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MONOTONE CONTRACTIVE MAPPINGS

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Abstract. We consider three classes of monotone contractive mappings defined on a complete metric space. For each mapping in one of these classes, we establish the existence of a unique fixed point which attracts all iterates.
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1. Introduction and preliminaries

Since the publication of Banach's classical fixed point theorem [2], metric fixed point theory has been and continues to be an important part of nonlinear operator theory [3, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17]. For example, several results regarding the existence of fixed points for general nonexpansive mappings in special Banach spaces were presented in [6, 7], while for self-mappings of general complete metric spaces existence results were established for classes of contractive mappings in [4, 10, 11]. An extension of the existence result of [11] and several other existence results for certain mappings of contractive type have also been presented in [18].

In the present paper, employing certain contractive type assumptions, we obtain existence results for monotone nonexpansive mappings – a class of nonlinear mappings which has been the subject of a rapidly growing area of research [1, 5].

Let (X,ρ) be a complete metric space equipped with a partial order $\leq$, that is, for all points $x,y,z\in X$, we have

$$x \leq x,$$

$$\text{if } x\leq y,\ y\leq x,\ \text{then } x=y,$$

and

$$\text{if } x\leq y,\ y\leq z,\ \text{then } x\leq z.$$

We also assume that

$$\{(x,y)\in X\times X:\ x\leq y\}$$

is a closed subset of $X\times X$.

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Let K be a nonempty closed subset of X which is not a singleton. Let $x_K \in K$ and assume that at least one of the following relations holds:

$$x_K \leq x \text{ for all } x \in K \tag{1.1}$$

or

$$x \leq x_K \text{ for all } x \in K. \tag{1.2}$$

For each $x \in K$ and each $r > 0$, set

$$B(x,r) := \{y \in X : \rho(x,y) \leq r\}.$$

Let $T : K \to X$. Denote by T^0 the identity operator $I : K \to K$, that is, $I(x) = x,\ x \in K$. Suppose that the graph of T

$$\text{graph}(T) = \{(x, T(x)) : x \in K\}$$

is a closed subset of $X \times X$,

$$T^i(x_K) \in K \text{ for all integers } i \geq 1 \tag{1.3}$$

and

$$T(x) \leq T(y) \text{ for all } x, y \in K \text{ such that } x \leq y. \tag{1.4}$$

In this paper we establish three theorems regarding the existence of a unique fixed point of such a mapping T under three different contractivity assumptions. In the first result we use contractivity in the sense of Rakotch [11], the second is in the spirit of Boyd and Wong [4], while in the third result T is a contractive mapping in the sense of Matkowski [10].

Theorem 1.1. *Assume that $\phi : [0,\infty) \to [0,1]$ is a decreasing function,*

$$\phi(t) < 1 \textit{ for all } t > 0 \tag{1.5}$$

and that for all $x, y \in K$ satisfying $x \leq y$, we have

$$\rho(T(x), T(y)) \leq \phi(\rho(x,y))\rho(x,y). \tag{1.6}$$

Then $\{T^i(x_K)\}_{i=1}^{\infty}$ converges, $\lim_{i\to\infty} T^i(x_K)$ is the unique fixed point of the mapping T and the following assertion holds.

Let M and ε be positive. Then there exists a natural number n_0 such that if

$$x \in K \cap B(x_K, M),$$

$n > n_0$ is an integer and $T^n(x)$ is defined, then

$$\rho(T^i(x), \lim_{j\to\infty} T^j(x_K)) \leq \varepsilon$$

for all $i = n_0 + 1, \ldots, n$.

