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THE STRONGEST SEQUENTIALLY COMPATIBLE TOPOLOGY ON A ν-GENERALIZED METRIC SPACE

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Dedicated to Professor Anthony To-Ming Lau on the occasion of his 75th birthday

Abstract. We introduce the strongest sequentially compatible topology on a ν-generalized metric space. We also discuss separation axioms for this topology.

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1. Introduction

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. Let A be an arbitrary set. Then we denote by $\#A$ the cardinal number of A. We define a subset $A^{(k)}$ of A^k as follows: $(a_1, a_2, \cdots, a_k) \in A^{(k)}$ iff $(a_1, a_2, \cdots, a_k) \in A^k$ and $a_1, a_2, \cdots, a_k$ are all different.

The following concept is recently called a semimetric space.

Definition 1.1. Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Then (X, d) is said to be a *semimetric space* if the following hold:

(N1) $d(x,y) = 0 \Leftrightarrow x = y$.
(N2) $d(x,y) = d(y,x)$.

This concept has been much important recently; see [3] and others. However, this concept is too weak. So we human beings have been studying this concept under some additional assumptions. ν-generalized metric space introduced by Branciari in 2000 is one of them. This concept is weaker than that of metric space and stronger than that of semimetric space. To study ν-generalized metric spaces is very important in order to understand the possibility and the impossibility of semimetric spaces.

Definition 1.2 (Branciari [1])**.** Let X be a nonempty set, let d be a function from $X \times X$ into $[0, \infty)$ and let $\nu \in \mathbb{N}$. Then (X, d) is said to be a *ν-generalized metric space* if (N1), (N2) and the following hold:

(N3)$_\nu$ $d(x,y) \le D(x, u_1, u_2, \cdots, u_\nu, y)$ for any $(x, u_1, u_2, \cdots, u_\nu, y) \in X^{(\nu+2)}$, where

$$D(x, u_1, u_2, \cdots, u_\nu, y) = d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y).$$

It is obvious that (X, d) is a metric space iff (X, d) is a 1-generalized metric space. We have studied the topological structure of this space. Indeed, recent studies tell the following:

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- 1- and 3-generalized metric spaces have the compatible topology (see [8]).
- For any $\nu \in \mathbb{N} \setminus \{1,3\}$, there exists a ν-generalized metric space which does not have the compatible topology (see [4, 8]).
- All ν-generalized metric spaces have the strongly compatible topology (see [6]).
- 2-generalized metric spaces have a sequentially compatible topology (see [7]).

In this paper, motivated by the above, we introduce the strongest sequentially compatible topology on a ν-generalized metric space. We also discuss separation axioms for this topology.

2. Preliminaries

In this section, we give some preliminaries.

Definition 2.1. Let X be a topological space with topology υ. Let d be a function from $X \times X$ into $[0,\infty)$.

- υ is said to be *compatible* with d if the following are equivalent for any net $\{x_\alpha\}$ in X and $x \in X$:
  - $\lim_\alpha d(x,x_\alpha) = 0$.
  - $\{x_\alpha\}$ converges to x in υ.
- υ is said to be *sequentially compatible* with d if the following are equivalent for any sequence $\{x_n\}$ in X and $x \in X$:
  - $\lim_n d(x,x_n) = 0$.
  - $\{x_n\}$ converges to x in υ.

Let X be a nonempty set and let d be a function from $X \times X$ into $[0,\infty)$. Define a subset $S(x,\delta)$ of X by

$$S(x,\delta) = \{y \in X : \ d(x,y) < \delta\} \tag{2.1}$$

for $x \in X$ and $\delta > 0$.

Theorem 2.1. *Let* (X,υ) *be a topological space and let* d *be a function from* $X \times X$ *into* $[0,\infty)$. *Then* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *holds*:

(i) υ *is sequentially compatible with* d.

(ii) For any open subset U *of* X, *the following* (A) *holds*:

(A) *For any* $x \in U$, *there exists* $\delta > 0$ *satisfying* $S(x,\delta) \subset U$.

(iii) If a net $\{x_\alpha\}$ *satisfies* $\lim_\alpha d(x,x_\alpha) = 0$ *for some* $x \in X$, *then* $\{x_\alpha\}$ *converges to* x *in* υ.

Proof. We first prove (i) $\Rightarrow$ (ii). Arguing by contradiction, we assume that there exist an open subset U of X and $x \in U$ satisfying $S(x,\delta) \setminus U \neq \varnothing$ for any $\delta > 0$. Then we can choose a sequence $\{x_n\}$ in X satisfying $x_n \in S(x,1/n) \setminus U$. It is obvious that $\lim_n d(x,x_n) = 0$ holds. However, $\{x_n\}$ does not converge to x in υ because $\{x_n : n \in \mathbb{N}\} \cap U = \varnothing$ holds. So υ is not sequentially compatible with d. This is a contradiction.

