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ON A CLASS OF SPLIT EQUALITY FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. The purpose of this paper is to study the split equality fixed point problem for quasi-asymptotically pseudo-contractive mappings via an iterative algorithm. Weak and strong convergence theorems are proved in the framework of infinite dimensional Hilbert spaces. The results presented in the article generalize and improve some recent results.

Keywords. Hilbert space; Split equality fixed point problem; Quasi-asymptotically pseudo-contractive mapping; Quasi-asymptotically nonexpansive mapping.

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1. INTRODUCTION-PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $||\cdot||$, respectively. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be an operator. We use $Fix(T)$ to denote the fixed point set of mapping T, i.e., $Fix(T) = \{x \in C : x = Tx\}$.

Recall that $T : C \to C$ is said to be nonexpansive iff

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

$T : C \to C$ is said to be quasi-nonexpansive iff $Fix(T) \neq \emptyset$ and

$$\|Tx - p\| \le \|x - p\|, \quad \forall x \in C, p \in Fix(T).$$

$T : C \to C$ is said to be asymptotically nonexpansive iff there exists a sequence $\{l_n\} \subset [1, +\infty)$ with $l_n \to 1$ such that

$$\|T^n x - T^n y\| \le l_n \|x - y\|, \quad \forall x, y \in C.$$

$T : C \to C$ is said to be quasi-asymptotically nonexpansive iff $Fix(T) \neq \emptyset$ and

$$\|T^n x - p\| \le l_n \|x - p\|, \quad \forall x \in C, p \in Fix(T).$$

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$T : C \to C$ is said to be pseudo-contractive if

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2, \quad \forall x, y \in C.$$

It is easy to know that T is pseudo-contractive iff

$$||Tx - Ty||^2 \le ||x - y||^2 + ||(I - T)x - (I - T)y||^2, \quad \forall x, y \in C.$$

$T : C \to C$ is said to be quasi-pseudo-contractive iff $Fix(T) \neq \emptyset$ and

$$||Tx - p||^2 \le ||x - p||^2 + ||Tx - x||^2, \quad \forall x \in C, \forall p \in Fix(T).$$

$T : C \to C$ is said to be asymptotically pseudo-contractive iff there exists a sequence $\{l_n\} \subset [1, +\infty)$ with $l_n \to 1$ such that

$$||T^n x - T^n y||^2 \le l_n ||x - y||^2 + ||(I - T^n)x - (I - T^n)y||^2,$$

for all $x, y \in C$ and for all $n \ge 1$.

It is well known that T is asymptotically pseudo-contractive iff

$$\langle T^n x - T^n y, x - y\rangle \le \frac{l_n + 1}{2} ||x - y||^2,$$

for all $x, y \in C$ and $n \ge 1$.

$T : C \to C$ is said to be quasi-asymptotically pseudo-contractive iff $Fix(T) \neq \emptyset$ and there exists a sequence $\{l_n\} \subset [1, +\infty)$ with $l_n \to 1$ such that

$$||T^n x - p||^2 \le l_n ||x - p||^2 + ||T^n x - x||^2,$$

for all $x \in C, p \in Fix(T)$ and for all $n \ge 1$.

Next we give an example of quasi-asymptotically pseudo-contractive mapping.

Example 1.1. [10] Let C be a unit ball in a real Hilbert space l^2 and let $S : C \to C$ be a mapping defined by

$$S : (x_1, x_2, \cdots) \to (0, x_1^2, a_2 x_2, a_3 x_3, \cdots).$$

It is proved in [10] that

(i) $||Sx - Sy||^2 \le 2^2 ||x - y||^2, \ \forall x, y \in C$;

(ii) $||S^n x - S^n y||^2 \le (2\prod_{j=2}^n a_j)^2 ||x - y||^2, \ \forall x, y \in C, \ n \ge 2$.

Taking $a_j = 2^{-\frac{1}{2^{j-1}}}$, $j \ge 2$, it is easy to see that $\prod_{j=2}^n a_j = \frac{1}{2}$. Letting $l_1 = 4$, $l_n = (2\prod_{j=2}^n a_j)^2$, $\forall n \ge 2$, then we have

$$\lim_{n\to\infty} l_n = \lim_{n\to\infty} (2\prod_{j=2}^n 2^{-\frac{1}{2^{j-1}}})^2 = 1,$$

and

$$||S^n x - S^n y||^2 \le l_n ||x - y||^2 \le l_n ||x - y||^2 + ||(I - S^n)x - (I - S^n)y||^2, \ \forall x, y, \in C.$$

This shows that $S : C \to C$ is an asymptotically pseudo-contractive mapping with $Fix(S) = \{(0, 0, 0, \cdots)\}$, so it is also a quasi-asymptotically pseudo-contractive mapping.

