J. Nonlinear Var. Anal. 1 (2017), No. 3, pp. 307-319
Available online at http://jnva.biemdas.com

GROUP ACTIONS WHOSE SPACE OF INVARIANT MEANS IS FINITE DIMENSIONAL

MATTHIAS NEUFANG[1], JAN PACHL[2], JURIS STEPRĀNS[3,*]

[1]*Department of Mathematics, Carleton University, 1125 Colonel By Drive, Ottawa, Ontario, Canada K1S 5B6*
[2]*Fields Institute, 222 College Street, Toronto, Ontario, Canada M5T 3J1*
[3]*Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario, Canada M3J 1P3*

Abstract. It is shown that under various set theoretic hypotheses there is an amenable subgroup F of the full symmetric group on $\mathbb{N}$ such that the space of means on $\mathbb{N}$ invariant under the natural action of F is finite dimensional, yet the Arens multiplication by invariant means in $\ell_\infty^*(F)$ is never weak* to weak* continuous.

Keywords. Amenable subgroup; Banach algebra; Group action; Invariant mean; Infinite permutation group.

2010 Mathematics Subject Classification. 43A10, 43A20.

1. Introduction

When a group G acts on a set X, the action naturally extends to an action $\odot$ of the convolution algebra $\ell_1(G)$ on the space $\ell_1(X)$. Then $\odot$ extends to an action of $\ell_\infty^*(G)$ on $\ell_\infty^*(X)$ in two canonical ways, called the *first (or left)* and the *second (or right) Arens product*, denoted here $\odot_1$ and $\odot_2$ (defined below). This paper is our contribution to the study of continuity properties of $\odot_1$ for the actions of subgroups of the infinite permutation group on a countable infinite set. We will point out relations to amenability of such groups and to the existence of invariant means. In this connection, our starting point are the papers of Foreman [4] and Yang [12].

We explore some ramifications of an argument of A. T.-M. Lau that can be found in Corollary 5 of [7]. Lau's argument will be used in Proposition 3.6 to show that if a group G acts on a set with a unique invariant mean then Arens multiplication by every invariant mean in $\ell_\infty^*(G)$ is weak* to weak* continuous. (The definitions of these terms will be supplied in §2.) The question then arises of whether the assumption of the action of the group having a unique invariant mean can be weakened to having a finite dimensional space of invariant elements of $\ell_\infty^*(\mathbb{N})$. This will be shown to be consistently false.

2. Arens products for Banach modules

By $\mathbb{S}_\infty$ we denote the *infinite permutation group*, the group of bijections from the set $\mathbb{N} = \{0, 1, 2, \dots\}$ onto itself. Further on we apply the following general definition to the specific instance of a group

*Corresponding author.
E-mail addresses: mneufang@math.carleton.ca (M. Neufang), jan.pachl@utoronto.ca (J. Pachl), steprans@yorku.ca (J. Steprāns).

Received October 23, 2017; Accepted November 20, 2017.

$G \subseteq \mathbb{S}_\infty$ acting on $\mathbb{N}$. In that instance, $\mathbb{A} = \ell_1(G)$, $\mathbb{B} = \ell_1(\mathbb{N})$, and the module multiplication $\odot$ is the canonical extension of the action of G on $\mathbb{N}$ to an action of $\ell_1(G)$ on $\ell_1(\mathbb{N})$.

Definition 2.1. Let $\mathbb{A}$ be a Banach algebra and let $\mathbb{B}$ be a left $\mathbb{A}$-module. Let the module multiplication of $\mathbb{A}$ acting on $\mathbb{B}$ be denoted by $\odot$. Define maps $\odot_1, \odot_2 \colon \mathbb{A}^{**} \times \mathbb{B}^{**} \to \mathbb{B}^{**}$ by

$$\langle \mathfrak{m} \odot_1 \mathfrak{n}, \psi \rangle = \langle \mathfrak{m}, a \mapsto \langle \mathfrak{n}, b \mapsto \psi(a \odot b) \rangle \rangle \tag{2.1}$$

$$\langle \mathfrak{m} \odot_2 \mathfrak{n}, \psi \rangle = \langle \mathfrak{n}, b \mapsto \langle \mathfrak{m}, a \mapsto \psi(a \odot b) \rangle \rangle \tag{2.2}$$

for $\mathfrak{m} \in \mathbb{A}^{**}, \mathfrak{n} \in \mathbb{B}^{**}, \psi \in \mathbb{B}^*$. The maps $\odot_1$ and $\odot_2$ are called the *first (or left)* and the *second (or right) Arens product* derived from the operation $\odot$ (with reference to [1]).

When $\mathbb{A}^{**}$ and $\mathbb{B}^{**}$ are equipped with their weak* topologies, for any mapping $\circledast \colon \mathbb{A}^{**} \times \mathbb{B}^{**} \to \mathbb{B}^{**}$ (in particular for $\odot_1$ and $\odot_2$) define

$$\Lambda_1(\circledast) := \{\mathfrak{m} \in \mathbb{A}^{**} \mid \text{ the mapping } \mathfrak{n} \mapsto \mathfrak{m} \circledast \mathfrak{n} \text{ is continuous from } \mathbb{B}^{**} \text{ to itself}\}$$

$$\Lambda_2(\circledast) := \{\, \mathfrak{n} \in \mathbb{B}^{**} \mid \text{ the mapping } \mathfrak{m} \mapsto \mathfrak{m} \circledast \mathfrak{n} \text{ is continuous from } \mathbb{A}^{**} \text{ to } \mathbb{B}^{**}\}$$

$\Lambda_1(\circledast)$ and $\Lambda_2(\circledast)$ are the *first (or left)* and the *second (or right) topological centre* defined by $\circledast$. □

Note that (2.1) and (2.2) imply that if $a \in \mathbb{A}$ and $b \in \mathbb{B}$ then $a \odot b = a \odot_1 b = a \odot_2 b$. It is also worth observing that, as in [1], the definition of $\odot_1$ and $\odot_2$ does not require that $\mathbb{A}$ be a Banach algebra — all that is needed is that the map $\odot$ be bilinear. A similar definition can be found in [3]. However, if $\mathbb{A}$ is an algebra with multiplication $\star$ then, applying Definition 2.1 to the left action of $\mathbb{A}$ on itself, we get two multiplications operations $\star_1$ and $\star_2$ on $\mathbb{A}^{**}$ that both make $\mathbb{A}^{**}$ into a Banach algebra. It is well known and easy to check that if $\mathbb{B}$ is a left $\mathbb{A}$ module as above then

$$(\mathfrak{m} \star_1 \mathfrak{m}') \odot_1 \mathfrak{n} = \mathfrak{m} \odot_1 (\mathfrak{m}' \odot_1 \mathfrak{n})$$

$$(\mathfrak{m} \star_2 \mathfrak{m}') \odot_2 \mathfrak{n} = \mathfrak{m} \odot_2 (\mathfrak{m}' \odot_2 \mathfrak{n})$$

for $\mathfrak{m}, \mathfrak{m}' \in \mathbb{A}^{**}$, $\mathfrak{n} \in \mathbb{B}^{**}$. Thus $\odot_i$ makes $\mathbb{B}^{**}$ into a left $(\mathbb{A}^{**}, \star_i)$ module for $i = 1, 2$.

The next proposition follows directly from the definitions (2.1) and (2.2). It shows that $\Lambda_2(\odot_1) = \mathbb{B}^{**}$, $\Lambda_1(\odot_2) = \mathbb{A}^{**}$, $\Lambda_1(\odot_1) \supseteq \mathbb{A}$ and $\Lambda_2(\odot_2) \supseteq \mathbb{B}$. Beyond that, characterizing $\Lambda_1(\odot_1)$ or $\Lambda_2(\odot_2)$ or even deciding whether $\Lambda_1(\odot_1) = \mathbb{A}$ or $\Lambda_2(\odot_2) = \mathbb{B}$ is often a difficult problem.

