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ON ALMOST PERIODIC FUNCTIONALS ON THE FIGÀ-TALAMANCA–HERZ ALEGBRAS $A_p(G)$

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Abstract. If G is a compact group of bounded representation type, then Dunkl and Ramirez have shown that the product of any two almost periodic functionals on the Fourier algebra $A(G)$ is again almost periodic. In this paper, we show that, under an additional condition, the same result holds for Figà-Talamanca–Herz algebras $A_p(G)$, $1 < p < \infty$.
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1. INTRODUCTION AND PRELIMINARIES

Dunkl and Ramirez have shown that if G is a compact group of bounded representation type, then the product of any two almost periodic functionals on the Fourier algebra $A(G)$ is again almost periodic, and therefore the space of all such functionals forms a Banach subalgebra of $A(G)^*$ ([4, Theorem 2]). Suppose that $\mathscr{C}$ denotes the set of all coordinate functions of continuous irreducible unitary representations of G. In this paper, we show that Dunkl–Ramirez's result can be extended to almost periodic functionals on the Figà-Talamanca–Herz algebras $A_p(G)$, $1 < p < \infty$, under the assumption that the absolute polar $\mathscr{C}^\diamond$ is bounded in $A_p(G)^*$ (this assumption will be shown to hold if $p = 2$). Our arguments in this paper have been inspired by the elegant proof of Dunkl and Ramirez in [4], in which the theory of Fourier transform on compact groups plays an important role. Among the extensive literature on almost periodic functionals related to algebras on locally compact groups we should mention Young [18], Granirer [7], Lau–Wong [14], Duncan–Ülger [2], and Mustafayev [17].

Throughout this paper (unless otherwise noted) G denotes a *compact* Hausdorff topological group equipped with its normalized Haar measure. Let $1 < p < \infty$ and p' be its conjugate exponent, that is, $1/p + 1/p' = 1$. The Figà-Talamanca–Herz algebra $A_p(G)$ consists of all functions $u \in C(G)$ that can be written as a sum $u = \sum_{k=1}^{\infty} g_k * f_k$, with $f_k \in L^p(G)$, $g_k \in L^{p'}(G)$, and

$$\sum_{k=1}^{\infty} \|f_k\|_p \|g_k\|_{p'} < \infty,$$

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where $*$ denotes the convolution product. With the pointwise operations of addition and multiplication and with the norm

$$\|u\|_{A_p} = \inf\left\{\sum_{k=1}^{\infty}\|f_k\|_p\|g_k\|_{p'} : u = \sum_{k=1}^{\infty} g_k * f_k\right\}, \tag{1.1}$$

the space $A_p(G)$ is a commutative Banach algebra (Figà-Talamanca [6], Herz [8]). We note that if $p = 2$, then $A_2(G) = A(G)$ is the Fourier algebra of G (Krein [13], Eymard [5], Kaniuth–Lau [11]). The dual space $A_p(G)^*$ is a Banach $A_p(G)$-module in which the module multiplication $u\cdot\varphi$ is defined by $\langle u\cdot\varphi, v\rangle = \langle\varphi, uv\rangle$, for all $\varphi\in A_p(G)^*$ and $u,v\in A_p(G)$. A functional $\varphi\in A_p(G)^*$ is called *almost periodic* if the continuous linear operator $M_\varphi\colon A_p(G)\longrightarrow A_p(G)^*$, defined by $M_\varphi(u) = u\cdot\varphi$, is compact. We denote the space of all almost periodic functionals on $A_p(G)$ by $AP(A_p(G))$.

In this paper we shall use a characterization of $A_p(G)^*$ in terms of the Fourier transform on compact groups. We shall briefly recall the terminology in order to establish the necessary notation (for further details we refer our readers to Hewitt and Ross [10] or Dunkl and Ramirez [3]). For G compact, let $\widehat{G}$ denote the set of all (equivalence classes) of continuous topologically irreducible unitary representations of G. If $\sigma\in\widehat{G}$, then σ is finite-dimensional, H_σ denotes its representation space, and d_σ denotes its dimension. If $\xi,\eta\in H_\sigma$, the *coordinate function* $\sigma_{\xi,\eta}\in C(G)$ is defined by the inner product $\sigma_{\xi,\eta}(s) = (\sigma(s)\xi|\eta)$ for all $s\in G$. If $\{e_1,\dots,e_{d_\sigma}\}$ is an orthonormal basis of H_σ, the coordinate function σ_{e_j,e_i} will be denoted by σ_{ij}. The linear span of all σ_{ij} ($1\le i,j\le d_\sigma$) as σ varies in $\widehat{G}$, is a translation invariant subalgebra of $C(G)$, which we denote by $\mathscr{E}_{\widehat{G}}$. It follows from the identity ([10, Theorem 27.20])

