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COINCIDENCE POINT AND COMMON FIXED POINT THEOREMS IN THE PRODUCT SPACES OF QUASI-ORDERED METRIC SPACES

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Abstract. The main aim of this paper is to study and establish some new coincidence point and common fixed point theorems in the product spaces of complete quasi-ordered metric spaces. The fixed point theorems in the product spaces will be the special case of coincidence point theorems in the product spaces. We also show that the concept of fixed point theorems in the product spaces extends the concept of coupled fixed point theorems.

Keywords. Function of contractive factor; Coincidence point; Common fixed point; Coupled fixed point; Quasi-ordered metric space.

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1. Introduction

The existence of coincidence points have been studied in [7, 13, 14, 15, 18, 27, 28] and the references therein. Also, the existence of common fixed points have been studied in [2, 3, 5, 6, 8, 9, 10, 11, 19, 21, 25] and the references therein. In this paper, we shall establish some new coincidence point and common fixed point theorems in the product spaces of so-called monotonically and mixed-monotonically complete quasi-ordered metric spaces. The coincidence point and common fixed point theorems in the product spaces have also been studied in Wu [29, 30]. In this paper, we consider the more succinct contractive inequalities that are different from Wu [29, 30] to study the coincidence point and common fixed point theorems in the product spaces. We also present the interesting application to the coupled fixed points.

The interesting concept of coupled fixed points was proposed by Bhaskar and Lakshmikantham [4]. In this paper, we are going to show that the coupled fixed point can be obtained from the fixed point in the product spaces of complete quasi-ordered metric spaces. In other words, the concept of fixed point in the product spaces extends the concept of coupled fixed points.

In Section 2, we separately derive the coincidence point theorems in the product spaces of mixed-monotonically complete quasi-ordered metric spaces and in the product spaces of monotonically complete quasi-ordered metric spaces. We also obtain the coincidence point theorems of functions having

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mixed-monotone property in the product space of monotonically complete quasi-ordered metric spaces, and the fixed point theorems of functions having comparable property in the product space of mixed-monotonically complete quasi-ordered metric spaces. In Section 3, we show that the coupled fixed point theorems can be obtained from the fixed point theorems in the product spaces of complete quasi-ordered metric spaces.

2. Main Results

Let X be a nonempty set. We consider the product set

$$X^m = \underbrace{X \times \cdots \times X}_{m \text{ times}}.$$

The element of X^m is represented by the vectorial notation $\mathbf{x} = (x^{(1)}, \cdots, x^{(m)})$, where $x^{(i)} \in X$ for $i = 1, \cdots, m$. We also consider the function $\mathbf{F} : X^m \to X^m$ defined by

$$\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), \cdots, F_m(\mathbf{x})),$$

where $F_k : X^m \to X$ for all $k = 1, 2, \cdots, m$. The vectorial element $\widehat{\mathbf{x}} = (\widehat{x}^{(1)}, \widehat{x}^{(2)}, \cdots, \widehat{x}^{(m)}) \in X^m$ is a *fixed point* of $\mathbf{F}$ if and only if $\mathbf{F}(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}$; that is,

$$F_k\left(\widehat{x}^{(1)}, \widehat{x}^{(2)}, \cdots, \widehat{x}^{(m)}\right) = \widehat{x}^{(k)}$$

for all $k = 1, 2, \cdots, m$.

Definition 2.1. For metric space (X, d), we consider a product metric space $(X^m, \mathfrak{d})$ in which the metric $\mathfrak{d}$ is induced by the original metric d.

- We say that metrics $\mathfrak{d}$ and d are compatible in the sense of *preserving convergence* if and only if, given a sequence $\{\mathbf{x}_n\}_{n \in \mathbb{N}}$ in X^m, the following statement holds:

$$\mathfrak{d}(\mathbf{x}_n, \widehat{\mathbf{x}}) \to 0 \text{ if and only if } d\left(x_n^{(k)}, \widehat{x}^{(k)}\right) \to 0 \text{ for all } k = 1, \cdots, m.$$

- We say that metrics $\mathfrak{d}$ and d are compatible in the sense of *preserving continuity* if and only if, given any $\varepsilon > 0$, there exists a positive constant $\mathfrak{k} > 0$ (which depends on ε) such that the following statement holds:

$$\mathfrak{d}(\mathbf{x}, \mathbf{y}) < \varepsilon \text{ if and only if } d\left(x^{(k)}, y^{(k)}\right) < \mathfrak{k} \cdot \varepsilon \text{ for all } k = 1, \cdots, m.$$

Definition 2.2. Let $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ and $\mathbf{f} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ be functions defined on $(X^m, \mathfrak{d})$ into itself. We say that $\mathbf{F}$ is *continuous with respect to* $\mathbf{f}$ at $\widehat{\mathbf{x}} \in X^m$ if and only if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathfrak{d}(\widehat{\mathbf{x}}, \mathbf{f}(\mathbf{x})) < \delta$ for $\mathbf{x} \in X^m$ implies $\mathfrak{d}(\mathbf{F}(\widehat{\mathbf{x}}), \mathbf{F}(\mathbf{x})) < \varepsilon$. We say that $\mathbf{F}$ is continuous with respect to $\mathbf{f}$ on X^m if and only if it is continuous with respect to $\mathbf{f}$ at each $\widehat{\mathbf{x}} \in X^m$.

It is obvious that if function $\mathbf{F}$ is continuous at $\widehat{\mathbf{x}}$ with respect to the identity function, then it is also continuous at $\widehat{\mathbf{x}}$.

2.1. **Quasi-Ordered Metric Spaces.** Let "$\preceq$" be a binary relation defined on X. We say that the binary relation "$\preceq$" is a quasi-order (pre-order or pseudo-order) if and only if it is reflexive and transitive. In this case, $(X,\preceq)$ is called a *quasi-ordered set*.

For any $\mathbf{x},\mathbf{y} \in X^m$, we say that $\mathbf{x}$ and $\mathbf{y}$ are $\preceq$*-mixed comparable* if and only if, for each $k = 1,\cdots,m$, one has either $x^{(k)} \preceq y^{(k)}$ or $y^{(k)} \preceq x^{(k)}$. Let I be a subset of $\{1,2,\cdots,m\}$ and $J = \{1,2,\cdots,m\} \setminus I$. In this case, we say that I and J are the *disjoint pair* of $\{1,2,\cdots,m\}$. We can define a binary relation on X^m as follows:

$$\mathbf{x} \preccurlyeq_I \mathbf{y} \text{ if and only if } x^{(k)} \preceq y^{(k)} \text{ for } k \in I \text{ and } y^{(k)} \preceq x^{(k)} \text{ for } k \in J. \tag{2.1}$$

It is obvious that $(X^m,\preccurlyeq_I)$ is a quasi-ordered set that depends on I. We also have

$$\mathbf{x} \preccurlyeq_I \mathbf{y} \text{ if and only if } \mathbf{y} \preccurlyeq_J \mathbf{x}, \tag{2.2}$$

where I or J is allowed to be an empty set.

Let I and J be a disjoint pair of $\{1,2,\cdots,m\}$. Given a quasi-ordered set $(X,\preceq)$, we also consider the quasi-ordered set $(X^m,\preccurlyeq_I)$ defined in (2.1).

- The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be a *mixed* $\preceq$*-monotone sequence* if and only if $x_n \preceq x_{n+1}$ or $x_{n+1} \preceq x_n$ (i.e., x_n and x_{n+1} are comparable with respect to "$\preceq$") for all $n \in \mathbb{N}$.
- The sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m is said to be a *mixed* $\preceq$*-monotone sequence* if and only if each sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ in X is a mixed $\preceq$-monotone sequence for all $k = 1,\cdots,m$.
- The sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m is said to be a *mixed* $\preccurlyeq_I$*-monotone sequence* if and only if $\mathbf{x}_n \preccurlyeq_I \mathbf{x}_{n+1}$ or $\mathbf{x}_{n+1} \preccurlyeq_I \mathbf{x}_n$ (i.e., $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ are comparable with respect to "$\preccurlyeq_I$") for all $n \in \mathbb{N}$.

Definition 2.3. Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order "$\preceq$". We say that $(X,d,\preceq)$ is *mixed-monotonically complete* if and only if each mixed $\preceq$-monotone Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is convergent.

It is obvious that if the quasi-ordered metric space $(X,d,\preceq)$ is complete, then it is also mixed-monotonically complete. However, the converse is not necessarily true. Next we are going to weaken the concept of mixed-monotone completeness for the quasi-ordered metric space.

We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X,\preceq)$ is $\preceq$-increasing if and only if $x_k \preceq x_{k+1}$ for all $k \in \mathbb{N}$. The concept of $\preceq$-decreasing sequence can be similarly defined. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in $(X,\preceq)$ is called $\preceq$*-monotone* if and only if $\{x_n\}_{n\in\mathbb{N}}$ is either $\preceq$-increasing or $\preceq$-decreasing.

Definition 2.4. Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order "$\preceq$". We say that $(X,d,\preceq)$ is *monotonically complete* if and only if each $\preceq$-monotone Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is convergent.

It is obvious that if $(X,d,\preceq)$ is a mixed-monotonically complete quasi-ordered metric space, then it is also a monotonically complete quasi-ordered metric space. However, the converse is not true. In other words, the concept of monotone completeness is weaker than that of mixed-monotone completeness.

Let I and J be a disjoint pair of $\{1,2,\cdots,m\}$. We say that the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in $(X^m,\preccurlyeq_I)$ is $\preccurlyeq_I$-increasing if and only if $\mathbf{x}_n \preccurlyeq_I \mathbf{x}_{n+1}$ for all $n \in \mathbb{N}$. The concept of $\preccurlyeq_I$-decreasing sequence can be similarly defined. The sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in $(X^m,\preccurlyeq_I)$ is called $\preccurlyeq_I$*-monotone* if and only if $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is either $\preccurlyeq_I$-increasing or $\preccurlyeq_I$-decreasing. Let $\mathbf{f} : (X^m, \preccurlyeq_I) \to (X^m, \preccurlyeq_I)$ be a function defined on $(X^m, \preccurlyeq_I)$ into itself.

- The function $\mathbf{f}$ is said to have the *sequentially mixed $\preceq$-monotone property* if and only if, given any mixed $\preceq$-monotone sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m, $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is also a mixed $\preceq$-monotone sequence.
- The function $\mathbf{f}$ is said to have the *sequentially mixed $\preccurlyeq_I$-monotone property* if and only if, given any mixed $\preccurlyeq_I$-monotone sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m, $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is also a mixed $\preccurlyeq_I$-monotone sequence.
- We say that $\mathbf{f}$ is $\preccurlyeq_I$-increasing if and only if $\mathbf{x} \preccurlyeq_I \mathbf{y}$ implies $\mathbf{f}(\mathbf{x}) \preccurlyeq_I \mathbf{f}(\mathbf{y})$. The concept of $\preccurlyeq_I$-decreasing function can be similarly defined.
- The function $\mathbf{f}$ is called *$\preccurlyeq_I$-monotone* if and only if $\mathbf{f}$ is either $\preccurlyeq_I$-increasing or $\preccurlyeq_I$-decreasing.

It is obvious that the identity function on X^m has the sequentially mixed $\preccurlyeq_I$-monotone and $\preceq$-monotone property.

Let X be a nonempty set. We consider the functions $\mathbf{F} : X^m \to X^m$ and $\mathbf{f} : X^m \to X^m$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$, where $\mathbf{F}^p(\mathbf{x}) = \mathbf{F}(\mathbf{F}^{p-1}(\mathbf{x}))$ for any $\mathbf{x} \in X^m$. Therefore, we have $F_k^p(\mathbf{x}) = F_k(\mathbf{F}^{p-1}(\mathbf{x}))$ for $k = 1, \cdots, m$. Given an initial element $\mathbf{x}_0 = (x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(m)}) \in X^m$, where $x_0^{(k)} \in X$ for $k = 1, \cdots, m$, since $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$, there exists $\mathbf{x}_1 \in X^m$ such that $\mathbf{f}(\mathbf{x}_1) = \mathbf{F}^p(\mathbf{x}_0)$. Similarly, there also exists $\mathbf{x}_2 \in X^m$ such that $\mathbf{f}(\mathbf{x}_2) = \mathbf{F}^p(\mathbf{x}_1)$. Continuing this process, we can construct a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ such that

$$\mathbf{f}(\mathbf{x}_n) = \mathbf{F}^p(\mathbf{x}_{n-1}) \tag{2.3}$$

for all $n \in \mathbb{N}$; that is,

$$f_k(\mathbf{x}_n) = f_k\left(x_n^{(1)}, \cdots, x_n^{(k)}, \cdots, x_n^{(m)}\right) = F_k^p\left(x_{n-1}^{(1)}, \cdots, x_{n-1}^{(k)}, \cdots, x_{n-1}^{(m)}\right) = F_k^p(\mathbf{x}_{n-1})$$

for all $k = 1, \cdots, m$. We introduce the concepts of seed element as follows.