Proposition 2.2. *Let $\mathbb{A}$ be a Banach algebra and $\mathbb{B}$ a left $\mathbb{A}$-module with module multiplication $\odot$. Equip $\mathbb{A}^{**}$ and $\mathbb{B}^{**}$ with their weak* topologies.*

(1) For every $\mathfrak{n} \in \mathbb{B}^{**}$ the mapping $\mathfrak{m} \mapsto \mathfrak{m} \odot_1 \mathfrak{n}$ is continuous from $\mathbb{A}^{**}$ to $\mathbb{B}^{**}$.
(2) For every $\mathfrak{m} \in \mathbb{A}^{**}$ the mapping $\mathfrak{n} \mapsto \mathfrak{m} \odot_2 \mathfrak{n}$ is continuous from $\mathbb{B}^{**}$ to itself.
(3) For every $a \in \mathbb{A}$ the mapping $\mathfrak{n} \mapsto a \odot_1 \mathfrak{n}$ is continuous from $\mathbb{B}^{**}$ to itself.
(4) For every $b \in \mathbb{B}$ the mapping $\mathfrak{m} \mapsto \mathfrak{m} \odot_2 b$ is continuous from $\mathbb{A}^{**}$ to $\mathbb{B}^{**}$. □

3. Arens products for group actions

In this paper we study the continuity properties of the operation $\odot_1$ that arises from a group G acting on a set X. The group operation defines the convolution operation $\star$ on the space $\ell_1(G)$ that makes $\ell_1(G)$ into a Banach algebra. The action of G on X defines an action $\odot$ of $(\ell_1(G), \star)$ on the Banach space $\ell_1(X)$. Then Definition 2.1 yields actions $\odot_i$ of $\ell_1(G)^{**} = \ell_\infty^*(G)$ with $\star_i$ on $\ell_1(X)^{**} = \ell_\infty^*(X)$ for $i = 1, 2$.

Notation 3.1. We identify each element of $\ell_\infty^*(G)$ (respectively $\ell_\infty^*(X)$) with a finitely additive measure on the algebra of all subsets of G (respectively X). Accordingly, for $\mathfrak{m} \in \ell_\infty^*(G)$ and $A \subseteq G$ we write $\mathfrak{m}(A)$ instead of the more cumbersome $\mathfrak{m}(1_A)$ or $\langle \mathfrak{m}, 1_A \rangle$. Similarly for $\mathfrak{n} \in \ell_\infty^*(X)$. $\square$

With the identification in Notation 3.1, formulas (2.1) and (2.2) become

$$\langle \mathfrak{m} \odot_1 \mathfrak{n}, \psi \rangle = \int_G \int_X \psi(gx)\, d\mathfrak{n}(x)\, d\mathfrak{m}(g)$$
$$\langle \mathfrak{m} \odot_2 \mathfrak{n}, \psi \rangle = \int_X \int_G \psi(gx)\, d\mathfrak{m}(g)\, d\mathfrak{n}(x)$$

hence questions about the continuity of $\odot_1$ and $\odot_2$ are questions about exchanging limits and integrals of certain functions with respect to finitely additive measures.

For the natural action of G on itself we have $G = X$ and $\star = \odot$, so that the centres $\Lambda_1(\odot_1) = \Lambda_1(\star_1)$ and $\Lambda_2(\odot_2) = \Lambda_2(\star_2)$ are subsets of $\ell_1(G)^{**} = \ell_\infty^*(G)$. In this case the weak* continuity of $\odot_1$ and $\odot_2$ is quite well understood. In particular:

Theorem 3.2 (Lau [7])**.** *Let G be any group. Then $\Lambda_1(\star_1) = \Lambda_2(\star_2) = \ell_1(G)$.*

For a general group action the situation is much less clear. Next we show that for uniquely amenable actions by amenable groups, such as those constructed by Yang [12] and Foreman [4], we have $\Lambda_1(\odot_1) \supsetneqq \ell_1(G)$.

When the elements of $\ell_\infty^*(G)$ are identified with finitely additive measures on G, every $g \in G$ is identified with the point mass at g, i.e. the measure in $\ell_\infty^*(G)$ that maps $A \subseteq G$ to $1_A(g)$.

Definition 3.3. Consider an action of a group G on a set X, with the corresponding action $\odot$ of $(\ell_1(G), \star)$ on $\ell_1(X)$. An element $\mathfrak{n} \in \ell_\infty^*(X)$ is *G-invariant* if $g \odot_1 \mathfrak{n} = \mathfrak{n}$ for every $g \in G$.

An element $\mathfrak{n} \in \ell_\infty^*(X)$ is a *mean* if $\mathfrak{n} \geq 0$ and $\mathfrak{n}(1) = 1$. The action is *amenable* if there exists a G-invariant mean in $\ell_\infty^*(X)$. It is *uniquely amenable* if there exists exactly one G-invariant mean in $\ell_\infty^*(X)$.

In the special case of the left natural action of G on itself, say that $\mathfrak{m} \in \ell_\infty^*(G)$ is *left-invariant* if $g \star_1 \mathfrak{m} = \mathfrak{m}$ for every $g \in G$. The group G is *amenable* or *uniquely amenable* if the left natural action of G on itself is such. $\square$

Obviously the set of G-invariant elements is a linear subspace of $\ell_\infty^*(X)$. By Corollary 3.5 below, the action is uniquely amenable if and only if the space of G-invariant elements has dimension 1.

The proof of the following proposition is essentially due to Namioka [9, 3.2].

Proposition 3.4. *Consider an action of a group G on a set X. Let $\mathfrak{n} \in \ell_\infty^*(X)$ be G-invariant. Then so are its positive and negative parts $\mathfrak{n}^+$ and $\mathfrak{n}^-$.* $\square$

Corollary 3.5. *Let an action of G on X be uniquely amenable, and let $\mathfrak{n}_0 \in \ell_\infty^*(X)$ be the unique G-invariant mean. Let $\mathfrak{n} \in \ell_\infty^*(X)$ be G-invariant. Then $\mathfrak{n} = \mathfrak{n}(1)\mathfrak{n}_0$.*

Proof. $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are G-invariant by Proposition 3.4. Hence $\mathfrak{n}^+ = \mathfrak{n}^+(1)\mathfrak{n}_0$ and $\mathfrak{n}^- = \mathfrak{n}^-(1)\mathfrak{n}_0$, and

$$\mathfrak{n} = \mathfrak{n}^+ - \mathfrak{n}^- = \mathfrak{n}^+(1)\mathfrak{n}_0 - \mathfrak{n}^-(1)\mathfrak{n}_0 = (\mathfrak{n}^+(1) - \mathfrak{n}^-(1))\mathfrak{n}_0 = \mathfrak{n}(1)\mathfrak{n}_0\,. \qquad \square$$

The proof of the following theorem is a variant of the argument that Lau used to prove that every locally compact group with a unique invariant mean is compact [7, Cor.5].

Proposition 3.6. *Let an action of G on X be uniquely amenable, and let $\mathfrak{m} \in \ell_\infty^*(G)$ be left-invariant. Then $\mathfrak{m} \in \Lambda_1(\odot_1)$.*

Proof. Let $\mathfrak{n}_0 \in \ell_\infty^*(X)$ be the unique G-invariant mean. For $g \in G$ and $\mathfrak{n} \in \ell_\infty^*(X)$ we have

$$g \odot_1 (\mathfrak{m} \odot_1 \mathfrak{n}) = (g \star_1 \mathfrak{m}) \odot_1 \mathfrak{n} = \mathfrak{m} \odot_1 \mathfrak{n}.$$

As $(\mathfrak{m} \odot_1 \mathfrak{n})(1) = \mathfrak{m}(1)\mathfrak{n}(1)$, from Corollary 3.5 we get $\mathfrak{m} \odot_1 \mathfrak{n} = \mathfrak{m}(1)\mathfrak{n}(1)\mathfrak{n}_0$.

Let $\{\mathfrak{n}_\gamma\}_\gamma$ be a net in $\ell_\infty^*(X)$ that weak* converges to $\mathfrak{n} \in \ell_\infty^*(X)$. Then $\lim_\gamma \mathfrak{n}_\gamma(1) = \mathfrak{n}(1)$, hence

$$\lim_\gamma \mathfrak{m} \odot_1 \mathfrak{n}_\gamma = \lim_\gamma \mathfrak{m}(1)\mathfrak{n}_\gamma(1)\mathfrak{n}_0 = \mathfrak{m}(1)\mathfrak{n}(1)\mathfrak{n}_0 = \mathfrak{m} \odot_1 \mathfrak{n}. \qquad \square$$

Proposition 3.6 does not yield anything interesting for a canonical action of a group G on itself, since such an action is uniquely amenable only if G is finite. Indeed, Lau and Paterson [8] have shown that if G is infinite amenable then the number of left-invariant means in $\ell_\infty^*(G)$ is always $2^{2^{|G|}}$.

