J. Nonlinear Var. Anal. 1 (2017), No. 3, pp. 345-356
Available online at http://jnva.biemdas.com

A NEW ITERATIVE SCHEME FOR THE SUM OF INFINITE M-ACCRETIVE MAPPINGS AND INVERSELY STRONGLY ACCRETIVE MAPPINGS AND ITS APPLICATIONS

LI WEI[1,*], YANWEI SHEN[1], YAQIN ZHENG[2], RUILIN TAN[1]

[1]*School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang 050061, China*
[2]*College of Science, Agricultural University of Hebei, Baoding 071001, China*

Abstract. In this paper, a new iterative scheme for approximating zero points of the sum of infinite m-accretive mappings and μ_i-inversely strongly accretive mappings in Hilbert spaces is presented. A strong convergence theorem is established. Computational experiments are conducted to verify the effectiveness of the iterative scheme. The applications of the new iterative scheme to nonlinear p_i-Laplacian parabolic and elliptic systems are demonstrated.
Keywords. Accretive mapping; Iterative scheme; Nonlinear parabolic system; Nonlinear elliptic system; Strong convergence; Zero point.

2010 Mathematics Subject Classification. 47H05, 47H09.

1. Introduction and Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. It is easy to check that, for $\forall x, y \in H$ and $\lambda \in [0,1]$,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

We denote by " $\to$ " and " $\rightharpoonup$ " strong and weak convergence in H, respectively. It is well-known that Hilbert space satisfies the Opial's condition [6] in the sense that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $y \neq x$.

Let C be a nonempty closed convex subset of H. For $\forall x \in H$, there exists a unique element in C, denoted by $P_C x$, such that

$$\|x - P_C x\| < \|x - y\|, \quad \forall y \in C.$$

In this case, P_C is called the metric projection from C onto H.

Assume $A : D(A) \subseteq H \to H$ is a mapping. Then A is said to be

(i) a contraction if there exists a constant $k \in (0,1)$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in D(A);$$

*Corresponding author.
E-mail address: diandianba@yahoo.com (L.Wei).
Received April 3, 2017; Accepted September 30, 2017.

(ii) nonexpansive if

$$\|Ax - Ay\| \le \|x - y\|, \quad \forall x, y \in D(A);$$

(iii) accretive if for $x, y \in D(A)$, $\langle x - y, Ax - Ay\rangle \ge 0$;
(iv) m-accretive if A is an accretive mapping and $R(I + \lambda A) = H$, $\forall \lambda > 0$;
(v) strongly accretive if $\langle x - y, Ax - Ay\rangle \ge \mu \|x - y\|^2$, $\forall x, y \in D(A)$ and $\mu > 0$;
(vi) μ-inversely strongly accretive if $\langle x - y, Ax - Ay\rangle \ge \mu \|Ax - Ay\|^2$, $\forall x, y \in D(A)$ and $\mu > 0$;
(vii) strongly positive ([4]) if there exists $\overline{\gamma} > 0$ such that

$$\langle x, Ax\rangle \ge \overline{\gamma}\|x\|^2, \quad \forall x \in D(A).$$

In this case,

$$\|aI - bA\| = \sup_{\|x\| \le 1} |\langle (aI - bA)x, x\rangle|,$$

where I is the identity mapping, $a \in [0,1]$ and $b \in [-1,1]$.

For nonlinear mapping A, we denote by $N(A)$ the set of zero points of A, that is, $N(A) = \{x \in D(A) : Ax = 0\}$. We denote by $F(A)$ the set of fixed points of A, that is, $F(A) = \{x \in D(A) : Ax = x\}$.

Many practical problems can be reduced to finding the solution of the following inclusion problem

$$Ax + Bx \ni 0,$$

where A is m-accretive and B is μ-inversely strongly accretive. In other words, finding zero point of the sum of an m-accretive mapping and a μ-inversely strongly accretive mapping. Later, this problem is extended to the following one:

$$A_i x + B_i x \ni 0,$$

where A_i is m-accretive and B_i is μ_i-inversely strongly accretive, for $i = 1, 2, \cdots, m$ or $i \in N = \{1, 2, \cdots\}$. Related work can be found, e.g., [5, 7, 8, 9, 10, 11, 12]. Inspired by these work, we shall present a new iterative scheme in this paper, do computational experiment to verify the feasibility of the iterative scheme and discuss the applications of the iterative scheme.