Theorem 1.2. *Assume that the function $\phi : [0,\infty) \to [0,\infty)$ is upper semicontinuous,*

$$\phi(t) < t \textit{ for all } t > 0 \tag{1.7}$$

and that for all $x, y \in K$ satisfying $x \leq y$, we have

$$\rho(T(x), T(y)) \leq \phi(\rho(x,y)). \tag{1.8}$$

Then $\{T^i(x_K)\}_{i=1}^{\infty}$ *converges,* $\lim_{i\to\infty} T^i(x_K)$ *is the unique fixed point of the mapping* T *and the following assertion holds.*

Let M *and* ε *be positive. Then there exists a natural number* n_0 *such that if*

$$x \in K \cap B(x_K, M),$$

$n > n_0$ *is an integer and* $T^n(x)$ *is defined, then*

$$\rho(T^i(x), \lim_{j\to\infty} T^j(x_K)) \le \varepsilon$$

for all $i = n_0 + 1, \dots, n$*.*

Theorem 1.3. *Assume that* $\phi : [0,\infty) \to [0,\infty)$ *is an increasing function,*

$$\lim_{n\to\infty} \phi^n(t) = 0 \text{ for all } t > 0 \tag{1.9}$$

and that for all $x, y \in K$ *satisfying* $x \le y$*, we have*

$$\rho(T(x), T(y)) \le \phi(\rho(x,y)). \tag{1.10}$$

Then $\{T^i(x_K)\}_{i=1}^{\infty}$ *converges,* $\lim_{i\to\infty} T^i(x_K)$ *is the unique fixed point of the mapping* T *and the following assertion holds.*

Let M *and* ε *be positive. Then there exists a natural number* n_0 *such that if*

$$x \in K \cap B(x_K, M),$$

$n > n_0$ *is an integer and* $T^n(x)$ *is defined, then*

$$\rho(T^i(x), \lim_{j\to\infty} T^j(x_K)) \le \varepsilon$$

for all $i = n_0 + 1, \dots, n$*.*

2. Proof of Theorem 1.1

If (1.1) holds, then $T^0(x_K) \le T^1(x_K)$. In view of (1.4), for all integers $i \ge 0$, we have

$$T^i(x_K) \le T^{i+1}(x_K). \tag{2.1}$$

If (1.2) holds, then $T^1(x_K) \le T^0(x_K)$. In view of (1.4), for all integers $i \ge 0$, we have

$$T^{i+1}(x_K) \le T^i(x_K). \tag{2.2}$$

By (1.6), (2.1) and (2.2), for all integers $i \ge 0$, we have

$$\rho(T^i(x_K), T^{i+1}(x_K)) \ge \rho(T^{i+1}(x_K), T^{i+2}(x_K)). \tag{2.3}$$

We claim that

$$\lim_{i\to\infty} \rho(T^i(x_K), T^{i+1}(x_K)) = 0. \tag{2.4}$$

Suppose to the contrary that (2.4) does not hold. In view of (2.3), one sees that there exists a number $r > 0$ such that

$$\rho(T^i(x_K), T^{i+1}(x_K)) > r \text{ for all integers } i \ge 0. \tag{2.5}$$

Since the function ϕ is decreasing, it follows from (1.6), (2.1), (2.2) and (2.5) that, for all integers $i \geq 0$,

$$\begin{aligned}
&\rho(T^i(x_K), T^{i+1}(x_K)) - \rho(T^{i+1}(x_K), T^{i+2}(x_K)) \\
&\geq \rho(T^i(x_K), T^{i+1}(x_K)) - \phi(\rho(T^i(x_K), T^{i+1}(x_K)))\rho(T^i(x_K), T^{i+1}(x_K)) \\
&\geq \rho(T^i(x_K), T^{i+1}(x_K))(1 - \phi(\rho(T^i(x_K), T^{i+1}(x_K)))) \\
&\geq \rho(T^i(x_K), T^{i+1}(x_K))(1 - \phi(r)) \\
&\geq r(1 - \phi(r)).
\end{aligned}$$

This implies, for every natural number n, that

$$\begin{aligned}
\rho(x_K, T^1(x_K)) &\geq \rho(x_K, T^1(x_K)) - \rho(T^n(x_K), T^{n+1}(x_K)) \\
&= \sum_{i=0}^{n-1} [\rho(T^i(x_K), T^{i+1}(x_K)) - \rho(T^{i+1}(x_K), T^{i+2}(x_K))] \\
&\geq nr(1 - \phi(r)) \to \infty \text{ as } n \to \infty.
\end{aligned}$$

The contradiction we have reached proves that (2.4) does hold.