However, Yang has shown in [12] under the Continuum Hypothesis — and Foreman under weaker hypotheses in [4, 3.1] — that there is an infinite amenable subgroup G of $\mathbb{S}_\infty$ whose canonical action on $\mathbb{N}$ has a unique invariant mean (we prove a version of Foreman's theorem in §5). Since G is amenable, there is a G-invariant mean $\mathfrak{m} \in \ell_\infty^*(G)$. As G is infinite, $\mathfrak{m} \notin \ell_1(G)$. It follows from Proposition 3.6 that $\mathfrak{m} \in \Lambda_1(\odot_1)$. Thus $\Lambda_1(\odot_1) \supsetneqq \ell_1(G)$.

One might ask whether Lau's argument in the proof of Proposition 3.6 can be used also when the space of G-invariant means is merely finite-dimensional rather than one-dimensional. It will be shown in §6 that under the Continuum Hypothesis (as well as various weaker assumptions) there is an amenable subgroup F of $\mathbb{S}_\infty$ such that the space of F-invariant means in $\ell_\infty^*(\mathbb{N})$ is two-dimensional yet no left-invariant element of $\ell_\infty^*(F)$ is in $\Lambda_1(\odot_1)$.

4. Group construction

Say that an element π of a group is an *involution* if $\pi^{-1} = \pi$. In §5 and §6 we construct certain subgroups of $\mathbb{S}_\infty$ generated by involutions. In this section we describe an abstract version of the construction, independent of the action on the underlying set $\mathbb{N}$.

Say that two subsets A and B of a group S *commute* if $ab = ba$ for all $a \in A$ and $b \in B$.

Lemma 4.1. *Let S be a group and G, H its subgroups, $G \subseteq H$. Let $\pi \in S$ be an involution such that H and $\pi H \pi$ commute. Then (i) $G\pi H\pi$ and (ii) $H\pi H \cup H\pi H\pi$ are subgroups of S.*

Proof. Since G and $\pi H \pi$ commute,

$$(G\pi H\pi)(G\pi H\pi) = GG\pi H\pi\pi H\pi = G\pi H\pi$$
$$(G\pi H\pi)^{-1} = \pi H^{-1}\pi G^{-1} = \pi H\pi G = G\pi H\pi$$

That proves (i). Next

$$(H\pi H)(H\pi H) = HH\pi HH\pi = H\pi H\pi$$
$$(H\pi H)(H\pi H\pi) = HH\pi HH\pi\pi = H\pi H$$
$$(H\pi H\pi)(H\pi H) = HH\pi H\pi\pi H = H\pi H$$
$$(H\pi H)^{-1} = H\pi H$$

Since $H\pi H\pi$ is a group by (i), that covers all cases for (ii). $\square$

Theorem 4.2. *Let S be a group with identity element e. Let κ be an infinite cardinal, and for $\xi \in \kappa$ let $\pi_\xi \in S$ be involutions, $\pi_\xi \neq e$. For $\xi \leq \kappa$ define F_ξ to be the group generated by π_ζ, $\zeta < \xi$. Assume that for every $\xi \in \kappa$*

(a) F_ξ and $\pi_\xi F_\xi \pi_\xi$ commute, and

(b) $F_\xi \cap \pi_\xi F_\xi \pi_\xi = \{e\}$.

Recursively define $G_{\zeta,\xi} \subseteq S$ for $\zeta \leq \xi \leq \kappa$ by

$$\begin{aligned} G_{\zeta,\zeta} &:= F_\zeta \\ G_{\zeta,\xi+1} &:= G_{\zeta,\xi}\pi_\xi F_\xi \pi_\xi \quad \text{for } \zeta \leq \xi < \kappa \\ G_{\zeta,\xi} &:= \bigcup_{\zeta<\eta<\xi} G_{\zeta,\eta} \quad \text{when } \xi \text{ is a limit ordinal, } \zeta < \xi \leq \kappa \end{aligned} \tag{4.1}$$

Then the following hold:

(i) For $\xi \in \kappa$, $F_{\xi+1}$ is a disjoint union of its normal subgroup $F_\xi \pi_\xi F_\xi \pi_\xi$ and its coset $F_\xi \pi_\xi F_\xi$.

(ii) For every $\xi \in \kappa$ and $f \in F_{\xi+1}$ there are unique $f_0, f_1 \in F_\xi$ such that $f = f_0 \pi_\xi f_1$ or $f = f_0 \pi_\xi f_1 \pi_\xi$.

(iii) For every $\xi \leq \kappa$ the group F_ξ is locally finite.

(iv) $F_\zeta \subseteq G_{\zeta,\xi} \subseteq F_\xi$ for $\zeta \leq \xi \leq \kappa$.

(v) $G_{\zeta,\xi} \cap F_\eta = G_{\zeta,\eta}$ for $\zeta \leq \eta \leq \xi \leq \kappa$.

(vi) $G_{\zeta,\xi}$ is a group for $\zeta \leq \xi \leq \kappa$.

(vii) For every $\zeta \in \kappa$ the index of $G_{\zeta,\kappa}$ in F_κ is κ.

Proof. Note that $\pi_\xi \notin F_\xi$ for all $\xi \in \kappa$, from (b) and $\pi_0 \neq e$.

We have $F_{\xi+1} = F_\xi \pi_\xi F_\xi \cup F_\xi \pi_\xi F_\xi \pi_\xi$ by Lemma 4.1(ii). To prove $F_\xi \pi_\xi F_\xi \cap F_\xi \pi_\xi F_\xi \pi_\xi = \emptyset$, suppose there are $f_0, f_1, f_2, f_3 \in F_\xi$ such that $f_0 \pi_\xi f_1 = f_2 \pi_\xi f_3 \pi_\xi$. Apply (a) to get

$$f_0 = f_2 \pi_\xi f_3 \pi_\xi f_1^{-1} \pi_\xi = f_2 \pi_\xi \pi_\xi f_1^{-1} \pi_\xi f_3 = f_2 f_1^{-1} \pi_\xi f_3$$

contradicting $\pi_\xi \notin F_\xi$. Next $F_\xi \pi_\xi F_\xi \pi_\xi$ is a subgroup by Lemma 4.1(i), and $F_\xi \pi_\xi F_\xi = F_\xi \pi_\xi F_\xi \pi_\xi \pi_\xi$ is its right coset. It follows that the index of $F_\xi \pi_\xi F_\xi \pi_\xi$ in $F_{\xi+1}$ is 2, hence it is a normal subgroup. That concludes the proof of (i).

To prove (ii), take any $f_0, f_1, f_2, f_3 \in F_\xi$ such that $f_0 \pi_\xi f_1 = f_2 \pi_\xi f_3$ or $f_0 \pi_\xi f_1 \pi_\xi = f_2 \pi_\xi f_3 \pi_\xi$. Then $f_2^{-1} f_0 = \pi_\xi f_3 f_1^{-1} \pi_\xi$, and from (b) we get $f_0 = f_2$ and $f_1 = f_3$.

Next we prove (iii) by induction on ξ. The only non-obvious part is the step from ξ to $\xi+1$. Assume that F_ξ is locally finite. Take any finite set $K \subseteq F_{\xi+1}$. By (i) and by the local finiteness of F_ξ there is a finite group $H \subseteq F_\xi$ such that $K \subseteq H\pi_\xi H \cup H\pi_\xi H \pi_\xi$. By Lemma 4.1(ii),

$$\langle K \rangle \subseteq \langle H\pi_\xi H \cup H\pi_\xi H\pi_\xi \rangle = H\pi_\xi H \cup H\pi_\xi H\pi_\xi$$

hence the group $\langle K \rangle$ is finite.

The first inclusion in (iv) is immediate from the definition (4.1). The second inclusion is proved by straightforward induction on ζ.