- We say that the initial element $\mathbf{x}_0$ is a *mixed $\preceq$-monotone seed element* of X^m if and only if the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (2.3) is a mixed $\preceq$-monotone sequence; that is, each sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ in X is a mixed $\preceq$-monotone sequence for $k = 1, \cdots, m$.
- Given a disjoint pair I and J of $\{1, 2, \cdots, m\}$, we say that the initial element $\mathbf{x}_0$ is a *mixed $\preccurlyeq_I$-monotone seed element* of X^m if and only if the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (2.3) is a mixed $\preccurlyeq_I$-monotone sequence.
- Given a disjoint pair I and J of $\{1, 2, \cdots, m\}$, we say that the initial element $\mathbf{x}_0$ is a *$\preccurlyeq_I$-monotone seed element* of X^m if and only if the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (2.3) is a $\preccurlyeq_I$-monotone sequence.

From observation (b) of Wu [29, Remark 2.2], it follows that if $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element, then it is a mixed $\preceq$-monotone seed element. It is also obvious that if $\mathbf{x}_0$ is a $\preccurlyeq_I$-monotone seed element, then it is a mixed $\preccurlyeq_I$-monotone seed element.

Definition 2.5. We say that $\varphi : [0,\infty) \to [0,1)$ is a *function of contractive factor* if and only if, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, we have

$$0 \le \sup_n \varphi(x_n) < 1 \tag{2.4}$$

Definition 2.6. Let X be a nonempty set. Consider the functions $\mathbf{F} : X^m \to X^m$ and $\mathbf{f} : X^m \to X^m$ by $\mathbf{F} = (F_1, F_2, \cdots, F_k)$ and $\mathbf{f} = (f_1, f_2, \cdots, f_k)$, where $F_k : X^m \to X$ and $f_k : X^m \to X$ for $k = 1,2,\cdots,m$.

- The element $\widehat{\mathbf{x}} \in X^m$ is a *coincidence point* of $\mathbf{F}$ and $\mathbf{f}$ if and only if $\mathbf{F}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}})$, i.e., $F_k(\widehat{\mathbf{x}}) = f_k(\widehat{\mathbf{x}})$ for all $k = 1,2,\cdots,m$.
- The element $\widehat{\mathbf{x}}$ is a *common fixed point* of $\mathbf{F}$ and $\mathbf{f}$ if and only if $\mathbf{F}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}$, i.e., $F_k(\widehat{\mathbf{x}}) = f_k(\widehat{\mathbf{x}}) = \widehat{x}^{(k)}$ for all $k = 1,2,\cdots,m$.
- The functions $\mathbf{F}$ and $\mathbf{f}$ are said to be *commutative* if and only if $\mathbf{f}(\mathbf{F}(\mathbf{x})) = \mathbf{F}(\mathbf{f}(\mathbf{x}))$ for all $\mathbf{x} \in X^m$.

2.2. **Fixed Point Theorems.** Now we are in a position to present the fixed point theorems in the product spaces. We shall separately study the fixed point theorems in the mixed-monotonically complete quasi-ordered metric space and monotonically complete quasi-ordered metric space.

Theorem 2.1. (**Mixed-Monotone Completeness**). *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is mixed-monotonically complete, and that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving continuity. Consider the functions $\mathbf{F} : (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ and $\mathbf{f} : (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preceq$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions:*

- *$\mathbf{F}$ and $\mathbf{f}$ are commutative;*
- *$\mathbf{f}$ has the sequentially mixed $\preceq$-monotone property;*
- *$\mathbf{F}^p$ is continuous with respect to $\mathbf{f}$ on X^m;*
- *each f_k is continuous on X^m for $k = 1,\cdots,m$.*

Suppose that there exists a function of contractive factor $\varphi : [0,\infty) \to [0,1)$ such that, for any two $\preceq$-mixed comparable elements $\mathbf{x}$ and $\mathbf{y}$ in X^m, the following inequalities

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \le \varphi\left(d\left(f_k(\mathbf{x}), f_k(\mathbf{y})\right)\right) \cdot d\left(f_k(\mathbf{x}), f_k(\mathbf{y})\right) \tag{2.5}$$

are satisfied for all $k = 1,\cdots,m$. Then, $\mathbf{F}^p$ has a fixed point $\widehat{\mathbf{x}}$ such that each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed from (2.3) for all $k = 1,\cdots,m$.

Proof. We consider the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (2.3). Since $\mathbf{x}_0$ is a mixed $\preceq$-monotone seed element in X^m, i.e., $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone sequence, from observation (d) of Wu [29, Remark 2.2], it follows that, for each $n \in \mathbb{N}$, $\mathbf{x}_n$ and $\mathbf{x}_{n+1}$ are $\preceq$-mixed comparable. According to the inequalities (2.5), we obtain

$$\begin{aligned} d\left(f_k(\mathbf{x}_{n+1}), f_k(\mathbf{x}_n)\right) &= d\left(F_k^p(\mathbf{x}_n), F_k^p(\mathbf{x}_{n-1})\right) \\ &\le \varphi\left(d\left(f_k(\mathbf{x}_n), f_k(\mathbf{x}_{n-1})\right)\right) \cdot d\left(f_k(\mathbf{x}_n), f_k(\mathbf{x}_{n-1})\right) \\ &< d\left(f_k(\mathbf{x}_n), f_k(\mathbf{x}_{n-1})\right) \end{aligned} \tag{2.6}$$

Let

$$\xi_n = d\left(f_k\left(\mathbf{x}_n\right), f_k\left(\mathbf{x}_{n-1}\right)\right).$$

Then, from (2.6), we see that the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is strictly decreasing and satisfies

$$\xi_{n+1} \le \varphi(\xi_n)\cdot\xi_n. \tag{2.7}$$

Let

$$0 < \gamma = \sup_n \varphi(\xi_n) < 1. \tag{2.8}$$

From (2.7) and (2.8), we obtain $\xi_{n+1} \le \gamma\cdot\xi_n$, which implies

$$d\left(f_k\left(\mathbf{x}_{n+1}\right), f_k\left(\mathbf{x}_n\right)\right) = \xi_{n+1} \le \gamma^n\cdot\xi_1. \tag{2.9}$$

For $n_1, n_2 \in \mathbb{N}$ with $n_1 > n_2$, since $0 < \gamma < 1$, it follows that

$$\begin{aligned} d\left(f_k\left(\mathbf{x}_{n_1}\right), f_k\left(\mathbf{x}_{n_2}\right)\right) &\le \sum_{j=n_2}^{n_1-1} d\left(f_k\left(\mathbf{x}_{j+1}\right), f_k\left(\mathbf{x}_j\right)\right) \text{ (by the triangle inequality)} \\ &\le \xi_1\cdot\sum_{j=n_2}^{n_1-1}\gamma^j \text{ (by (2.9))} \\ &\le \frac{\xi_1\cdot\gamma^{n_2}\cdot(1-\gamma^{n_1-n_2})}{1-\gamma} < \frac{\xi_1\cdot\gamma^{n_2}}{1-\gamma} \to 0 \text{ as } n_2\to\infty, \end{aligned}$$

which also says that $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X for any fixed k. Since $\mathbf{f}$ has the sequentially mixed $\preceq$-monotone property, i.e., $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone Cauchy sequence for $k = 1,\cdots,m$, by the mixed $\preceq$-monotone completeness of X, there exists $\widehat{x}^{(k)} \in X$ such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ as $n\to\infty$ for $k = 1,\cdots,m$. By Wu [30, Proposition 2.8], since the metrics $\mathfrak{d}$ and d are also compatible in the sense of preserving continuity, it follows that $\mathbf{f}(\mathbf{x}_n) \to \widehat{\mathbf{x}}$ as $n\to\infty$. Since each f_k is continuous on X^m, we also have

$$f_k\left(\mathbf{f}(\mathbf{x}_n)\right) \to f_k(\widehat{\mathbf{x}}) \text{ as } n\to\infty.$$

Since $\mathbf{F}^p$ is continuous with respect to $\mathbf{f}$ on X^m, by Wu [30, Proposition 2.12], given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathbf{x}\in X^m$ with $d(\widehat{x}^{(k)}, f_k(\mathbf{x})) < \delta$ for all $k = 1,\cdots,m$ imply

$$d\left(F_k^p(\widehat{\mathbf{x}}), F_k^p(\mathbf{x})\right) < \frac{\varepsilon}{2} \text{ for all } k = 1,\cdots,m. \tag{2.10}$$

Since $f_k(\mathbf{x}_n) \to \hat{x}^{(k)}$ as $n\to\infty$ for all $k = 1,\cdots,m$, given $\zeta = \min\{\varepsilon/2,\delta\} > 0$, there exists $n_0\in\mathbb{N}$ such that

$$d\left(f_k(\mathbf{x}_n), \hat{x}^{(k)}\right) < \zeta \le \delta \text{ for all } n\in\mathbb{N} \text{ with } n\ge n_0 \text{ and for all } k = 1,\cdots,m. \tag{2.11}$$

For each $n\ge n_0$, by (2.10) and (2.11), it follows that

$$d\left(F_k^p(\hat{\mathbf{x}}), F_k^p(\mathbf{x}_n)\right) < \frac{\varepsilon}{2} \text{ for all } k = 1,\cdots,m. \tag{2.12}$$

Therefore, we obtain

$$\begin{aligned}
d\left(F_k^p(\hat{\mathbf{x}}),\hat{x}^{(k)}\right) &\leq d\left(F_k^p(\hat{\mathbf{x}}),f_k(\mathbf{x}_{n_0+1})\right)+d\left(f_k(\mathbf{x}_{n_0+1}),\hat{x}^{(k)}\right)\\
&= d\left(F_k^p(\hat{\mathbf{x}}),F_k^p(\mathbf{x}_{n_0})\right)+d\left(f_k(\mathbf{x}_{n_0+1}),\hat{x}^{(k)}\right)\\
&< \frac{\varepsilon}{2}+\zeta \text{ (by (2.11) and (2.12))}\\
&\leq \varepsilon \text{ for all } k=1,\cdots,m.
\end{aligned}$$

Since ε is any positive number, we conclude that $d(F_k^p(\hat{\mathbf{x}}),\hat{x}^{(k)})=0$ for all $k=1,\cdots,m$, which also says that $F_k^p(\hat{\mathbf{x}})=\hat{x}^{(k)}$ for all $k=1,\cdots,m$, i.e., $\mathbf{F}^p(\widehat{\mathbf{x}})=\widehat{\mathbf{x}}$. This completes the proof. □

By taking $\mathbf{f}$ as the identity function in Theorem 2.1, we have the following interesting result.

Corollary 2.1. *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is mixed-monotonically complete, and that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving continuity. Consider the function $\mathbf{F}:(X^m,\mathfrak{d})\to(X^m,\mathfrak{d})$. Given an initial element $\mathbf{x}_0$ in X^m, assume that the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from*

$$\mathbf{x}_n=\mathbf{F}^p(\mathbf{x}_{n-1}) \text{ for some } p\in\mathbb{N} \tag{2.13}$$

is a mixed $\preceq$-monotone sequence. Suppose that there exists a function of contractive factor $\varphi:[0,\infty)\to[0,1)$ such that, for any two $\preceq$-mixed comparable elements $\mathbf{x}$ and $\mathbf{y}$ in X^m, the following inequalities

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right)\leq\varphi\left(d\left(x^{(k)},y^{(k)}\right)\right)\cdot d\left(x^{(k)},y^{(k)}\right)$$

are satisfied for all $k=1,\cdots,m$. If $\mathbf{F}^p$ is continuous on X^m, then $\mathbf{F}^p$ has a fixed point $\widehat{\mathbf{x}}$ such that each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (2.13) for all $k=1,\cdots,m$.

By considering the mixed $\preccurlyeq_I$-monotone seed element instead of mixed $\preceq$-monotone seed element, the assumption for the inequality (2.5) can be weaken, which is shown below.

Theorem 2.2. (**Mixed-Monotone Completeness**). *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is mixed-monotonically complete, and that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving continuity. Given a disjoint pair I and J of $\{1,2,\cdots,m\}$, consider the functions $\mathbf{F}:(X^m,\mathfrak{d},\preccurlyeq_I)\to(X^m,\mathfrak{d},\preccurlyeq_I)$ and $\mathbf{f}:(X^m,\mathfrak{d},\preccurlyeq_I)\to(X^m,\mathfrak{d},\preccurlyeq_I)$ satisfying $\mathbf{F}^p(X^m)\subseteq\mathbf{f}(X^m)$ for some $p\in\mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preccurlyeq_I$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions:*

- *$\mathbf{F}$ and $\mathbf{f}$ are commutative;*
- *$\mathbf{f}$ has the sequentially mixed $\preccurlyeq_I$-monotone property or the $\preceq$-monotone property;*
- *$\mathbf{F}^p$ is continuous with respect to $\mathbf{f}$ on X^m;*
- *each f_k is continuous on X^m for $k=1,\cdots,m$.*

Suppose that there exists a function of contractive factor $\varphi:[0,\infty)\to[0,1)$ such that, for any $\mathbf{x},\mathbf{y}\in X^m$ with $\mathbf{y}\preccurlyeq_I\mathbf{x}$ or $\mathbf{x}\preccurlyeq_I\mathbf{y}$, the following inequalities

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right)\leq\varphi\left(d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right)\right)\cdot d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right) \tag{2.14}$$

are satisfied for all $k = 1, \cdots, m$. Then, $\mathbf{F}^p$ has a fixed point $\widehat{\mathbf{x}}$ such that each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed from (2.3) for all $k = 1, \cdots, m$.