Next, we prove that $\{T^i(x_K)\}_{i=1}^\infty$ is a Cauchy sequence. To this end, let $\varepsilon > 0$. By (2.4), there exists a natural number n_0 such that, for all integers $i \geq n_0$,

$$\rho(T^i(x_K), T^{i+1}(x_K)) \leq (\varepsilon/4)(1 - \phi(\varepsilon)). \tag{2.6}$$

Assume that

$$n_2 > n_1 \geq n_0 \tag{2.7}$$

are integers. We now show that

$$\rho(T^{n_1}(x_K), T^{n_2}(x_K)) \leq \varepsilon.$$

Suppose to the contrary that

$$\rho(T^{n_1}(x_K), T^{n_2}(x_K)) > \varepsilon. \tag{2.8}$$

By (1.6), (2.1), (2.2) and (2.8), one sees that

$$\begin{aligned}
&\rho(T^{n_1+1}(x_K), T^{n_2+1}(x_K)) \\
&\leq \phi(\rho(T^{n_1}(x_K), T^{n_2}(x_K)))\rho(T^{n_1}(x_K), T^{n_2}(x_K)) \\
&\leq \phi(\varepsilon)\rho(T^{n_1}(x_K), T^{n_2}(x_K)).
\end{aligned} \tag{2.9}$$

In view of (2.9), one has

$$\begin{aligned}
&\phi(\varepsilon)\rho(T^{n_1}(x_K), T^{n_2}(x_K)) \\
&\geq \rho(T^{n_1+1}(x_K), T^{n_2+1}(x_K)) \\
&\geq \rho(T^{n_1}(x_K), T^{n_2}(x_K)) - \rho(T^{n_1}(x_K), T^{n_1+1}(x_K)) - \rho(T^{n_2}(x_K), T^{n_2+1}(x_K)).
\end{aligned} \tag{2.10}$$

It follows from (2.6)–(2.8) and (2.10) that

$$\begin{aligned}
(1 - \phi(\varepsilon))\varepsilon/2 &\geq \rho(T^{n_1}(x_K), T^{n_1+1}(x_K)) + \rho(T^{n_2}(x_K), T^{n_2+1}(x_K)) \\
&\geq (1 - \phi(\varepsilon))\rho(T^{n_1}(x_K), T^{n_2}(x_K)) \\
&\geq \varepsilon(1 - \phi(\varepsilon)),
\end{aligned}$$

which is a contradiction. The contradiction we have reached proves that

$$\rho(T^{n_1}(x_K), T^{n_2}(x_K)) \le \varepsilon$$

for all integers $n_1, n_2 \ge n_0$. Thus $\{T^i(x_K)\}_{i=1}^{\infty}$ is indeed a Cauchy sequence, as claimed, and there exists

$$\widehat{x} = \lim_{i\to\infty} T^i(x_K). \tag{2.11}$$

Since graph(T) is a closed set, one sees that (2.11) implies that $T(\widehat{x}) = \widehat{x}$. Next we show that $\widehat{x}$ is the unique fixed point of T.

To this end, we assume that $z \in K$ and

$$T(z) = z. \tag{2.12}$$

If (1.1) holds, then $T^i(x_K) \le z$ for all integers $i \ge 1$ and $\widehat{x} \le z$. If (1.2) holds, then $z \le T^i(x_K)$ for all integers $i \ge 1$ and $z \le \widehat{x}$. In both cases, (1.6) implies that $\rho(z,\widehat{x}) \le \phi(\rho(z,\widehat{x}))\rho(z,\widehat{x})$. If $z \ne \widehat{x}$, then we have reached a contradiction. Therefore $z = \widehat{x}$, as claimed.