In (v), the inclusion $G_{\zeta,\xi} \cap F_\eta \supseteq G_{\zeta,\eta}$ for $\zeta \leq \eta \leq \xi \leq \kappa$ follows from the definition (4.1) and from (iv). The opposite inclusion holds for $\xi = \eta$ by (iv). For $\xi > \eta$ it is proved by induction on ξ, with the help of (iv) and (b): If $\xi \geq \eta$ and $f \in G_{\zeta,\xi+1} \cap F_\eta = G_{\zeta,\xi}\pi_\xi F_\xi \pi_\xi \cap F_\eta$ then there is $g \in G_{\zeta,\xi}$ such that

$$g^{-1} f \in \pi_\xi F_\xi \pi_\xi \cap g^{-1} F_\eta \subseteq \pi_\xi F_\xi \pi_\xi \cap F_\xi = \{e\}$$

so that $f \in G_{\zeta,\xi} \cap F_\eta$.

Statement (vi) is proved by induction on ξ, using (iv) and Lemma 4.1(i).

Statement (vii) follows from the following, which we prove by contradiction: If $\zeta < \eta < \xi < \kappa$ then $\pi_\eta G_{\zeta,\kappa} \neq \pi_\xi G_{\zeta,\kappa}$. Suppose there are ζ, η and ξ such that $\zeta < \eta < \xi < \kappa$ and $\pi_\eta G_{\zeta,\kappa} = \pi_\xi G_{\zeta,\kappa}$. Then $\pi_\eta \pi_\xi \in G_{\zeta,\kappa}$. From (iv), (v) and (4.1) we get

$$\pi_\eta \pi_\xi \in G_{\zeta,\kappa} \cap F_{\xi+1} = G_{\zeta,\xi+1} = G_{\zeta,\xi} \pi_\xi F_\xi \pi_\xi \subseteq F_\xi \pi_\xi F_\xi \pi_\xi$$

so that $\pi_\eta \in F_\xi \pi_\xi F_\xi$, in contradiction with $\pi_\xi \notin F_\xi$. □

In section 6 we need a slightly more general version of Theorem 4.2 in which identities hold only up to a normal subgroup.

Definition 4.3. Let S be a group and S_0 its normal subgroup. For $g_0, g_1 \in S$ write $g_0 \equiv g_1 \mod S_0$ if $g_0 g_1^{-1} \in S_0$ (or equivalently $g_0^{-1} g_1 \in S_0$). Say that two subsets A and B of S *commute* $\mod S_0$ if $ab \equiv ba \mod S_0$ for all $a \in A$ and $b \in B$.

Theorem 4.4. *Let S be a group with a normal subgroup S_0. Let κ be an infinite cardinal, and for $\xi \in \kappa$ let $\pi_\xi \in S \setminus S_0$ be involutions. For $\xi \leq \kappa$ define F_ξ to be the group generated by S_0 and π_ζ, $\zeta < \xi$. Assume that for every $\xi \in \kappa$*

(a) F_ξ and $\pi_\xi F_\xi \pi_\xi$ commute $\mod S_0$, and
(b) $F_\xi \cap \pi_\xi F_\xi \pi_\xi = S_0$.

Define $G_{\zeta,\xi} \subseteq S$ for $\zeta \leq \xi \leq \kappa$ by (4.1) as in Theorem 4.2. Then the following hold:

(i) For $\xi \in \kappa$, $F_{\xi+1}$ is a disjoint union of its normal subgroup $F_\xi \pi_\xi F_\xi \pi_\xi$ and its coset $F_\xi \pi_\xi F_\xi$.
(ii) For $\xi \in \kappa$, if $f_0, f_1, f_3, f_4 \in F_\xi$ are such that $f_0 \pi_\xi f_1 = f_3 \pi_\xi f_4$ or $f_0 \pi_\xi f_1 \pi_\xi = f_3 \pi_\xi f_4 \pi_\xi$ then $f_0 \equiv f_3 \mod S_0$ and $f_1 \equiv f_4 \mod S_0$.
(iii) If S_0 is locally finite then so is F_ξ for every $\xi \leq \kappa$.
(iv) $F_\zeta \subseteq G_{\zeta,\xi} \subseteq F_\xi$ for $\zeta \leq \xi \leq \kappa$.
(v) $G_{\zeta,\xi} \cap F_\eta = G_{\zeta,\eta}$ for $\zeta \leq \eta \leq \xi \leq \kappa$.
(vi) $G_{\zeta,\xi}$ is a group for $\zeta \leq \xi \leq \kappa$.
(vii) For every $\zeta \in \kappa$ the index of $G_{\zeta,\kappa}$ in F_κ is κ.

Proof. Let $q\colon S \to S/S_0$ be the quotient mapping. Define $\pi'_\xi := q(\pi_\xi)$ and apply Theorem 4.2 with π'_ξ, F'_ξ and $G'_{\xi,\zeta}$ in place of π_ξ, F_ξ and $G_{\xi,\zeta}$. Since $F_\xi = q^{-1}(F'_\xi)$ and $G_{\xi,\zeta} = q^{-1}(G'_{\xi,\zeta})$, we obtain (i) – (vii) from the corresponding statements in Theorem 4.2. □

5. A UNIQUELY AMENABLE ACTION

Definition 5.1. The relation $\subseteq^*$ on subsets of $\mathbb{N}$ is defined by $A \subseteq^* B$ if $A \setminus B$ is finite. Write $A =^* B$ when $A \subseteq^* B$ and $B \subseteq^* A$. Two sets $A, B \subseteq \mathbb{N}$ are *almost disjoint* if $A \cap B =^* \emptyset$. Denote by COF the set of cofinite subsets of $\mathbb{N}$.

Say that a family $\mathscr{F}$ of subsets of $\mathbb{N}$ has the *strong finite intersection property (SFIP)* if $\bigcap \mathscr{F}_0$ is infinite for every finite $\mathscr{F}_0 \subseteq \mathscr{F}$.

The cardinal invariant $\mathfrak{p}$ is defined to be the least cardinality of a family $\mathscr{F}$ of subsets of $\mathbb{N}$ that has the SFIP yet there is no infinite $Y \subseteq \mathbb{N}$ with $Y \subseteq^* F$ for $F \in \mathscr{F}$.

In the following we identify ultrafilters on $\mathbb{N}$ with points of $\beta\mathbb{N}$; thus $\beta\mathbb{N}\setminus\mathbb{N}$ is the set of non-principal ultrafilters on $\mathbb{N}$. An ultrafilter on $\mathbb{N}$ will be called a *tower ultrafilter* (or more precisely a $\mathfrak{p}$-*tower ultrafilter*) if it is generated by a family $\{U_\xi \mid \xi \in \mathfrak{p}\}$ such that $U_\zeta \supseteq^* U_\xi$ for $\zeta < \xi < \mathfrak{p}$.

Notation 5.2. For $g \in \mathbb{S}_\infty$, the *support of* g, denoted $\mathsf{supp}(g)$, is the set of all $x \in \mathbb{N}$ such that $g(x) \neq x$. $\mathbb{S}_{<\aleph_0}$ will denote the group of all $g \in \mathbb{S}_\infty$ with finite support. Given $A \subseteq^* B \subseteq \mathbb{N}$, an (A,B)-*involution* is an involution $\pi \in \mathbb{S}_\infty$ such that

- $\mathsf{supp}(\pi) = B$
- $\pi \restriction A$ is a bijection from A onto $B\setminus A$.

Tower ultrafilters (which exist in some but not all models of set theory) play a key role in this and the next section. They are generalizations of P-points which were defined originally by Gillman and Henriksen in Definition 4.1 of [5] but in the current context of ultrafilters, by Rudin in [10], coincidentally, also in Definition 4.1. As a first application of Theorem 4.4, we prove a slightly weaker version of Foreman's theorem [4, 3.1]:

Theorem 5.3. *Let $\mathfrak{u}$ be a tower ultrafilter on $\mathbb{N}$. Then there is a locally finite group $G \subseteq \mathbb{S}_\infty$ for which $\mathfrak{u}$ is the only G-invariant mean.*

Proof. Take a generating family of $\mathfrak{u}$ of the form $\{\mathbb{N}\setminus Y_\xi \mid \xi < \mathfrak{p}\}$, where $Y_\zeta \subseteq^* Y_\xi$ and $|Y_\xi| = |Y_{\xi+1}\setminus Y_\xi| = \aleph_0$ for $\zeta < \xi < \mathfrak{p}$. For every $\xi < \mathfrak{p}$ pick an $(Y_\xi, Y_{\xi+1})$-involution π_ξ. Define G to be the group generated by $\{\pi_\xi \mid \xi < \mathfrak{p}\} \cup \mathbb{S}_{<\aleph_0}$.