Proof. We consider the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (2.3). Since $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element in X^m, it follows that $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a mixed $\preccurlyeq_I$-monotone sequence, i.e., for each $n \in \mathbb{N}$,

$$\mathbf{x}_{n-1} \preccurlyeq_I \mathbf{x}_n \text{ or } \mathbf{x}_n \preccurlyeq_I \mathbf{x}_{n-1}. \tag{2.15}$$

According to (2.15) and the inequalities (2.14), we obtain

$$\begin{aligned} d\left(f_k\left(\mathbf{x}_{n+1}\right), f_k\left(\mathbf{x}_n\right)\right) &= d\left(F_k^p(\mathbf{x}_n), F_k^p(\mathbf{x}_{n-1})\right) \\ &\leq \varphi\left(d\left(f_k\left(\mathbf{x}_n\right), f_k\left(\mathbf{x}_{n-1}\right)\right)\right) \cdot d\left(f_k\left(\mathbf{x}_n\right), f_k\left(\mathbf{x}_{n-1}\right)\right) \\ &< d\left(f_k\left(\mathbf{x}_n\right), f_k\left(\mathbf{x}_{n-1}\right)\right) \end{aligned}$$

Using the argument in the proof of Theorem 2.1, we can show that $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X for any fixed k. We consider the following cases.

- Suppose that $\mathbf{f}$ has the sequentially mixed $\preceq$-monotone property. Since $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a mixed $\preccurlyeq_I$-monotone sequence, by observation (b) of Wu [29, Remark 2.2], it follows that $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ in X is a mixed $\preceq$-monotone sequence for all $k = 1, \cdots, m$. Therefore, we obtain that $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone Cauchy sequence for $k = 1, \cdots, m$.
- Suppose that $\mathbf{f}$ has the sequentially mixed $\preccurlyeq_I$-monotone property. Then, by definition, it follows that $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preccurlyeq_I$-monotone sequence, from observation (b) of Wu [29, Remark 2.2] again, we also see that $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone Cauchy sequence for $k = 1, \cdots, m$.

By the mixed $\preceq$-monotone completeness of X, there exists $\widehat{x}^{(k)} \in X$ such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ as $n \to \infty$ for $k = 1, \cdots, m$. The remaining proof follows from the same argument in the proof of Theorem 2.1. This completes the proof. □

Next we consider the fixed point theorems in the monotonically complete quasi-ordered metric space.

Theorem 2.3. (**Monotone Completeness**). *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is monotonically complete, and that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving continuity. Consider the functions $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ and $\mathbf{f} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a $\preccurlyeq_I$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions:*

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative;*
- $\mathbf{f}$ *is $\preccurlyeq_I$-monotone;*
- $\mathbf{F}^p$ *is continuous with respect to* $\mathbf{f}$ *on X^m;*
- *each f_k is continuous on X^m for $k = 1, \cdots, m$.*

Suppose that there exists a function of contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any $\mathbf{x}, \mathbf{y} \in X^m$ with $\mathbf{y} \preccurlyeq_I \mathbf{x}$ or $\mathbf{x} \preccurlyeq_I \mathbf{y}$, the following inequalities

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \leq \varphi\left(d\left(f_k\left(\mathbf{x}\right), f_k\left(\mathbf{y}\right)\right)\right) \cdot d\left(f_k\left(\mathbf{x}\right), f_k\left(\mathbf{y}\right)\right) \tag{2.16}$$

are satisfied for all $k = 1, \cdots, m$. Then, $\mathbf{F}^p$ has a fixed point $\widehat{\mathbf{x}}$ such that each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed from (2.3) for all $k = 1, \cdots, m$.

Proof. We consider the sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ constructed from (2.3). Since $\mathbf{x}_0$ is a $\preccurlyeq_I$-monotone seed element in X^m, i.e., $\mathbf{x}_n \preccurlyeq_I \mathbf{x}_{n+1}$ for all $n \in \mathbb{N}$ or $\mathbf{x}_{n+1} \preccurlyeq_I \mathbf{x}_n$ for all $n \in \mathbb{N}$, according to the inequalities (2.16), we obtain

$$\begin{aligned} d\left(f_k\left(\mathbf{x}_{n+1}\right), f_k\left(\mathbf{x}_n\right)\right) &= d\left(F_k^p(\mathbf{x}_n), F_k^p(\mathbf{x}_{n-1})\right) \\ &\leq \varphi\left(d\left(f_k\left(\mathbf{x}_n\right), f_k\left(\mathbf{x}_{n-1}\right)\right)\right) \cdot d\left(f_k\left(\mathbf{x}_n\right), f_k\left(\mathbf{x}_{n-1}\right)\right) \\ &< d\left(f_k\left(\mathbf{x}_n\right), f_k\left(\mathbf{x}_{n-1}\right)\right) \end{aligned}$$

Using the argument in the proof of Theorem 2.1, we can show that $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X for any fixed $k = 1, \cdots, n$. Since $\mathbf{f}$ is $\preccurlyeq_I$-monotone and $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-monotone sequence, it follows that $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-monotone sequence.

- If $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-increasing sequence, then $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preceq$-increasing Cauchy sequence for $k \in I$, and is a $\preceq$-decreasing Cauchy sequence for $k \in J$.
- If $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-decreasing sequence, then $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preceq$-decreasing Cauchy sequence for $k \in I$, and is a $\preceq$-increasing Cauchy sequence for $k \in J$.

By the monotone completeness of X, there exists $\widehat{x}^{(k)} \in X$ such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ as $n \to \infty$ for $k = 1, \cdots, m$. The remaining proof follows from the same argument in the proof of Theorem 2.1. This completes the proof. □

Next we shall study the fixed points of functions having mixed-monotone property in the product spaces. Recall that the concept of mixed-monotone property for functions was adopted for presenting the coupled fixed point theorems. In this paper, we consider the extended concept in the product spaces.

Let $(X, \preceq)$ be a quasi-order set, and let I and J be the disjoint pair of $\{1, 2, \cdots, m\}$. Consider the quasi-order set $(X^m, \preccurlyeq_I)$ and the function $\mathbf{F} : (X^m, \preccurlyeq_I) \to (X^m, \preccurlyeq_I)$. We say that $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_I)$-*increasing* if and only if $\mathbf{x} \preccurlyeq_I \mathbf{y}$ implies $\mathbf{F}(\mathbf{x}) \preccurlyeq_I \mathbf{F}(\mathbf{y})$. Many other monotonic concepts are defined in Wu [30, Definition 4.4]. On the other hand, the I-mixed-monotone property and J-mixed-monotone property for $\mathbf{F}$ are also defined in Wu [30, Definition 4.6].

Theorem 2.4. (**Monotone Completeness and Mixed Monotone Property**). *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is monotonically complete, and that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving continuity. Given a disjoint pair I and J of $\{1, 2, \cdots, m\}$, assume that the function $\mathbf{F} : (X^m, \mathfrak{d}, \preccurlyeq_I) \to (X^m, \mathfrak{d}, \preccurlyeq_I)$ is continuous on X^m and has the I-mixed-monotone property, and that there exists a function of contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any $\mathbf{x}, \mathbf{y} \in X^m$ with $\mathbf{y} \preccurlyeq_I \mathbf{x}$ or $\mathbf{x} \preccurlyeq_I \mathbf{y}$, the following inequalities*

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \leq \varphi\left(d\left(x^{(k)}, y^{(k)}\right)\right) \cdot d\left(x^{(k)}, y^{(k)}\right) \tag{2.17}$$

are satisfied for all $k = 1, \cdots, m$ and for some $p \in \mathbb{N}$. If there exists $\mathbf{x}_0 \in X^m$ such that $\mathbf{x}_0 \preccurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$ or $\mathbf{x}_0 \succcurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$, then the function $\mathbf{F}^p$ has a fixed point $\widehat{\mathbf{x}} \in X^m$, where each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit *of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed below*

$$\mathbf{x}_n = \mathbf{F}^p(\mathbf{x}_{n-1}) \tag{2.18}$$

for all $k = 1, \cdots, m$.

Proof. According to (2.18), we have $\mathbf{x}_0 \preccurlyeq_I \mathbf{x}_1$ or $\mathbf{x}_0 \succcurlyeq_I \mathbf{x}_1$. From Wu [30, Lemma 4.10], the initial element $\mathbf{x}_0$ is a $\preccurlyeq_I$-monotone seed element in X^m. Therefore, the results follow immediately from Theorem 2.3 by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Theorem 2.5. (**Monotone Completeness and Monotone Property**). *Suppose that the quasi-ordered metric space* $(X, d, \preceq)$ *is monotonically complete, and that the metrics* $\mathfrak{d}$ *and* d *are compatible in the sense of preserving continuity. Given a disjoint pair* I *and* J *of* $\{1, 2, \cdots, m\}$*, assume that the function* $\mathbf{F} : (X^m, \mathfrak{d}, \preccurlyeq_I) \to (X^m, \mathfrak{d}, \preccurlyeq_I)$ *is continuous on* X^m *and satisfies any one of the following conditions:*

- (a) $\mathbf{F}$ *is* $(\preccurlyeq_I, \preccurlyeq_I)$*-increasing.*
- (b) p *is an even integer and* $\mathbf{F}$ *is* $(\preccurlyeq_I, \preccurlyeq_I)$*-decreasing.*

Assume that there exists a function of contractive factor $\varphi : [0, \infty) \to [0, 1)$ *such that, for any* $\mathbf{x}, \mathbf{y} \in X^m$ *with* $\mathbf{y} \preccurlyeq_I \mathbf{x}$ *or* $\mathbf{x} \preccurlyeq_I \mathbf{y}$*, the following inequalities*

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \leq \varphi\left(d\left(x^{(k)}, y^{(k)}\right)\right) \cdot d\left(x^{(k)}, y^{(k)}\right)$$

are satisfied for all $k = 1, \cdots, m$ *and for some* $p \in \mathbb{N}$*. If there exists* $\mathbf{x}_0 \in X^m$ *such that* $\mathbf{x}_0 \preccurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$ *or* $\mathbf{x}_0 \succcurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$*, then the function* $\mathbf{F}^p$ *has a fixed point* $\widehat{\mathbf{x}} \in X^m$*, where each component* $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed from* (2.18) *for all* $k = 1, \cdots, m$.

Proof. We consider the following cases.

- If $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_I)$-increasing, then it follows that $\mathbf{F}^p$ is $(\preccurlyeq_I, \preccurlyeq_I)$-increasing.
- If $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_I)$-decreasing and p is an even integer, then $\mathbf{F}^p$ is also $(\preccurlyeq_I, \preccurlyeq_I)$-increasing.

According to (2.18), we have $\mathbf{x}_0 \preccurlyeq_I \mathbf{x}_1$ or $\mathbf{x}_0 \succcurlyeq_I \mathbf{x}_1$. Since $\mathbf{x}_1 = \mathbf{F}^p(\mathbf{x}_0)$ and $\mathbf{x}_2 = \mathbf{F}^p(\mathbf{x}_1)$, it follows that $\mathbf{x}_0 \preccurlyeq_I \mathbf{x}_1$ implies $\mathbf{x}_1 \preccurlyeq_I \mathbf{x}_2$, and $\mathbf{x}_0 \succcurlyeq_I \mathbf{x}_1$ implies $\mathbf{x}_1 \succcurlyeq_I \mathbf{x}_2$. Therefore, if $\mathbf{x}_0 \preccurlyeq_I \mathbf{x}_1$, then we can generate a $\preccurlyeq_I$-increasing sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, and if $\mathbf{x}_0 \succcurlyeq_I \mathbf{x}_1$, then we can generate a $\preccurlyeq_I$-decreasing sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, which also says that the initial element $\mathbf{x}_0$ is a $\preccurlyeq_I$-monotone seed element in X^m. Therefore, the results follow immediately from Theorem 2.3 by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Theorem 2.6. (**Mixed-Monotone Completeness and Decreasing Property**). *Suppose that the quasi-ordered metric space* $(X, d, \preceq)$ *is mixed-monotonically complete, and that the metrics* $\mathfrak{d}$ *and* d *are compatible in the sense of preserving continuity. Given a disjoint pair* I *and* J *of* $\{1, 2, \cdots, m\}$*, assume that the function* $\mathbf{F} : (X^m, \mathfrak{d}, \preccurlyeq_I) \to (X^m, \mathfrak{d}, \preccurlyeq_I)$ *is continuous on* X^m *and* $(\preccurlyeq_I, \preccurlyeq_I)$*-decreasing, and that there exists a function of contractive factor* $\varphi : [0, \infty) \to [0, 1)$ *such that, for any* $\mathbf{x}, \mathbf{y} \in X^m$ *with* $\mathbf{y} \preccurlyeq_I \mathbf{x}$ *or* $\mathbf{x} \preccurlyeq_I \mathbf{y}$*, the following inequalities*

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \leq \varphi\left(d\left(x^{(k)}, y^{(k)}\right)\right) \cdot d\left(x^{(k)}, y^{(k)}\right)$$

are satisfied for all $k = 1,\cdots,m$ *and for some odd integer* $p \in \mathbb{N}$. *If there exists* $\mathbf{x}_0 \in X^m$ *such that* $\mathbf{x}_0 \preccurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$ *or* $\mathbf{x}_0 \succcurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$*, then the function* $\mathbf{F}^p$ *has a fixed point* $\widehat{\mathbf{x}} \in X^m$*, where each component* $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed from* (2.18) *for all* $k = 1,\cdots,m$.

