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FUNCTIONAL ANALYTIC PROPERTIES OF SOME BANACH ALGEBRAS RELATED TO THE FOURIER ALGEBRA

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Dedicated to Professor Anthony To-Ming Lau on the occasions of his 75th birthday with admiration and respect

Abstract. This is a survey of some functional analytic properties of the Banach algebras $A_p^r(G) = A_p \cap L^r(G)$ for locally compact groups G obtained in [18, 19, 20, 21]. These include Arens regularity, strict containment for fixed p and increasing r, the RNP and optimisation in $A_p^r(G)$, weak sequential completeness, spectral synthesis, etc. It is found that $A_p^r(G)$ behaves very differently from $A_p(G)$, if $r < \infty$. The results are new even if $G = R$ or $G = Z$, the real line or the additive integers.
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1. INTRODUCTION

Let G be a locally compact group and let $A_p(G)$ denote the Figa-Talamanca-Herz Banach algebra of G, as defined in [23], thus generated by $L^{p'} * L^{\vee p}(G)$, where $1 < P < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, see sequel. Hence $A_2(G)$ is the Fourier algebra of G as defined and studied by Eymard in [10]. If G is abelian, then $A_2(G) = L^1(\hat{G})^\wedge$. Denote $A_p^r(G) = A_p \cap L^r(G)$, for $1 \leq r \leq \infty$, equipped with the norm

$$\|u\|_{A_p^r} = \|u\|_{A_p} + \|u\|_{L^r}.$$

We note that

$$\|u\|_{A_p} \leq \|u\|_{A_p} + \|u\|_{L^\infty} \leq 2\|u\|_{A_p}.$$

Hence, if $r = \infty$, we denote $A_p^r(G) = A_p(G)$ with the $A_p(G)$ norm. If G is abelian, then

$$A_2^r(G) = L^1(\hat{G})^\wedge \cap L^r(G)$$

with the norm

$$\|u\| = \|f\|_{L^1(\hat{G})} + \|\hat{f}\|_{L^r(G)},$$

if $u = \hat{f}$, the Fourier transform of f.

The study of these Banach Algebras started in a beautiful paper of Larsen, Liu and Wang [28] in the abelian case, and continued in [18, 19, 20, 21, 27, 30, 31].

The paper is divided into the following points:

(1) The strict containment theorem.
(2) Arens regularity.
(3) Approximate identities and Banach algebra amenability.

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(4) The dual of $A_p^r(G)$ and weak sequential completeness.
(5) Optimisation in $A_p^r(G)$ and the RNP.
(6) Spectral synthesis for $A_p^r(G)$.

2. Definitions and notations

Denote as in [23]

$$A_p(G) = \{u = \sum u_n * v_n^\vee : u_n \in L^{p'}, v_n \in L^p, \sum \|u_n\|_{L^{p'}} \|v_n\|_{L^p} < \infty\},$$

where the norm of $u \in A_p$ is the infimum of the last sum over all the representations of u as as above. We will omit at times G and write L^p, A_p, etc. instead of $L^p(G)$, $A_p(G)$, etc. Denote by $PM_p = A_p(G)^*$, and by $PF_p(G)$, the norm closure in $PM_p(G)$ of $L^1(G)$, (as a space of convolutors on $L_p(G)$).

Let $W_p(G) = PF_p(G)^*$. Then $W_p(G)$ is a Banach algebra of bounded continuous functions on G containing the ideal $A_p(G)$, see Cowling [5].

If G is abelian and $p = 2$, then $W_2(G) = M(\hat{G})^\wedge$, where $M(G)$ is the space of bounded Borel measures on G. Let $C_0(G)$ $[C_c(G)]$ denote the continuous functions which tend to 0 at ∞, [with compact support], with norm

$$\|u\|_\infty = \sup\{|u(x)| : x \in G\}.$$

Let $A_p^r(G)$ be as above, and define $W_p^r(G) = W_p \cap L^r(G)$, with the sum norm. The group G is weakly amenable if $A_2(G)$ has an approximate identity $\{v_\alpha\}$ bounded in the norm of $B_2(G)$, the space of Herz-Schur multipliers, see [11, 20, 23, 24] and the references therein..