We next prove (ii) $\Rightarrow$ (iii). Let $\{x_\alpha\}$ be a net satisfying $\lim_\alpha d(x,x_\alpha) = 0$ for some $x \in X$. Let U be an open neighborhood at x. Then from (ii), there exists $\delta > 0$ satisfying $S(x,\delta) \subset U$. From $\lim_\alpha d(x,x_\alpha) = 0$, there exists α_0 such that $\alpha \geq \alpha_0$ implies $d(x,x_\alpha) < \delta$. Thus, $\alpha \geq \alpha_0$ implies $x_\alpha \in S(x,\delta) \subset U$. Therefore $\{x_\alpha\}$ converges to x in υ. □

Lemma 2.1 (Proposition 4.1 in [5])**.** *Let* (X,d) *be a* ν*-generalized metric space. Then* $(\text{N3})_{k\nu}$ *holds for any* $k \in \mathbb{N}$. *Thus,* (X,d) *is a* $(k\nu)$*-generalized metric space.*

Lemma 2.2. *Let $\{x_n\}$ be a sequence in a ν-generalized metric space (X,d). Then there exist a subsequence $\{y_n\}$ of $\{x_n\}$ and a subset Z of X such that either* (a) *or* (b) *holds*:

(a): (a1)–(a3) *hold.*

(a1) $\liminf_n d(u,y_n) > 0$ *for any* $u \in X \setminus Z$.

(a2) $\lim_n d(z,y_n) = 0$ *for any* $z \in Z$.

(a3) $\#Z < \infty$.

(b): (b1)–(b2) *hold.*

(b1) $\lim_n d(z,y_n) = 0$ *for any* $z \in Z$.

(b2) $\#Z = \infty$.

Proof. Define a subsequence $\{x^{(1)}{}_n\}$ of $\{x_n\}$ by $x^{(1)}{}_n = x_n$ and a subset Z_1 of X by $Z_1 = \varnothing$. We assume that we have defined a subsequence $\{x^{(k)}{}_n\}$ and a subset Z_k of X for some $k \in \mathbb{N}$. We consider the following two cases:

- $\liminf_n d(u,x^{(k)}{}_n) > 0$ for any $u \in X \setminus Z_k$.
- $\liminf_n d(v,x^{(k)}{}_n) = 0$ for some $v \in X \setminus Z_k$.

In the first case, we define $\{x^{(k+1)}{}_n\}$ and Z^{k+1} by

$$x^{(k+1)}{}_n = x^{(k)}{}_n \quad \text{and} \quad Z_{k+1} = Z_k.$$

In the second case, we choose a subsequence $\{x^{(k+1)}{}_n\}$ of $\{x^{(k)}{}_n\}$ satisfying $\lim_n d(v,x^{(k+1)}{}_n) = 0$ and we define Z_{k+1} by $Z_{k+1} = Z_k \cup \{v\}$. By induction, we can define a sequence $\{x^{(k)}{}_n\}$ of subsequences of $\{x_n\}$ and a sequence $\{Z_k\}$ of subsets of X. Define a subsequence $\{y_n\}$ of $\{x_n\}$ and a subset Z of X by

$$y_n = x^{(n)}{}_n \quad \text{and} \quad Z = \bigcup \{Z_k : k \in \mathbb{N}\}.$$

Then in the case where $\#Z < \infty$, (a1)–(a3) obviously hold. In the other case, where $\#Z = \infty$, (b1)–(b2) obviously hold. □

Lemma 2.3. *Let (X,d) be a ν-generalized metric space and let $\{y_n\}$ and Z satisfy* (b1)–(b2) *of Lemma* 2.2. *Let $x,y,z \in X$ satisfy $z \in Z$ and*

$$d(x,y) < \inf \big\{ d(x,v) : v \in \{y_n : n \in \mathbb{N}\} \cup \{z\} \big\}.$$

Then

$$\inf \big\{ d(y,v) : v \in \{y_n : n \in \mathbb{N}\} \cup \{z\} \big\} > 0 \tag{2.2}$$

holds.