Recall that $T: C \to C$ is said to be uniformly L-Lipschtzian iff there exists some $L > 0$ such that

$$||T^n x - T^n y|| \leq L||x-y||,$$

for all $x, y \in C$ and for all $n \geq 1$.

The split equality fixed point problem, which is a generalization of the split feasibility problem and of the convex feasibility problem, has been extensively investigated recently due to its extraordinary utility and broad applicability in many areas of applied mathematics. Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A: H_1 \to H_2$ be a bounded linear operator.

Recall that the *split feasibility problem* (SFP) is formulated as to find a point $q \in H_1$ such that:

$$q \in C \ \ and \ \ Aq \in Q. \tag{1.1}$$

Let P_C and P_Q be the metric projections from H_1 and H_2 to C and Q, respectively. It is easy to see that $q \in C$ solves equation (1.1) iff it solves the following fixed point equation

$$q = P_C(I - \gamma A^*(I - P_Q)A)q,$$

where $\gamma > 0$ is a real constant and A^* is the adjoint of A.

In 1994, Censor and Elfving [5] first introduced the (SFP) in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [1]. It has been found that the (SFP) can also be used in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning [6, 7, 8]. The (SFP) in an infinite dimensional Hilbert space can be found in [2, 3, 14, 15, 16].

Recently, Moudafi [11], Moudafi and Al-Shemas [12] and Moudafi [13] introduced the following *split equality feasibility problem* (SEFP):

$$to\ find \quad x \in C, \quad y \in Q \quad such\ that \quad Ax = By, \tag{1.2}$$

where $A: H_1 \to H_3$ and $B: H_2 \to H_3$ are two bounded linear operators. Obviously, if $B = I$ (identity mapping on H_2) and $H_3 = H_2$, then (1.2) reduces to (1.1).

In order to solve split equality feasibility problem (1.2), Moudafi [11] introduced the following simultaneous iterative algorithm:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \beta B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.3}$$

and under suitable conditions some weak convergence of the sequence $\{(x_n, y_n)\}$ to a solution of (1.2) in Hilbert spaces is proved.

In order to avoid using the projection, recently, Moudafi [12] introduced and studied the following problem: Let $T: H_1 \to H_1$ and $S: H_2 \to H_2$ be nonlinear operators such that $Fix(T) \neq \emptyset$ and $Fix(S) \neq \emptyset$. If $C = Fix(T)$ and $Q = Fix(S)$, then the split equality feasibility problem (1.2) reduces to:

$$find \ \ x \in Fix(T) \ \ and \ \ y \in Fix(S) \ \ such\ that\ Ax = By, \tag{1.4}$$

which is called *split equality fixed point problem (in short, (SEFPP))*. In the sequel, we denote by Γ the solution set of split equality fixed point problem (1.4).

Recently Moudafi [13] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.4):

$$\begin{cases} x_{n+1} = T(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = S(y_n + \beta_n B^*(Ax_{n+1} - By_n)). \end{cases} \tag{1.5}$$

He also studied the weak convergence of the sequences generated by scheme (1.5) under the condition that T and S are firmly quasi-nonexpansive mappings.

In 2015, Che and Li [9] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.4):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tu_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)Sv_n. \end{cases} \tag{1.6}$$

They also established the weak convergence of scheme (1.6) under the condition that the operators T and S are quasi-nonexpansive mappings.

Recently, Chang, Wang and Qin [4] proposed the following iterative algorithm for finding a solution of (SEFPP) (1.4):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)((1-\xi_n)I + \xi_n T((1-\eta_n)I + \eta_n T))u_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)((1-\xi_n)I + \xi_n S((1-\eta_n)I + \eta_n S))v_n. \end{cases} \tag{1.7}$$

They established the weak convergence of scheme (1.7) under the condition that operators T and S are quasi-pseudo-contractive mappings which is more general than the classes of quasi-nonexpansive mappings, directed mappings and demi-contractive mappings.

Motived by above results, the purpose of this paper is to consider split equality fixed point problem (1.4) for a class of quasi-asymptotically pseudo-contractive mappings which is more general than the classes of quasi-nonexpansive mappings and quasi-pseudo-contractive mappings. Under suitable conditions, some weak and strong convergence theorems are proved.