$$\sigma_{ij} = d_\sigma\sigma_{ik}*\sigma_{kj} \qquad (i,j,k\in\{1,\dots,d_\sigma\}), \tag{1.2}$$

that $\sigma_{ij}\in A_p(G)$ and hence $\mathscr{E}_{\widehat{G}}\subset A_p(G)$. Since $\mathscr{E}_{\widehat{G}}$ is dense in $C(G)$, it is also dense in $L^p(G)$, and hence $\mathscr{E}_{\widehat{G}}$ is dense in $A_p(G)$.

Next consider the product space $\prod_{\sigma\in\widehat{G}}\mathscr{L}(H_\sigma)$, where $\mathscr{L}(H_\sigma)$ is the space of all continuous linear operators on H_σ. This space is an involutive algebra under the coordinatewise operations. We define $\mathfrak{E}^\infty(\widehat{G})$ as the set of all $\varphi\in\prod_{\sigma\in\widehat{G}}\mathscr{L}(H_\sigma)$ such that $\sup_{\sigma\in\widehat{G}}\|\varphi(\sigma)\|<\infty$; $\mathfrak{E}_{00}(\widehat{G})$ as the set of all φ such that

$$\{\sigma\in\widehat{G}\colon\ \varphi(\sigma)\neq 0\}$$

is finite; and $\mathfrak{E}_0(\widehat{G})$ as the set of all φ such that

$$\{\sigma\in\widehat{G}\colon\ \|\varphi(\sigma)\|\ge\varepsilon\}$$

is finite, for all $\varepsilon>0$, where $\|\varphi(\sigma)\|$ denotes the operator norm of $\varphi(\sigma)\in\mathscr{L}(H_\sigma)$. The *Fourier transform* of $f\in L^1(G)$ is the function $\widehat{f}\in\mathfrak{E}_0(\widehat{G})$ defined by $\widehat{f}(\sigma) = \int_G\overline{\sigma}(x)f(x)\,dx$ for all $\sigma\in\widehat{G}$, where $\overline{\sigma}$ is the conjugate representation of σ.

An element $\varphi\in\prod_{\sigma\in\widehat{G}}\mathscr{L}(H_\sigma)$ is a *multiplier* of $L^p(G)$ if $\varphi\widehat{f}\in\widehat{L^p(G)}$ for all $f\in L^p(G)$. The set of all multipliers of $L^p(G)$ is a subalgebra of $\prod_{\sigma\in\widehat{G}}\mathscr{L}(H_\sigma)$ denoted by $\mathscr{M}(L^p(G))$. If $\varphi\in\mathscr{M}(L^p(G))$, then the mapping $T_\varphi\colon L^p(G)\longrightarrow L^p(G)$, defined by $T_\varphi f = (\varphi\widehat{f})^\vee$, where $\vee$ is the inverse Fourier transform, is a continuous linear operator on $L^p(G)$ ([10, Lemma 35.2]). Furthermore, one can easily verify that $T_{\varphi\psi} = T_\varphi T_\psi$ for all $\varphi,\psi\in\mathscr{M}(L^p(G))$. We denote the operator norm of T_φ by $\|T_\varphi\|$. If $h\in\mathscr{E}_{\widehat{G}}$, then $\widehat{h}\in\mathfrak{E}_{00}(\widehat{G})$ and $T_\varphi h$ can be expressed as

$$T_\varphi h = (\varphi\widehat{h})^\vee = \sum_{\sigma\in\widehat{G}} d_\sigma\mathrm{tr}([\varphi(\sigma)\widehat{h}(\sigma)]^t\sigma), \tag{1.3}$$

(where $[\,]^t$ denotes the transpose; see [10, (34.32.i), (35.2.vi)]). Since $\widehat{h} \in \mathfrak{E}_{00}(\widehat{G})$, the above sum is finite and hence $T_\varphi h \in \mathscr{E}_{\widehat{G}}$. The dual of $A_p(G)$ is identified with the multiplier algebra $\mathscr{M}(L^{p'}(G))$ as follows: each $\varphi \in \mathscr{M}(L^{p'}(G))$ defines an element in $A_p(G)^*$ by the duality

$$\langle \varphi, h\rangle_{A_p^*,A_p} = (T_\varphi h)(e) = \sum_{\sigma\in\widehat{G}} d_\sigma \mathrm{tr}(\varphi(\sigma)\widehat{h}(\sigma)) \quad \text{for all } h \in \mathscr{E}_{\widehat{G}}. \tag{1.4}$$