Proof. In the case where $y = x$, the conclusion obviously holds. So we assume $y \neq x$. Then it is obvious that $(x,y,z) \in X^{(3)}$ holds. Arguing by contradiction, we assume that the left hand side of (2.2) equals 0. From the assumption, we have $y \notin \{y_n : n \in \mathbb{N}\} \cup \{z\}$. So there exists a subsequence $\{y_{f(n)}\}$ of $\{y_n\}$ such that $y_{f(n)}$ $(n \in \mathbb{N})$ are all different and $\lim_n d(y,y_{f(n)}) = 0$ holds. Since $\#Z = \infty$ holds, we can choose $z_2,\cdots,z_\nu \in Z$ satisfying $(x,y,z_1,\cdots,z_\nu) \in X^{(\nu+2)}$, where we put $z_1 = z$. Fix $\varepsilon > 0$. Then we can choose $\mu \in \mathbb{N}$ satisfying

$$\{x,y,z_1,\cdots,z_\nu\} \cap \{y_{f(m)} : m \geq \mu\} = \varnothing,$$
$$\max\{d(z_j,y_{f(n)}) : j = 1,\cdots,\nu\} < \varepsilon \quad \text{and} \quad d(y,y_{f(n)}) < \varepsilon$$

for any $n \in \mathbb{N}$ with $n \geq \mu$. By Lemma 2.1, we note that $(\mathrm{N3})_{2\nu}$ holds. Since

$$(x, y, y_{f(\mu+1)}, \cdots, y_{f(\mu+\nu)}, z_1, \cdots, z_\nu) \in X^{(2\nu+2)}$$

holds, we have by $(\mathrm{N3})_{2\nu}$

$$d(x,z) \leq D(x, y, y_{f(\mu+\nu)}, z_\nu, \cdots, y_{f(\mu+2)}, z_2, f_{f(\mu+1)}, z_1) < d(x,y) + 2\nu\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $d(x,z) \leq d(x,y)$, which implies a contradiction. Therefore we obtain (2.2). □

3. Main results

Throughout this section, we let (X,d) be a ν-generalized metric space with $\nu \geq 2$. Let F be the set of all functions from X into $(0,\infty)$. Define $S(x,\delta)$ and $T(x,\delta)$ by (2.1) and

$$T(x,\delta) = S(x,\delta) \setminus \{x\}$$

for $x \in X$ and $\delta > 0$. Define $S(x,f,\delta)$ and $T(x,f,\delta)$ by

$$S(x,f,\delta) = S\big(x, \min\{f(x),\delta\}\big) \quad \text{and} \quad T(x,f,\delta) = S(x,f,\delta) \setminus \{x\}$$

for $x \in X$, $f \in F$ and $\delta > 0$. Define $U(x,f,\delta,n)$ and $U(x,f)$ by

$$U(x,f,\delta,n) = \begin{cases} S(x,f,\delta), & \text{if } n = 1, \\ S(x,f,\delta) \cup \bigcup \big\{U\big(y,f,\delta - d(x,y), n-1\big) : y \in T(x,f,\delta)\big\}, & \text{if } n > 1, \end{cases}$$

and

$$U(x,f) = U\big(x,f,f(x),\nu\big)$$

for $x \in X$, $f \in F$, $\delta > 0$ and $n \in \mathbb{N}$. Let τ be a topology on X induced by a subbase

$$\{U(x,f) : x \in X, f \in F\}.$$

Lemma 3.1. *Let $x \in X$ and $f,g \in F$. Then the following hold*:

(i) If $f \leq g$ holds, then $U(x,f) \subset U(x,g)$ holds.
(ii) If the restrictions of f and g to $U(x,f)$ coincide, then $U(x,f) = U(x,g)$ holds.

Proof. Obvious. □

Lemma 3.2. *Define $V(x,f,n)$ and $V(x,f)$ by*

$$V(x,f,n) = \begin{cases} S\big(x,f(x)\big), & \text{if } n = 1, \\ S\big(x,f(x)\big) \cup \bigcup \big\{V\big(y,f,n-1\big) : y \in T\big(x,f(x)\big)\big\}, & \text{if } n > 1, \end{cases}$$

and

$$V(x,f) = V\big(x,f,\nu\big)$$

for $x \in X$, $f \in F$ and $n \in \mathbb{N}$. Then $U(x,f) \subset V(x,f)$ holds.

Proof. Obvious. □

Lemma 3.3. *Let $x \in X$ and $f \in F$. Then the following hold*:

(i) For any $z \in U(x,f)$, there exists $\varepsilon > 0$ satisfying $S(z,\varepsilon) \subset U(x,f)$.
(ii) For any $z \in U(x,f)$, there exists $g \in F$ satisfying $U(z,g) \subset U(x,f)$.