To prove our main results, the following definitions and tools are essential.

Recall that an operator $T : C \to C$ is said to be demiclosed at 0 if, for any sequence $\{x_n\} \subset C$ which converges weakly to x and $||x_n - T(x_n)|| \to 0$, then $Tx = x$.

$T : H \to H$ is said to be semi-compact if, for any bounded sequence $\{x_n\} \subset H$ with $||x_n - Tx_n|| \to 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x \in H$.

Lemma 1.1. *Let H be a real Hilbert space. For any $x, y \in H$, the following conclusions hold:*

$$||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x-y||^2, \; t \in [0,1]; \tag{1.8}$$

$$||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle. \tag{1.9}$$

Recall that a Banach space X is said to satisfy Opial's condition, if for any sequence $\{x_n\}$ in X which converges weakly to x^*, we have

$$\liminf_{n\to\infty} ||x_n - x^*|| < \liminf_{n\to\infty} ||x_n - y||, \ \forall y \in X \text{with } y \neq x^*,$$

which is equivalent to

$$\limsup_{n\to\infty} ||x_n - x^*|| < \limsup_{n\to\infty} ||x_n - y||, \ \forall y \in X \text{with } y \neq x^*.$$

Lemma 1.2. *Let $\{p_n\}, \{q_n\}$ and $\{r_n\}$ be the nonnegative real sequences satisfying the following conditions*

$$p_{n+1} \leq (1+q_n)p_n + r_n, \ n \geq 0, \ \sum_{n=1}^{\infty} q_n < \infty, \text{ and } \sum_{n=1}^{\infty} r_n < \infty.$$

Then (i) $\lim_{n\to\infty} p_n$ exists; (ii) if $\liminf_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

Lemma 1.3. *Let H be a real Hilbert space, $T : H \to H$ be a uniformly L-Lipschitzian and $\{l_n\}-$ quasi-asymptotically pseudocontractive mapping with $L \geq 1$, $\{l_n\} \subset [1,\infty)$ and $\lim_{n\to\infty} l_n = 1$. Let $\{K_n : H \to H\}$ be a sequence of mappings defined by:*

$$K_n := \xi T^n(\eta T^n + (1-\eta)I) + (1-\xi)I. \tag{1.10}$$

If $0 < \xi < \eta < \frac{1}{\frac{M+1}{2} + \sqrt{\frac{(M+1)^2}{4} + L^2}}$, where $M = \sup_{n\geq 1} l_n$, then the following conclusions hold:

(i) $Fix(T) = Fix(T^n((1-\eta)I + \eta T^n)) = Fix(K_n)$ for all $n \geq 1$;

(ii) If T is demiclosed at 0, then K_1 is also demiclosed at 0;

(iii) For all $n \geq 1$ and for all $x \in H, u^* \in Fix(T) = Fix(K_n)$,

$$||K_n x - u^*|| \leq k_n ||x - u^*||,$$

where $k_n = 1 + \xi(l_n - 1)(1 + \eta l_n), \{k_n\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_n = 1$.

Proof. (i) If $x^* \in Fix(T)$, i.e., $x^* = Tx^*$, we have

$$T^n((1-\eta)I + \eta T^n)x^* = T^n((1-\eta)x^* + \eta T^n x^*) = T^n x^* = x^*.$$

This shows that $x^* \in Fix(T^n((1-\eta)I + \eta T^n))$. Conversely, if $x^* \in Fix(T^n((1-\eta)I + \eta T^n))$ for all $n \geq 1$, i.e., $x^* = T^n((1-\eta)I + \eta T^n)x^*$, letting $U^n = (1-\eta)I + \eta T^n$, we have $T^n U^n x^* = x^*$. Put $U^n x^* = y^*$, then $T^n y^* = x^*$. Now we prove that $x^* = y^*$. In fact, we have

$$\begin{aligned} ||x^* - y^*|| =& ||x^* - U^n x^*|| = ||x^* - ((1-\eta)I + \eta T^n)x^*|| \\ =& \eta ||x^* - T^n x^*|| = \eta ||T^n y^* - T^n x^*|| \leq L\eta ||x^* - y^*||. \end{aligned}$$

Since $0 < L\eta < 1$, we have $x^* = y^*$, i.e.,$x^* \in Fix(T)$. This shows that $Fix(T) = Fix(T^n((1-\eta)I + \eta T^n))$ for all $n \geq 1$. It is obvious that $x \in Fix(K_n)$ if and only if $x \in Fix(T^n((1-\eta)I + \eta T^n))$. The conclusion (i) is proved.