2. Main results

Lemma 2.1. [1] *If H is a Hilbert space and $f : H \to H$ is a contraction, then there exists a unique element $u \in H$ such that $f(u) = u$.*

Lemma 2.2. [13] *For $\forall x \in H$ and $\forall y \in C$,*

$$\|P_C x - y\|^2 + \|P_C x - x\|^2 \le \|y - x\|^2.$$

Lemma 2.3. [13] *If $A : H \to H$ is m-accretive, then $(I + rA)^{-1}$ is nonexpansive and $F((I + rA)^{-1}) = N(A)$.*

Lemma 2.4. [2] *Let C be a nonempty closed and convex subset of Hilbert space H. Let $T_m : C \to C$ be a nonexpansive mapping for each $m \in N$. Let $\{a_m\}$ be a real number sequence in (0,1) such that $\sum_{m=1}^{\infty} a_m = 1$. Suppose that $\bigcap_{m=1}^{\infty} F(T_m) \ne \emptyset$. Then the mapping $\sum_{m=1}^{\infty} a_m T_m$ is nonexpansive with $F(\sum_{m=1}^{\infty} a_m T_m) = \bigcap_{m=1}^{\infty} F(T_m)$.*

Theorem 2.1. *Let H be a Hilbert space. Let $A_i : H \to H$ be m-accretive mappings and let $B_i : H \to H$ be μ_i-inversely strongly accretive mappings, for $i \in N$. Let $f : H \to H$ be a contraction with $k \in (0,1)$ and let $T : H \to H$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma}$. Suppose $\bigcap_{i=1}^{\infty} N(A_i + B_i) \neq \emptyset$. Let $\{x_n\}$ be generated by the following iterative scheme:*

$$\begin{cases} x_1 \in H \text{ is given arbitrarily}, \\ u_n = \alpha_n \eta f(x_n) + (I - \alpha_n T)x_n, \\ y_n = \beta_n u_n + \delta_n \sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n + y_n}{2}) + \lambda_n \varepsilon_n, \\ H_1 := H, \\ H_{n+1} = \{v \in H_n : \|y_n - v\|^2 \leq \frac{1+\beta_n - \lambda_n}{2-\delta_n}\|u_n - v\|^2 + \frac{2\lambda_n}{2-\delta_n}\|\varepsilon_n - v\|^2\}, \\ x_{n+1} = P_{H_{n+1}}(x_1), \ n \in N. \end{cases} \tag{2.1}$$

Suppose that $\{\varepsilon_n\} \subset H$ is the error sequence, $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$ and $\{\lambda_n\}$ are four sequences in (0,1), $\{r_{n,i}\}$ and $\{\mu_i\}$ are two sequences in $(0, +\infty)$. Furthermore, the following conditions are satisfied:
(i) $\beta_n + \delta_n + \lambda_n \equiv 1$;
(ii) $\alpha_n \to 0$, $\lambda_n \to 0$, $\delta_n \not\to 0$,
(iii) $0 < \eta < \frac{\overline{\gamma}}{2k}$;
(iv) $\sum_{i=1}^{\infty} a_{n,i} = 1$, $n \in N$; *(v)* $r_{n,i} \leq 2\mu_i$, $i \in N$, $n \in N$; *(vi)* $\|\varepsilon_n\| \leq M$, *where $M > 0$ is a constant. Then $x_n \to p_0$, $u_n \to p_0$, $y_n \to p_0$, as $n \to \infty$, where $p_0 = P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1)$.*

Proof. We split the proof into nine steps.

Step 1. Show that $\sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)$ is nonexpansive.

Since $r_{n,i} \leq 2\mu_i$ we find that, for $x, y \in H$,

$$\begin{aligned} \|(I - r_{n,i}B_i)x - (I - r_{n,i}B_i)y\|^2 &= \|(x-y) - r_{n,i}(B_i x - B_i y)\|^2 \\ &= \|x-y\|^2 - 2r_{n,i}\langle x-y, B_i x - B_i y\rangle + r_{n,i}^2\|B_i x - B_i y\|^2 \\ &\leq \|x-y\|^2 + r_{n,i}^2\|B_i x - B_i y\|^2 - 2r_{n,i}\mu_i\|B_i x - B_i y\|^2 \\ &\leq \|x-y\|^2, \end{aligned}$$

which ensures that $(I - r_{n,i}B_i)$ is nonexpansive, $\forall i, n \in N$. Using Lemma 2.3 and Lemma 2.4, one sees that $\sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i) : H \to H$ is nonexpansive since $\sum_{i=1}^{\infty} a_{n,i} = 1$, for all $n \in N$.