Proof. We first show (i). We fix $z \in U(x,f)$. Then there exist $k \in \mathbb{N} \cup \{0\}$ and $(u_0, \cdots, u_k) \in X^{k+1}$ satisfying $x = u_0$, $z = u_k$,

$$\begin{aligned} d(u_0,u_1) &< f(x), \\ d(u_1,u_2) &< \min\{f(u_1), f(x) - D(u_0,u_1)\}, \\ d(u_2,u_3) &< \min\{f(u_2), f(x) - D(u_0,u_1,u_2)\}, \\ &\vdots \\ d(u_{k-1},u_k) &< \min\{f(u_{k-1}), f(x) - D(u_0,\cdots,u_{k-1})\}. \end{aligned}$$

We consider the following four cases:

- $k = 0$ holds.
- $1 \le k < \nu$ holds.
- $k = \nu$ holds and there exist i, j satisfying $i < j$ and $u_i = u_j$.
- $k = \nu$ and $(u_0, \cdots, u_\nu) \in X^{(\nu+1)}$ hold.

In the first case, we have $S\big(z, f(z)\big) = S\big(x, f(x)\big) \subset U(x,f)$. In the second case, we have

$$S\big(z, \min\{f(z), f(x) - D(u_0,\cdots,u_k)\}\big) \subset U(x,f).$$

In the third case, since either the first or the second case holds for another $(u_0, \cdots, u_k)$, there exists $\varepsilon > 0$ satisfying $S(z,\varepsilon) \subset U(x,f)$. In the fourth case, we put

$$\varepsilon := f(x) - D(u_0, \cdots, u_k) > 0.$$

Let $w \in S(z,\varepsilon)$. In the case where $w \in \{u_0, \cdots, u_k\}$, it is obvious that $w \in U(x,f)$. In the other case, where $w \notin \{u_0, \cdots, u_k\}$, we note that $u_0, \cdots, u_k, w$ are all different. So we have by (N3)$_\nu$

$$d(x,w) \le D(u_0, \cdots, u_\nu, w) < D(u_0, \cdots, u_\nu) + \varepsilon = f(x).$$

Thus $w \in S\big(x, f(x)\big) \subset U(x,f)$. We have shown (i).

Let us prove (ii). By (i), there exists $g \in F$ satisfying $S\big(y, g(y)\big) \subset U(x,f)$ for any $y \in U(x,f)$. For $z \in U(x,f)$, we have $U(z,g) \subset U(x,f)$. □

Lemma 3.4. *Let U be an open subset of (X,τ). Then the following hold*:

(i) For any $x \in U$, there exists $\delta > 0$ satisfying $S(x,\delta) \subset U$.
(ii) For any $x \in U$, there exists $f \in F$ satisfying $U(x,f) \subset U$.

Remark 3.1. From (ii), $\{U(x,f) : f \in F\}$ is a neighborhood basis at x.

Proof. There exist $k \in \mathbb{N}$, $y_j \in X$ and $g_j \in F$ $(j = 1, \cdots, k)$ satisfying

$$x \in \bigcap_{j=1}^{k} U(y_j, g_j) \subset U.$$

By Lemma 3.3 (ii), there exists $f_j \in F$ satisfying $U(x, f_j) \subset U(y_j, g_j)$. Define $f \in F$ by $f(u) = \min\{f_j(u) : j = 1, \cdots, k\}$. Then we have

$$U(x,f) \subset \bigcap_{j=1}^{k} U(x, f_j) \subset \bigcap_{j=1}^{k} U(y_j, g_j) \subset U.$$

We have shown (ii). It is obvious that (i) follows from (ii). □

Lemma 3.5. *Let U be a subset of X. Then U is open in τ iff* (A) *holds.*

Proof. We assume that U is open in τ. Then by Lemma 3.4 (i), (A) holds. Let us prove the converse implication. We assume that (A) holds, that is, there exists $f \in F$ satisfying $S\big(x, f(x)\big) \subset U$ for any $x \in U$. Then it is obvious that $U(x, f) \subset U$ for any $x \in U$. Thus we obtain

$$\bigcup \{U(x, f) : x \in U\} = U.$$

Therefore U is open in τ. □

Let f be a function from X into $\mathbb{R}$ and define $\bar{f} \in F$ by

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } f(x) > 0, \\ 1, & \text{if } f(x) \le 0. \end{cases}$$

We extend the domain of U as follows: For $x \in X$, we define $U(x, f)$ by $U(x, f) = U(x, \bar{f})$ provided $f(y) > 0$ holds for any $y \in U(x, \bar{f})$. By Lemma 3.1 (ii), we note that the characteristics of U does not change at all. Similarly we extend the domain of V.

The following two lemmas play a very important role in the proof of our main result.