Furthermore, $\|\varphi\|_{A_p^*} = \|T_\varphi\|$ (see [10, (35.16.f)], keeping in mind that $A_p(G)$ coincides with $\mathfrak{J}_{p'}(G)$ in [10]). With multiplication induced from $\mathscr{M}(L^{p'}(G))$ and with the norm $\|\cdot\|_{A_p^*}$, the space $A_p(G)^*$ is a Banach algebra. Henceforth we shall identify $\mathscr{M}(L^{p'}(G))$ with $A_p(G)^*$.

2. Main results

Unless otherwise noted, throughout this section G is a compact Hausdorff topological group. In preparation for our main theorem, we need a few preliminary results. First, we note that if $f \in L^1(G)$ and R_x is the right translation defined by $(R_x f)(y) = f(yx)$ for $x, y \in G$, then for every $\sigma \in \widehat{G}$,

$$\begin{aligned}\widehat{R_x f}(\sigma) &= \int_G \overline{\sigma}(y) f(yx)\,dy = \int_G \overline{\sigma}(yx^{-1}) f(y)\,dy \\ &= \int_G \overline{\sigma}(y) f(y)\,dy \cdot \sigma(x)^t = \widehat{f}(\sigma)\sigma(x)^t.\end{aligned} \tag{2.1}$$

Lemma 2.1. *Let $h \in \mathscr{E}_{\widehat{G}}$, $\varphi \in A_p(G)^*$, $\sigma \in \widehat{G}$ and $x \in G$. Then*

(i) $R_x(T_\varphi h) = T_\varphi(R_x h)$,

(ii) $(T_\varphi h)(x) = \langle \varphi, R_x h\rangle_{A_p^*,A_p}$,

(iii) $T_\varphi(\sigma_{ij} h) = \sum_{k=1}^{d_\sigma} \sigma_{kj} T_{\sigma_{ik}\cdot\varphi} h$, *for all $i, j \in \{1, 2, \ldots, d_\sigma\}$.*

Proof. (i) Commutation with right translations is a known property of T_φ (see, for example, Derighetti [1]). It is, however, interesting to give a direct proof of this fact using (2.1) and the definition of T_φ in (1.3). For all $y \in G$, one has

$$\begin{aligned} R_x(T_\varphi h)(y) = (T_\varphi h)(yx) &= \sum_{\sigma\in\widehat{G}} d_\sigma \mathrm{tr}(\widehat{h}(\sigma)^t \varphi(\sigma)^t \sigma(y)\sigma(x)) \\ \text{(cyclic permutation)} \quad &= \sum_{\sigma\in\widehat{G}} d_\sigma \mathrm{tr}(\varphi(\sigma)^t \sigma(y)\sigma(x)\widehat{h}(\sigma)^t) \\ &= \sum_{\sigma\in\widehat{G}} d_\sigma \mathrm{tr}(\varphi(\sigma)^t \sigma(y)[\widehat{h}(\sigma)\sigma(x)^t]^t). \end{aligned}$$

By (2.1), $\widehat{R_x h}(\sigma) = \widehat{h}(\sigma)\sigma(x)^t$, and thus

$$\begin{aligned} R_x(T_\varphi h)(y) &= \sum_{\sigma\in\widehat{G}} d_\sigma \mathrm{tr}(\varphi(\sigma)^t \sigma(y)\widehat{R_x h}(\sigma)^t) \\ &= \sum_{\sigma\in\widehat{G}} d_\sigma \mathrm{tr}(\widehat{R_x h}(\sigma)^t \varphi(\sigma)^t \sigma(y)) \\ &= \sum_{\sigma\in\widehat{G}} d_\sigma \mathrm{tr}([\varphi(\sigma)\widehat{R_x h}(\sigma)]^t \sigma(y)) \\ &= T_\varphi(R_x h)(y), \end{aligned}$$

which proves (i).