Proof. Since $\mathbf{F}$ is $(\preccurlyeq_I,\preccurlyeq_I)$-decreasing and p is an odd integer, it follow that $\mathbf{F}^p$ is $(\preccurlyeq_I,\preccurlyeq_I)$-decreasing, which says that $\mathbf{x}_0 \preccurlyeq_I \mathbf{x}_1$ implies $\mathbf{x}_1 \succcurlyeq_I \mathbf{x}_2$, and $\mathbf{x}_0 \succcurlyeq_I \mathbf{x}_1$ implies $\mathbf{x}_1 \preccurlyeq_I \mathbf{x}_2$. Therefore, we can generate a $\preccurlyeq_I$-mixed-monotone sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, which also says that the initial element $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element in X^m. Therefore, the results follow immediately from Theorem 2.2 by taking $\mathbf{f}$ as the identity function. This completes the proof. ☐

In the mixed-monotonically complete quasi-ordered metric space, we can also study the fixed points of functions having comparable property in the product spaces. The *$\preceq$-mixed comparable property* for the function $\mathbf{F}: X^m \to X^m$ and the *$\preccurlyeq_I$-comparable property* for the function $\mathbf{F}: (X^m,\preccurlyeq_I) \to (X^m,\preccurlyeq_I)$ are defined in Wu [30, Definition 5.1].

Theorem 2.7. (**$\preceq$-mixed comparable property**). *Suppose that the quasi-ordered metric space* $(X,d,\preceq)$ *is mixed-monotonically complete, and that the metrics* $\mathfrak{d}$ *and* d *are compatible in the sense of preserving continuity. Assume that the function* $\mathbf{F}: (X^m,\mathfrak{d}) \to (X^m,\mathfrak{d})$ *is continuous on* X^m *and has the* $\preceq$*-mixed comparable property, and that there exists a function of contractive factor* $\varphi: [0,\infty) \to [0,1)$ *such that, for any two* $\preceq$*-mixed comparable elements* $\mathbf{x}$ *and* $\mathbf{y}$ *in* X^m*, the following inequalities*

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right) \le \varphi\left(d\left(x^{(k)},y^{(k)}\right)\right)\cdot d\left(x^{(k)},y^{(k)}\right)$$

are satisfied for all $k = 1,\cdots,m$ *and for some* $p \in \mathbb{N}$. *If there exists* $\mathbf{x}_0 \in X^m$ *such that* $\mathbf{x}_0$ *and* $\mathbf{F}^p(\mathbf{x}_0)$ *are* $\preceq$*-mixed comparable, then* $\mathbf{F}^p$ *has a fixed point* $\widehat{\mathbf{x}}$ *such that each component* $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed from* (2.18) *for all* $k = 1,\cdots,m$.

Proof. According to (2.18), we see that $\mathbf{x}_0$ and $\mathbf{x}_1$ are $\preceq$-mixed comparable. Since $\mathbf{F}$ has the $\preceq$-mixed comparable property, we see that $\mathbf{F}^p$ has also the $\preceq$-mixed comparable property. It follows that $\mathbf{x}_1 = \mathbf{F}^p(\mathbf{x}_0)$ and $\mathbf{x}_2 = \mathbf{F}^p(\mathbf{x}_1)$ are also $\preceq$-mixed comparable. Therefore, we can generate a mixed $\preceq$-monotone sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ by observation (d) of Wu [29, Remark 2.2], which also says that the initial element $\mathbf{x}_0$ is a mixed $\preceq$-monotone seed element in X^m. Since $\mathbf{F}$ is continuous on X^m, it follows that $\mathbf{F}^p$ is also continuous on X^m. Therefore, the result follows from Theorem 2.1 immediately by taking $\mathbf{f}$ as the identity function. This completes the proof. ☐

Theorem 2.8. (**$\preccurlyeq_I$-comparable property**). *Suppose that the quasi-ordered metric space* $(X,d,\preceq)$ *is mixed-monotonically complete, and that the metrics* $\mathfrak{d}$ *and* d *are compatible in the sense of preserving continuity. Assume that there exists a disjoint pair* I *and* J *of* $\{1,\cdots,m\}$ *such that the following conditions are satisfied:*

- *the function* $\mathbf{F}: (X^m,\mathfrak{d},\preccurlyeq_I) \to (X^m,\mathfrak{d},\preccurlyeq_I)$ *is continuous on* X^m *and has the* $\preccurlyeq_I$*-comparable property;*
- *there exists* $\mathbf{x}_0 \in X^m$ *such that* $\mathbf{x}_0$ *and* $\mathbf{F}^p(\mathbf{x}_0)$ *are comparable with respect to the quasi-order "*$\preccurlyeq_I$*" for some* $p \in \mathbb{N}$.

If there exists a function of contractive factor $\varphi:[0,\infty)\to[0,1)$ *such that, for any* $\mathbf{x},\mathbf{y}\in X^m$ *with* $\mathbf{y}\preccurlyeq_I\mathbf{x}$ *or* $\mathbf{x}\preccurlyeq_I\mathbf{y}$*, the following inequalities*

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right)\leq\varphi\left(d\left(x^{(k)},y^{(k)}\right)\right)\cdot d\left(x^{(k)},y^{(k)}\right)$$

are satisfied for all $k=1,\cdots,m$*, then* $\mathbf{F}^p$ *has a fixed point* $\widehat{\mathbf{x}}$ *such that each component* $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ *constructed from* (2.18) *for all* $k=1,\cdots,m$.

Proof. According to (2.18), we see that $\mathbf{x}_0$ and $\mathbf{x}_1$ are comparable with respect to "$\preccurlyeq_I$". Since $\mathbf{F}$ has the $\preccurlyeq_I$-comparable property, we see that $\mathbf{F}^p$ has also the $\preccurlyeq_I$-comparable property. It follows that $\mathbf{x}_1=\mathbf{F}^p(\mathbf{x}_0)$ and $\mathbf{x}_2=\mathbf{F}^p(\mathbf{x}_1)$ are also comparable with respect to '$\preccurlyeq_I$". Therefore, we can generate a mixed $\preccurlyeq_I$-monotone sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, which also says that the initial element $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element in X^m. Since $\mathbf{F}$ is continuous on X^m, it follows that $\mathbf{F}^p$ is also continuous on X^m. Therefore, the result follows from Theorem 2.2 immediately by taking $\mathbf{f}$ as the identity function. This completes the proof. □

2.3. **Coincidence Point Theorems.** Now we study the coincidence point in the product spaces of mixed-monotonically complete quasi-ordered metric spaces. Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order "$\preceq$". We say that $(X,d,\preceq)$ preserves the *mixed-monotone convergence* if and only if, for each mixed $\preceq$-monotone sequence $\{x_n\}_{n\in\mathbb{N}}$ that converges to $\widehat{x}$, we have either $x_n\preceq\widehat{x}$ or $\widehat{x}\preceq x_n$ for each $n\in\mathbb{N}$, .

Theorem 2.9. (**Mixed-Monotone Completeness**). *Suppose that the quasi-ordered metric space* $(X,d,\preceq$ $)$ *is mixed-monotonically complete and preserves the mixed-monotone convergence. Assume that the metrics* $\mathfrak{d}$ *and* d *are compatible in the sense of preserving convergence. Consider the functions* $\mathbf{F}:(X^m,\mathfrak{d})\to(X^m,\mathfrak{d})$ *and* $\mathbf{f}:(X^m,\mathfrak{d})\to(X^m,\mathfrak{d})$ *satisfying* $\mathbf{F}^p(X^m)\subseteq\mathbf{f}(X^m)$ *for some* $p\in\mathbb{N}$*. Let* $\mathbf{x}_0$ *be a mixed* $\preceq$*-monotone seed element in* X^m*. Assume that the functions* $\mathbf{F}$ *and* $\mathbf{f}$ *satisfy the following conditions:*

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative;*
- $\mathbf{f}$ *has the sequentially mixed* $\preceq$*-monotone property;*
- *each* f_k *is continuous on* X^m *for* $k=1,\cdots,m$.

Suppose that there exists a function of contractive factor $\varphi:[0,\infty)\to[0,1)$ *such that, for any two* $\preceq$*-mixed comparable elements* $\mathbf{x}$ *and* $\mathbf{y}$ *in* X^m*, the following inequalities*

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right)\leq\varphi\left(d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right)\right)\cdot d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right)\tag{2.19}$$

are satisfied for all $k=1,\cdots,m$*. Then, the following statements hold true.*

(i) *There exists* $\widehat{\mathbf{x}}\in X^m$ *of* $\mathbf{F}$ *such that* $\mathbf{F}^p(\widehat{\mathbf{x}})=\mathbf{f}(\widehat{\mathbf{x}})$*. If* $p=1$*, then* $\widehat{\mathbf{x}}$ *is a coincidence point of* $\mathbf{F}$ *and* $\mathbf{f}$.

(ii) *If there exists another* $\widehat{\mathbf{y}}\in X^m$ *such that* $\widehat{\mathbf{x}}$ *and* $\widehat{\mathbf{y}}$ *are* $\preceq$*-mixed comparable satisfying* $\mathbf{F}^p(\widehat{\mathbf{y}})=\mathbf{f}(\widehat{\mathbf{y}})$*, then* $\mathbf{f}(\widehat{\mathbf{x}})=\mathbf{f}(\widehat{\mathbf{y}})$.

(iii) *Suppose that* $\widehat{\mathbf{x}}$ *is obtained from part* (i)*. If* $\widehat{\mathbf{x}}$ *and* $\mathbf{F}(\widehat{\mathbf{x}})$ *are* $\preceq$*-mixed comparable, then* $\mathbf{f}^q(\widehat{\mathbf{x}})$ *is a fixed point of* $\mathbf{F}$ *for any* $q\in\mathbb{N}$.

Moreover, each component $\widehat{x}^{(k)}$ *of* $\widehat{\mathbf{x}}$ *is the limit of the sequence* $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ *constructed from* (2.3) *for all* $k = 1,\cdots,m$.

Proof. From the proof of Theorem 2.1, we can construct a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ and $f_k(\mathbf{f}(\mathbf{x}_n)) \to f_k(\widehat{\mathbf{x}})$ as $n\to\infty$, where $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone sequence for all $k = 1,\cdots,n$. Since $f_k(\mathbf{f}(\mathbf{x}_n)) \to f_k(\widehat{\mathbf{x}})$ as $n\to\infty$, given any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d\left(f_k\left(\mathbf{f}(\mathbf{x}_n)\right), f_k(\widehat{\mathbf{x}})\right) < \frac{\varepsilon}{2} \tag{2.20}$$

for all $n\in\mathbb{N}$ with $n \geq n_0$ and for all $k = 1,\cdots,m$. Since $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone convergent sequence for all $k = 1,\cdots,n$, from observation (a) of Wu [29, Remark 2.6], we see that, for each $n\in\mathbb{N}$, $\mathbf{f}(\mathbf{x}_n)$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable. For each $n \geq n_0$, from (2.19) and (2.20), it follows that

$$\begin{aligned} d\left(F_k^p(\widehat{\mathbf{x}}), F_k^p\left(\mathbf{f}(\mathbf{x}_n)\right)\right) &\leq \varphi\left(d\left(f_k(\widehat{\mathbf{x}}), f_k\left(\mathbf{f}(\mathbf{x}_n)\right)\right)\right)\cdot d\left(f_k(\widehat{\mathbf{x}}), f_k\left(\mathbf{f}(\mathbf{x}_n)\right)\right) \\ &< d\left(f_k(\widehat{\mathbf{x}}), f_k\left(\mathbf{f}(\mathbf{x}_n)\right)\right) < \frac{\varepsilon}{2} \end{aligned} \tag{2.21}$$

Since $\mathbf{F}$ and $\mathbf{f}$ are commutative, we have $\mathbf{f}(\mathbf{F}^p(\mathbf{x})) = \mathbf{F}^p(\mathbf{f}(\mathbf{x}))$ for all $\mathbf{x}\in X^m$, which also implies

$$f_k\left(\mathbf{f}(\mathbf{x}_n)\right) = f_k\left(\mathbf{F}^p(\mathbf{x}_{n-1})\right) = F_k^p\left(\mathbf{f}(\mathbf{x}_{n-1})\right).$$

Now, we obtain

$$\begin{aligned} d\left(F_k^p(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{x}})\right) &\leq d\left(F_k^p(\widehat{\mathbf{x}}), F_k^p(\mathbf{f}(\mathbf{x}_{n_0}))\right) + d\left(F_k^p(\mathbf{f}(\mathbf{x}_{n_0})), f_k(\widehat{\mathbf{x}})\right) \\ &= d\left(F_k^p(\widehat{\mathbf{x}}), F_k^p(\mathbf{f}(\mathbf{x}_{n_0}))\right) + d\left(f_k\left(\mathbf{f}(\mathbf{x}_{n_0+1})\right), f_k(\widehat{\mathbf{x}})\right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ (by (2.20) and (2.21))} \\ &= \varepsilon. \end{aligned}$$

Since ε is any positive number, we conclude that $d(F_k^p(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{x}})) = 0$, which says that $F_k^p(\widehat{\mathbf{x}}) = f_k(\widehat{\mathbf{x}})$ for all $k = 1,\cdots,m$, i.e., $\mathbf{F}^p(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}})$. This proves part (i).