Lemma 3.6. *Let $x \in X$, let $\{y_n\}$ be a sequence in X and let Z be a subset of X satisfying* (b1)–(b2) *of Lemma 2.2. Fix $z \in Z$. Define a function f from X into $[0, \infty)$ by*

$$f(u) = \inf\big\{d(u, v) : v \in \{y_n : n \in \mathbb{N}\} \cup \{z\}\big\}/2.$$

Then the following hold:

(i) *If $f(x) > 0$ holds, then $f(y) > 0$ holds for any $y \in S\big(x, f(x)\big)$.*

(ii) *If $f(x) > 0$ holds, then $f(u) > 0$ holds for any $u \in U(x, f)$ and $\{y_n : n \in \mathbb{N}\} \cap U(x, f) = \varnothing$ holds.*

Proof. We first prove (i). Fix $y \in S\big(x, f(x)\big)$. Since

$$d(x, y) < f(x) < \inf\big\{d(x, v) : v \in \{y_n : n \in \mathbb{N}\} \cup \{z\}\big\}$$

holds, by Lemma 2.3, we have

$$2 f(y) = \inf\big\{d(y, v) : v \in \{y_n : n \in \mathbb{N}\} \cup \{z\}\big\} > 0.$$

Thus, (i) holds.

Let us prove (ii). Define V as in Lemma 3.2. From (i), $f(u_1) > 0$ holds for any $u_1 \in S\big(x, f(x)\big)$. So by (i) again, $f(u_2) > 0$ holds for any $u_2 \in S\big(u_1, f(u_1)\big)$. Continuing this argument, we have $f(u) > 0$ for any $u \in V(x, f)$. Also, we have shown that (x, f) belongs to the extended domain of V. By Lemma 3.2, we have $f(u) > 0$ holds for any $u \in U(x, f)$, which implies $(\{y_n : n \in \mathbb{N}\} \cup \{z\}) \cap U(x, f) = \varnothing$. Hence $\{y_n : n \in \mathbb{N}\} \cap U(x, f) = \varnothing$ holds. □

Lemma 3.7. *Let $x \in X$, let $\{y_n\}$ be a sequence in X and let Z be a subset of X satisfying* (a1)–(a3) *of Lemma 2.2. Define a function f from X into $[0, \infty)$ by*

$$f(u) = \inf\big\{d(u, v) : v \in \{y_n : n \in \mathbb{N}\} \cup Z\big\}/2.$$

Then the following hold:

(i) *If $\inf\{d(x, y_n) : n \in \mathbb{N}\} > 0$ holds, then $f(x) > 0$ holds.*

(ii) *If $f(x) > 0$ holds, then $f(y) > 0$ holds for any $y \in S\big(x, f(x)\big)$.*

(iii) If $\inf\{d(x,y_n) : n \in \mathbb{N}\} > 0$ *holds, then* $f(u) > 0$ *holds for any* $u \in U(x,f)$ *and* $\{y_n : n \in \mathbb{N}\} \cap U(x,f) = \varnothing$ *holds.*

Proof. In order to prove (i), we assume $\inf\{d(x,y_n) : n \in \mathbb{N}\} > 0$. Then $x \in X \setminus Z$ holds. From (a3), we have $f(x) > 0$. Thus, we have shown (i).

We next prove (ii). Assume $f(x) > 0$ and fix $y \in S\big(x, f(x)\big)$. Then since

$$d(x,y) < f(x) < \inf\big\{d(x,v) : v \in \{y_n : n \in \mathbb{N}\} \cup Z\big\}$$

holds, $y \notin \{y_n : n \in \mathbb{N}\} \cup Z$ holds. Since $y \in X \setminus Z$ holds, from (a1), we have $\liminf_n d(y,y_n) > 0$. Hence $\inf\{d(y,y_n) : n \in \mathbb{N}\} > 0$ holds. So by (i), we obtain $f(y) > 0$. Thus (ii) holds.

Let us prove (iii). Assume $\inf\{d(x,y_n) : n \in \mathbb{N}\} > 0$. Then by (i), we have $f(x) > 0$. Using (ii), we can obtain the desired result as in the proof of Lemma 3.6. □

Now we can prove our main result.

Theorem 3.1. *τ is sequentially compatible with d.*

Remark 3.2.

- By Theorem 2.1 and Lemma 3.5, τ is the strongest topology that is sequentially compatible with d.
- Theorem 3.1 is independent of Lemma 9.3 in [2] because in page 426 of [2] we assume that X is regular and T_1.

Proof. Let $\{x_n\}$ be a sequence in X and let $x \in X$. Let us prove that the following are equivalent:

(i) $\lim_n d(x_n,x) = 0$.
(ii) $\{x_n\}$ converges to x in τ.

We first prove (i) $\Rightarrow$ (ii). We assume $\lim_n d(x_n,x) = 0$. Let U be an open subset of (X,τ) containing x. Then by Lemma 3.4 (i), there exists $\delta > 0$ satisfying $S(x,\delta) \subset U$. From (i), there exists $\mu \in \mathbb{N}$ such that $n \geq \mu$ implies $d(x_n,x) < \delta$. So

$$x_n \in S(x,\delta) \subset U$$

holds for $n \in \mathbb{N}$ with $n \geq \mu$. Therefore $\{x_n\}$ converges to x in τ.