Now let M and ε be positive. There exists a natural number n_0 such that

$$n_0 > M((1-\phi(\varepsilon/2))(\varepsilon/2))^{-1} \tag{2.13}$$

and

$$\rho(T^i(x_K),\widehat{x}) \le \varepsilon/2 \text{ for all integers } i \ge n_0. \tag{2.14}$$

Assume that

$$x \in B(x_K, M) \cap K, \tag{2.15}$$

$n > n_0$ is an integer and that $T^n(x)$ is defined. If (1.1) holds, then we find from (1.6) that

$$T^i(x_K) \le T^i(x) \text{ for all integers } i = 1, \dots, n. \tag{2.16}$$

If (1.2) holds, then we find from (1.6) that

$$T^i(x) \le T^i(x_K) \text{ for all integers } i = 1, \dots, n. \tag{2.17}$$

In both cases, (1.6) implies, for all $i = 0, \dots, n-1$, that

$$\rho(T^{i+1}(x), T^{i+1}(x_K)) \le \phi(\rho(T^i(x), T^i(x_K)))\rho(T^i(x), T^i(x_K)). \tag{2.18}$$

We now show that there exists an integer $i \in [0, n_0]$ such that $\rho(T^i(x), T^i(x_K)) \le \varepsilon/2$. Suppose to the contrary that this does not hold. For all $i = 0, \dots, n_0$, one has

$$\rho(T^i(x), T^i(x_K)) > \varepsilon/2. \tag{2.19}$$

In view of (2.18) and (2.19), for all $i = 0, \dots, n_0$, one has

$$\begin{aligned}\rho(T^{i+1}(x), T^{i+1}(x_K)) &\le \phi(\rho(T^i(x), T^i(x_K)))\rho(T^i(x), T^i(x_K)) \\ &\le \phi(\varepsilon/2)\rho(T^i(x), T^i(x_K)).\end{aligned}$$

By (2.9), (2.15) and the relation above, we have

$$\begin{aligned}
M &\geq \rho(x, x_K) \\
&\geq \rho(x, x_K) - \rho(T^{n_0}(x), T^0(x_K)) \\
&= \sum_{i=0}^{n_0-1} [\rho(T^i(x), T^i(x_K)) - \rho(T^{i+1}(x), T^{i+1}(x_K))] \\
&\geq \sum_{i=0}^{n_0-1} (1 - \phi(\varepsilon/2))\rho(T^i(x), T^i(x_K)) \\
&\geq n_0(1 - \phi(\varepsilon/2))\varepsilon/2
\end{aligned}$$

and

$$n_0 \leq M(1 - \phi(\varepsilon/2))^{-1}(\varepsilon/2)^{-1}.$$

This inequality contradicts (2.13). The contradiction we have reached proves that there indeed exists an integer $i_0 \in \{0, \ldots, n_0\}$ such that

$$\rho(T^{i_0}(x), T^{i_0}(x_K)) \leq \varepsilon/2. \tag{2.20}$$

Assume now that an integer i satisfies $i_0 \leq i \leq n$. In view of (2.18) and (2.20), one has

$$\rho(T^i(x), T^i(x_K)) \leq \varepsilon/2. \tag{2.21}$$

Since $i \geq n_0$, we find from (2.14) and (2.21) that

$$\rho(\widehat{x}, T^i(x)) \leq \rho(\widehat{x}, T^i(x_K)) + \rho(T^i(x), T^i(x_K)) \leq \varepsilon.$$