To prove part (ii), if $f_k(\widehat{\mathbf{x}}) \neq f_k(\widehat{\mathbf{y}})$, i.e., $d\left(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})\right) \neq 0$, then, from (2.19), we obtain

$$d\left(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})\right) = d\left(F_k^p(\widehat{\mathbf{x}}), F_k^p(\widehat{\mathbf{y}})\right) \leq \varphi\left(d\left(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})\right)\right) d\left(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})\right) < d\left(f_k(\widehat{\mathbf{x}}), f_k(\widehat{\mathbf{y}})\right),$$

since $\varphi(t) < 1$ for any $t\in[0,\infty)$. This contradiction says that $f_k(\widehat{\mathbf{x}}) = f_k(\widehat{\mathbf{y}})$ for all $k = 1,\cdots,m$, i.e., $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$.

To prove part (iii), using the commutativity of $\mathbf{F}$ and $\mathbf{f}$, we have

$$\mathbf{f}(\mathbf{F}(\widehat{\mathbf{x}})) = \mathbf{F}(\mathbf{f}(\widehat{\mathbf{x}})) = \mathbf{F}(\mathbf{F}^p(\widehat{\mathbf{x}})) = \mathbf{F}^p(\mathbf{F}(\widehat{\mathbf{x}})). \tag{2.22}$$

By taking $\widehat{\mathbf{y}} = \mathbf{F}(\widehat{\mathbf{x}})$, the equalities (2.22) says that $\mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}^p(\widehat{\mathbf{y}})$. Since $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}} = \mathbf{F}(\widehat{\mathbf{x}})$ are $\preceq$-mixed comparable by the assumption, part (ii) says that

$$\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{f}(\mathbf{F}(\widehat{\mathbf{x}})) = \mathbf{F}(\mathbf{f}(\widehat{\mathbf{x}})),$$

which says that $\mathbf{f}(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$. Given any $q \in \mathbb{N}$, we have

$$\begin{aligned}\mathbf{F}\left(\mathbf{f}^q(\widehat{\mathbf{x}})\right) &= \mathbf{f}^{q-1}\left(\mathbf{F}(\mathbf{f}(\widehat{\mathbf{x}}))\right) \text{ (by the commutativity of } \mathbf{F} \text{ and } \mathbf{f})\\ &= \mathbf{f}^{q-1}\left(\mathbf{f}(\widehat{\mathbf{x}})\right) = \mathbf{f}^q(\widehat{\mathbf{x}}),\end{aligned}$$

which says that $\mathbf{f}^q(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$. This completes the proof. □

Theorem 2.10. (**Mixed-Monotone Completeness**). *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is mixed-monotonically complete and preserves the mixed-monotone convergence. Assume that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving convergence. Given a disjoint pair I and J of $\{1,2,\cdots,m\}$, consider the functions $\mathbf{F} : (X^m,\mathfrak{d},\preccurlyeq_I) \to (X^m,\mathfrak{d},\preccurlyeq_I)$ and $\mathbf{f} : (X^m,\mathfrak{d},\preccurlyeq_I) \to (X^m,\mathfrak{d},\preccurlyeq_I)$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preccurlyeq_I$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions:*

- *$\mathbf{F}$ and $\mathbf{f}$ are commutative;*
- *$\mathbf{f}$ has the sequentially mixed $\preccurlyeq_I$-monotone property or the $\preceq$-monotone property;*
- *each f_k is continuous on X^m for $k = 1,\cdots,m$.*

Suppose that there exists a function of contractive factor $\varphi : [0,\infty) \to [0,1)$ such that, for any $\mathbf{x},\mathbf{y} \in X^m$ and any disjoint pair I° and J° of $\{1,\cdots,m\}$ with $\mathbf{y} \preccurlyeq_{I^\circ} \mathbf{x}$ or $\mathbf{x} \preccurlyeq_{I^\circ} \mathbf{y}$, the following inequalities

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right) \le \varphi\left(d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right)\right)\cdot d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right) \tag{2.23}$$

are satisfied for all $k = 1,\cdots,m$. Then, the following statements hold true.

(i) *There exists $\widehat{\mathbf{x}} \in X^m$ of $\mathbf{F}$ such that $\mathbf{F}^p(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}})$. If $p = 1$, then $\widehat{\mathbf{x}}$ is a coincidence point of $\mathbf{F}$ and $\mathbf{f}$.*

(ii) *If there exist a disjoint pair I° and J° of $\{1,\cdots,m\}$ and another $\widehat{\mathbf{y}} \in X^m$ such that $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{y}}$ are comparable with respect to the quasi-order "$\preccurlyeq_{I^\circ}$" satisfying $\mathbf{F}^p(\widehat{\mathbf{y}}) = \mathbf{f}(\widehat{\mathbf{y}})$, then $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$.*

(iii) *Suppose that $\widehat{\mathbf{x}}$ is obtained from part* (i). *If there exists a disjoint pair I° and J° of $\{1,\cdots,m\}$ such that $\widehat{\mathbf{x}}$ and $\mathbf{F}(\widehat{\mathbf{x}})$ are comparable with respect to the quasi-order "$\preccurlyeq_{I^\circ}$", then $\mathbf{f}^q(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$ for any $q \in \mathbb{N}$.*

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed from (2.3) *for all $k = 1,\cdots,m$.*

Proof. From the proof of Theorem 2.2, we can construct a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m such that $f_k(\mathbf{x}_n) \to \widehat{x}^{(k)}$ and $f_k(\mathbf{f}(\mathbf{x}_n)) \to f_k(\widehat{\mathbf{x}})$ as $n \to \infty$, where $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a mixed $\preceq$-monotone sequence for all $k = 1,\cdots,m$. From observation (b) of Wu [29, Remark 2.6], we see that, for each $n \in \mathbb{N}$, there exists a subset I_n of $\{1,\cdots,m\}$ such that

$$\mathbf{f}(\mathbf{x}_n) \preccurlyeq_{I_n} \widehat{\mathbf{x}} \text{ or } \widehat{\mathbf{x}} \preccurlyeq_{I_n} \mathbf{f}(\mathbf{x}_n). \tag{2.24}$$

From (2.24), (2.23) and the same argument in the proof of part (i) of Theorem 2.9, we can obtain $\mathbf{F}^p(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{x}})$. This proves part (i). By referring to (2.23), parts (ii) and (iii) can be similarly obtained by the same argument in the proof of Theorem 2.9. This completes the proof. □

Remark 2.1. Suppose that the inequalities (2.19) and (2.23) in Theorems 2.9 and 2.10 are satisfied for any $\mathbf{x},\mathbf{y}\in X^m$. Then, from the proofs of Theorems 2.9 and 2.10, we can see that parts (ii) and (iii) can be changed as follows.

(ii)′ If there exists another $\widehat{\mathbf{y}}\in X^m$ satisfying $\mathbf{F}^p(\widehat{\mathbf{y}})=\mathbf{f}(\widehat{\mathbf{y}})$, then $\mathbf{f}(\widehat{\mathbf{x}})=\mathbf{f}(\widehat{\mathbf{y}})$.

(iii)′ Suppose that $\widehat{\mathbf{x}}$ is obtained from part (i). Then $\mathbf{f}^q(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$ for any $q\in\mathbb{N}$.

Next we study the coincidence point in the product spaces of monotonically complete quasi-ordered metric spaces. Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order "$\preceq$". We say that $(X,d,\preceq)$ p-reserves the *monotone convergence* if and only if, for each $\preceq$-monotone sequence $\{x_n\}_{n\in\mathbb{N}}$ that converges to $\widehat{x}$, either one of the following conditions is satisfied:

- if $\{x_n\}_{n\in\mathbb{N}}$ is a $\preceq$-increasing sequence, then $x_n\preceq\widehat{x}$ for each $n\in\mathbb{N}$;
- if $\{x_n\}_{n\in\mathbb{N}}$ is a $\preceq$-decreasing sequence, then $\widehat{x}\preceq x_n$ for each $n\in\mathbb{N}$.

Theorem 2.11. (**Monotone Completeness**). *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is monotonically complete and preserves the monotone convergence. Assume that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving convergence. Given a disjoint pair I and J of $\{1,2,\cdots,m\}$, consider the functions $\mathbf{F}:(X^m,\mathfrak{d},\preccurlyeq_I)\to(X^m,\mathfrak{d},\preccurlyeq_I)$ and $\mathbf{f}:(X^m,\mathfrak{d},\preccurlyeq_I)\to(X^m,\mathfrak{d},\preccurlyeq_I)$ satisfying $\mathbf{F}^p(X^m)\subseteq\mathbf{f}(X^m)$ for some $p\in\mathbb{N}$. Let $\mathbf{x}_0$ be a $\preccurlyeq_I$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions:*

- $\mathbf{F}$ *and* $\mathbf{f}$ *are commutative;*
- $\mathbf{f}$ *is $\preccurlyeq_I$-monotone;*
- *each f_k is continuous on X^m for $k=1,\cdots,m$.*

Suppose that there exists a function of contractive factor $\varphi:[0,\infty)\to[0,1)$ such that, for any $\mathbf{x},\mathbf{y}\in X^m$ with $\mathbf{y}\preccurlyeq_I\mathbf{x}$ or $\mathbf{x}\preccurlyeq_I\mathbf{y}$, the following inequalities

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right)\leq\varphi\left(d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right)\right)\cdot d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right) \tag{2.25}$$

are satisfied for all $k=1,\cdots,m$. Then, the following statements hold true.

(i) *There exists $\widehat{\mathbf{x}}\in X^m$ of $\mathbf{F}$ such that $\mathbf{F}^p(\widehat{\mathbf{x}})=\mathbf{f}(\widehat{\mathbf{x}})$. If $p=1$, then $\widehat{\mathbf{x}}$ is a coincidence point of $\mathbf{F}$ and $\mathbf{f}$.*

(ii) *If there exists another $\widehat{\mathbf{y}}\in X^m$ such that $\mathbf{F}^p(\widehat{\mathbf{y}})=\mathbf{f}(\widehat{\mathbf{y}})$ with $\widehat{\mathbf{x}}\preccurlyeq_I\widehat{\mathbf{y}}$ or $\widehat{\mathbf{y}}\preccurlyeq_I\widehat{\mathbf{x}}$, then $\mathbf{f}(\widehat{\mathbf{x}})=\mathbf{f}(\widehat{\mathbf{y}})$.*

(iii) *Suppose that $\widehat{\mathbf{x}}$ is obtained from part* (i). *If $\widehat{\mathbf{x}}$ and $\mathbf{F}(\widehat{\mathbf{x}})$ are comparable with respect to "$\preccurlyeq_I$, then $\mathbf{f}^q(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$ for any $q\in\mathbb{N}$.*

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed from (2.3) *for all $k=1,\cdots,m$.*

Proof. From the proof of Theorem 2.3, we can construct a sequence $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ in X^m such that $f_k(\mathbf{x}_n)\to\widehat{x}^{(k)}$ and $f_k(\mathbf{f}(\mathbf{x}_n))\to f_k(\widehat{\mathbf{x}})$ as $n\to\infty$ for all $k=1,\cdots,m$, where $\{\mathbf{f}(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ is a $\preccurlyeq_I$-monotone sequence. From Wu [29, Remark 2.12], it follows that, for each $n\in\mathbb{N}$, $\mathbf{f}(\mathbf{x}_n)\preccurlyeq_I\widehat{\mathbf{x}}$ or $\mathbf{f}(\mathbf{x}_n)\succcurlyeq_I\widehat{\mathbf{x}}$. Since $f_k(\mathbf{f}(\mathbf{x}_n))\to f_k(\widehat{\mathbf{x}})$ as $n\to\infty$, given any $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$d\left(f_k\left(\mathbf{f}(\mathbf{x}_n)\right),f_k(\widehat{\mathbf{x}})\right)<\frac{\varepsilon}{2} \tag{2.26}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$ and for all $k = 1, \cdots, m$. For each $n \geq n_0$, from (2.25) and (2.26), it follows that

$$d\left(F_k^p(\widehat{\mathbf{x}}), F_k^p(\mathbf{f}(\mathbf{x}_n))\right) \leq \varphi\left(d\left(f_k(\widehat{\mathbf{x}}), f_k(\mathbf{f}(\mathbf{x}_n))\right)\right) \cdot d\left(f_k(\widehat{\mathbf{x}}), f_k(\mathbf{f}(\mathbf{x}_n))\right) < d\left(f_k(\widehat{\mathbf{x}}), f_k(\mathbf{f}(\mathbf{x}_n))\right) < \frac{\varepsilon}{2}.$$

The remaining proof follows from the similar argument of Theorem 2.9, □

Remark 2.2. Suppose that the inequalities (2.25) in Theorem 2.11 are satisfied for any $\mathbf{x}, \mathbf{y} \in X^m$. Then, from the proof of Theorem 2.11, we can see that parts (ii) and (iii) can be changed as follows.