(ii) For any sequence $\{x_n\} \subset H$ satisfying $x_n \rightharpoonup x^*$ and $||x_n - Kx_n|| \to 0$. Next we show that $x^* \in Fix(K)$. From conclusion (i), it suffices to prove $x^* \in Fix(T)$.

In fact, since T is L-Lipschizian, we have

$$\begin{aligned}\xi||x_n - Tx_n|| \leq& \xi||T((1-\eta)I+\eta T)x_n - Tx_n|| + \xi||x_n - T((1-\eta)I+\eta T)x_n||\\ \leq& \xi L\eta||x_n - Tx_n|| + ||x_n - (1-\xi)x_n - \xi T((1-\eta)I+\eta T)x_n||\\ =& \xi L\eta||x_n - Tx_n|| + ||x_n - K_1x_n||.\end{aligned}$$

Simplifying it, we have

$$||x_n - Tx_n|| \leq \frac{1}{\xi(1-L\eta)}||x_n - K_1x_n|| \to 0. \tag{1.11}$$

Since T is demiclosed at 0, we have $x^* \in F(T) = F(K)$. The conclusion (ii) is proved.

(iii) For all $u^* \in Fix(T)$, we have

$$\begin{aligned}&||T^n((1-\eta)I+\eta T^n)x - u^*||^2\\ \leq& l_n||(1-\eta)x+\eta T^n x - u^*||^2 + ||((1-\eta)I+\eta T^n)x - T^n((1-\eta)I+\eta T^n)x||^2\\ =&\eta T^n)x - T^n((1-\eta)I+\eta T^n)x||^2 + l_n||(1-\eta)(x-u^*)+\eta(T^n x - u^*)||^2 + ||((1-\eta)I\end{aligned} \tag{1.12}$$

and

$$||T^n x - u^*||^2 \leq l_n||x-u^*||^2 + ||x - T^n x||^2. \tag{1.13}$$

Since T is L-Lip and $x - ((1-\eta)x+\eta T^n x) = \eta(x - T^n x)$ we have

$$||T^n x - T^n((1-\eta)x+\eta T^n x)|| \leq L||x - ((1-\eta)x+\eta T^n x)|| = L\eta||x - T^n x||. \tag{1.14}$$

From (1.8) and (1.14), we have

$$\begin{aligned}&||(1-\eta)(x-u^*)+\eta(T^n x - u^*)||^2\\ =&(1-\eta)||x-u^*||^2 + \eta||T^n x - u^*||^2 - \eta(1-\eta)||x - T^n x||^2\\ \leq&(1-\eta)||x-u^*||^2 + \eta(l_n||x-u^*||^2 + ||x-T^n x||^2) - \eta(1-\eta)||x-T^n x||^2\\ =&(1+\eta(l_n-1))||x-u^*||^2 + \eta^2||x-T^n x||^2.\end{aligned} \tag{1.15}$$

From (1.8) and (1.14), we have

$$\begin{aligned}&||((1-\eta)I+\eta T^n)x - T^n((1-\eta)I+\eta T^n)x||^2\\ =&(1-\eta)||x - T^n((1-\eta)x+\eta T^n x)||^2 + \eta||T^n x - T^n((1-\eta)x+\eta T^n x)||^2\\ &-\eta(1-\eta)||x-T^n x||^2\\ \leq&(1-\eta)||x - T^n((1-\eta)x+\eta T^n x)||^2 - \eta(1-\eta-\eta^2L^2)||x-T^n x||^2.\end{aligned} \tag{1.16}$$

Substituting (1.15) and (1.16) into (1.12), we obtain

$$\begin{aligned} &||T^n((1-\eta)I+\eta T^n)x-u^*||^2 \\ \le& l_n(1+\eta(l_n-1))||x-u^*||^2+l_n\eta^2||T^nx-x||^2 \\ &+(1-\eta)||x-T^n((1-\eta)x+\eta T^nx)||^2-\eta(1-\eta-\eta^2L^2)||T^nx-x||^2 \\ =& l_n(1+\eta(l_n-1))||x-u^*||^2+(1-\eta)||x-T^n((1-\eta)x+\eta T^nx)||^2 \\ &+\eta(\eta+\eta^2L^2+l_n\eta-1)||T^nx-x||^2. \end{aligned} \tag{1.17}$$