Next, in order to prove (ii) $\Rightarrow$ (i), we assume that $\limsup_n d(x_n,x) > 0$. We choose a subsequence $\{x_{g(n)}\}$ of $\{x_n\}$ satisfying $\inf\{d(x,x_{g(n)}) : n \in \mathbb{N}\} > 0$. By Lemma 2.2, there exists a subsequence of $\{y_n\}$ of $\{x_{g(n)}\}$ and a subset Z of X satisfying either (a1)–(a3) or (b1)–(b2) hold. We note that $\{y_n\}$ is also a subsequence of $\{x_n\}$. We will show the following:

(B) There exists a function f from X into $[0,\infty)$ satisfying $U(x,f) \cap \{y_n : n \in \mathbb{N}\} = \varnothing$.

In the case where (a1)–(a3) hold, define a function f as in Lemma 3.7. Then since $\inf\{d(x,y_n) : n \in \mathbb{N}\} > 0$ holds, we obtain (B) by Lemma 3.7 (iii). In the other case, where (b1)–(b2) hold, since $\#Z = \infty \geq 2$ holds, we can choose $z \in Z$ with $z \neq x$. Define a function f as in Lemma 3.6. Then $f(x) > 0$ obviously holds. So, by Lemma 3.6 (ii), we obtain (B). Therefore we have shown (B) in all cases. Since $U(x,f)$ is a neighborhood of x, $\{x_n\}$ does not converge to x in τ. Therefore we have shown (ii) $\Rightarrow$ (i). Thus, τ is sequentially compatible with d. □

Theorem 3.2. *Every ν-generalized metric space (X,d) has a sequentially compatible topology with d.*

Theorem 3.3. *Let (X,d) be a ν-generalized metric space. Then the following are equivalent*:

(i) X has the compatible topology with d.

(ii) τ is the compatible topology with d.

(iii) For any $x \in X$ and $\delta > 0$, there exists an open neighborhood U at x in τ satisfying $U \subset S(x,\delta)$.

Proof. (i) $\Rightarrow$ (ii): We assume that a topology υ is compatible with d. Then since τ is the strongest topology that is sequentially compatible with d, we note that τ is stronger than υ. Let $\{x_\alpha\}$ be a net converging to $x \in X$ in τ. Then $\{x_\alpha\}$ converges to x in υ. From (i), $\lim_\alpha d(x,x_\alpha) = 0$ holds. On the other hand, by Theorem 2.1 (iii), we obtain the converse implication. Thus, τ is compatible with d.

(ii) $\Rightarrow$ (i): Obvious.

(ii) $\Rightarrow$ (iii): Arguing by contradiction, we assume that (iii) does not hold. Then there exist $x \in X$ and $\delta > 0$ such that for any open neighborhood U at x in τ, there exists $x_U \in U \setminus S(x,\delta)$. Then the net $\{x_U\}$ converges to x in τ. So we have by (ii)

$$0 < \delta \le \lim_U d(x,x_U) = 0,$$

which implies a contradiction.

(iii) $\Rightarrow$ (ii): Let $\{x_\alpha\}$ be a net converging to x in τ. Fix $\delta > 0$. Then from (iii), there exists an open neighborhood U at x in τ satisfying $U \subset S(x,\delta)$. There exists α_0 such that for any $\alpha \ge \alpha_0$, we have $x_\alpha \in U \subset S(x,\delta)$ and hence $d(x,x_\alpha) < \delta$. Therefore we obtain $\lim_\alpha d(x,x_\alpha) = 0$. On the other hand, by Theorem 2.1 (iii), we obtain the converse implication. Thus, τ is compatible with d. □

4. SEPARATION AXIOMS

In this section, we discuss separation axioms.

We recall that a topological space X is said to be a T_1 *space* if for any two distinct points x and y in X, there exist open subsets U and V of X satisfying $x \in U$, $y \in V$, $y \notin U$ and $x \notin V$. A T_1 space is also called an *accessible space* or a *Fréchet space*. It is well known that X is T_1 iff for any $x \in X$, the singleton set $\{x\}$ is closed.

We recall that X is said to be a T_2 *space* if for any two distinct points x and y in X, there exist open subsets U and V of X satisfying $x \in U$, $y \in V$ and $U \cap V = \varnothing$. A T_2 space is also called a *separated space* or a *Hausdorff space*.