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

If (1.1) holds, then $T^0(x_K) \leq T^1(x_K)$. In view of (1.4), for all integers $i \geq 0$, one has

$$T^i(x_K) \leq T^{i+1}(x_K). \tag{3.1}$$

If (1.2) holds, then $T^1(x_K) \leq T^0(x_K)$. In view of (1.4), for all integers $i \geq 0$, one has

$$T^{i+1}(x_K) \leq T^i(x_K). \tag{3.2}$$

By (1.7), (1.8), (3.1) and (3.2), for all integers $i \geq 0$, one has

$$\rho(T^{i+1}(x_K), T^{i+2}(x_K)) \leq \phi(\rho(T^i(x_K), T^{i+1}(x_K))) \leq \rho(T^i(x_K), T^{i+1}(x_K)). \tag{3.3}$$

We claim that $\lim_{i\to\infty} \rho(T^i(x_K), T^{i+1}(x_K)) = 0$. Suppose to the contrary that this does not hold. In view of (3.3), one sees that there exists a number $r > 0$ such that

$$\rho(T^i(x_K), T^{i+1}(x_K)) > r \text{ for all integers } i \geq 0. \tag{3.4}$$

Since the function $t - \phi(t)$ is positive for all $t > 0$ and lower semicontinuous, there exists a number $\gamma > 0$ such that

$$t - \phi(t) > \gamma \text{ for all } t \in [r/4, \rho(T^0(x_K), T^1(x_K)) + 1]. \tag{3.5}$$

It now follows from (3.3), (3.4) and (3.5) that

$$\rho(T^{i+1}(x_K), T^{i+2}(x_K)) \leq \phi(\rho(T^i(x_K), T^{i+1}(x_K))) \leq \rho(T^i(x_K), T^{i+1}(x_K)) - \gamma, \quad \forall i \geq 0. \tag{3.6}$$

For all integers $n \geq 1$, we find from (3.6) that

$$\begin{aligned}
\rho(T^0(x_K), T^1(x_K)) &\geq \rho(T^0(x_K), T^1(x_K)) - \rho(T^n(x_K), T^{n+1}(x_K)) \\
&= \sum_{i=0}^{n-1} [\rho(T^i(x_K), T^{i+1}(x_K)) - \rho(T^{i+1}(x_K), T^{i+2}(x_K))] \\
&\geq \gamma n \to \infty \text{ as } n \to \infty.
\end{aligned}$$

The contradiction we have reached proves that

$$\lim_{i\to\infty} \rho(T^i(x_K), T^{i+1}(x_K)) = 0, \tag{3.7}$$

as claimed.

Next, we prove that $\{T^i(x_K)\}_{i=1}^\infty$ is a Cauchy sequence. To this end, let $\delta > 0$. We show that there exists a natural number n_0 such that for each pair of integers $i, j \geq n_0$, we have $\rho(T^i(x_K), T^j(x_K)) \leq \delta$. Suppose to the contrary that this does not hold. For each natural number k, there exist integers i_k, j_k such that $k \leq i_k < j_k$ and

$$\rho(T^{i_k}(x_K), T^{j_k}(x_K)) > \delta. \tag{3.8}$$

We may assume without any loss of generality that for each natural number k, the following property holds:

(P1) if an integer j satisfies $i_k \leq j < j_k$, then

$$\rho(T^{i_k}(x_K), T^j(x_K)) \leq \delta. \tag{3.9}$$

Let k be a natural number. By (3.8) and (3.9), one has

$$\begin{aligned}
\delta &< \rho(T^{i_k}(x_K), T^{j_k}(x_K)) \\
&\leq \rho(T^{j_k}(x_K), T^{j_k-1}(x_K)) + \rho(T^{j_k-1}(x_K), T^{i_k}(x_K)) \\
&\leq \rho(T^{j_k}(x_K), T^{j_k-1}(x_K)) + \delta.
\end{aligned} \tag{3.10}$$

In view of (3.7), one has

$$\lim_{k\to\infty} \rho(T^{j_k}(x_K), T^{j_k-1}(x_K)) = 0. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\lim_{k\to\infty} \rho(T^{i_k}(x_K), T^{j_k}(x_K)) = \delta. \tag{3.12}$$