(ii) Using (i) and (1.4) we have

$$(T_\varphi h)(x) = R_x(T_\varphi h)(e) = T_\varphi(R_x h)(e) = \langle \varphi, R_x h\rangle_{A_p^*,A_p}.$$

(iii) Since $\mathscr{E}_{\widehat{G}}$ is an algebra, $\sigma_{ij}h \in \mathscr{E}_{\widehat{G}}$, and so using (ii), one has

$$\begin{aligned} T_\varphi(\sigma_{ij}h)(x) &= \langle \varphi, (R_x\sigma_{ij})(R_x h)\rangle_{A_p^*,A_p} \\ &= \langle \varphi, (\sum_{k=1}^{d_\sigma} \sigma_{kj}(x)\sigma_{ik})(R_x h)\rangle_{A_p^*,A_p} \\ &= \sum_{k=1}^{d_\sigma} \sigma_{kj}(x)\langle \sigma_{ik}\cdot\varphi, R_x h\rangle_{A_p^*,A_p} \\ &= \sum_{k=1}^{d_\sigma} \sigma_{kj}(x)(T_{\sigma_{ik}\cdot\varphi}h)(x), \end{aligned}$$

which proves (iii). □

Lemma 2.2. *Let $h \in \mathscr{E}_{\widehat{G}}$, $\varphi \in A_p(G)^*$, $\psi \in A_p(G)^*$ and $\sigma \in \widehat{G}$. Then*

(i) $\langle \varphi\psi, h\rangle_{A_p^*,A_p} = \langle \varphi, T_\psi h\rangle_{A_p^*,A_p}$,

(ii) $\sigma_{ij}\cdot(\varphi\psi) = \sum_{k=1}^{d_\sigma}(\sigma_{kj}\cdot\varphi)(\sigma_{ik}\cdot\psi)$, *for all* $i,j \in \{1,2,\dots,d_\sigma\}$.

Proof. To prove (i), we use (1.4) and write

$$\langle \varphi\psi, h\rangle_{A_p^*,A_p} = (T_{\varphi\psi}h)(e) = (T_\varphi T_\psi h)(e) = \langle \varphi, T_\psi h\rangle_{A_p^*,A_p}.$$

To prove (ii), we use (i) and Lemma 2.1(iii):

$$\begin{aligned} \langle \sigma_{ij}\cdot(\varphi\psi), h\rangle_{A_p^*,A_p} &= \langle \varphi\psi, \sigma_{ij}h\rangle_{A_p^*,A_p} = \langle \varphi, T_\psi(\sigma_{ij}h)\rangle_{A_p^*,A_p} \\ &= \langle \varphi, \sum_{k=1}^{d_\sigma} \sigma_{kj}T_{\sigma_{ik}\cdot\psi}h\rangle_{A_p^*,A_p} \\ &= \sum_{k=1}^{d_\sigma} \langle \sigma_{kj}\cdot\varphi, T_{\sigma_{ik}\cdot\psi}h\rangle_{A_p^*,A_p} \\ &= \sum_{k=1}^{d_\sigma} \langle (\sigma_{kj}\cdot\varphi)(\sigma_{ik}\cdot\psi), h\rangle_{A_p^*,A_p} \\ &= \langle \sum_{k=1}^{d_\sigma} (\sigma_{kj}\cdot\varphi)(\sigma_{ik}\cdot\psi), h\rangle_{A_p^*,A_p}. \end{aligned}$$

Since $h \in \mathscr{E}_{\widehat{G}}$ is arbitrary and $\mathscr{E}_{\widehat{G}}$ is dense in $A_p(G)$, (ii) follows. □

To state our next lemma, let

$$\mathscr{C} = \{\sigma_{ij} \colon \sigma \in \widehat{G},\ 1 \le i,j \le d_\sigma\} \subset A_p(G), \tag{2.2}$$

and let $\mathscr{C}^\diamond$ be the absolute polar of $\mathscr{C}$ in $A_p(G)^*$ defined by

$$\mathscr{C}^\diamond = \{\psi \in A_p(G)^* \colon |\psi(\sigma_{ij})| \le 1 \text{ for all } \sigma_{ij} \in \mathscr{C}\}.$$

A group G is called of *bounded representation type* if $\sup_{\sigma\in\widehat{G}} d_\sigma < \infty$. Such groups are characterized by Moore [16] as groups with a closed Abelian subgroup of finite index (also called *virtually Abelian* groups).

Lemma 2.3. *Let G be a compact group of bounded representation type and let $N = \sup_{\sigma\in\widehat{G}} d_\sigma$. Then the following statements hold.*

(i) $\|\sigma_{ij}\|_{A_p} \le d_\sigma \le N$, *for every* $\sigma_{ij} \in \mathscr{C}$.
(ii) *If* $\psi \in \mathscr{C}^\diamond$, *then* $\psi \in \mathfrak{E}^\infty(\widehat{G})$ *with* $\|\psi\|_\infty \le N$.
(iii) *If* $p = 2$, *then* $\mathscr{C}^\diamond$ *is norm bounded in* $A(G)^*$.