Theorem 4.1. *Let (X,d) be a ν-generalized metric space and let υ be a topology on X which is sequentially compatible with d. Then (X,υ) is T_1.*

Proof. Fix $x \in X$ and define a sequence $\{x_n\}$ in X by $x_n = x$. Then we have $\lim_n d(y,x_n) = d(y,x) > 0$ for any $y \in X \setminus \{x\}$. Hence $\{x_n\}$ does not converge to y in υ. There exist an open neighborhood V_y at y and a subsequence $\{f(n)\}$ of $\{n\}$ satisfying $x_{f(n)} \notin V_y$. Thus, $x \notin V_y$ holds. We have

$$X \setminus \{x\} = \bigcup \{V_y : y \in X \setminus \{x\}\}.$$

Hence $X \setminus \{x\}$ is open. We have shown that $\{x\}$ is closed. □

Theorem 4.2. *Let (X,d) and υ be as in Theorem 4.1. Then* (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) *holds*:

(i) (X,υ) is T_2.

(ii) If $\lim_n d(x_n,x) = 0$ holds for some $x \in X$, then $\liminf_n d(x_n,y) > 0$ holds for any $y \in X \setminus \{x\}$.

(iii) If $\lim_n d(x_n,x) = \lim_n d(x_n,y) = 0$ *holds for some* $x,y \in X$*, then* $x = y$ *holds.*

Proof. We first show (i) $\Rightarrow$ (ii). We assume (i), $\lim_n d(x_n,x) = 0$ and $y \in X \setminus \{x\}$. Then we note that from the assumption, $\{x_n\}$ converges to x in (X,υ). From (i), there exist open subsets U and V of X satisfying $x \in U$, $y \in V$ and $U \cap V = \varnothing$. By Theorem 2.1 (ii), we can choose $\delta > 0$ satisfying $S(y,\delta) \subset V$. Since $x_n \in U$ holds for sufficiently large $n \in \mathbb{N}$, we have $x_n \notin S(y,\delta)$ for sufficiently large $n \in \mathbb{N}$. Thus

$$\liminf_{n\to\infty} d(x_n,y) \geq \delta > 0.$$

We have shown (i) $\Rightarrow$ (ii). We can easily prove (ii) $\Leftrightarrow$ (iii). $\square$

In the case where $\nu \leq 3$ and $\upsilon = \tau$, (i)–(iii) of Theorem 4.2 are equivalent. See [7, 8]. Motivated by this fact, we will show that there exists a counterexample in the case where $\nu \geq 8$.

Lemma 4.1. *Let X be a set. Let $a,b \in X$ and let A and B be two nonempty subsets of X with*

$$X = \{a\} \sqcup \{b\} \sqcup A \sqcup B,$$

where '$\sqcup$' represents 'disjoint union'. Let S be a mapping from B into A. Let M be a positive real number and let f be a function from A into $(0,M]$. Let g and h be functions from B into $(0,M]$. Define a function d from $X \times X$ into $[0,\infty)$ by

$$\begin{aligned}
&d(x,x) = 0, && \\
&d(a,x) = d(x,a) = f(x), && \text{if } x \in A, \\
&d(b,y) = d(y,b) = g(y), && \text{if } y \in B, \\
&d(Sy,y) = d(y,Sy) = h(y), && \text{if } y \in B, \\
&d(x,y) = M, && \text{otherwise.}
\end{aligned}$$

Then (X,d) is a ν-generalized metric space for $\nu \geq 8$.

Remark 4.1. Let $(a,b,y_1,y_1',y_2,y_3,Sy_1,Sy_2,Sy_3) \in X^{(9)}$ satisfy $\{y_1,y_1',y_2,y_3\} \subset B$ and $Sy_1' = Sy_1$. Then we have

$$\begin{aligned}
&D(y_1,Sy_1,y_1',b,y_2,Sy_2,a,Sy_3,y_3) \\
&= h(y_1) + h(y_1') + g(y_1') + g(y_2) + h(y_2) + f(Sy_2) + f(Sy_3) + h(y_3).
\end{aligned}$$

So, there is possibility that (X,d) is not a ν-generalized metric space for $\nu \leq 7$.

Proof. Fix $\nu \in \mathbb{N}$ with $\nu \geq 8$. From the above observation, for any $(u_1,\cdots,u_{\nu+2}) \in X^{(\nu+2)}$, we can easily prove

$$d(u_1,u_{\nu+2}) \leq M \leq D(u_1,\cdots,u_{\nu+2}),$$

though its proof is not short. Thus, (X,d) is a ν-generalized metric space for $\nu \geq 8$. $\square$

Example 4.1. Put $X = \{a\} \sqcup \{b\} \sqcup \big((0,1] \times [0,1]\big)$. Define a function d from $X \times X$ into $[0,\infty)$ by

$$d(x,y) = d(y,x) = \begin{cases} 0, & \text{if } x = y, \\ s, & \text{if } x = (s,0),\ y = a, \\ s+t, & \text{if } x = (s,t),\ y = b, \\ t, & \text{if } x = (s,0),\ y = (s,t), \\ 2, & \text{otherwise,} \end{cases}$$

where $s,t \in (0,1]$. Then the following hold:

