x})\ \Leftrightarrow\ \tilde{\xi}\notin\mathbb{R}^n\setminus C(\bar{x})$ or, $(\tilde{v},\tilde{z})\notin D\times D$. Now we split the rest of the proof into two cases:

Case 1. If $\tilde{\xi} \notin \mathbb{R}^n \setminus C(\bar{x})$, then $\tilde{\xi} \in C(\bar{x})$. By Theorem 2.1 (f), we get $z^{-C(\bar{x}),k}(\tilde{\xi}) \leq 0$. Then, for $\lambda = 1$, one has

$$\omega(\tilde{\xi}, \tilde{v}, \tilde{z}; \lambda) = \lambda\, z^{-C(\bar{x}),k}(\tilde{\xi}) - \delta_{D\times D}(\tilde{v}, \tilde{z}) \leq z^{-C(\bar{x}),k}(\tilde{\xi}) \leq 0,$$

which contradicts (3.2).

Case 2. If $(\tilde{v}, \tilde{z}) \notin D \times D$, then $\delta_{D\times D}(\tilde{v}, \tilde{z}) = +\infty$. Then, for any $\lambda \in \mathbb{R}_{++}$,

$$\omega(\tilde{\xi}, \tilde{v}, \tilde{z}; \lambda) = \lambda\, z^{-C(\bar{x}),k}(\tilde{\xi}) - \delta_{D\times D}(\tilde{v}, \tilde{z}) = -\infty,$$

by Theorem 2.1 (f) and because the functional $z^{-C(\bar{x}),k}$ is finite-valued. But this is a contradiction to (3.2). Consequently, the desired result follows from Cases 1 and 2. □

Remark 3.1. Recall from Theorem 2.1 (h) that the functional $z^{-C(\bar{x}),k}$ is finite-valued if and only if $-C(\bar{x})$ does not contain lines parallel to k and $\mathbb{R}k + C(\bar{x}) = \mathbb{R}^n$. According to Remark 2.1, $z^{-C(\bar{x}),k}$ is finite-valued if $k \in -\operatorname{int} C(\bar{x})$.

A first theorem of the alternative is presented below.

Theorem 3.1. *The system* (2.12) *in the unknown x, and the system:*

$$\exists \bar{\lambda} \geq 0 \text{ s.t. } \omega(\mathscr{A}_{\bar{x}}(x); \bar{\lambda}) < 0,\ \forall x \in \mathbb{R}^n, \tag{3.3}$$

are not simultaneously possible.

Proof. Assume that the system (2.12) in the unknown x is possible. Then there exists $\hat{x} \in \mathbb{R}^n$ such that $f(\hat{x}) - f(\bar{x}) \notin C(\bar{x}),\ g(\hat{x}) \in D$, and $g(\bar{x}) \in D$. By the definition of $\mathscr{H}(\bar{x})$, given in (2.10), this means that

$$\mathscr{A}_{\bar{x}}(\hat{x}) = (f(\hat{x}) - f(\bar{x}),\, g(\hat{x}),\, g(\bar{x})) \in \mathscr{H}(\bar{x}).$$

It follows from Proposition 3.1 (i) that for every $\bar{\lambda} \geq 0$

$$\omega(\mathscr{A}_{\bar{x}}(\hat{x}); \bar{\lambda}) \geq 0.$$

Therefore, the system (3.3) is impossible.

Conversely, assume that the system (3.3) is possible. Then there exists some $\bar{\lambda} \geq 0$ such that

$$\mathscr{K}(\bar{x}) \subseteq \left\{(\xi, v, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m : \omega(\xi, v, z; \bar{\lambda}) < 0\right\},$$

where $\mathscr{K}(\bar{x})$ is defined by (2.11). This implies that

$$\mathscr{K}(\bar{x}) \bigcap \left\{(\xi, v, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m : \omega(\xi, v, z; \bar{\lambda}) \geq 0\right\} = \emptyset.$$

Proposition 3.1 (i) implies that $\mathscr{H}(\bar{x}) \subseteq \operatorname{lev}_{\geq 0} \omega(\cdot, \cdot, \cdot; \bar{\lambda})$. Moreover, it holds

$$\mathscr{A}_{\bar{x}}(x) \notin \mathscr{H}(\bar{x}),\ \forall x \in \mathbb{R}^n.$$

Therefore, the system (2.12) in the unknown x is impossible. □

The following result is established in a similar manner from Proposition 3.1 (ii).

Theorem 3.2. *The system* (2.12) *in the unknown x, and the system:*

$$\exists \bar{\lambda} > 0 \text{ s.t. } \omega(\mathscr{A}_{\bar{x}}(x); \bar{\lambda}) \leq 0,\ \forall x \in \mathbb{R}^n,$$

are not simultaneously possible.

The nonlinear separation approach will now be used to propose the following sufficient optimality conditions for CIVVI (2.1) in the image space as a consequence of Theorems 3.1 and 3.2.