First note that if $\xi < \mathfrak{p}$ and g_0, g_1 are in the group generated by $\{\pi_\zeta \mid \zeta < \xi\} \cup \mathbb{S}_{<\aleph_0}$ then $\mathsf{supp}(g_0)$ and $\mathsf{supp}(\pi_\xi g_1 \pi_\xi)$ are almost disjoint. It follows that properties (a) and (b) in Theorem 4.4 hold with $S_0 = \mathbb{S}_{<\aleph_0}$. Hence, by (iii) in the theorem, the group G is locally finite.

We have $\mathsf{supp}(g) \notin \mathfrak{u}$ for every $g \in \{\pi_\xi \mid \xi < \mathfrak{p}\} \cup \mathbb{S}_{<\aleph_0}$, hence also for every $g \in G$. Thus $\mathfrak{u}$ is G-invariant.

Take any G-invariant mean $\mathfrak{n}$ on $\mathbb{N}$. Then $\mathfrak{n}(K) = 0$ for every finite set K. For $\xi < \mathfrak{p}$ and $n \in \mathbb{N}$ we have

$$\mathfrak{n}(Y_\xi) = \mathfrak{n}(Y_{\xi+1}\setminus Y_\xi) = \mathfrak{n}(Y_{\xi+2}\setminus Y_{\xi+1}) = \cdots = \mathfrak{n}(Y_{\xi+n+1}\setminus Y_{\xi+n})$$

and the sets $Y_\xi, Y_{\xi+1}\setminus Y_\xi, Y_{\xi+2}\setminus Y_{\xi+1}, \ldots, Y_{\xi+n+1}\setminus Y_{\xi+n}$ are pairwise almost disjoint. Therefore $\mathfrak{n}(Y_\xi) = 0$. That proves $\mathfrak{n}(A) = 0$ for every $A \notin \mathfrak{u}$, hence also $\mathfrak{n}(A) = 1$ for every $A \in \mathfrak{u}$. But that means $\mathfrak{n} = \mathfrak{u}$. ☐

If G is the group in Theorem 5.3 and $\mathfrak{m}$ is a left-invariant element of $\ell^*_\infty(G)$ then $\mathfrak{m} \in \Lambda_1(\odot_1)$ by Proposition 3.6. In the next section we construct a group for which the opposite holds.

6. A NON-UNIQUELY AMENABLE ACTION

The next theorem will be used in the proof of Lemma 6.2.

Theorem 6.1 (M. Bell [2])**.** *For any Polish space X without isolated points and any cardinal κ the following are equivalent:*

- $\kappa < \mathfrak{p}$.
- *Every family $\mathscr{D}$ of dense open subsets of X such that $|\mathscr{D}| \leq \kappa$ has non-empty intersection.*

Lemma 6.2. *Let $\mathfrak{u}_0, \mathfrak{u}_1$ be tower ultrafilters on $\mathbb{N}$, $\mathfrak{u}_0 \neq \mathfrak{u}_1$. Let $\mathscr{B}$ be a family of infinite subsets of $\mathbb{N}$ such that $|\mathscr{B}| \leq \mathfrak{p}$. Then there is a locally finite subgroup F of $\mathbb{S}_\infty$ satisfying the following properties.*

(A) $\mathfrak{u}_0$ *and* $\mathfrak{u}_1$ *are both* F*-invariant means on* $\mathbb{N}$.
(B) *Every* F*-invariant element of* $\ell^*_\infty(\mathbb{N})$ *is in the linear space spanned by* $\mathfrak{u}_0$ *and* $\mathfrak{u}_1$.
(C) *For every* $i \in 2$, $U \in \mathfrak{u}_i$ *and* $B \in \mathscr{B}$ *there exists a subgroup* $H \subseteq F$ *of infinite index in* F *such that*

$$\left|\bigcap_{g\in K} B \cap gU\right| = \aleph_0$$

for every finite $K \subseteq F \setminus H$.

Proof. There are infinite sets P_0 and P_1 that partition $\mathbb{N}$ and a family $\{Y_\xi\}_{\xi\in\mathfrak{p}}$ of infinite subsets of $\mathbb{N}$ such that $Y_\zeta \subseteq^* Y_\xi$ and $\left|P_i \cap Y_{\xi+1} \setminus Y_\xi\right| = \aleph_0$ for $\zeta < \xi < \mathfrak{p}$ and $i \in 2$, and $\left\{P_i \setminus Y_\xi \mid \zeta < \xi < \mathfrak{p}\right\}$ generates $\mathfrak{u}_i$ for $i \in 2$.

As in the proof of Theorem 5.3, if π_ξ is an $(Y_\xi, Y_{\xi+1})$-involution for every $\xi \in \mathfrak{p}$ then assumptions (a) and (b) in Theorem 4.4 are satisfied with $S = \mathbb{S}_\infty$, $S_0 = \mathbb{S}_{<\aleph_0}$ and $\kappa = \mathfrak{p}$, and therefore properties (i) to (vii) in that theorem hold with F_ξ and $G_{\zeta,\xi}$ defined therein. The group $F := F_\mathfrak{p}$ is locally finite by 4.4(iii), and Conditions (A) and (B) follow as before. Indeed, from the definition of F we get $\mathsf{supp}(g) \notin \mathfrak{u}_i$ for every $g \in F$, and that implies (A). And if $\mathfrak{n} \in \ell^*_\infty(\mathbb{N})$ is any F-invariant mean then, as in the proof of Theorem 5.3, $\mathfrak{n}(P_i \cap Y_\xi) \le \mathfrak{n}(Y_\xi) = 0$ for $\xi \in \mathfrak{p}$ and $i \in 2$, hence $\mathfrak{n} = \mathfrak{n}(P_0)\mathfrak{u}_0 + \mathfrak{n}(P_1)\mathfrak{u}_1$, and (B) follows from Proposition 3.4.

Next we show that Condition (C) holds when involutions π_ξ are chosen in a certain way. Let $\{B_\xi\}_{\xi\in\mathfrak{p}}$ enumerate $\mathscr{B}$ so that each set occurs cofinally often. Assume, without loss of generality, that $B_\xi \cap Y_\xi$ is infinite for every ξ. We shall show that $(Y_\xi, Y_{\xi+1})$-involutions π_ξ can be chosen so that there exist families $\mathscr{B}^i_{\zeta,\xi}$ of subsets of $\mathbb{N}$ for $i \in 2$, $\zeta \le \xi \in \mathfrak{p}$, and the following hold for $i \in 2$ and $\zeta \le \xi \in \mathfrak{p}$:

(1) $\left|\mathscr{B}^i_{\zeta,\xi}\right| \le \max(|\xi|, \aleph_0)$
(2) $\{B_\zeta \cap Y_\zeta\} \cup \mathsf{COF} \subseteq \mathscr{B}^i_{\zeta,\eta} \subseteq \mathscr{B}^i_{\zeta,\xi}$ for $\zeta \le \eta \le \xi$
(3) $\mathscr{B}^i_{\zeta,\xi}$ has the SFIP
(4) if $g \in F_\xi \setminus G_{\zeta,\xi}$ then $g(P_i \cap Y_\xi \setminus Y_\zeta)$ is in the filter generated by $\mathscr{B}^i_{\zeta,\xi}$

Assuming such π_ξ and $\mathscr{B}^i_{\zeta,\xi}$ exist, take any $i \in 2$, $U \in \mathfrak{u}_i$ and $B \in \mathscr{B}$. There is $\zeta \in \mathfrak{p}$ such that $B = B_\zeta$ and $P_i \setminus Y_\zeta \subseteq^* U$. The index of $H := G_{\zeta,\mathfrak{p}}$ in F is infinite by 4.4(vii). Take any finite subset K of $F \setminus H$. Let $\xi \in \mathfrak{p}$ be so large that $K \subseteq F_\xi$ and $\zeta < \xi$. Since $H \cap F_\xi = G_{\zeta,\xi}$ by 4.4(v), it follows that $K \subseteq F_\xi \setminus G_{\zeta,\xi}$.

If $g \in K$ then $g(P_i \setminus Y_\zeta) \subseteq^* gU$, hence gU is in the filter generated by $\mathscr{B}^i_{\zeta,\xi}$ by (2) and (4). So is $B = B_\zeta$ by (2). Therefore $\left|\bigcap_{g\in K} B \cap gU\right| = \aleph_0$ by (3). That proves Condition (C).