Any closed subgroup G of any finite extension of the general Lorenz group $SO_0(n,1)$, for $n > 1$, (hence the free group on $n > 1$ generators F_n) is weakly amenable. Clearly amenable groups are weakly amenable, yet F_n is weakly amenable but not amenable.

3. Main results

3.1. The strict containment theorem (SCT).

Theorem 3.1. *Let G be a noncompact locally compact group and $1 < p < \infty$.*

(1) If $1 \le r < p'$ and $r < s \le \infty$, then $A_p^r \subsetneq A_p^s$.
(2) If G is in addition unimodular this is the case if $1 \le r < \max\{p, p'\}$.
(3) If G is amenable the above holds with no restriction on r thus for $1 \le r < \infty$.
(4) If $G = SL(2,R)$, then, by the Kunze-Stein phenomenon, $A_2^r = A_2$, $\forall r > 2$ (see [5, 26]). Hence the theorem does not hold if $p = p = 2$ and $r > 2$.

Proof. In [18], A_p denotes $A_{p'}$ a la [23]. Replacing p by p' in SCT [18], one gets that if $1 \le r < p'$ and $r < s$, then $A_{p'}^r \neq A_{p'}^s$. It follows that $A_p^r \neq A_p^s$ in the present (namely [23]) notation. For detailed proofs see [18] and the references therein. □

Remark 3.1. (4) in Theorem 3.1 holds true for any connected semisimple Lie group with finite center by Cowling's [5] impressive improvement of Kunze-Stein [26].

The case which is not covered by this theorem is $r = p'$.

Question 3.1. (1) If $p' < s \le \infty$, is $A_p^{p'} \neq A_p^s$? (2) In particular, is it possible that $A_p^{p'} \neq A_p$?

Proposition 3.1. *Assume that G is unimodular. If $2 < p$, then $A_p^{p'} \neq A_p^s$, if $p' < s \leq \infty$.*

Proof. If $p < 2 < p'$, then $A_p^p \neq A_p^s$, if $p < s$ since $p < p'$. Hence $\check{A_p^p} \neq \check{A_p^s}$, i.e., $A_{p'}^p \neq A_{p'}^s$, $\forall p < s \leq \infty$ by unimodularity and since by [6] (pp. 91) $\check{A_p} = A_{p'}$. Replace now p by p'. If $p' < 2 < p$ and $p' < s$, then $A_p^{p'} \neq A_p^s$. This completes the proof. □

Using a result of Saeki [33], suggested to us by John J.F. Fournier, one can prove the following

Proposition 3.2. *Let G be any noncompact group and $p \geq 2$. Then $A_p^{p'} \neq A_p$.*

3.2. **Arens regularity.** A commutative Banach algebra A is Arens regular [AR] if A^{**} with Arens multiplication is commutative, see [9] and the references therein.

Let $I \subset J \subset A$ be closed ideals and let $B \subset A$ be a closed subalgebra. If A is AR, then so are B and A/J. Hence, if A/J (or B) is not AR then A/I (A) is not AR.

In particular if $A = A_p^r(G)$ or) or $A = A_p(G)$ and $F_1 \subset F_2 \subset G$ are closed, and $A(F_2)$ is not AR then neither is $A(F_1)$. For the above and more, see [9] and the references therein. In view of this, it is of interest to know how thin can F be and yet $A(F)$ be non AR, even if $G = R$. Results of Colin Graham [13, 14, 15] address this question. This is also addressed in part in [18].

Theorem 3.2. *Let G be a locally compact group, H a closed nondiscrete subgroup, $F \subset G$ closed and $a, b \in G$. If the set $int_{aHb}F$ is not empty, then $A_p^r(F)$ is NOT AR for any $1 \leq r \leq \infty$ and $1 < p < \infty$.*