Proof. (i) Since $\sigma \in \widehat{G}$ is unitary, $|\sigma_{ij}(x)| = |(\sigma(x)e_j|e_i)| \le 1$ for all $x \in G$. It follows from compactness of G that $\|\sigma_{ij}\|_p \le 1$. By (1.2),

$$\|\sigma_{ij}\|_{A_p} \le d_\sigma \|\sigma_{kj}\|_p \|\sigma_{ik}\|_{p'} \le d_\sigma \le N.$$

(ii) A simple calculation shows that for each $\tau \in \widehat{G}$,

$$\widehat{\sigma_{ij}}(\tau) = \begin{cases} 0, & \text{if } \tau \ne \sigma, \\ \frac{1}{d_\sigma} E_{ij}, & \text{if } \tau = \sigma, \end{cases} \tag{2.3}$$

where E_{ij} is the $d_\sigma \times d_\sigma$ matrix, whose (i, j) entry is 1 and all other entries are 0. It follows from (2.3) and (1.4) that for every $\psi \in A_p(G)^*$ and every $\sigma_{ij} \in \mathscr{C}$,

$$\langle \psi, \sigma_{ij}\rangle = \sum_{\tau\in\widehat{G}} d_\tau \mathrm{tr}(\psi(\tau)\widehat{\sigma_{ij}}(\tau)) = \mathrm{tr}(\psi(\sigma)E_{ij}) = \psi(\sigma)_{ji}. \tag{2.4}$$

Thus if $\psi \in \mathscr{C}^\diamond$, then $|\psi(\sigma)_{ji}| = |\langle\psi, \sigma_{ij}\rangle| \le 1$. If $\|\psi(\sigma)\|$ is the operator norm of $\psi(\sigma) \in \mathscr{L}(H_\sigma)$, then

$$\|\psi(\sigma)\| \le (\sum_{i,j}^{d_\sigma} |\psi(\sigma)_{ij}|^2)^{1/2} \le (\sum_{i,j}^{d_\sigma} 1)^{1/2} = d_\sigma \le N,$$

from which we obtain $\|\psi\|_\infty = \sup_{\sigma\in\widehat{G}} \|\psi(\sigma)\| \le N$.

(iii) We shall prove that $\mathscr{C}^\diamond$ is norm bounded by N. To this end, it suffices to show that for all $\psi \in \mathscr{C}^\diamond$ and all $h \in \mathscr{E}_{\widehat{G}}$, $|\langle\psi, h\rangle_{A^*,A}| \le N\|h\|_A$. Letting $\mathfrak{E}^1(\widehat{G})$ to denote the set of all $\varphi \in \prod_{\sigma\in\widehat{G}} \mathscr{L}(H_\sigma)$ such that

$$\|\varphi\|_1 := \sum_{\sigma\in\widehat{G}} d_\sigma \|\varphi(\sigma)\|_1 < \infty,$$

where $\|\varphi(\sigma)\|_1$ is the von Neumann norm of the matrix $\varphi(\sigma)$, then it is known that $A(G) \cong \mathfrak{E}^1(\widehat{G})$ (an isometric isomorphism via the Fourier transform $f \mapsto \widehat{f}$), and $\mathfrak{E}^1(\widehat{G})^* = \mathfrak{E}^\infty(\widehat{G})$ ([10, Theorem 28.31, Corollary 34.7]). Hence using (ii) and Hölder's inequality, we can write

$$|\langle\psi, h\rangle_{A^*,A}| = |\langle\psi, \widehat{h}\rangle_{\mathfrak{E}^\infty,\mathfrak{E}^1}| \le \|\psi\|_\infty \|\widehat{h}\|_1 \le N\|\widehat{h}\|_1 = N\|h\|_A, \tag{2.5}$$

which is what we wanted to show. □

Remark 2.1. We do not know if Lemma 2.3(iii) still holds for $p \ne 2$. It would be interesting to know the answer to this question even for the special case that G is a compact Abelian group.

Now we are ready to prove our main theorem. We keep the notation of the previous lemma.