It remains to construct π_ξ and $\mathscr{B}^i_{\zeta,\xi}$ that satisfy (1)–(4) for $i \in 2$ and $\zeta \le \xi < \mathfrak{p}$. The construction relies on the following claim.

Claim 1. Fix $\kappa \in \mathfrak{p}$. Let $\mathbb{P}$ be the Polish space of all $(Y_\kappa, Y_{\kappa+1})$-involutions with the topology of pointwise convergence. Note that $\mathbb{P}$ has no isolated points. For any finite $K \subseteq F_\kappa$, infinite $X \subseteq Y_\kappa$ and $i \in 2$ the set

$$D := \left\{\pi \in \mathbb{P} \;\middle|\; X \cap \bigcap_{g\in K} g\pi(P_i \cap Y_{\kappa+1} \setminus Y_\kappa) \ne \emptyset\right\}$$

is dense open in $\mathbb{P}$.

Proof of the claim. Given a non-empty open set $U \subseteq \mathbb{P}$, there are some $\sigma \in \mathbb{P}$ and a finite $A \subseteq Y_{\kappa+1} \setminus Y_\kappa$ such that $\{\pi \in \mathbb{P} \mid \pi \restriction A = \sigma \restriction A\} \subseteq U$. As $\bigcup_{g\in K} \mathsf{supp}(g) \subseteq^* Y_\kappa$, there are only finitely many $n \in Y_\kappa$ for which there is some $g \in K$ such that $g^{-1}(n) \notin Y_\kappa$. Hence there exists

$$x \in X \setminus \{g(\sigma(n)) \mid n \in A \text{ and } g \in K\}$$

such that $g^{-1}(x) \in Y_\kappa$ for each $g \in K$. Let $Z := \{g^{-1}(x) \mid g \in K\}$ and let $W \subseteq P_i \cap Y_{\kappa+1} \setminus (Y_\kappa \cup A)$ be such that $|W| = |Z|$. As $A \cap W = \emptyset$ and $\sigma(A) \cap Z = \emptyset$, there is $\pi \in \mathbb{P}$ such that $\pi \restriction A = \sigma \restriction A$ and $\pi(W) = Z$. Then $\pi \in D$ because $x \in X \cap \bigcap_{g\in K} g\pi(P_i \cap Y_{\kappa+1} \setminus Y_\kappa)$. Hence D is dense in $\mathbb{P}$.

To prove that D is open, take any $\sigma \in D$. Fix any $x \in X \cap \bigcap_{g\in K} g\sigma(P_i \cap Y_{\kappa+1} \setminus Y_\kappa)$ and define $A := \{g^{-1}(x) \mid g \in K\}$. Then $\{\pi \in \mathbb{P} \mid \pi \restriction A = \sigma \restriction A\} \subseteq D$. □

Returning to the proof of Lemma 6.2, we construct π_ξ and $\mathscr{B}^i_{\zeta,\xi}$ satisfying (1)–(3) by recursion in ξ. Property (4) will be proved separately.

Define $\mathscr{B}^0_{0,0} := \mathscr{B}^1_{0,0} := \{B_0 \cap Y_0\} \cup \mathsf{COF}$, and let π_0 be any (Y_0, Y_1)-involution. Evidently (1)–(3) hold for $\zeta = \xi = 0$.

Fix $\kappa \in \mathfrak{p}$, $\kappa > 0$, and assume there are π_ξ and $\mathscr{B}^i_{\zeta,\xi}$ such that (1)–(3) hold for $i \in 2$, $\zeta \le \xi < \kappa$. Write

$$\mathscr{B}^{i*}_{\zeta,\kappa} := \begin{cases} \{B_\zeta \cap Y_\zeta\} \cup \mathsf{COF} & \text{when} \quad \kappa = \zeta \\ \bigcup \left\{ \mathscr{B}^i_{\zeta,\xi} \mid \zeta \le \xi < \kappa \right\} & \text{when} \quad \kappa > \zeta \end{cases}$$

Since $|\kappa| < \mathfrak{p}$ and (1)–(3) hold for $i \in 2$, $\zeta \le \xi < \kappa$, it follows that for every $i \in 2$ and $\zeta \le \kappa$ there exists an infinite set $D^i_\zeta \subseteq Y_\zeta \subseteq^* Y_\kappa$ such that $D^i_\zeta \subseteq^* B$ for every $B \in \mathscr{B}^{i*}_{\zeta,\kappa}$.

By Theorem 6.1, $|\kappa| < \mathfrak{p}$ and Claim 1, there is an $(Y_\kappa, Y_{\kappa+1})$-involution π_κ such that for every $i \in 2$, $\zeta \le \kappa$ and finite $K \subseteq F_\xi$ the set

$$D^i_\zeta \cap \bigcap_{g\in K} g\pi_\kappa(P_i \cap Y_{\kappa+1} \setminus Y_\kappa)$$

is infinite. Then for $\zeta \le \kappa$ define

$$\mathscr{B}^i_{\zeta,\kappa} := \mathscr{B}^{i*}_{\zeta,\kappa} \cup \{g\pi_\kappa(P_i \cap Y_{\kappa+1} \setminus Y_\kappa) \mid g \in F_\kappa\}.$$

For $\xi = \kappa$, statement (1) follows from $|F_\kappa| \le \max(|\kappa|, \aleph_0)$, (2) from the definition of $\mathscr{B}^{i*}_{\zeta,\kappa}$ and $\mathscr{B}^i_{\zeta,\kappa}$, and (3) from the choice of π_κ. That completes the recursive definition of π_ξ and $\mathscr{B}^i_{\zeta,\xi}$ satisfying (1)–(3).

Next we prove property (4) for $i \in 2$ and $\zeta \le \xi \in \mathfrak{p}$, by induction on ξ. In the following arguments we use $\mathsf{COF} \subseteq \mathscr{B}^i_{\zeta,\xi}$ to conclude that if $\mathbb{N} \supseteq A \supseteq^* B \in \mathscr{B}^i_{\zeta,\xi}$ then A is in the filter generated by $\mathscr{B}^i_{\zeta,\xi}$.

Induction basis: (4) is trivially true for $\xi = \zeta$ because $F_\xi \setminus G_{\zeta,\xi} = \emptyset$ in that case. If $\eta \in \mathfrak{p}$ is a limit ordinal then (4) for $\xi = \eta$ follows from (4) for all smaller ξ.

Fix $i \in 2$ and assume that $\eta \in \mathfrak{p}$ is such that (4) holds for $\zeta \le \xi = \eta$, with the goal of proving that (4) holds for $\zeta < \xi = \eta + 1$.

Take any $\zeta \le \eta$ and $g \in F_{\eta+1} \setminus G_{\zeta,\eta+1}$. By 4.4(i) there are two possibilities.

Case One. $g \in F_\eta \pi_\eta F_\eta$, so that $g = g_1\pi_\eta g_2$ where $g_1, g_2 \in F_\eta$. From $\mathsf{supp}(g_2) \subseteq^* Y_\eta$ we get

$$g_2(P_i \cap Y_{\eta+1} \setminus Y_\eta) =^* P_i \cap Y_{\eta+1} \setminus Y_\eta$$
$$g(P_i \cap Y_{\eta+1} \setminus Y_\zeta) \supseteq^* g_1\pi_\eta g_2(P_i \cap Y_{\eta+1} \setminus Y_\eta) =^* g_1\pi_\eta(P_i \cap Y_{\eta+1} \setminus Y_\eta) \in \mathscr{B}^i_{\zeta,\eta} \subseteq \mathscr{B}^i_{\zeta,\eta+1}$$

Case Two. $g \in F_\eta \pi_\eta F_\eta \pi_\eta$, so that $g = g_1\pi_\eta g_2\pi_\eta$ where $g_1, g_2 \in F_\eta$.