Remark 3.2. $int_{aHb}F$ denotes the interior of $aHb \cap F$ in aHb. $A_p^r(F)$ is the quotient algebra $A_p^r(G)/I_F$, where

$$I_F = \{u \in A_p^r(G) : u = 0 \text{ on } F\}.$$

Proof. In fact this is Theorem 4 of [18]. The proof there contains a gap which can be fixed as follows: There is some open nonvoid open $V \subset G$ with compact closure such that

$$V \cap aHb \subset F \cap aHb.$$

Let now K denote the closure of $V \cap aHb$, which is also included in $F \cap aHb$. Using Theorem 5 (ii) in [17], one sees that there exist $\psi_1, \psi_2 \in A_p(K)^{**}$ such that $\psi_1 \neq \psi_2$. But by Lemma 2(b) in [17], $\psi_i \Box \psi_j = \psi_j$ if $i \neq j$, where $\Box$ is the Arens multiplication in $A_p(K)^{**}$, which is hence not commutative. Thus $A_p(K)$ is not AR. However, since the identity $I : A_p(K) \to A_p^r(K)$ is one to one onto and continuous, it is an isomorphism. Hence $A_p(K)$, afotiori $A_p^r(F)$ is not AR. For $A_p^r(G)$ take $aHb = F = G$. □

Corollary 3.1. *If G is not discrete, then $A_p^r(G)$is NOT AR for any $1 < p < \infty$, $1 \leq r \leq \infty$.*

Remark 3.3. Let G be discrete. If G is amenable and infinite then $A_2(G)$ is NOT AR by [29]. If G is abelian and infinite then $A_p(G)$ is NOT AR by [12].

In marked contrast, for any discrete G, $A_p^r(G)$ IS AR for any $1 \leq r \leq \max\{p, p'\}$, $1 < p < \infty$ by the following.

Theorem 3.3. *Let G be any discrete infinite group and. Then $A_p^r(G) = l^r(G)$, if $1 \leq r \leq \max\{p, p'\}$ with pointwise multiplication. For such r, $A_p^r(G)$ and all their even duals are AR Banach algebras.*

See Theorem 7 of [18] and the references therein.

Question 3.2. It is not clear to us what happens if $r > \max\{p, p'\}$.

$A_p(F)$ for thin sets F has been studied in [13, 14, 15].

3.3. **Approximate identities and Banach algebra amenability.** Let G be infinite and commutative. Then $A_2(G)$ is an amenable Banach algebra since $L^1(\hat{G})$ is amenable by Johnsons theorem, see [25] for definition and proof. There exist though *compact* groups G for which this is not the case [25]. It is of interest to note, that the Banach algebras $A_p^r(G)$ for $1 \leq r < \infty$ behave very differently, see [18] and the references therein.

Theorem 3.4. *Let G be a non compact locally compact group. Then for any $1 < p < \infty$ and $1 \leq r < \infty$, $A_p^r(G)$, and any of its even duals with Arens multiplication, are NON AMENABLE Banach algebras.*

The proof is based on the next result, which is basically due to a more general theorem of Burnham [4]. It is directly proved in [18].

Proposition 3.3. *If G is non compact and $\{w_\alpha\}$ is an approximate identity in $A_p^r(G)$, for some $1 \leq r < \infty$, then $\|w_\alpha\|_{A_p^r} \to \infty$.*

This is again in marked contrast to the fact that for any amenable group G, $A_p(G)$ has a bounded approximate identity.

3.4. **The dual of $A_p^r(G)$ and weak sequential completeness.** The following theorem is a particular case of Theorem 1.7 of Liu and van Rooij [32] who proved it for any Banach spaces. A simple and direct proof that is more explicit than that given in Theorem 2 of [28] is given in [18] (pp.411).

Theorem 3.5. *Let $1 \leq r < \infty$ and $1 < p < \infty$. Then $A_p^r(G)^* = L^{r'}(G) + PM_p(G)$ with the action $\langle f + \phi, u\rangle = \int f u d_\lambda + \langle f, u\rangle$, $u \in A_p^r$. The norm of $F = f + \phi$ being*

$$\|F\|_{A_p^{r*}} = \inf\{\max\{\|f\|_{L^{r'}}, \|\phi\|_{PM_p}\}\}$$

the infimum (which is attained) being over all $f \in L^{r'}$, $\phi \in PM_p$ such that $F = f + \phi$. Note that $r' = \infty$, if $r = 1$.