By (3.3) and (3.8), one has

$$\begin{aligned}
\delta &< \rho(T^{i_k}(x_K), T^{j_k}(x_K)) \\
&\leq \rho(T^{i_k}(x_K), T^{i_k+1}(x_K)) + \rho(T^{i_k+1}(x_K), T^{j_k+1}(x_K)) + \rho(T^{j_k+1}(x_K), T^{j_k}(x_K)) \\
&\leq \phi(\rho(T^{i_k}(x_K), T^{j_k}(x_K))) + \rho(T^{i_k}(x_K), T^{i_k+1}(x_K)) + \rho(T^{j_k+1}(x_K), T^{j_k}(x_K)).
\end{aligned} \tag{3.13}$$

Using (3.7), (3.12) and (3.13), one has

$$\begin{aligned}
\delta &= \lim_{k\to\infty} \rho(T^{i_k}(x_K), T^{j_k}(x_K)) \\
&\leq \liminf_{k\to\infty} \phi(\rho(T^{i_k}(x_K), T^{j_k}(x_K))) \\
&= \phi(\delta).
\end{aligned}$$

It follows that $\delta \le \phi(\delta)$. The contradiction we have reached proves that $\{T^i(x_K)\}_{i=1}^{\infty}$ is indeed a Cauchy sequence, as claimed, and there exists $\widehat{x} = \lim_{i\to\infty} T^i(x_K)$. Since graph$(T)$ is a closed set, we have

$$T(\widehat{x}) = \widehat{x}. \tag{3.14}$$

Now we show that $\widehat{x}$ is the unique fixed point of T.

To this end, assume that $z \in K$ and

$$T(z) = z. \tag{3.15}$$

If (1.2) holds, then $T^i(x_K) \le z$ for all integers $i \ge 1$ and $\widehat{x} \le z$. If (1.2) holds, then $z \le T^i(x_K)$ for all integers $i \ge 1$ and $z \le \widehat{x}$. In both cases, $\rho(z,\widehat{x}) \le \phi(\rho(z,\widehat{x}))$. This implies that $z = \widehat{x}$, as claimed. Let M and ε be positive. Since the function $t - \phi(t)$ is positive for all $t > 0$ and lower semicontinuous, there exists a number $\gamma > 0$ such that

$$t - \phi(t) > \gamma \text{ for all } t \in [\varepsilon/4, M + \varepsilon + 1]. \tag{3.16}$$

There also exists a natural number n_0 such that

$$n_0 > M\gamma^{-1} \tag{3.17}$$

and

$$\rho(T^i(x_K),\widehat{x}) \le \varepsilon/2 \text{ for all integers } i \ge n_0. \tag{3.18}$$

Assume that

$$x \in B(x_K, M), \tag{3.19}$$

$n > n_0$ is an integer and that $T^n(x)$ is defined. If (1.1) holds, then we find from (1.4) that

$$T^i(x_K) \le T^i(x) \text{ for all integers } i = 1,\dots,n. \tag{3.20}$$

If (1.2) holds, then we find from (1.4) that

$$T^i(x) \le T^i(x_K) \text{ for all integers } i = 1,\dots,n. \tag{3.21}$$

In both cases, inequality (1.8) implies that, for all $i = 0,\dots,n-1$,

$$\rho(T^{i+1}(x),T^{i+1}(x_K)) \le \phi(\rho(T^i(x),T^i(x_K))). \tag{3.22}$$

We now show that there exists an integer $i \in [0,n_0]$ such that $\rho(T^i(x),T^i(x_K)) \le \varepsilon/2$. Suppose to the contrary that this does not hold. In view of (3.19) and (3.22), we have, for all $i = 0,\dots,n_0$, that