Step 2. Show that $\{y_n\}$ is well defined.

Assume that $U : H \to H$ is nonexpansive. Define $W : H \to H$ by

$$W(x) = sy + tU(\frac{x+y}{2}) + (1-s-t)v.$$

For $\forall x, z \in H$ and $t, s \in (0,1)$, we have

$$\|W(x) - W(z)\| \leq t\|\frac{x+y}{2} - \frac{z+y}{2}\| = \frac{t}{2}\|x - z\|.$$

It follows that W is a contraction, which implies from Lemma 2.1 that there exists a unique $x \in H$ such that $W(x) = x$. From Step 1, we know that $\{y_n\}$ is well defined.

Step 3. Show that $\{x_n\}$ is well defined.

It suffices to prove that H_n is nonempty closed and convex subset of H, for all $n\in N$. In fact, one has

$$\begin{aligned}
&\|y_n-v\|^2\le \frac{1+\beta_n-\lambda_n}{2-\delta_n}\|u_n-v\|^2+\frac{2\lambda_n}{2-\delta_n}\|\varepsilon_n-v\|^2\\
&\Leftrightarrow \|y_n\|^2-2\langle y_n,v\rangle+\|v\|^2\\
&\quad\le \frac{1+\beta_n-\lambda_n}{2-\delta_n}(\|u_n\|^2-2\langle u_n,v\rangle+\|v\|^2)+\frac{2\lambda_n}{2-\delta_n}(\|\varepsilon_n\|^2-2\langle\varepsilon_n,v\rangle+\|v\|^2)\\
&\Leftrightarrow \frac{1+\beta_n-\lambda_n}{2-\delta_n}\|u_n\|^2+\frac{2\lambda_n}{2-\delta_n}\|\varepsilon_n\|^2-\|y_n\|^2\ge\langle\frac{4\lambda_n}{2-\delta_n}\varepsilon_n+\frac{2(1+\beta_n-\lambda_n)}{2-\delta_n}u_n-2y_n,v\rangle,
\end{aligned}$$

which implies that H_n is closed and convex subset of H for all $n\in N$.

Next, we shall use the inductive method to show that $\emptyset\neq\bigcap_{i=1}^{\infty}N(A_i+B_i)\subset H_n$, for all $n\in N$. Note that $\bigcap_{i=1}^{\infty}N(A_i+B_i)\neq\emptyset$. Taking $p\in\bigcap_{i=1}^{\infty}N(A_i+B_i)$, we find from Step 1 that

$$\begin{aligned}
\|y_1-p\|^2&\le\beta_1\|u_1-p\|^2+\delta_1\sum_{i=1}^{\infty}a_{1,i}\|(I+r_{1,i}A_i)^{-1}(I-r_{1,i}B_i)(\frac{u_1+y_1}{2})-p\|^2+\lambda_1\|\varepsilon_1-p\|^2\\
&\le\frac{2\beta_1+\delta_1}{2}\|u_1-p\|^2+\frac{\delta_1}{2}\|y_1-p\|^2+\lambda_1\|\varepsilon_1-p\|^2.
\end{aligned}$$

Thus $\|y_1-p\|^2\le\frac{1+\beta_1-\lambda_1}{2-\delta_1}\|u_1-p\|^2+\frac{2\lambda_1}{2-\delta_1}\|\varepsilon_1-p\|^2$, which implies that $p\in H_2$.

Suppose that $p\in H_{n+1}$. From (2.1), one has

$$\begin{aligned}
\|y_{n+1}-p\|^2&\le\beta_{n+1}\|u_{n+1}-p\|^2+\lambda_{n+1}\|\varepsilon_{n+1}-p\|^2\\
&\quad+\delta_{n+1}\|\sum_{i=1}^{\infty}a_{n+1,i}(I+r_{n+1,i}A_i)^{-1}(I-r_{n+1,i}B_i)(\frac{u_{n+1}+y_{n+1}}{2})\\
&\quad-\sum_{i=1}^{\infty}a_{n+1,i}(I+r_{n+1,i}A_i)^{-1}(I-r_{n+1,i}B_i)p\|^2\\
&\le\frac{2\beta_{n+1}+\delta_{n+1}}{2}\|u_{n+1}-p\|^2+\frac{\delta_{n+1}}{2}\|y_{n+1}-p\|^2+\lambda_{n+1}\|\varepsilon_{n+1}-p\|^2.
\end{aligned}$$

It follows that

$$\|y_{n+1}-p\|^2\le\frac{1+\beta_{n+1}-\lambda_{n+1}}{2-\delta_{n+1}}\|x_{n+1}-p\|^2+\frac{2\lambda_{n+1}}{2-\delta_{n+1}}\|\varepsilon_{n+1}-p\|^2,$$

which implies that $p\in H_{n+2}$. Therefore, by induction, one has $\emptyset\neq\bigcap_{i=1}^{\infty}N(A_i+B_i)\subset\bigcap_{n=1}^{\infty}H_n$. All of the above ensures that $\{x_n\}$ is well defined.