Since $\eta < \frac{1}{\frac{M+1}{2}+\sqrt{\frac{(M+1)^2}{4}+L^2}}$, we find from (1.17) that

$$\begin{aligned} &||T^n((1-\eta)x+\eta T^nx)-u^*||^2 \\ \le& l_n(1+\eta(l_n-1))||x-u^*||^2+(1-\eta)||x-T^n((1-\eta)x+\eta T^nx)||^2. \end{aligned} \tag{1.18}$$

Combine (1.8) and (1.18) we have that

$$\begin{aligned} &||K_nx-u^*||^2 \\ =&(1-\xi)||x-u^*||^2+\xi||T^n((1-\eta)x+\eta T^nx)-u^*||^2-\xi(1-\xi)||x-T^n((1-\eta)x+\eta T^nx)||^2 \\ \le&(1-\xi)||x-u^*||^2+\xi l_n(1+\eta(l_n-1))||x-u^*||^2 \\ &+(\xi(1-\eta)-\xi(1-\xi))||x-T^n((1-\eta)x+\eta T^nx)||^2 \\ =&(1+\xi(l_n-1)(1+\eta l_n))||x-u^*||^2-\xi(\eta-\xi)||x-T^n((1-\eta)x+\eta T^nx)||^2. \end{aligned}$$

This together with $\xi < \eta$ implies that

$$||K_nx-u^*||^2 \le k_n||x-u^*||^2$$

for all $x \in H, u^* \in Fix(K_n)$ and $n \ge 1$, where

$$k_n = \xi(l_n-1)(1+\eta l_n)+1.$$

In view of that $\{l_n\} \subset [1,+\infty)$ and $\lim_{n\to\infty} l_n = 1$ we have $\{k_n\} \subset [1,+\infty)$ and $\lim_{n\to\infty} k_n = 1$. The conclusion (iii) is proved. □

2. Main results

Now we are in a position to give the following main result.

Theorem 2.1. *Let H_1, H_2 and H_3 be three real Hilbert spaces, $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators, A^* and B^* be the adjoint operators of A and B, respectively. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two uniformly L-Lipschitzian and $\{l_n\}$-quasi-asymptotically pseudo-contractive mappings with $L \ge 1$, $l_n \in [1,\infty)$, $l_n \to 1$ and $\Sigma_{n=1}^{\infty}(l_n^2-1) < \infty$, $Fix(T) \ne \emptyset$, and $Fix(S) \ne \emptyset$. Let $\{\alpha_{n,i}\}$ be the sequence in $(0,1)$ satisfying the following conditions:*

$$(i)\ \sum_{i=0}^{n}\alpha_{n,i} = 1,\ for\ each\ n \ge 1;\ \liminf_{n\to\infty}\alpha_{n,i} > 0,\ \sum_{n=1}^{\infty}\alpha_{n,i} < \infty,\ for\ each\ i \ge 0.$$

Suppose further that for any given $x_0 \in H_1, y_0 \in H_2$, $\{(x_n, y_n)\}$ is the sequence generated by:

$$\begin{cases} (a)\ u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ (b)\ x_{n+1} = \alpha_{n,0} x_n + \sum_{i=1}^{n} \alpha_{n,i} K_i u_n, \\ (c)\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ (d)\ y_{n+1} = \alpha_{n,0} y_n + \sum_{i=1}^{n} \alpha_{n,i} G_i v_n, \end{cases} \tag{2.1}$$

where

$$K_i = (1-\xi)I + \xi T^i((1-\eta)I + \eta T^i); \ \ and \ \ G_i = (1-\xi)I + \xi S^i((1-\eta)I + \eta S^i). \tag{2.2}$$

If T and S are demiclosed at 0, and the solution set Γ of problem (1.4)

$$\Gamma = \{(x^*, y^*) \in Fix(T) \times Fix(S) \ such \ \ that\ Ax^* = By^*\} \tag{2.3}$$

is nonempty, and the following conditions are satisfied:

(ii) $\gamma_n \in (0, \min(\frac{1}{||A||^2}, \frac{1}{||B||^2}))$, $\forall n \geq 1$ *with* $\liminf \gamma_n > 0$*;*

(iii) $0 < a < \xi < \eta < b < \frac{1}{\frac{M+1}{2} + \sqrt{\frac{(M+1)^2}{4} + L^2}}$, $\forall n \geq 1$,

where $M = \sup_{n\geq 1} l_n$, then the following conclusions hold:

(I) the sequence $\{(x_n, y_n)\}$ converges weakly to a solution of problem (1.4);

(II) In addition, if S, T both are semi-compact, then $\{(x_n, y_n)\}$ converges strongly to a solution of problem (1.4).