(ii)′ If there exists another $\widehat{\mathbf{y}} \in X^m$ satisfying $\mathbf{F}^p(\widehat{\mathbf{y}}) = \mathbf{f}(\widehat{\mathbf{y}})$, then $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$.

(iii)′ Suppose that $\widehat{\mathbf{x}}$ is obtained from part (i). Then $\mathbf{f}^q(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$ for any $q \in \mathbb{N}$.

Next, we are going to apply Theorems 2.10 and 2.11 by taking $\mathbf{f}$ as the identity function to study the chain-uniqueness of the fixed point in which we can drop the assumption of continuity of $\mathbf{F}$ by assuming that $(X, d, \preceq)$ preserves the mixed-monotone convergence.

Since we consider a metric space $(X, d, \preceq)$ endowed with a quasi-order "$\preceq$", given any disjoint pair I and J of $\{1, \cdots, p\}$, we can define a quasi-order "$\preccurlyeq_I$" on X^m as given in (2.1). Now, given any $\mathbf{x} \in X^m$, we define the chain $\mathfrak{C}(\preccurlyeq_I, \mathbf{x})$ containing $\mathbf{x}$ as follows:

$$\mathfrak{C}(\preccurlyeq_I, \mathbf{x}) = \{\mathbf{y} \in X^m : \mathbf{y} \preccurlyeq_I \mathbf{x} \text{ or } \mathbf{x} \preccurlyeq_I \mathbf{y}\}.$$

Definition 2.7. Consider the function $\mathbf{F} : X^m \to X^m$ defined on the product set X^m into itself. The fixed point $\widehat{\mathbf{x}} \in X^m$ of $\mathbf{F}$ is called *$\preccurlyeq_I$-chain-unique* if and only if, given any other fixed point $\mathbf{x}$ of $\mathbf{F}$, if $\mathbf{x} \in \mathfrak{C}(\preccurlyeq_I, \widehat{\mathbf{x}})$, then $\mathbf{x} = \widehat{\mathbf{x}}$.

Theorem 2.12. (**$\preccurlyeq_I$-Chain-Uniqueness for the Case of Monotone Completeness and Mixed Monotone Property**). *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is monotonically complete and preserves the monotone convergence, and that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving convergence. Given a disjoint pair I and J of $\{1, 2, \cdots, m\}$, assume that the function $\mathbf{F} : (X^m, \mathfrak{d}, \preccurlyeq_I) \to (X^m, \mathfrak{d}, \preccurlyeq_I)$ has the I-mixed-monotone property, and that there exists a function of contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any $\mathbf{x}, \mathbf{y} \in X^m$ with $\mathbf{y} \preccurlyeq_I \mathbf{x}$ or $\mathbf{x} \preccurlyeq_I \mathbf{y}$, the following inequalities*

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \leq \varphi\left(d\left(x^{(k)}, y^{(k)}\right)\right) \cdot d\left(x^{(k)}, y^{(k)}\right)$$

are satisfied for all $k = 1, \cdots, m$ and for some $p \in \mathbb{N}$. If there exists $\mathbf{x}_0 \in X^m$ such that $\mathbf{x}_0 \preccurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$ or $\mathbf{x}_0 \succcurlyeq_I \mathbf{F}^p(\mathbf{x}_0)$, then the function $\mathbf{F}^p$ has a $\preccurlyeq_I$-chain-unique fixed point $\widehat{\mathbf{x}} \in X^m$, where each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{x_n^{(k)}\}_{n \in \mathbb{N}}$ constructed from (2.18) for all $k = 1, \cdots, m$.

Proof. According to (2.18), we have $\mathbf{x}_0 \preccurlyeq_I \mathbf{x}_1$ or $\mathbf{x}_0 \succcurlyeq_I \mathbf{x}_1$. From Wu [30, Lemma 4.10], the initial element $\mathbf{x}_0$ is a $\preccurlyeq_I$-monotone seed element in X^m. Therefore, the results follow immediately from Theorem 2.11 by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Theorem 2.13. (**$\preccurlyeq_I$-Chain-Uniqueness for the Case of Monotone Completeness and Monotone Property**). *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is monotonically complete and preserves the monotone convergence, and that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving* *convergence. Given a disjoint pair I and J of $\{1,2,\cdots,m\}$, assume that the function $\mathbf{F}:(X^m,\mathfrak{d},\preccurlyeq_I)\to(X^m,\mathfrak{d},\preccurlyeq_I)$ satisfies any one of the following conditions:*

(a) *$\mathbf{F}$ is $(\preccurlyeq_I,\preccurlyeq_I)$-increasing.*

(b) *p is an even integer and $\mathbf{F}$ is $(\preccurlyeq_I,\preccurlyeq_I)$-decreasing.*

Assume that there exists a function of contractive factor $\varphi:[0,\infty)\to[0,1)$ such that, for any $\mathbf{x},\mathbf{y}\in X^m$ with $\mathbf{y}\preccurlyeq_I\mathbf{x}$ or $\mathbf{x}\preccurlyeq_I\mathbf{y}$, the following inequalities

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right)\leq\varphi\left(d\left(x^{(k)},y^{(k)}\right)\right)\cdot d\left(x^{(k)},y^{(k)}\right)$$

are satisfied for all $k=1,\cdots,m$ and for some $p\in\mathbb{N}$. If there exists $\mathbf{x}_0\in X^m$ such that $\mathbf{x}_0\preccurlyeq_I\mathbf{F}^p(\mathbf{x}_0)$ or $\mathbf{x}_0\succcurlyeq_I\mathbf{F}^p(\mathbf{x}_0)$, then the function $\mathbf{F}^p$ has a $\preccurlyeq_I$-chain-unique fixed point $\widehat{\mathbf{x}}\in X^m$, where each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ constructed from (2.18) for all $k=1,\cdots,m$.

Proof. From the proof of Theorem 2.5, we see that the initial element $\mathbf{x}_0$ is a $\preccurlyeq_I$-monotone seed element in X^m. Therefore, the results follow immediately from Theorem 2.11 by taking $\mathbf{f}$ as the identity function. This completes the proof. □

Theorem 2.14. ($\preccurlyeq_I$**-Chain-Uniqueness for the Case of Mixed-Monotone Completeness and Decreasing Property**). *Let the quasi-ordered metric space $(X,d,\preceq)$ be mixed-monotonically complete and preserves the mixed-monotone convergence. Suppose that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving convergence. Given a disjoint pair I and J of $\{1,2,\cdots,m\}$, suppose that there exists $\mathbf{x}_0\in X^m$ such that the following conditions are satisfied:*

- *the function $\mathbf{F}:(X^m,\mathfrak{d},\preccurlyeq_I)\to(X^m,\mathfrak{d},\preccurlyeq_I)$ is $(\preccurlyeq_I,\preccurlyeq_I)$-decreasing.*
- *$\mathbf{x}_0\preccurlyeq_I\mathbf{F}^p(\mathbf{x}_0)$ or $\mathbf{x}_0\succcurlyeq_I\mathbf{F}^p(\mathbf{x}_0)$;*

Assume that there exists a function of contractive factor $\varphi:[0,\infty)\to[0,1)$ such that, for any $\mathbf{x},\mathbf{y}\in X^m$ and any disjoint pair I° and J° of $\{1,\cdots,m\}$ with $\mathbf{y}\preccurlyeq_{I^\circ}\mathbf{x}$ or $\mathbf{x}\preccurlyeq_{I^\circ}\mathbf{y}$, the following inequalities

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right)\leq\varphi\left(d\left(x^{(k)},y^{(k)}\right)\right)\cdot d\left(x^{(k)},y^{(k)}\right)$$

are satisfied for all $k=1,\cdots,m$ and for some odd integer $p\in\mathbb{N}$. Then, the function $\mathbf{F}^p$ has a chain-unique fixed point $\widehat{\mathbf{x}}\in X^m$, where each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{x_n^{(k)}\}_{n\in\mathbb{N}}$ constructed from (2.18) for all $k=1,\cdots,m$.

Proof. From the proof of Theorem 2.6, we see that the initial element $\mathbf{x}_0$ is a mixed $\preccurlyeq_I$-monotone seed element in X^m. Therefore, the results follow immediately from Theorem 2.10 by taking $\mathbf{f}$ as the identity function. This completes the proof. □

2.4. **Common Fixed Point.** Next we shall consider two different sense of uniqueness of common fixed point theorems in the product spaces. One is called the uniqueness in the $\preceq$-mixed comparable sense, and another one is called the chain-uniqueness.

Definition 2.8. Let $(X,\preceq)$ be a quasi-order set. Consider the functions $\mathbf{F}:X^m\to X^m$ and $\mathbf{f}:X^m\to X^m$ defined on the product set X^m into itself. The common fixed point $\widehat{\mathbf{x}}\in X^m$ of $\mathbf{F}$ and $\mathbf{f}$ is *unique in the*

$\preceq$-mixed comparable sense if and only if, for any other common fixed point $\mathbf{x}$ of $\mathbf{F}$ and $\mathbf{f}$, if $\mathbf{x}$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable, then $\mathbf{x} = \widehat{\mathbf{x}}$.

Theorem 2.15. (**$\preceq$-Mixed Comparable Sense**). *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is mixed-monotonically complete and preserves the mixed-monotone convergence. Assume that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving continuity. Consider the functions $\mathbf{F} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ and $\mathbf{f} : (X^m, \mathfrak{d}) \to (X^m, \mathfrak{d})$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preceq$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions:*

- *$\mathbf{F}$ and $\mathbf{f}$ are commutative;*
- *$\mathbf{f}$ has the sequentially mixed $\preceq$-monotone property;*
- *$\mathbf{F}^p$ is continuous with respect to $\mathbf{f}$ on X^m;*
- *each f_k is continuous on X^m for $k = 1, \cdots, m$.*

Suppose that there exists a function of contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any two $\preceq$-mixed comparable elements $\mathbf{x}$ and $\mathbf{y}$ in X^m, the following inequalities

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \leq \varphi\left(d\left(f_k\left(\mathbf{x}\right), f_k\left(\mathbf{y}\right)\right)\right) \cdot d\left(f_k\left(\mathbf{x}\right), f_k\left(\mathbf{y}\right)\right) \tag{2.27}$$

are satisfied for all $k = 1, \cdots, m$. Then, the following statements hold true.

(i) *$\mathbf{F}^p$ and $\mathbf{f}$ have a unique common fixed point $\widehat{\mathbf{x}}$ in the $\preceq$-mixed comparable sense. Equivalently, if $\widehat{\mathbf{y}}$ is another common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$, and is $\preceq$-mixed comparable with $\widehat{\mathbf{x}}$, then $\widehat{\mathbf{y}} = \widehat{\mathbf{x}}$.*

(ii) *For $p \neq 1$, suppose that $\mathbf{F}(\widehat{\mathbf{x}})$ and $\widehat{\mathbf{x}}$ obtained in* (i) *are $\preceq$-mixed comparable. Then, $\mathbf{F}$ and $\mathbf{f}$ have a unique common fixed point $\widehat{\mathbf{x}}$ in the $\preceq$-mixed comparable sense.*

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed from (2.3) *for all $k = 1, \cdots, m$.*

Proof. To prove part (i), from Wu [30, Proposition 2.8] and part (i) of Theorem 2.9, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{F}^p(\widehat{\mathbf{x}})$. From Theorem 2.1, we also have $\mathbf{F}^p(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}$. Therefore, we obtain

$$\widehat{\mathbf{x}} = \mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{F}^p(\widehat{\mathbf{x}}).$$

This shows that $\widehat{\mathbf{x}}$ is a common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$. For the uniqueness in the $\preceq$-mixed comparable sense, let $\widehat{\mathbf{y}}$ be another common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$ such that $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable, i.e., $\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}^p(\widehat{\mathbf{y}})$. By part (ii) of Theorem 2.9, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$. Therefore, by the triangle inequality, we have

$$d\left(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}\right) \leq d\left(\widehat{\mathbf{x}}, \mathbf{f}(\widehat{\mathbf{x}})\right) + d\left(\mathbf{f}(\widehat{\mathbf{x}}), \mathbf{f}(\widehat{\mathbf{y}})\right) + d\left(\mathbf{f}(\widehat{\mathbf{y}}), \widehat{\mathbf{y}}\right) = 0, \tag{2.28}$$

which says that $\widehat{\mathbf{x}} = \widehat{\mathbf{y}}$. This proves part (i).