Step 4. Show that $\{x_n\}$,$\{y_n\}$ and $\{u_n\}$ are all bounded.

Note that $x_{n+1}=P_{H_{n+1}}(x_1)$. From definition of the metric projection, we have, for $\forall p\in\bigcap_{i=1}^{\infty}N(A_i+B_i)\subset\bigcap_{n=1}^{\infty}H_n$, that $\|x_{n+1}-x_1\|\le\|p-x_1\|$, which implies that $\{x_n\}$ is bounded. Since

$$\begin{aligned}
\|u_n-p\|&=\|\alpha_n\eta f(x_n)-\alpha_n\eta f(p)\|+\|\alpha_n\eta f(p)\|+\|x_n\|+\alpha_n\|Tx_n\|+\|p\|\\
&\le\alpha_n\eta k\|x_n-p\|+\alpha_n\eta\|f(p)\|+\|x_n\|+\alpha_n\|Tx_n\|+\|p\|,
\end{aligned}$$

one obtains that $\{u_n\}$ is bounded. Since

$$\begin{aligned}
\|y_n-p\|^2&\le\frac{1+\beta_n-\lambda_n}{2-\delta_n}\|u_n-p\|^2+\frac{2\lambda_n}{2-\delta_n}\|\varepsilon_n-p\|^2\\
&\le\|u_n-p\|^2+\|\varepsilon_n-p\|^2,
\end{aligned}$$

one obtains that $\{y_n\}$ is bounded.

Step 5. Show that $x_n-x_{n+1}\to 0$, $u_n-x_n\to 0$, $y_n-u_n\to 0$, as $n\to\infty$.

Since $x_n = P_{H_n}(x_1)$ and $x_{n+1} \in H_{n+1} \subset H_n$, one sees that $\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$ for all $n \in N$. Combining with the fact that $\{x_n\}$ is bounded, we have $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Using lemma 2.2, we find that

$$\|x_n - x_{n+1}\|^2 + \|x_n - x_1\|^2 \leq \|x_{n+1} - x_1\|^2.$$

Thus $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Since $\alpha_n \to 0$ and both $\{x_n\}$ and $\{Tx_n\}$ are bounded, then $u_n - x_n = \alpha_n(\eta f(x_n) - Tx_n) \to 0$ as $n \to \infty$. Since $\lambda_n \to 0$ and $u_n - x_n \to 0$, we find from 2.1 that

$$\|x_{n+1} - y_n\|^2 \leq \frac{1 - \lambda_n + \beta_n}{2 - \delta_n} \|u_n - x_{n+1}\|^2 + \frac{2\lambda_n}{2 - \delta_n} \|\varepsilon_n - x_{n+1}\|^2 \to 0,$$

as $n \to \infty$. Thus $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Step 6. Show that $\{x_n\}$ is a Cauchy sequence.

Lemma 2.2 implies that,

$$\|x_n - x_{n+m}\|^2 \leq \|x_{n+m} - x_1\|^2 - \|x_n - x_1\|^2, \quad \forall m \in N. \tag{2.2}$$

If, $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon_0 > 0$ and two subsequences $\{n_k\}$ and $\{m_k\}$ of $\{n\}$ such that

$$\|x_{n_k+m_k} - x_{n_k}\| \geq \varepsilon_0, \quad \forall k \in N.$$

Noticing (2.2) and the fact that $\lim_{n\to\infty} \|x_n - x_1\|$ exists, we have

$$\begin{aligned}\|x_{n_k} - x_{n_k+m_k}\|^2 &\leq \|x_{n_k+m_k} - x_1\|^2 - \|x_{n_k} - x_1\|^2 = \|x_{n_k+m_k} - x_1\|^2 - \lim_{k\to\infty} \|x_{n_k+m_k} - x_1\|^2 \\ &+ \lim_{k\to\infty} \|x_{n_k} - x_1\|^2 - \|x_{n_k} - x_1\|^2 \to 0,\end{aligned}$$

as $k \to \infty$. This makes a contradiction! Thus $\{x_n\}$ is a Cauchy sequence.