(i) (X,d) is a ν-generalized metric space with $\nu \geq 8$.
(ii) $U(a,f) \cap U(b,g) \neq \varnothing$ for any $f,g \in F$.
(iii) (X,τ) is not T_2, where τ is the topology defined in Section 3.
(iv) (X,υ) is not T_2 for any sequentially compatible topology υ with d.
(v) (iii) of Theorem 4.2 holds.
(vi) $U(a,f) \subset S(a,1)$ does not hold for any $f \in F$.
(vii) The compatible topology with d does not exist.

Proof. By Lemma 4.1, (i) holds. (iii) follows from (ii). Also (iv) follows from Remark 3.2. We can easily prove (iii) of Theorem 4.2. (vii) follows from (vi) and Theorem 3.3.

Let us prove (ii). Fix $f,g \in F$. Then by Lemma 3.3 (i), there exist $\delta_1, \delta_2 \in (0,1)$ satisfying $S(b,\delta_1) \subset U(b,g)$ and $S(a,\delta_2) \subset U(a,f)$. Fix s with $0 < s < \min\{\delta_1,\delta_2\}$. Using Lemma 3.3 (i) again, there exists $\delta_3 \in (0,s)$ satisfying $S\big((s,0),\delta_3\big) \subset U(a,f)$. We have

$$\begin{aligned}\{s\} \times (0, \min\{\delta_1 - s, \delta_3\}) &= S(b,\delta_1) \cap S\big((s,0),\delta_3\big) \\ &\subset U(b,g) \cap U(a,f).\end{aligned}$$

We have shown (ii). We also obtain (vi). □

Remark 4.2. We do not know whether (i)–(iii) of Theorem 4.2 are equivalent in the case where $4 \leq \nu \leq 7$ and $\upsilon = \tau$.

5. Observation

We finally observe how f of $U(x,f)$ works.

Let (X,d) be as in Example 4.1 and define a topology τ as in Section 3. Define $W(x,\delta)$ by

$$W(x,\delta) = \bigcup \big\{ S\big(y, \delta - d(x,y)\big) : y \in S(x,\delta) \big\}$$

for $x \in X$ and $\delta > 0$. $W(x,\delta)$ seems to be more natural than $U(x,f)$, however $W(x,\delta)$ does not work in our purpose. Indeed, let $\{s_n\}$ and $\{t_n\}$ be sequences in $(0,1]$ converging to 0. Define a sequence $\{y_n\}$ in X by $y_n = (s_n,t_n)$. Then $d(a,y_n) = 2$ holds. However, for any $\delta > 0$, there exists $\mu \in \mathbb{N}$ satisfying $y_n \in W(a,\delta)$ for $n \geq \mu$ because

$$D\big(a,(s_n,0),y_n\big) = s_n + t_n$$

holds. On the other hand, let $f \in F$ satisfy $f(a) = s_1$, $f\big((s,t)\big) < t$ and

$$f\big((s,0)\big) < \min\big\{ s, s_1 - s, \min\{t_n : s_n \geq s\} \big\}$$

for $s \in (0, s_1)$ and $t \in (0, 1]$. Then we have

$$U(a, f) = S(a, s_1) \cup \bigcup \{S\big((s, 0), f(s, 0)\big) : 0 < s < s_1\}.$$

Therefore $U(a, f) \cap \{y_n : n \in \mathbb{N}\} = \varnothing$ holds.

References

[1] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen 57 (2000), 31-37.

[2] G. Gruenhage, "Generalized metric spaces" in Handbook of set-theoretic topology, pp. 423-501, North-Holland, Amsterdam, 1984.

[3] W.A. Kirk, N. Shahzad, Fixed points and Cauchy sequences in semimetric spaces, J. Fixed Point Theory Appl. 17 (2015), 541-555.

[4] T. Suzuki, Generalized metric spaces do not have the compatible topology, Abstr. Appl. Anal. 2014 (2014), Article ID 458098.

[5] T. Suzuki, Another generalization of Edelstein's fixed point theorem in generalized metric spaces, Linear Nonlinear Anal. 2 (2016), 271-279.

[6] T. Suzuki, The strongly compatible topology on ν-generalized metric spaces, RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. (2017), DOI: 10.1007/s13398-017-0380-0.

[7] T. Suzuki, Every generalized metric space has a sequentially compatible topology, Linear Nonlinear Anal. in press.

[8] T. Suzuki, B. Alamri, M. Kikkawa, Only 3-generalized metric spaces have a compatible symmetric topology, Open Math. 13 (2015), 510-517.