To prove part (ii), since $\mathbf{F}(\widehat{\mathbf{x}})$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable, part (iii) of Theorem 2.9 says that $\mathbf{f}(\widehat{\mathbf{x}})$ is a fixed point of $\mathbf{F}$, i.e., $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{F}(\mathbf{f}(\widehat{\mathbf{x}}))$, which implies $\widehat{\mathbf{x}} = \mathbf{F}(\widehat{\mathbf{x}})$, since $\widehat{\mathbf{x}} = \mathbf{f}(\widehat{\mathbf{x}})$. This shows that $\widehat{\mathbf{x}}$ is a common fixed point of $\mathbf{F}$ and $\mathbf{f}$. For the uniqueness in the $\preceq$-mixed comparable sense, let $\widehat{\mathbf{y}}$ be another common fixed point of $\mathbf{F}$ and $\mathbf{f}$ such that $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{x}}$ are $\preceq$-mixed comparable, i.e., $\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}(\widehat{\mathbf{y}})$. Then, we have

$$\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}(\widehat{\mathbf{y}}) = \mathbf{F}(\mathbf{f}(\widehat{\mathbf{y}})) = \mathbf{F}^2(\widehat{\mathbf{y}}) = \cdots = \mathbf{F}^p(\widehat{\mathbf{y}}).$$

By part (ii) of Theorem 2.9, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$. From (2.28), we can similarly obtain $\widehat{\mathbf{x}} = \widehat{\mathbf{y}}$. This completes the proof. □

Let us recall that the chain $\mathfrak{C}(\preccurlyeq_I, \mathbf{x})$ containing $\mathbf{x}$ is given by

$$\mathfrak{C}(\preccurlyeq_I, \mathbf{x}) = \{\mathbf{y} \in X^m : \mathbf{y} \preccurlyeq_I \mathbf{x} \text{ or } \mathbf{x} \preccurlyeq_I \mathbf{y}\}.$$

We shall introduce the concept of chain-uniqueness for common fixed point as follows.

Definition 2.9. Let $(X, \preceq)$ be a quasi-order set. Consider the functions $\mathbf{F} : X^m \to X^m$ and $\mathbf{f} : X^m \to X^m$ defined on the product set X^m into itself.

- Given a disjoint pair I and J of $\{1, \cdots, m\}$, the common fixed point $\widehat{\mathbf{x}} \in X^m$ of $\mathbf{F}$ and $\mathbf{f}$ is called $\preccurlyeq_I$*-chain-unique* if and only if, given any other common fixed point $\mathbf{x}$ of $\mathbf{F}$ and $\mathbf{f}$, if $\mathbf{x} \in \mathfrak{C}(\preccurlyeq_I, \widehat{\mathbf{x}})$, then $\mathbf{x} = \widehat{\mathbf{x}}$.
- The common fixed point $\widehat{\mathbf{x}} \in X^m$ of $\mathbf{F}$ and $\mathbf{f}$ is called *chain-unique* if and only if, given any other common fixed point $\mathbf{x}$ of $\mathbf{F}$ and $\mathbf{f}$, if $\mathbf{x} \in \mathfrak{C}(\preccurlyeq_{I^\circ}, \widehat{\mathbf{x}})$ for some disjoint pair I° and J° of $\{1, \cdots, m\}$, then $\mathbf{x} = \widehat{\mathbf{x}}$.

Theorem 2.16. (**Chain-Uniqueness**). *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is mixed-monotonically complete and preserves the mixed-monotone convergence. Assume that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving continuity. Given a disjoint pair I and J of $\{1, 2, \cdots, m\}$, consider the functions $\mathbf{F} : (X^m, \mathfrak{d}, \preccurlyeq_I) \to (X^m, \mathfrak{d}, \preccurlyeq_I)$ and $\mathbf{f} : (X^m, \mathfrak{d}, \preccurlyeq_I) \to (X^m, \mathfrak{d}, \preccurlyeq_I)$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a mixed $\preccurlyeq_I$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions:*

- *$\mathbf{F}$ and $\mathbf{f}$ are commutative;*
- *$\mathbf{f}$ has the sequentially mixed $\preccurlyeq_I$-monotone property or the $\preceq$-monotone property;*
- *$\mathbf{F}^p$ is continuous with respect to $\mathbf{f}$ on X^m;*
- *each f_k is continuous on X^m for $k = 1, \cdots, m$.*

Suppose that there exists a function of contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any $\mathbf{x}, \mathbf{y} \in X^m$ and any disjoint pair I° and J° of $\{1, \cdots, m\}$ with $\mathbf{y} \preccurlyeq_{I^\circ} \mathbf{x}$ or $\mathbf{x} \preccurlyeq_{I^\circ} \mathbf{y}$, the following inequalities

$$d\left(F_k^p(\mathbf{x}), F_k^p(\mathbf{y})\right) \leq \varphi\left(d\left(f_k(\mathbf{x}), f_k(\mathbf{y})\right)\right) \cdot d\left(f_k(\mathbf{x}), f_k(\mathbf{y})\right) \tag{2.29}$$

are satisfied for all $k = 1, \cdots, m$. Then, the following statements hold true.

(i) *$\mathbf{F}^p$ and $\mathbf{f}$ have a chain-unique common fixed point $\widehat{\mathbf{x}}$. Equivalently, if $\widehat{\mathbf{y}} \in \mathfrak{C}(\preccurlyeq_{I^\circ}, \widehat{\mathbf{x}})$ for some disjoint pair I° and J° of $\{1, \cdots, m\}$ is another common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$, then $\widehat{\mathbf{y}} = \widehat{\mathbf{x}}$.*

(ii) *For $p \neq 1$, suppose that $\mathbf{F}(\widehat{\mathbf{x}})$ and $\widehat{\mathbf{x}}$ obtained in* (i) *are comparable with respect to the quasi-order "$\preccurlyeq_{I^\circ}$" for some disjoint pair I° and J° of $\{1, \cdots, m\}$. Then, $\mathbf{F}$ and $\mathbf{f}$ have a chain-unique common fixed point $\widehat{\mathbf{x}}$.*

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n \in \mathbb{N}}$ constructed from (2.3) *for all $k = 1, \cdots, m$.*

Proof. To prove part (i), from Wu [30, Proposition 2.8] and part (i) of Theorem 2.10, we can show that $\widehat{\mathbf{x}}$ is a common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$. For the chain-uniqueness, let $\widehat{\mathbf{y}}$ be another common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$ with $\widehat{\mathbf{y}} \preccurlyeq_{I^\circ} \widehat{\mathbf{x}}$ or $\widehat{\mathbf{x}} \preccurlyeq_{I^\circ} \widehat{\mathbf{y}}$ for some disjoint pair I° and J° of $\{1,\cdots,m\}$, i.e., $\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}^p(\widehat{\mathbf{y}})$. By part (ii) of Theorem 2.10, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$. Therefore, by the triangle inequality, we have

$$d(\widehat{\mathbf{x}},\widehat{\mathbf{y}}) \leq d(\widehat{\mathbf{x}},\mathbf{f}(\widehat{\mathbf{x}})) + d(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}(\widehat{\mathbf{y}})) + d(\mathbf{f}(\widehat{\mathbf{y}}),\widehat{\mathbf{y}}) = 0,$$

which says that $\widehat{\mathbf{x}} = \widehat{\mathbf{y}}$. This proves part (i). Part (ii) can be similarly obtained by applying Theorem 2.10 to the argument in the proof of part (ii) of Theorem 2.15. This completes the proof. □

Remark 2.3. Suppose that the inequalities (2.27) and (2.29) in Theorems 2.15 and 2.16 are satisfied for any $\mathbf{x},\mathbf{y} \in X^m$. Then, from Remark 2.1 and the proofs of Theorems 2.15 and 2.16, we can see that parts (i) and (ii) can be combined together to conclude that $\mathbf{F}$ and $\mathbf{f}$ have a unique common fixed point $\widehat{\mathbf{x}}$.

Theorem 2.17. ($\preccurlyeq_I$**-Chain-Uniqueness**). *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is monotonically complete and preserves the monotone convergence. Assume that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving continuity. Given a disjoint pair I and J of $\{1,2,\cdots,m\}$, consider the functions $\mathbf{F} : (X^m,\mathfrak{d},\preccurlyeq_I) \to (X^m,\mathfrak{d},\preccurlyeq_I)$ and $\mathbf{f} : (X^m,\mathfrak{d},\preccurlyeq_I) \to (X^m,\mathfrak{d},\preccurlyeq_I)$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a $\preccurlyeq_I$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions:*

- *$\mathbf{F}$ and $\mathbf{f}$ are commutative;*
- *$\mathbf{f}$ is $\preccurlyeq_I$-monotone;*
- *$\mathbf{F}^p$ is continuous with respect to $\mathbf{f}$ on X^m;*
- *each f_k is continuous on X^m for $k = 1,\cdots,m$.*

Suppose that there exists a function of contractive factor $\varphi : [0,\infty) \to [0,1)$ such that, for any $\mathbf{x},\mathbf{y} \in X^m$ with $\mathbf{y} \preccurlyeq_I \mathbf{x}$ or $\mathbf{x} \preccurlyeq_I \mathbf{y}$, the following inequalities

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right) \leq \varphi\left(d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right)\right) \cdot d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right) \tag{2.30}$$

are satisfied for all $k = 1,\cdots,m$. Then, the following statements hold true.

(i) *$\mathbf{F}^p$ and $\mathbf{f}$ have a $\preccurlyeq_I$-chain-unique common fixed point $\widehat{\mathbf{x}}$.*
(ii) *For $p \neq 1$, suppose that $\mathbf{F}(\widehat{\mathbf{x}})$ and $\widehat{\mathbf{x}}$ obtained in* (i) *are comparable with respect to "$\preccurlyeq_I$". Then, $\mathbf{F}$ and $\mathbf{f}$ have a $\preccurlyeq_I$-chain-unique common fixed point $\widehat{\mathbf{x}}$.*

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed from (2.3) *for all $k = 1,\cdots,m$.*

Proof. To prove part (i), from Wu [30, Proposition 2.8] and part (i) of Theorem 2.11, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{F}^p(\widehat{\mathbf{x}})$. From Theorem 2.3, we also have $\mathbf{F}^p(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}$. Therefore, we obtain

$$\widehat{\mathbf{x}} = \mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{F}^p(\widehat{\mathbf{x}}).$$

This shows that $\widehat{\mathbf{x}}$ is a common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$. For the $\preccurlyeq_I$-chain-uniqueness, let $\widehat{\mathbf{y}}$ be another common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$ such that $\widehat{\mathbf{y}}$ and $\widehat{\mathbf{x}}$ are comparable with respect to "$\preccurlyeq_I$', i.e., $\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) =$

$\mathbf{F}^p(\widehat{\mathbf{y}})$. By part (ii) of Theorem 2.11, we have $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$. Therefore, by the triangle inequality, we have

$$d(\widehat{\mathbf{x}},\widehat{\mathbf{y}}) \leq d(\widehat{\mathbf{x}},\mathbf{f}(\widehat{\mathbf{x}})) + d(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}(\widehat{\mathbf{y}})) + d(\mathbf{f}(\widehat{\mathbf{y}}),\widehat{\mathbf{y}}) = 0,$$

which says that $\widehat{\mathbf{x}} = \widehat{\mathbf{y}}$. This proves part (i). Part (ii) can be obtained by applying part (iii) of Theorem 2.11 to the similar argument in the proof of Theorem 2.15. This completes the proof. □

Theorem 2.18. (**Chain-Uniqueness**). *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is monotonically complete and preserves the monotone convergence. Assume that the metrics $\mathfrak{d}$ and d are compatible in the sense of preserving continuity. Given a disjoint pair I and J of $\{1,2,\cdots,m\}$, consider the functions $\mathbf{F} : (X^m,\mathfrak{d},\preccurlyeq_I) \to (X^m,\mathfrak{d},\preccurlyeq_I)$ and $\mathbf{f} : (X^m,\mathfrak{d},\preccurlyeq_I) \to (X^m,\mathfrak{d},\preccurlyeq_I)$ satisfying $\mathbf{F}^p(X^m) \subseteq \mathbf{f}(X^m)$ for some $p \in \mathbb{N}$. Let $\mathbf{x}_0$ be a $\preccurlyeq_I$-monotone seed element in X^m. Assume that the functions $\mathbf{F}$ and $\mathbf{f}$ satisfy the following conditions:*

- *$\mathbf{F}$ and $\mathbf{f}$ are commutative;*
- *$\mathbf{f}$ is $\preccurlyeq_I$-monotone;*
- *$\mathbf{F}^p$ is continuous with respect to $\mathbf{f}$ on X^m;*
- *each f_k is continuous on X^m for $k = 1,\cdots,m$.*

Suppose that there exists a function of contractive factor $\varphi : [0,\infty) \to [0,1)$ such that, for any $\mathbf{x},\mathbf{y} \in X^m$ and any disjoint pair I° and J° of $\{1,\cdots,m\}$ with $\mathbf{y} \preccurlyeq_{I^\circ} \mathbf{x}$ or $\mathbf{x} \preccurlyeq_{I^\circ} \mathbf{y}$, the following inequalities

$$d\left(F_k^p(\mathbf{x}),F_k^p(\mathbf{y})\right) \leq \varphi\left(d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right)\right) \cdot d\left(f_k(\mathbf{x}),f_k(\mathbf{y})\right) \tag{2.31}$$

are satisfied for all $k = 1,\cdots,m$. Then, the following statements hold true.