Proof. First we prove the conclusion (I).

For any given $(p, q) \in \Gamma$, then $p \in Fix(T), q \in Fix(S)$ and $Ap = Bq$. From (2.1) (a) and Lemma 1.1, we have

$$\begin{aligned} ||u_n - p||^2 =& ||x_n - \gamma_n A^*(Ax_n - By_n) - p||^2 \\ =& \gamma_n^2 ||A^*(Ax_n - By_n)||^2 - 2\gamma_n \langle x_n - p, A^*(Ax_n - By_n)\rangle + ||x_n - p||^2 \\ \leq& \gamma_n^2 ||A||^2 ||Ax_n - By_n||^2 - 2\gamma_n \langle Ax_n - Ap, Ax_n - By_n\rangle + ||x_n - p||^2. \end{aligned} \tag{2.4}$$

Similarly, from (2.1) (c) and Lemma 1.1, we have

$$||v_n - q||^2 \leq \gamma_n^2 ||B||^2 ||Ax_n - By_n||^2 + 2\gamma_n \langle By_n - Bq, Ax_n - By_n\rangle + ||y_n - q||^2. \tag{2.5}$$

By condition (iii) and Lemma 1.3, the mappings $\{K_i\}$ and $\{G_i\}$ have the following properties:

(a) $Fix(T) = Fix(K_i)$ and $Fix(S) = Fix(G_i)$ for each $i \geq 1$;

(b) K_1 and G_1 are demiclosed at 0;

(c) For each $i \geq 1$ and for all $x \in H_1,\ y \in H_2, u^* \in Fix(T) = Fix(K_i), v^* \in Fix(S) = Fix(G_i)$,

$$||K_i x - u^*|| \leq k_i ||x - u^*||, \quad ||G_i y - v^*|| \leq k_i ||y - v^*||,$$

where $k_i = 1+\xi(l_i-1)(1+\eta l_i)$, $\{k_i\} \subset [1,+\infty)$ and $\lim_{i\to\infty} k_i = 1$. By the assumption that $\sum_{i=1}^{\infty}(l_i^2-1) < \infty$, we have

$$\sum_{i=1}^{\infty}(k_i-1) \le \sum_{i=1}^{\infty}\xi(l_i-1)(l_i+1) \le \sum_{i=1}^{\infty}(l_i^2-1) < \infty. \tag{2.6}$$

On the other hand, it follows from (2.1) (b), (2.4) and a well-known result that for any positive integer $1 \le l \le n$ we have

$$\begin{aligned} ||x_{n+1}-p||^2 &\le \alpha_{n,0}||x_n-p||^2 + \sum_{i=1}^{n}\alpha_{n,i}||K_i u_n - p||^2 - \alpha_{n,0}\alpha_{n,l}||x_n - K_l u_n||^2 \\ &\le \alpha_{n,0}||x_n-p||^2 + \sum_{i=1}^{n}\alpha_{n,i}k_i^2||u_n - p||^2 - \alpha_{n,0}\alpha_{n,l}||x_n - K_l u_n||^2 \\ &\le \alpha_{n,0}||x_n-p||^2 + \sum_{i=1}^{n}\alpha_{n,i}k_i^2\{||x_n-p||^2 + \gamma_n^2||A||^2||Ax_n - By_n||^2 \\ &\quad - 2\gamma_n\langle Ax_n - Ap, Ax_n - By_n\rangle\} - \alpha_{n,0}\alpha_{n,l}||x_n - K_l u_n||^2. \end{aligned} \tag{2.7}$$

Similarly, it follows from (2.1) (d) and (2.5) that for any positive integer $1 \le l \le n$

$$\begin{aligned} ||y_{n+1}-q||^2 &\le \alpha_{n,0}||y_n-p||^2 + \sum_{i=1}^{n}\alpha_{n,i}k_i^2\{||y_n-q||^2 + \gamma_n^2||B||^2||Ax_n - By_n||^2 \\ &\quad + 2\gamma_n\langle By_n - Bq, Ax_n - By_n\rangle\} - \alpha_{n,0}\alpha_{n,l}||y_n - G_l v_n||^2. \end{aligned} \tag{2.8}$$