Definition (4.1) yields $g_1 \notin G_{\zeta,\eta}$ because $g \notin G_{\zeta,\eta+1} = G_{\zeta,\eta}\pi_\eta F_\eta \pi_\eta$, and $\zeta < \eta$ because for $\zeta = \eta$ we have $F_\eta \pi_\eta F_\eta \pi_\eta \setminus G_{\zeta,\eta+1} = \emptyset$. From $\operatorname{supp}(\pi_\eta g_2 \pi_\eta) \cap Y_\eta =^* \emptyset$ we get

$$\pi_\eta g_2 \pi_\eta (P_i \cap Y_\eta \setminus Y_\zeta) =^* P_i \cap Y_\eta \setminus Y_\zeta$$
$$g(P_i \cap Y_{\eta+1} \setminus Y_\zeta) \supseteq^* g_1 \pi_\eta g_2 \pi_\eta (P_i \cap Y_\eta \setminus Y_\zeta) =^* g_1(P_i \cap Y_\eta \setminus Y_\zeta)$$

As $g_1 \in F_\eta \setminus G_{\zeta,\eta}$, the last set is in the filter generated by $\mathscr{B}^i_{\zeta,\eta} \subseteq \mathscr{B}^i_{\zeta,\eta+1}$ because (4) holds for $\xi = \eta$ by the induction hypothesis. □

Definition 6.3. Given $\mathfrak{u}$ and $\mathfrak{v}$ in $\beta\mathbb{N} \setminus \mathbb{N}$ define the partial order $\prec$ on $\mathfrak{u} \times \mathfrak{v}$ by $(U,V) \prec (U',V')$ if $U \subseteq^* U'$ and $V \subseteq^* V'$. A net $\{\mathfrak{n}_{A,B}\}_{(A,B)\in\mathfrak{u}\times\mathfrak{v}}$ in $\ell^*_\infty(\mathbb{N})$ will be said to *converge* if it converges with respect to the downward directed partial order $\prec$ in the weak* topology.

Corollary 6.4. *Let $\mathfrak{u}_0$ and $\mathfrak{u}_1$ be tower ultrafilters on $\mathbb{N}$, $\mathfrak{u}_0 \neq \mathfrak{u}_1$. Let $\mathfrak{n} \in \beta\mathbb{N} \setminus \mathbb{N}$ be generated by a set of cardinality $\mathfrak{p}$. Then there are an amenable group $F \subseteq \mathbb{S}_\infty$ and $\mathfrak{n}_{i,A,B} \in \beta\mathbb{N} \setminus \mathbb{N}$ for $i \in 2$, $(A,B) \in \mathfrak{u}_i \times \mathfrak{n}$, such that $\mathfrak{u}_0$ and $\mathfrak{u}_1$ are F-invariant, every F-invariant element of $\ell^*_\infty(\mathbb{N})$ is in the linear space spanned by $\mathfrak{u}_0$ and $\mathfrak{u}_1$ and*

- *the net $\{\mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ converges to $\mathfrak{n}$*
- *for every left-invariant element $\mathfrak{m}$ of $\ell^*_\infty(F)$ the net $\{\mathfrak{m} \odot_1 \mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ converges to $\mathfrak{u}_i$.*

Thus no left-invariant element of $\ell^*_\infty(F)$ is in $\Lambda_1(\odot_1)$.

Proof. Let $\mathscr{B} \subseteq \mathfrak{n}$ be a generating set of cardinality $\mathfrak{p}$ and let F be as in Lemma 6.2. Using (C), for each $i \in 2$, $A \in \mathfrak{u}_i$ and $B \in \mathfrak{n}$ let $H_{i,A,B}$ be a subgroup of infinite index in F such that

$$\{B \cap g^{-1}A \mid g \in F \setminus H_{i,A,B}\}$$

can be extended to an ultrafilter $\mathfrak{n}_{i,A,B}$. Then $\{\mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ converges to $\mathfrak{n}$ for each $i \in 2$.

Let $\mathfrak{m} \in \ell^*_\infty(F)$ be left-invariant. In view of Proposition 3.4 we may assume that $\mathfrak{m}$ is a left-invariant mean. Since each $H_{i,A,B}$ has infinite index, it follows that $\mathfrak{m}(H_{i,A,B}) = 0$ for all i, A and B. Moreover, $g^{-1}A \in \mathfrak{n}_{i,A,B}$ for $i \in 2$, $A \in \mathfrak{u}_i$, $B \in \mathfrak{n}$ and $g \in F \setminus H_{i,A,B}$. Hence for $D \supseteq A \in \mathfrak{u}_i$ we have

$$\mathfrak{m} \odot_1 \mathfrak{n}_{i,A,B}(D) = \int_F \int_{\mathbb{N}} 1_D(gk)\, \mathrm{d}\mathfrak{n}_{i,A,B}(k)\, \mathrm{d}\mathfrak{m}(g) = \int_{F\setminus H_{i,A,B}} \int_{\mathbb{N}} 1_D(gk)\, \mathrm{d}\mathfrak{n}_{i,A,B}(k)\, \mathrm{d}\mathfrak{m}(g) = 1$$

which proves that $\{\mathfrak{m} \odot_1 \mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ converges to $\mathfrak{u}_i$. □

Corollary 6.5. *Assume the Continuum Hypothesis. Let $\mathfrak{u}_0$ and $\mathfrak{u}_1$ be tower ultrafilters on $\mathbb{N}$, $\mathfrak{u}_0 \neq \mathfrak{u}_1$. Then there is an amenable group $F \subseteq \mathbb{S}_\infty$ such that $\mathfrak{u}_0$ and $\mathfrak{u}_1$ are F-invariant, every F-invariant element of $\ell^*_\infty(\mathbb{N})$ is in the linear space spanned by $\mathfrak{u}_0$ and $\mathfrak{u}_1$ and for every $\mathfrak{n} \in \beta\mathbb{N} \setminus \mathbb{N}$ there are $\mathfrak{n}_{i,A,B} \in \beta\mathbb{N} \setminus \mathbb{N}$ such that*

- *the net $\{\mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ converges to $\mathfrak{n}$*
- *for every left-invariant element $\mathfrak{m}$ of $\ell^*_\infty(F)$ the net $\{\mathfrak{m} \odot_1 \mathfrak{n}_{i,A,B}\}_{(A,B)\in\mathfrak{u}_i\times\mathfrak{n}}$ converges to $\mathfrak{u}_i$.*

In particular, for such $\mathfrak{m}$ the mapping $\mathfrak{v} \mapsto \mathfrak{m} \odot_1 \mathfrak{v}$ is not weak continuous at any point of $\beta\mathbb{N} \setminus \mathbb{N}$.*

Proof. Let $\mathscr{B}$ be the set of all infinite subsets of $\mathbb{N}$. Then $|\mathscr{B}| = 2^{\aleph_0} = \aleph_1$, and $\mathfrak{n} \subseteq \mathscr{B}$ for every $\mathfrak{n} \in \beta\mathbb{N} \setminus \mathbb{N}$. The argument in the proof of Corollary 6.4 applies, now with F independent of $\mathfrak{n}$. □

7. Remarks and questions

The constructions in §5 and §6 rely on the existence of tower ultrafilters. While it was shown by Rudin in [10] that these exist under the Continuum Hypothesis, it should be noted that there are many examples of models of set theory in which tower ultrafilters exist, yet the Continuum Hypothesis fails. The model obtained by iteratively adding ω_2 Sacks reals with countable support to a model of the Continuum Hypothesis provides a prototypical example. On the other hand, the model obtained by adding $\aleph_2$ Cohen reals to a model of the Continuum Hypothesis yields a model where $\mathfrak{p} = \aleph_1$ and there are no ultrafilters generated by a set of cardinality $\aleph_1$. But even more is true in this model.

Proposition 7.1 (M. Foreman [4]). *In the model obtained by adding $\aleph_2$ Cohen reals to a model of the Continuum Hypothesis there is no locally finite subgroup of $\mathbb{S}_\infty$ such that the set of means invariant under the natural action of this group on $\mathbb{N}$ is finite dimensional.*

Proof. This follows from a simple modification of the argument of [4], so familiarity with the proof of Theorem 4.1 from [4] will be assumed. Instead of defining $c_\alpha = \{n \mid p(\alpha,n) = 1\}$ as in the second sentence of the proof of Theorem 4.1, define Cohen forcing $\mathbb{P}$ to be all partial functions from $\kappa \times \mathbb{N}$ to $\mathbb{N}$ with finite domain and let $c_{\alpha,j} = \{n \mid p(\alpha,n) = j\}$.