Theorem 3.6. *(a) If for some p, A_p is weak sequentially complete, then A_p^r is weak sequentially complete for all $1 \leq r < \infty$. (b) $A_2^r(G)$ is weak sequentially complete for all $1 \leq r \leq \infty$. (c) If $K \subset G$ is compact then $S_p^r(K) = \{u \in A_p^r : spt\ u \subset K\}$ is weak sequentially complete $\forall 1 < p < \infty$ and $1 \leq r \leq \infty$.*

Note that $A_2(G)$ is the predual of a W^* algebra and as such is weak sequentially complete, see [34] (pp.148) and the references therein. It is not clear to us, if $A_p(G)$ is weak sequentially complete if $p \neq 2$, even if G is an amenable group.

3.5. **Optimization in $A_p^r(G)$.**

Definition 3.1. Let X be a Banach space. X has the *Krein-Milman Property (KMP)*, if each norm closed convex bounded subset B of X is the norm closed convex hull of its extreme points, ext(B), namely, $B = c\bar{o}\ ext(B)$.

Hence Linear Optimisation Problems in such Sets B have Solutions.

The closed unit ball B_1 of $L^1(\mu)$, for a nonatomic measure μ, has no extreme points, hence it does not have the KMP. And yet, if μ is atomic (for example in case of l^1) then B_1 has many extreme points. In fact this space does even have the KMP and is in addition a dual space. Known results (see [7]) then imply that this space has a stronger property denoted RNP, namely:

Definition 3.2. Let X be a Banach space. X has the *Radon-Nikodym Property (RNP)* if each B as above is the norm closed convex hull of its strongly exposed points (strexp(B)), namely $B = c\bar{o}\ strext(B)$ see sequel or [7] (pp. 190 and pp. 218).

Points in strexp(B) are points of ext(B) that have beautiful smoothness properties. In particular they are weak to norm continuity points of B and are peak points of B.

Hence: Linear Optimisation Problems in such sets B, Have Solutions but Moreover, it is Easy to Test if an Algorithm Converges to a Solution.

Quoting Jerry Uhl: A Banach space has the RNP if its unit ball wants to be weakly compact, but just cannot make it.

Definition 3.3. Let B be a bounded subset of the Banach space X and $b \in B$. b is a strongly exposed point of B (and strexp(B) denotes the set of all such), if $\exists b^* \in X^*$ such that (see [7])

$$Re\ b^*(x) < Re\ b^*(b), \quad x \in B, x \neq b$$

and

$$Re\ b^*(x_n) \to Re\ b^*(b),\ for\ x_n \in B\ implies\ \|x_n - b\| \to 0.$$

Hence in order to Test an Algorithm for some b in strexp(B) it is Enough to Test it on One Element of X^*.

Any X which is norm isomorphic to l^1 has the RNP. If X a dual Banach space, and B is w^* compact convex, then the functional b^* can be chosen in the predual of X, see [7] and the references therein. It follows from above that if G is abelian, then $A_2(G)$ has the RNP if G is compact and does not have the RNP it if G is not compact. And yet, for any abelian G and any compact subset K

$$A_K^2(G) = \{u \in A_2(G) : spt\ u \subset K\}$$

does have the RNP, where spt denotes support.

In fact we have proved in [16] that for any G and any compact subset K and any $1 < p < \infty$, $A_K^p(G) = \{u \in A_p(G) : spt\ u \subset K\}$ has the RNP. Tools for abelian G are not available in this case. It has been proved by Braun, in an unpublished preprint [2], that if G is amenable, then $A_p^1(G)$ is a dual Banach space with the RNP.

The result in [2] uses the method in [16] and the involved machinery of [3], which is avoided in our proofs. The following results have been obtained in our paper [21], and are improvements of results in [19] and [20].

Theorem 3.7. *Let $p = 2$ or G be weakly amenable and $1 < p < \infty$. Then*

$$W_p \cap L^r(G) = A_p \cap L^r(G), \quad \forall 1 \leq r \leq p'.$$

Hence (by [20] Theorem 2.2), $A_p^r(G)$ is a dual Banach space for such r. If G is in addition unimodular, then the above holds for $1 \leq r \leq \max\{p, p'\}$.