(i) *$\mathbf{F}^p$ and $\mathbf{f}$ have a chain-unique common fixed point $\widehat{\mathbf{x}}$.*

(ii) *For $p \neq 1$, suppose that $\mathbf{F}(\widehat{\mathbf{x}})$ and $\widehat{\mathbf{x}}$ obtained in* (i) *are comparable with respect to the quasi-order "$\preccurlyeq_{I^\circ}$" for some disjoint pair I° and J° of $\{1,\cdots,m\}$. Then, $\mathbf{F}$ and $\mathbf{f}$ have a chain-unique common fixed point $\widehat{\mathbf{x}}$.*

Moreover, each component $\widehat{x}^{(k)}$ of $\widehat{\mathbf{x}}$ is the limit of the sequence $\{f_k(\mathbf{x}_n)\}_{n\in\mathbb{N}}$ constructed from (2.3) *for all $k = 1,\cdots,m$.*

Proof. To prove part (i), from the proof of Theorem 2.17, we can show that $\widehat{\mathbf{x}}$ is a common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$. For the chain-uniqueness, let $\widehat{\mathbf{y}}$ be another common fixed point of $\mathbf{F}^p$ and $\mathbf{f}$ with $\widehat{\mathbf{y}} \preccurlyeq_{I^\circ} \widehat{\mathbf{x}}$ or $\widehat{\mathbf{x}} \preccurlyeq_{I^\circ} \widehat{\mathbf{y}}$ for some disjoint pair I° and J° of $\{1,\cdots,m\}$, i.e., $\widehat{\mathbf{y}} = \mathbf{f}(\widehat{\mathbf{y}}) = \mathbf{F}^p(\widehat{\mathbf{y}})$. If $f_k(\widehat{\mathbf{x}}) \neq f_k(\widehat{\mathbf{y}})$, i.e., $d(f_k(\widehat{\mathbf{x}}),f_k(\widehat{\mathbf{y}})) \neq 0$, then, from (2.31), we obtain

$$d(f_k(\widehat{\mathbf{x}}),f_k(\widehat{\mathbf{y}})) = d\left(F_k^p(\widehat{\mathbf{x}}),F_k^p(\widehat{\mathbf{y}})\right) \leq \varphi\left(d(f_k(\widehat{\mathbf{x}}),f_k(\widehat{\mathbf{y}}))\right) d(f_k(\widehat{\mathbf{x}}),f_k(\widehat{\mathbf{y}})) < d(f_k(\widehat{\mathbf{x}}),f_k(\widehat{\mathbf{y}})),$$

since $\varphi(t) < 1$ for any $t \in [0,\infty)$. This contradiction says that $f_k(\widehat{\mathbf{x}}) = f_k(\widehat{\mathbf{y}})$ for all $k = 1,\cdots,m$, i.e., $\mathbf{f}(\widehat{\mathbf{x}}) = \mathbf{f}(\widehat{\mathbf{y}})$. Therefore, by the triangle inequality, we have

$$d(\widehat{\mathbf{x}},\widehat{\mathbf{y}}) \leq d(\widehat{\mathbf{x}},\mathbf{f}(\widehat{\mathbf{x}})) + d(\mathbf{f}(\widehat{\mathbf{x}}),\mathbf{f}(\widehat{\mathbf{y}})) + d(\mathbf{f}(\widehat{\mathbf{y}}),\widehat{\mathbf{y}}) = 0,$$

which says that $\widehat{\mathbf{x}} = \widehat{\mathbf{y}}$. This proves part (i). Part (ii) can be similarly obtained by applying Theorem 2.10 to the argument in the proof of part (ii) of Theorem 2.15. This completes the proof. □

Remark 2.4. Suppose that the inequalities (2.30) and (2.31) in Theorems 2.17 and 2.18 are satisfied for any $\mathbf{x}, \mathbf{y} \in X^m$. Then, from Remark 2.2 and the proofs of Theorems 2.17 and 2.18, we can see that parts (i) and (ii) can be combined together to say that $\mathbf{F}$ and $\mathbf{f}$ have a unique common fixed point $\widehat{\mathbf{x}}$.

3. Applications to Coupled Fixed Points

Let us consider the function $G : X^2 \to X$. The concept of coupled fixed point was proposed by Bhaskar and Lakshmikantham [4]. We say that $(\widehat{x}^{(1)}, \widehat{x}^{(2)}) \in X^2$ is a *coupled fixed point* of G if and only if $G(\widehat{x}^{(1)}, \widehat{x}^{(2)}) = \widehat{x}^{(1)}$ and $G(\widehat{x}^{(2)}, \widehat{x}^{(1)}) = \widehat{x}^{(2)}$.

Now, we assume that the function $G : X^2 \to X$ has the I-mixed-monotone property by taking $I = \{1\}$ and $J = \{2\}$. This says that $G(x^{(1)}, x^{(2)})$ is increasing with respect to the first variable $x^{(1)}$ and decreasing with respect to the second variable $x^{(2)}$. Of course, we can take $I = \{2\}$ and $J = \{1\}$. However, in this case, there will be no significant difference compared with the case of $I = \{1\}$ and $J = \{2\}$. In this case, we can simply say that $G : X^2 \to X$ has the mixed-monotone property if and only if $G(x^{(1)}, x^{(2)})$ is increasing with respect to the first variable and decreasing with respect to the second variable. The following useful result is not hard to prove.

Proposition 3.1. *Let X be a universal set. Consider the mapping $G : X^2 \to X$. Define a mapping $\mathbf{F} : X^2 \to X^2$ by $F_1(x^{(1)}, x^{(2)}) = G(x^{(1)}, x^{(2)})$ and $F_2(x^{(1)}, x^{(2)}) = G(x^{(2)}, x^{(1)})$. Then $(\widehat{x}^{(1)}, \widehat{x}^{(2)})$ is a fixed point of $\mathbf{F}$ if and only if it is a coupled fixed point of G.*

Theorem 3.1. *Suppose that the quasi-ordered metric space $(X, d, \preceq)$ is monotonically complete, and that the metrics $\mathfrak{d}$ on X^2 and d are compatible in the sense of preserving continuity. Assume that the function $G : (X^2, \mathfrak{d}) \to (X, d)$ is continuous and owns the mixed-monotone property on X^2, and that there exists a function of contractive factor $\varphi : [0, \infty) \to [0, 1)$ such that, for any $(x^{(1)}, x^{(2)}), (y^{(1)}, y^{(2)}) \in X^2$ with $x^{(1)} \preceq y^{(1)}$ and $x^{(2)} \succeq y^{(2)}$, the following inequality*

$$d\left(G\left(x^{(1)}, x^{(2)}\right), G\left(y^{(1)}, y^{(2)}\right)\right) \leq \varphi\left(d\left(x^{(1)}, y^{(1)}\right)\right) \cdot d\left(x^{(1)}, y^{(1)}\right) \tag{3.1}$$

is satisfied. Suppose that there exist $x_0, y_0 \in X$ such that $x_0 \preceq G(x_0, y_0)$ and $G(y_0, x_0) \succeq y_0$. Then there exists a coupled fixed point of G.

Proof. We define a mapping $\mathbf{F} : (X^2, \mathfrak{d}) \to (X^2, \mathfrak{d})$ by $F_1(x, y) = G(x, y)$ and $F_2(x, y) = G(y, x)$. Since G owns the I-mixed-monotone property on X^2 for $I = \{1\}$ and $J = \{2\}$, it follows that F_1 owns the I-mixed-monotone property and F_2 owns the J-mixed-monotone property on X^2. From Wu [30, Remark 4.3], we see that F_1 is $(\preccurlyeq_I, \preceq)$-increasing and F_2 is $(\preccurlyeq_I, \preceq)$-decreasing. This shows that $\mathbf{F}$ is $(\preccurlyeq_I, \preccurlyeq_I)$-increasing. Since G is continuous and the metrics $\mathfrak{d}$ on X^2 and d are compatible in the sense of preserving continuity, it follows that $\mathbf{F}$ is also continuous. Since there exist $x_0, y_0 \in X$ such that $x_0 \preceq G(x_0, y_0)$ and $G(y_0, x_0) \succeq y_0$, it follows that (x_0, y_0) and $\mathbf{F}(x_0, y_0)$ are comparable with respect to "$\preccurlyeq_I$". From (3.1), we see that

$$d\left(G\left(x^{(2)}, x^{(1)}\right), G\left(y^{(2)}, y^{(1)}\right)\right) \leq \varphi\left(d\left(x^{(2)}, y^{(2)}\right)\right) \cdot d\left(x^{(2)}, y^{(2)}\right). \tag{3.2}$$

From (3.1) and (3.2), if $(x^{(1)},x^{(2)}) \preccurlyeq_I (y^{(1)},y^{(2)})$ or $(y^{(1)},y^{(2)}) \preccurlyeq_I (x^{(1)},x^{(2)})$, then

$$d\left(F_1\left(x^{(1)},x^{(2)}\right),F_1\left(y^{(1)},y^{(2)}\right)\right) \leq \varphi\left(d\left(x^{(1)},y^{(1)}\right)\right)\cdot d\left(x^{(1)},y^{(1)}\right)$$

and

$$d\left(F_2\left(x^{(1)},x^{(2)}\right),F_2\left(y^{(1)},y^{(2)}\right)\right) \leq \varphi\left(d\left(x^{(2)},y^{(2)}\right)\right)\cdot d\left(x^{(2)},y^{(2)}\right).$$

The result follows from Proposition 3.1 and Theorem 2.5 immediately. □

Next, we can consider the chain-uniqueness and drop the assumption of continuity of G by assuming that $(X,d,\preceq)$ preserves the monotone convergence. We first introduce the chain-uniqueness for the coupled fixed point.

For any $(x^{(1)},x^{(2)}),(y^{(1)},y^{(2)}) \in X^2$, we write

$$\left(x^{(1)},x^{(2)}\right) \precsim \left(y^{(1)},y^{(2)}\right) \text{ if and only if } x^{(1)} \preceq y^{(1)} \text{ and } x^{(2)} \succeq y^{(2)}.$$

Given any $(x^{(1)},x^{(2)}) \in X^2$, we define the chain $\mathfrak{C}(\precsim,(x^{(1)},x^{(2)}))$ containing $(x^{(1)},x^{(2)})$ as follows:

$$\begin{aligned}\mathfrak{C}\left(\precsim,\left(x^{(1)},x^{(2)}\right)\right) &= \left\{\left(y^{(1)},y^{(2)}\right) \in X^2 : \left(y^{(1)},y^{(2)}\right) \precsim \left(x^{(1)},x^{(2)}\right) \text{ or } \left(x^{(1)},x^{(2)}\right) \precsim \left(y^{(1)},y^{(2)}\right)\right\}\\ &= \left\{\left(y^{(1)},y^{(2)}\right) \in X^2 : \left(x^{(1)},x^{(2)}\right) \text{ and } \left(y^{(1)},y^{(2)}\right) \text{ are comparable with respect to ``}\precsim\text{''}\right\}.\end{aligned}$$

Definition 3.1. Let $(X,d,\preceq)$ be a metric space endowed with a quasi-order "$\preceq$". Consider the function $G : X^2 \to X$. The coupled fixed point $(\widehat{x}^{(1)},\widehat{x}^{(2)}) \in X^2$ of G is called *chain-unique* if and only if, given any other coupled fixed point $(x^{(1)},x^{(2)})$ of G, if $(x^{(1)},x^{(2)}) \in \mathfrak{C}(\precsim,(\widehat{x}^{(1)},\widehat{x}^{(2)}))$, then $(x^{(1)},x^{(2)}) = (\widehat{x}^{(1)},\widehat{x}^{(2)})$.

Theorem 3.2. *Suppose that the quasi-ordered metric space $(X,d,\preceq)$ is monotonically complete and preserves the monotone convergence, and that the metrics $\mathfrak{d}$ on X^2 and d are compatible in the sense of preserving convergence. Assume that the function $G : X^2 \to X$ owns the mixed-monotone property on X^2, and that there exists a function of contractive factor $\varphi : [0,\infty) \to [0,1)$ such that, for any $(x^{(1)},x^{(2)}),(y^{(1)},y^{(2)}) \in X^2$ with $x^{(1)} \preceq y^{(1)}$ and $x^{(2)} \succeq y^{(2)}$, the following inequality*

$$d\left(G\left(x^{(1)},x^{(2)}\right),G\left(y^{(1)},y^{(2)}\right)\right) \leq \varphi\left(d\left(x^{(1)},y^{(1)}\right)\right)\cdot d\left(x^{(1)},y^{(1)}\right)$$

is satisfied. Suppose that there exist $x_0,y_0 \in X$ such that $x_0 \preceq G(x_0,y_0)$ and $G(y_0,x_0) \succeq y_0$. Then there exists a chain-unique coupled fixed point of G.

Proof. From the proof of Theorem 3.1, since $I = \{1\}$ and $J = \{2\}$ is a disjoint pair of $\{1,2\}$. It follows that the partial order "$\precsim$" is equivalent to the partial order "$\preccurlyeq_I$". By applying Theorem 2.13 to the similar argument in the proof of Theorem 3.1, the mapping $\mathbf{F} : X^2 \to X^2$ defined by $F_1(x,y) = G(x,y)$ and $F_2(x,y) = G(y,x)$ has a $\preccurlyeq_I$-chain-unique fixed point, which, equivalently, says that G has a chain-unique coupled fixed point. This completes the proof. □

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