Adding up (2.7) and (2.8) and noting $Ap = Bq$, we have that

$$\begin{aligned} ||y_{n+1}-p||^2 + ||x_{n+1}-q||^2 &\le (\alpha_{n,0} + \sum_{i=1}^{n}\alpha_{n,i}k_i^2)\{||x_n-p||^2 + ||y_n-q||^2\} \\ &\quad + \sum_{i=1}^{n}\alpha_{n,i}k_i^2\gamma_n^2(||A||^2 + ||B||^2)||Ax_n - By_n||^2 \\ &\quad - 2\gamma_n\sum_{i=1}^{n}\alpha_{n,i}k_i^2\langle Ax_n - Ap - By_n + Bq, Ax_n - By_n\rangle \\ &\quad - \alpha_{n,0}\alpha_{n,l}\{||x_n - K_l u_n||^2 + ||y_n - G_l v_n||^2\} \\ &= (1 + \sum_{i=1}^{n}\alpha_{n,i}(k_i^2-1)\{||x_n-p||^2 + ||y_n-q||^2\} \\ &\quad + \sum_{i=1}^{n}\alpha_{n,i}k_i^2\gamma_n(\gamma_n(||A||^2 + ||B||^2) - 2)||Ax_n - By_n||^2 \\ &\quad - \alpha_{n,0}\alpha_{n,l}\{||x_n - K_l u_n||^2 + ||y_n - G_l v_n||^2\}. \end{aligned} \tag{2.9}$$

Since $\gamma_n \in (0, \min\{\frac{1}{||A||^2}, \frac{1}{||B||^2}\})$, $\gamma_n||A||^2 < 1$ and $\gamma_n||B||^2 < 1$. This implies that

$$\gamma_n(||A||^2 + ||B||^2) - 2 < 0.$$

Therefore (2.9) can be written as

$$\begin{aligned} ||y_{n+1}-q||^2 + ||x_{n+1}-p||^2 &\le (1 + \sum_{i=1}^{n}\alpha_{n,i}(k_i^2-1))(||x_n-p||^2 + ||y_n-q||^2). \\ &= (1+\sigma_n)(||x_n-p||^2 + ||y_n-q||^2), \end{aligned} \tag{2.10}$$

where $\sigma_n = \sum_{i=1}^{n} \alpha_{n,i}(k_i^2 - 1)$. Since $k_n \to 1$ and by (2.6) $\sum_{i=1}^{\infty}(k_i - 1) < \infty$, this implies that $\sum_{i=1}^{\infty}(k_i^2 - 1) < \infty$. Again since

$$\sum_{n=1}^{\infty} \sigma_n = \sum_{n=1}^{\infty}\sum_{i=1}^{n} \alpha_{n,i}(k_i^2 - 1) \le \sum_{i=1}^{\infty}(k_i^2 - 1) \sum_{n=1}^{\infty} \alpha_{n,i} \le \sum_{i=1}^{\infty}(k_i^2 - 1) < \infty$$

$\sigma_n \to 0$. By virtue of Lemma 1.2 and (2.10), it gets that the limit $\lim_{n\to\infty}(||x_n - p||^2 + ||y_n - q||^2)$ exists, so the limits $\lim_{n\to\infty} ||x_n - p||$ and $\lim_{n\to\infty} ||y_n - q||$ exist for all $(p,q) \in \Gamma$. Now rewrite (2.9) as

$$\begin{aligned}
&\sum_{i=1}^{n} \alpha_{n,i} k_i^2 \gamma_n(2 - \gamma_n(||A||^2 + ||B||^2))||Ax_n - By_n||^2 \\
&\quad + \alpha_{n,0}\alpha_{n,l}\{||x_n - K_l u_n||^2 + ||y_n - G_l v_n||^2\} \\
&\quad \le (1+\sigma_n)(||x_n - p||^2 + ||y_n - q||^2) - (||x_{n+1} - p||^2 \\
&\quad - (||x_{n+1} - p||^2 + ||y_{n+1} - q||^2) \to 0 \ (as\ n \to \infty)\ (since\ \ \sigma_n \to 0).
\end{aligned} \tag{2.11}$$

It follows from conditions (i), (ii) and (2.11) that for each $l = 1, 2, 3, \cdots$

$$\lim_{n\to\infty} ||Ax_n - By_n|| = 0, \ \lim_{n\to\infty} ||K_l u_n - x_n|| = 0, \ \lim_{n\to\infty} ||G_l v_n - y_n|| = 0. \tag{2.12}$$