$$M \ge \rho(x,x_K) \ge \rho(T^i(x),T^i(x_K)) > \varepsilon/2. \tag{3.23}$$

By (3.16) and (3.23), for all $i = 0,\dots,n_0$, one has

$$\rho(T^i(x),T^i(x_K)) - \phi(\rho(T^i(x),T^i(x_K))) > \gamma.$$

When combined with (3.22), this inequality implies that

$$\rho(T^i(x),T^i(x_K)) - \rho(T^{i+1}(x),T^{i+1}(x_K)) > \gamma. \tag{3.24}$$

By (3.19) and (3.24), one has

$$\begin{aligned} M &\geq \rho(x, x_K) \\ &\geq \rho(x, x_K) - \rho(T^{n_0}(x), T^0(x_K)) \\ &= \sum_{i=0}^{n_0-1} [\rho(T^i(x), T^i(x_K)) - \rho(T^{i+1}(x), T^{i+1}(x_K))] \\ &\geq \gamma n_0 \end{aligned}$$

and $n_0 \leq M\gamma^{-1}$. This contradicts (3.17). The contradiction we have reached proves that there indeed exists $i_0 \in \{0, \dots, n_0\}$ such that

$$\rho(T^{i_0}(x), T^{i_0}(x_K)) \leq \varepsilon/2. \tag{3.25}$$

It follows from (3.22) and (3.25) that

$$\rho(T^i(x), T^i(x_K)) \leq \varepsilon/2, \quad \forall i \in \{i_0, \dots, n\}.$$

Since $i \geq n_0$, we find from (3.18) that

$$\rho(\widehat{x}, T^i(x)) \leq \rho(\widehat{x}, T^i(x_K)) + \rho(T^i(x), T^i(x_K)) < \varepsilon.$$

This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

If (1.1) holds, then $T^0(x_K) \leq T^1(x_K)$ and for all integers $i \geq 0$,

$$T^i(x_K) \leq T^{i+1}(x_K). \tag{4.1}$$

If (1.2) holds, then $T^1(x_K) \leq T^0(x_K)$ and for all integers $i \geq 0$,

$$T^{i+1}(x_K) \leq T^i(x_K). \tag{4.2}$$

Since $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$ and ϕ is increasing, we have

$$\phi(t) < t \text{ for all } t > 0. \tag{4.3}$$

By (1.9), (1.10) and (4.3), for all integers $i \geq 0$, one has

$$\rho(T^{i+1}(x_K), T^{i+2}(x_K)) \leq \phi(\rho(T^i(x_K), T^{i+1}(x_K))) \leq \rho(T^i(x_K), T^{i+1}(x_K)) \tag{4.4}$$

and for all integers $i \geq 1$, one has

$$\rho(T^i(x_K), T^{i+1}(x_K)) \leq \phi^i(\rho(x_K, T^1(x_K))) \to 0 \text{ as } i \to \infty.$$

It follows that

$$\lim_{i \to \infty} \rho(T^i(x_K), T^{i+1}(x_K)) = 0. \tag{4.5}$$

We claim that $\{T^i(x_K)\}_{i=1}^{\infty}$ is a Cauchy sequence. Let $\delta > 0$. In view of (4.3), one has

$$\phi(\delta) < \delta. \tag{4.6}$$

By (4.5) and (4.6), one sees that there exists a natural number i_0 such that

$$\rho(T^i(x_K), T^{i+1}(x_K)) \leq \delta - \phi(\delta) \tag{4.7}$$

for all integers $i \geq i_0$.