The interval $[1, p']$ cannot be improved even if $p = 2$ and $G = Z$, the additive integers, see [19, 22], and the sequel. Use of Theorem 3.1 is made in proving the main result of this section, namely:

Theorem 3.8. *(see [21]) Let G be a weakly amenable locally compact group. Then*

(1) $\forall 1 \leq r \leq p'$, $A_p^r(G) = W_p^r(G)$ and $A_p^r(G)$ is a dual Banach Algebra with the RNP.

(2) *If G is in addition unimodular this is the case $1 \le r \le \max\{p, p'\}$.*
(3) *If G is any connected semisimple Lie group with finite center, $p = p' = 2$ and $r > 2$, then $A_2^r(G) = A_2(G)$ (by Cowling [5]) and $A_2^r(G)$ does not have the RNP and is not a dual space.*

Remark 3.4. (2) was proved in [20]. (3) with $G = SL(2,R)$ was proved in [19], by considering the support of the Plancherel measure on the dual of G. The proof of (3) in the present form uses results of Baggett [1].

Question 3.3. Does $A_2^r(Z)$ fail to have the RNP if $r > 2$?

3.6. **Spectral synthesis for $A_p^r(G)$.**

Theorem 3.9. *(see [18]) Assume that for some p, $A_p(G)$ has an approximate identity $\{u_\alpha\}$ such that $\|u_\alpha\|_{m_p} < \infty$. Then, for this p, the Banach algebras $A_p^r(G)$ have the same sets of synthesis for all $1 \le r \le \infty$.*

The multiplier norm is defined by

$$\|u\|_{m_p} = \sup\{\|uv\|_{A_p} : v \in A_p, \|v\|_{A_p} \le 1\}.$$

Recall that $A_p^r(G) = A_p(G)$, if $r = \infty$.

The above improves results of Burnham [4] and of Lai and Chen [27].

Remark 3.5. The empty set $\emptyset$ is a set of synthesis for $A_p(G)$ for all G, i.e. $A_p \cap C_c(G)$ is norm dense in $A_p(G)$. And yet we are not able to prove that $\emptyset$ is a set of synthesis for $A_p^r(G)$, if $r < \infty$, for all G. This is the reason for the need of the approximate identity in the theorem above. In fact it is not clear if Theorem 3.9 holds for the group $G = SL(2,R) \lhd R^n$ if $n > 1$. In this case $A_2(G)$ has no multiplier bounded approximate identity, by Doroffaeff [8]. Is $\emptyset$ a synthesis set for $A_2^r(G)$ if $r < \infty$?

If p varies then the sets of synthesis do not remain the same, even if $G = R^n$. In fact let S^{n-1} denote the unit sphere in R^n.

(1) If $1 < p < 3/2$, then S^3 IS a synthesis set for $A_p^r(R^4)$, $1 \le r \le \infty$.
(2) If $3/2 < p \le 2$, then S^3 IS NOT a synthesis set for $A_p^r(R^4)$, $1 \le r \le \infty$.

Eymard [11] proved the above for $A_p(R^4)$, (i.e. for $r = \infty$). Eymards result together with Theorem 3.9 imply (1) and (2).

Theorem 3.10. *Assume that for some p, $A_p(G)$ has an approximate identity $\{u_\alpha\}$ such that $\|u_\alpha\|_{m_p} < \infty$. Let H be a closed subgroup of G. Then, for this p, one has (i) If $p = 2$, then any coset of H is a set of synthesis for $A_2^r(G)$, $\forall 1 \le r \le \infty$. (ii) If $p \ne 2$ and H is amenable or is normal in G or a coset of such (singletons $\{x\}$ for x in G are such) then it is a synthesis set for $A_p^r(G)$, $\forall 1 \le r \le \infty$.*

Proof. (i) By Takesaki-Tatsuma [35], H is a synthesis set for $A_2(G)$. (ii) By Herz [23] H is a synthesis set for $A_p(G)$. □

Addendum. If G is second countable and weakly amenable and $A_p^t(G)$ has the RNP then so does $A_p^s(G)$, $\forall s \le t$, by [21]. Hence, if $A_2(G)$ has the RNP, (i.e. G is a Fell group see [1]) then so does $A_2^s(G)$, $\forall 1 \le s \le \infty$, see [21].

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