This together with (2.1) and the condition (i) shows that

$$\begin{cases}
\lim_{n\to\infty} ||u_n - x_n|| = 0; \\
\lim_{n\to\infty} ||v_n - y_n|| = 0. \\
\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_{n,i} ||K_i u_n - x_n|| \\
\qquad\qquad\qquad\qquad \le \lim_{n\to\infty} \sup_{i\ge 1} ||K_i u_n - x_n|| = 0. \\
\lim_{n\to\infty} ||y_{n+1} - y_n|| = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_{n,i} ||G_i v_n - y_n|| = 0 \\
\qquad\qquad\qquad\qquad \le \lim_{n\to\infty} \sup_{i\ge 1} ||G_i v_n - y_n|| = 0.
\end{cases} \tag{2.13}$$

From (2.12) and (2.13) we have

$$\begin{cases}
||K_1 u_n - u_n|| \le ||K_1 u_n - x_n|| + ||x_n - u_n|| \to 0; \\
||G_1 v_n - v_n|| \le ||G_1 v_n - y_n|| + ||y_n - v_n|| \to 0.
\end{cases} \tag{2.14}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded sequences, there exist some weakly convergent subsequences, say $\{x_{n_i}\} \subset \{x_n\}$ and $\{y_{n_i}\} \subset \{y_n\}$ such that $x_{n_i} \rightharpoonup x^*$ and $y_{n_i} \rightharpoonup y^*$. Since every Hilbert space has the Opial's property which guarantees that the weak limit of $\{(x_n, y_n)\}$ is unique. Therefore we have $x_n \rightharpoonup x^*$, and $y_n \rightharpoonup y^*$.

On the other hand, it follows from (2.13) that $u_n \rightharpoonup x^*$ and $v_n \rightharpoonup y^*$. Since K_1 and G_1 both are demiclosed at 0 (by Lemma 1.3 (ii), from (2.14) it gets $K_1 x^* = x^*$ and $G_1 y^* = y^*$. This implies that $x^* \in Fix(T)$ and $y^* \in Fix(S)$.

Now we show that $Ax^* = By^*$. In fact, since $Ax_n - By_n \rightharpoonup Ax^* - By^*$, by using the weakly lower semi-continuity of norm, we have

$$||Ax^* - By^*||^2 \le \liminf_{n\to\infty} ||Ax_n - By_n||^2 = \lim_{n\to\infty} ||Ax_n - By_n||^2 = 0.$$

Thus $Ax^* = By^*$. This completes the proof of the conclusion (I).

Now we prove the conclusion (II).

In fact, since K_1 is uniformly continuous, we have $\lim_{\to\infty} ||K_1 u_n - K_1 x_n|| = 0$. Hence from (2.13), we have $||K_1 x_n - x_n|| \le ||x_n - K_1 u_n|| + ||K_1 u_n - K_1 x_n||$. It follows that

$$\lim_{n\to\infty} ||K_1 x_n - x_n|| = 0. \tag{2.15}$$

Similarly, we can also prove that

$$\lim_{\to\infty} ||G_1 y_n - y_n|| = 0. \tag{2.16}$$

By virtue of (1.11), (2.14), (2.15) and (2.16), we have

$$\begin{cases} ||x_n - Tx_n|| \le \dfrac{1}{\xi(1-L\eta)} ||x_n - K_1 x_n|| \to 0 \ (n \to \infty); \\ ||y_n - Sy_n|| \le \dfrac{1}{\xi(1-L\eta)} ||y_n - G_1 y_n|| \to 0 \ (n \to \infty). \end{cases} \tag{2.17}$$

Since S, T are semi-compact, it follows from (2.17) that there exist subsequences $\{x_{n_i}\} \subset \{x_n\}$ and $\{y_{n_j}\} \subset \{y_n\}$ such that $x_{n_i} \to x$ (some point in $Fix(T)$) and $y_{n_j} \to y$ (some point in $Fix(S)$). Since $x_n \rightharpoonup x^*$ and $y_n \rightharpoonup y^*$ and the limits $\lim_{n\to\infty} ||x_n - p||$ and $\lim_{n\to\infty} ||y_n - q||$ exist for all $(p,q) \in \Gamma$, it follows from (2.12) that $x_n \to x^*$ and $y_n \to y^*$ and $Ax^* = By^*$. This completes the proof of Theorem 2.1. □

Remark 2.1. Theorem 2.1 is an improvement and generalization of the corresponding results in Chang, Wang and Qin [4], Che and Li [9], Moudafi [11], Moudafi and Al-Shemas [12], Moudafi [13].

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