Corollary 3.1. *Assume that any one of the following conditions holds:*

(i) *there exists* $\bar{\lambda} \geq 0$ *such that*

$$\omega(\mathscr{A}_{\bar{x}}(x); \bar{\lambda}) < 0,\ \forall x \in \mathbb{R}^n; \tag{3.4}$$

(ii) *there exists* $\bar{\lambda} > 0$ *such that*

$$\omega(\mathscr{A}_{\bar{x}}(x); \bar{\lambda}) \leq 0,\ \forall x \in \mathbb{R}^n. \tag{3.5}$$

Then $\bar{x}$ *is a nonzero solution of CIVVI (2.1).*

Proof. The assertions follow directly from Theorems 3.1 and 3.2 and the definition of $\bar{x}$ being a nonzero solution of CIVVI (2.1). □

We now present a necessary optimality condition for CIVVI (2.1).

Theorem 3.3. *If* $\bar{x}$ *is a nonzero solution of CIVVI (2.1), then there exists* $\bar{\lambda} \geq 0$ *such that*

$$\omega(\mathscr{A}_{\bar{x}}(x); \bar{\lambda}) \leq 0,\ \forall x \in \mathbb{R}^n. \tag{3.6}$$

Proof. Let $\bar{x}$ be a nonzero solution of CIVVI (2.1). Then $g(\bar{x}) \in D$ and the system (2.12) is impossible. Therefore, for each $x \in \mathbb{R}^n$, $f(x) - f(\bar{x}) \in C(\bar{x})$ or, $g(x) \notin D$.

If $f(x) - f(\bar{x}) \in C(\bar{x})$, then, by Theorem 2.1 (f), $z^{-C(\bar{x}),k}(f(x) - f(\bar{x})) \leq 0$, and therefore

$$\forall \lambda \in \mathbb{R}_+ :\ \lambda z^{-C(\bar{x}),k}(f(x) - f(\bar{x})) \leq 0.$$

Furthermore, it holds for all $\lambda \in \mathbb{R}_+$

$$\begin{aligned} \omega(\mathscr{A}_{\bar{x}}(x); \lambda) &= \lambda z^{-C(\bar{x}),k}(f(x) - f(\bar{x})) - \delta_{D \times D}(g(x), g(\bar{x})) \\ &\leq \lambda z^{-C(\bar{x}),k}(f(x) - f(\bar{x})) \\ &\leq 0. \end{aligned}$$

If $g(x) \notin D$, $-\delta_{D \times D}(g(x), g(\bar{x})) = -\infty$. Taking $\bar{\lambda} = 0$, we have

$$\begin{aligned} \omega(\mathscr{A}_{\bar{x}}(x); \bar{\lambda}) &= \bar{\lambda} z^{-C(\bar{x}),k}(f(x) - f(\bar{x})) - \delta_{D \times D}(g(x), g(\bar{x})) \\ &= -\delta_{D \times D}(g(x), g(\bar{x})) \\ &= -\infty. \end{aligned}$$

This means that there exists $\bar{\lambda} \geq 0$ such that (3.6) holds true. □

In the following definition, we present the nonlinear separation notion for the sets $\mathscr{K}(\bar{x})$ and $\mathscr{H}(\bar{x})$, given by (2.11) and (2.10), via the function ω defined by (3.1). This definition is motivated by the nonlinear separation notion given in [7].

Definition 3.2. The sets $\mathscr{K}(\bar{x})$ and $\mathscr{H}(\bar{x})$ admit a *regular nonlinear separation* if there exists $\bar{\lambda} > 0$ such that

$$\omega(\mathscr{A}_{\bar{x}}(x);\bar{\lambda}) \leq 0,\ \forall x \in \mathbb{R}^n.$$

The next result easily follows from Definition 3.2 and Corollary 3.1 (ii).

Corollary 3.2. *If the sets $\mathscr{K}(\bar{x})$ and $\mathscr{H}(\bar{x})$ admit a regular nonlinear separation, then $\bar{x}$ is a nonzero solution of CIVVI* (2.1).

We next give a saddle point conditions for the regular nonlinear separation between the sets $\mathscr{K}(\bar{x})$ and $\mathscr{H}(\bar{x})$.