Now we show that $\rho(T^i(x_K), T^j(x_K)) \leq \delta$ for all integers $j > i \geq i_0$. To this end, assume that $i \geq i_0$ is an integer,

$$x \in K \cap B(T^i(x_K), \delta) \tag{4.8}$$

and

$$\text{either } x \leq T^i(x_K) \text{ or } T^i(x_K) \leq x. \tag{4.9}$$

It follows from (4.3), (4.7), (4.9) and (4.10) that

$$\begin{aligned}\rho(T(x), T^i(x_K)) &\leq \rho(T(x), T^{i+1}(x_K)) + \rho(T^{i+1}(x_K), T^i(x_K)) \\ &\leq \phi(\rho(x, T^i(x_K))) + \delta - \phi(\delta) \leq \delta.\end{aligned}$$

Therefore

$$T(K \cap B(T^i(x_K), \delta) \cap (\{x \in K:\ x \leq T^i(x_K)\} \cup \{x \in K:\ T^i(x_K) \leq x\})) \subset B(T^i(x_K), \delta). \tag{4.10}$$

By (4.10), if (1.1) holds, then $T^{i_1}(x_K) \leq T^{i_2}(x_K)$ for all integers $i_2 \geq i_1 \geq 0$ and $T^j(x_K) \in K \cap B(T^i(x_K), \delta)$ for all integers $j > i$. By (4.10), if (1.2) holds, then $T^{i_2}(x_K) \leq T^{i_1}(x_K)$ for all integers $i_2 \geq i_1 \geq 0$ and $T^j(x_K) \in K \cap B(T^i(x_K), \delta)$ for all integers $j > i$. Thus in both cases, $\rho(T^i(x_K), T^j(x_K)) \leq \delta$ for all integers $j > i \geq i_0$, as claimed. Therefore $\{T^i(x_K)\}_{i=1}^{\infty}$ is indeed a Cauchy sequence and there exists

$$\widehat{x} = \lim_{i \to \infty} T^i(x_K). \tag{4.11}$$

Since graph(T) is a closed set, we have $T(\widehat{x}) = \widehat{x}$.

Next we show that $\widehat{x}$ is the unique fixed point of T. To this end, assume that $z \in K$ and $T(z) = z$. If (1.1) holds, then $T^i(x_K) \leq z$ for all integers $i \geq 1$ and $\widehat{x} \leq z$, and if (1.2) holds, then $z \leq T^i(x_K)$ for all integers $i \geq 1$ and $z \leq \widehat{x}$. In both cases, $\rho(z, \widehat{x}) \leq \phi(\rho(z, \widehat{x}))$. This implies that $z = \widehat{x}$.

Now let M and ε be positive. By (1.9) and (4.11), there exists a natural number n_0 such that

$$\phi^{n_0}(M) < \varepsilon/2 \tag{4.12}$$

and

$$\rho(T^i(x_K), \widehat{x}) \leq \varepsilon/2 \text{ for all integers } i \geq n_0. \tag{4.13}$$

Assume that

$$x \in B(x_K, M) \cap K, \tag{4.14}$$

$n > n_0$ is an integer and $T^n(x)$ is defined. If (1.1) holds, then we find from (1.4) that $T^i(x_K) \leq T^i(x)$ for all integers $i = 1, \ldots, n$. If (1.2) holds, then we find from (1.4) that $T^i(x) \leq T^i(x_K)$ for all integers $i = 1, \ldots, n$. In both cases, inequality (1.4) implies that, for all $i = 0, \ldots, n-1$,

$$\rho(T^{i+1}(x), T^{i+1}(x_K)) \leq \phi(\rho(T^i(x), T^i(x_K))). \tag{4.15}$$

It follows from (4.12), (4.14) and (4.15) that

$$\rho(T^{n_0}(x), T^{n_0}(x_K)) \leq \phi^{n_0}(\rho(x, x_K)) \leq \phi^{n_0}(M) < \varepsilon/2. \tag{4.16}$$

By (4.13), (4.15) and (4.16), for $i = n_0, \ldots, n$, one has $\rho(T^i(x), T^i(x_K)) \leq \varepsilon/2$ and

$$\rho(\widehat{x}, T^i(x)) \leq \rho(\widehat{x}, T^i(x_K)) + \rho(T^i(x), T^i(x_K)) < \varepsilon.$$

This completes the proof of Theorem 1.3.

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