Now suppose that $\dot{G}$ is a $\mathbb{P}$ name for a locally finite group and that

$$1 \Vdash_{\mathbb{P}} \text{“the set of means invariant under } \dot{G} \text{ is contained in the linear span of } \{\mathfrak{m}_i\}_{i=1}^k\text{”}.$$

For each $\alpha \in \kappa$ it is possible to find $j(\alpha) \in \mathbb{N}$ and $p_\alpha \in \mathbb{P}$ such that $p_\alpha \Vdash_{\mathbb{P}}$ “$(\forall i \leq k)\ \mathfrak{m}_j(c_{\alpha,j(\alpha)}) < 1$”. Using Proposition 1 of [12] (see also the Lemma of Yang on page 11 of [4]) it follows that for each α there is $r_\alpha < 1$ and $p^*_\alpha \supseteq p_\alpha$ such that $p^*_\alpha \Vdash_{\mathbb{P}}$ “$c_{\alpha,j(\alpha)}$ is not r_α thick”. The argument of [4] can now be applied to yield a contradiction. □

It is important to observe that the argument of Proposition 7.1 applies only to locally finite groups. It will be shown in the doctoral thesis of D. Kalajdzievski [6] that in this same model any group acting on $\mathbb{N}$ with a unique invariant mean cannnot have locally sub-exponential growth and or be locally solvable.

Foreman's construction [4, 3.1] shows, in essence, that Theorem 5.3 holds more generally when $\mathfrak{u}$ is an ultrafilter of character $\mathfrak{p}$.

Question 7.2. Do Lemma 6.2 and its corollaries hold more generally when $\mathfrak{u}_0$ and $\mathfrak{u}_1$ are ultrafilters of character $\mathfrak{p}$?

Throughout the paper we have dealt with the natural action of an individual group $G \subseteq \mathbb{S}_\infty$ on $\mathbb{N}$ and its extension to the actions $\odot$ of $\ell_1(G)$ on $\ell_1(\mathbb{N})$ and $\odot_1$ of $\ell^*_\infty(G)$ on $\ell^*_\infty(\mathbb{N})$. Now consider each such action of G as the restriction of the action $\circ$ of the whole group $\mathbb{S}_\infty$ on $\mathbb{N}$, which in turn extends to the actions $\circledcirc$ of $\ell_1(\mathbb{S}_\infty)$ on $\ell_1(\mathbb{N})$ and $\circledcirc_1$ of $\ell^*_\infty(\mathbb{S}_\infty)$ on $\ell^*_\infty(\mathbb{N})$. With the usual natural identifications we have $\ell_1(G) \subseteq \ell_1(\mathbb{S}_\infty)$, $\ell^*_\infty(G) \subseteq \ell^*_\infty(\mathbb{S}_\infty)$ and $\ell_1(\mathbb{S}_\infty) \cap \ell^*_\infty(G) = \ell_1(G)$. Then $\odot$ is a restriction of $\circledcirc$ and $\odot_1$ is a restriction of $\circledcirc_1$. From the definition of Λ_1 we immediately get $\Lambda_1(\circledcirc_1) \cap \ell^*_\infty(G) = \Lambda_1(\odot_1)$.

By Proposition 3.6, if $G \subseteq \mathbb{S}_\infty$ is a (necessarily infinite) amenable group whose action on $\mathbb{N}$ is uniquely amenable and if $\mathfrak{m} \in \ell^*_\infty(G)$ is a left-G-invariant mean then $\mathfrak{m} \in \Lambda_1(\odot_1) \setminus \ell_1(G) \subseteq \Lambda_1(\circledcirc_1) \setminus \ell_1(\mathbb{S}_\infty)$. We conjecture that the whole space $\Lambda_1(\circledcirc_1) \subseteq \ell^*_\infty(\mathbb{S}_\infty)$ may be obtained from such means and from $\ell_1(\mathbb{S}_\infty)$:

Conjecture 7.1. Let $M \subseteq \ell_\infty^*(\mathbb{S}_\infty)$ be the union of the sets

$$\{\mathfrak{m} \in \ell_\infty^*(G) \subseteq \ell_\infty^*(\mathbb{S}_\infty) \mid \mathfrak{m} \text{ is left-}G\text{-invariant mean}\}$$

over all amenable groups $G \subseteq \mathbb{S}_\infty$ whose action on $\mathbb{N}$ is uniquely amenable. Then $\Lambda_1(\circledcirc_1)$ is the norm closure of the linear span of $M \cup \ell_1(\mathbb{S}_\infty)$.

It should be easier to describe $\Lambda_1(\odot_1)$ for the specific subgroups described in this paper, before approaching the conjecture in its full generality:

Question 7.3. What is the $\Lambda_1(\odot_1)$ for Foreman's group [4, 3.1] and group F in Theorem 5.3? What about the groups constructed in §6?

It has already been mentioned that in the model obtained by adding $\aleph_2$ Cohen reals to a model of the Continuum Hypothesis no locally finite subgroup of the full symmetric group acts with a unique invariant mean.

Question 7.4. Is it true that in this model for the action of $\mathbb{S}_\infty$ on $\mathbb{N}$ we have $\Lambda_1(\circledcirc_1) = \ell_1(\mathbb{S}_\infty)$?

In view of Proposition 3.6, if the answer to Question 7.4 is yes then in the given model there is no amenable subgroup of $\mathbb{S}_\infty$ whose action is uniquely amenable. Conversely, if there is no amenable subgroup of $\mathbb{S}_\infty$ whose action is uniquely amenable and if Conjecture 7.1 holds then $\Lambda_1(\circledcirc_1) = \ell_1(\mathbb{S}_\infty)$.

Question 7.5. Is there a model where there is an amenable subgroup of the full symmetric group acting on $\mathbb{N}$ with a $k+1$ dimensional subspace of invariant means, but there is no such action of an amenable group with a k dimensional subspace of invariant means?

Question 7.6. What can be said about $\Lambda_2(\odot_2)$?

This article has focussed on actions of amenable groups because of their relevance to Proposition 3.6. However, it may be of interest to look at the topological centres of actions of non-amenable groups as well. While it is easy to construct for $\mathfrak{u} \in \beta\mathbb{N} \setminus \mathbb{N}$ a subgroup $G \subseteq \mathbb{S}_\infty$ such that $\mathfrak{u}$ is the unique invariant mean for the action of G — simply let G be all permutations that preserve $\mathfrak{u}$ — the following result from [11] is less obvious.

Theorem 7.7 (E. van Douwen)**.** *There is a transitive action of $\mathbb{F}_2$, the free group of rank two, on $\mathbb{N}$ such that each non-identity element of $\mathbb{F}_2$ fixes only finitely many points, yet the action admits an $\mathbb{F}_2$-invariant element of $\ell_\infty^*(\mathbb{N})$.*

There are a number of other examples of group actions in [11] for which it might be of interest to calculate the topological centres.

Acknowledgement
Research for this paper was partially supported by NSERC of Canada.

REFERENCES

[1] R. Arens, The adjoint of a bilinear operation, Proc. Amer. Math. Soc. 2 (1951), 839-848.
[2] M.G. Bell, On the combinatorial principle $P(\mathfrak{c})$, Fund. Math. 114 (1981), 149-1571.
[3] M.E. Gordji, M. Filali, Arens regularity of module actions, Studia Math. 181 (2007), 237-254.

[4] M. Foreman, Amenable groups and invariant means, J. Funct. Anal. 126 (1994), 7-25.

[5] L. Gillman, M. Henriksen, Concerning rings of continuous functions, Trans. Amer. Math. Soc. 77 (1954), 340-362.

[6] D. Kalajdzievski, TBA. PhD thesis, York University.

[7] A.T.M. Lau, Continuity of Arens multiplication on the dual space of bounded uniformly continuous functions on locally compact groups and topological semigroups, Math. Proc. Cambridge Philos. Soc. 99 (1986), 273-283.

[8] A.T.M. Lau, A.L.T. Paterson, The exact cardinality of the set of topological left invariant means on an amenable locally compact group, Proc. Amer. Math. Soc. 98 (1986), 75-80.

[9] I. Namioka, On certain actions of semi-groups on L-spaces, Studia Math. 29 (1967), 63-77.

[10] W. Rudin, Homogeneity problems in the theory of Čech compactifications, Duke Math. J. 23 (1956), 409-419.

[11] E.K. van Douwen, Measures invariant under actions of F_2, Topology Appl. 34 (1990), 53-68.

[12] Z. Yang, Action of amenable groups and uniqueness of invariant means, J. Funct. Anal. 97 (1991), 50-63.