Theorem 3.4. *$\mathscr{K}(\bar{x})$ and $\mathscr{H}(\bar{x})$ admit a regular nonlinear separation if and only if there exists $\bar{\lambda} > 0$ such that $(\bar{x};\bar{\lambda})$ is a saddle point of $\omega(\mathscr{A}_{\bar{x}}(\cdot);\cdot)$ on $\mathbb{R}^n \times \mathbb{R}_+$, i.e.,*

$$\omega(\mathscr{A}_{\bar{x}}(\bar{x});\lambda) \geq \omega(\mathscr{A}_{\bar{x}}(\bar{x});\bar{\lambda}) \geq \omega(\mathscr{A}_{\bar{x}}(x);\bar{\lambda}),\ \forall x \in \mathbb{R}^n,\ \lambda \in \mathbb{R}_+.$$

Proof. Assume that $\mathscr{K}(\bar{x})$ and $\mathscr{H}(\bar{x})$ admit a regular nonlinear separation. Then there exists $\bar{\lambda} > 0$ such that

$$\omega(\mathscr{A}_{\bar{x}}(x);\bar{\lambda}) \leq 0,\ \forall x \in \mathbb{R}^n. \tag{3.7}$$

It follows by Corollary 3.1 (ii) that $\bar{x}$ is a nonzero solution of CIVVI (2.1). Therefore, $g(\bar{x}) \in D$. Furthermore, we have for all $\lambda \in \mathbb{R}_+$

$$\begin{aligned}\omega(\mathscr{A}_{\bar{x}}(\bar{x});\bar{\lambda}) &= \bar{\lambda}\, z^{-C(\bar{x}),k}(f(\bar{x}) - f(\bar{x})) - \delta_{D\times D}(g(\bar{x}), g(\bar{x}))\\ &= 0 = \omega(\mathscr{A}_{\bar{x}}(\bar{x});\lambda).\end{aligned}$$

Combining this equality with (3.7) implies that $(\bar{x};\bar{\lambda})$ is a saddle point of $\omega(\mathscr{A}_{\bar{x}}(\cdot);\cdot)$ on $\mathbb{R}^n \times \mathbb{R}_+$.

Conversely, assume that there exists $\bar{\lambda} > 0$ such that $(\bar{x};\bar{\lambda})$ is a saddle point of $\omega(\mathscr{A}_{\bar{x}}(\cdot);\cdot)$ on $\mathbb{R}^n \times \mathbb{R}_+$. Then

$$\omega(\mathscr{A}_{\bar{x}}(\bar{x});\bar{\lambda}) \geq \omega(\mathscr{A}_{\bar{x}}(x);\bar{\lambda}),\ \forall x \in \mathbb{R}^n. \tag{3.8}$$

Suppose by contradiction that there exists $\hat{x} \in \mathbb{R}^n$ such that

$$\omega(\mathscr{A}_{\bar{x}}(\hat{x});\bar{\lambda}) > 0.$$

Consequently, one has

$$0 < \omega(\mathscr{A}_{\bar{x}}(\hat{x});\bar{\lambda}) \overset{(3.8)}{\leq} \omega(\mathscr{A}_{\bar{x}}(\bar{x});\bar{\lambda}) = -\delta_{D\times D}(g(\bar{x}), g(\bar{x})),$$

i.e., $\delta_{D\times D}(g(\bar{x}), g(\bar{x})) < 0$, which is a contradiction. Therefore, $\mathscr{K}(\bar{x})$ and $\mathscr{H}(\bar{x})$ admit a regular nonlinear separation. □

The following result is a direct consequence of Corollary 3.2 and Theorem 3.4.

Corollary 3.3. *If $\bar{\lambda} > 0$ and $(\bar{x};\bar{\lambda})$ is a saddle point of $\omega(\mathscr{A}_{\bar{x}}(\cdot);\cdot)$ on $\mathbb{R}^n \times \mathbb{R}_+$, then $\bar{x}$ is a nonzero solution of CIVVI* (2.1).

We next illustrate that the separation theorem for not necessarily convex sets in [11] cannot be applied to solve CIVVI (2.1).

Remark 3.2. By using the Gerth (Tammer) type nonlinear scalarization function, a separation theorem [11, Theorem 2.3.6, page 44] is established for two not necessarily convex sets. It is worth mentioning that this separation theorem [11, Theorem 2.3.6, page 44] cannot be applied to solve CIVVI (2.1). The reason is that the cone D in this paper does not require $\operatorname{int} D \neq \emptyset$. This means that it is possible that $\operatorname{int}\mathscr{H}(\bar{x}) = \operatorname{int}[(\mathbb{R}^n \setminus C(\bar{x})) \times D \times D] = \emptyset$. On the other hand, even if we further add additional assumptions on the parameters involved in the nonconvex separation theorem [11, Theorem 2.3.6, page 44], we can only deduce that the separation function $z_{\mathrm{cl}(\mathscr{H}(\bar{x})),k^0}$ defined by (2.33) in [11, page 39] is a weak separation function by [11, Theorem 2.3.1 and 2.3.6, pages 40,44], rather than a regular weak separation function, where $\mathrm{cl}(\mathscr{H}(\bar{x}))$ denotes the closure of $\mathscr{H}(\bar{x})$.

4. Conclusions

This work examines constrained inverse vector variational inequalities using image space analysis via a prominent nonlinear scalarizing functional. Our results leave various avenues for future research. Refining our results, it would be very valuable to obtain existence results for constrained inverse vector variational inequalities based on appropriate scalarization. Moreover, it would be interesting to apply the concepts to real-world rather than academic problems and derive corresponding algorithms for solving such inequalities.

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