Theorem 2.1. *Suppose G is a compact group of bounded representation type and $1 < p < \infty$. If $\mathscr{C}^\diamond$ is norm bounded in $A_p(G)^*$, then $AP(A_p(G))$ is a Banach subalgebra of $A_p(G)^*$.*

Proof. We assume that $A_p(G)$ and $A_p(G)^*$ are equipped with their respective norm topologies (thus the closures and compactness assertions are with respect to the norm topologies). Let B be the closed unit ball in $A_p(G)$. We need to show that if $\varphi,\psi\in AP(A_p(G))$, then $\varphi\psi\in AP(A_p(G))$, or equivalently,

$$B\cdot(\varphi\psi)=\{u\cdot(\varphi\psi):\,u\in B\}$$

is relatively compact in $A_p(G)^*$.

Let $E\subset A_p(G)^*$ be defined by

$$E=\mathscr{C}\cdot(\varphi\psi)=\{\sigma_{ij}\cdot(\varphi\psi):\,\sigma_{ij}\in\mathscr{C}\}.$$

We will show that $\overline{E}$ is compact in $A_p(G)^*$. To this end, let $N=\sup_{\sigma\in\widehat{G}}d_\sigma$, and

$$K_1=\{\sigma_{ij}\cdot\varphi:\,\sigma_{ij}\in\mathscr{C}\},\quad K_2=\{\sigma_{ij}\cdot\psi:\,\sigma_{ij}\in\mathscr{C}\}.$$

Since φ and ψ are almost periodic, and since by Lemma 2.3(i), $\|\sigma_{ij}\|_{A_p}\le N$ for all $\sigma_{ij}\in\mathscr{C}$, it follows that K_1 and K_2 are relatively compact in $A_p(G)^*$. By Lemma 2.2(ii), one has

$$\sigma_{ij}\cdot(\varphi\psi)=\sum_{k=1}^{d_\sigma}(\sigma_{kj}\cdot\varphi)(\sigma_{ik}\cdot\psi)\in K_1K_2+\cdots+K_1K_2,$$

where the number of terms on the right hand side is d_σ. Since both K_1 and K_2 are relatively compact, $K_1\times K_2$ is relatively compact in $A_p(G)^*\times A_p(G)^*$, hence K_1K_2 is relatively compact in $A_p(G)^*$. Thus $\overline{K_1K_2}$ is compact. So, $S_{d_\sigma}:=\overline{K_1K_2}+\cdots+\overline{K_1K_2}$ (d_σ-times) is compact. Since $E\subset\bigcup_{k=1}^N S_k$, it follows that $\overline{E}$ is compact.

Let us denote the convex balanced hull of a set X by $cbh(X)$. Then $cbh(E)=cbh(\mathscr{C})\cdot(\varphi\psi)$, and thus we may write

$$\overline{cbh(\overline{E})}=\overline{cbh(E)}=\overline{cbh(\mathscr{C})\cdot(\varphi\psi)}\supset\overline{cbh(\mathscr{C})}\cdot(\varphi\psi)=\mathscr{C}^{\diamond\diamond}\cdot(\varphi\psi),$$

(the first equality follows easily from definitions, and the last equality is the bipolar theorem). However, the compactness of $\overline{E}$ implies that $\overline{cbh(\overline{E})}$ is compact in $A_p(G)^*$ ([12, (20.6.3)]), and hence $\mathscr{C}^{\diamond\diamond}\cdot(\varphi\psi)$ is relatively compact.

We know that if the closed convex balanced hull of $\mathscr{C}$ is absorbing, then it is a neighborhood of the origin ([15, Theorem 1.3.14]) and hence contains a nonzero multiple of B. By the bipolar theorem, the closed convex balanced hull of $\mathscr{C}$ is equal to $\mathscr{C}^{\diamond\diamond}$. It is easy to check that $\mathscr{C}^{\diamond\diamond}$ is absorbing if and only if $\mathscr{C}^{\diamond}$ is w^*-bounded in $A_p(G)^*$ (in fact, since $A_p(G)$ is a Banach space, $\mathscr{C}^{\diamond}$ is w^*-bounded if and only if it is norm bounded, [15, Theorem 2.6.7]). Therefore by our assumption, $\mathscr{C}^{\diamond\diamond}$ must be a neighborhood of 0, and thus for a suitable $\rho>0$, $\rho B\subset\mathscr{C}^{\diamond\diamond}$. Hence $(\rho B)\cdot(\varphi\psi)\subset\mathscr{C}^{\diamond\diamond}\cdot(\varphi\psi)$ is relatively compact and therefore so is $B\cdot(\varphi\psi)$. This completes the proof of the theorem. □

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