Step 7. Show that $x_n \to p_0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$ as $n \to \infty$.

Since $\{x_n\}$ is a Cauchy sequence, one sees that there exists a unique $p_0 \in H$ such that $x_n \to p_0$ as $n \to \infty$. If, $(A_i + B_i)p_0 \neq 0$, then Lemma 2.4 implies that $p_0 \neq \sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)p_0$, $n \in N$. Since $\lambda_n \to 0$, $y_n - u_n \to 0$, and $\delta_n \nrightarrow 0$, we find from (2.1) that

$$[\sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{y_n + u_n}{2}) - y_n] = \frac{\beta_n}{\delta_n}(y_n - u_n) + \frac{\lambda_n}{\delta_n}(y_n - \varepsilon_n) \to 0,$$

as $n \to \infty$. Combining with Step 5, we arrive at

$$\sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{y_n + u_n}{2}) - \frac{y_n + u_n}{2} \to 0,$$

as $n \to \infty$. Moreover,

$$\begin{aligned}&\|\sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{x_n + y_n}{2}) - \frac{x_n + y_n}{2}\| \\ &\leq \|\sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{x_n + y_n}{2}) - \sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{y_n + u_n}{2})\| \\ &+ \|\sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{y_n + u_n}{2}) - \frac{u_n + y_n}{2}\| \\ &+ \|\frac{u_n + y_n}{2} - \frac{x_n + y_n}{2}\| \\ &\leq \|x_n - u_n\| + \|\sum_{i=1}^{\infty} a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{y_n + u_n}{2}) - \frac{u_n + y_n}{2}\| \to 0,\end{aligned}$$

as $n \to \infty$. Using the Opial's condition, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \|\frac{x_n+y_n}{2} - p_0\| \\
&< \liminf_{n\to\infty} \|\frac{x_n+y_n}{2} - \sum_{i=1}^{\infty} a_{n,i}(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)p_0\| \\
&\leq \liminf_{n\to\infty} \|\frac{x_n+y_n}{2} - \sum_{i=1}^{\infty} a_{n,i}(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)(\frac{x_n+y_n}{2})\| \\
&+ \liminf_{n\to\infty} \|\sum_{i=1}^{\infty} a_{n,i}(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)(\frac{x_n+y_n}{2}) - \sum_{i=1}^{\infty} a_{n,i}(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)p_0\| \\
&\leq \liminf_{n\to\infty} \|\frac{x_n+y_n}{2} - p_0\|,
\end{aligned}$$

which makes a contradiction! Thus

$$p_0 = \sum_{i=1}^{\infty} a_{n,i}(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)p_0, n \in N.$$

Then $p_0 \in \bigcap_{i=1}^{\infty} N(A_i+B_i)$.

Step 8. Show that $p_0 = P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1)$, where p_0 is the same as that in Step 7.

Since $H_j \subset H_i$, for $\forall j \geq i \geq 0$, one has

$$\|P_{H_1}(x_1) - x_1\|^2 \leq \|P_{H_2}(x_1) - x_1\|^2 \leq \cdots \leq \|P_{H_n}(x_1) - x_1\|^2 \leq \|P_{\bigcap_{n=1}^{\infty} H_n}(x_1) - x_1\|^2.$$

Since $\bigcap_{i=1}^{\infty} N(A_i+B_i) \subset \bigcap_{n=1}^{\infty} H_n$, one has

$$\begin{aligned}
\|p_0 - x_1\| &= \lim_{n\to\infty} \|x_n - x_1\| = \lim_{n\to\infty} \|P_{H_n}(x_1) - x_1\| \\
&\leq \|P_{\bigcap_{n=1}^{\infty} H_n}(x_1) - x_1\| = \min_{x \in \bigcap_{n=1}^{\infty} H_n} \|x - x_1\| \\
&\leq \|P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1) - x_1\|.
\end{aligned}$$

Since $p_0 \in \bigcap_{i=1}^{\infty} N(A_i+B_i)$, we find from Lemma 2.2 that

$$\|P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1) - p_0\|^2 + \|P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1) - x_1\|^2 \leq \|x_1 - p_0\|^2 \leq \|P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1) - x_1\|^2.$$

Thus $p_0 = P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1)$.

Step 9. Show that $u_n \to p_0$ and $y_n \to p_0$, as $n \to \infty$, where $p_0 = P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1)$ is the same as that in Step 7 or Step 8.

It follows from the results of Step 5, Step 7 and Step 8.

This completes the proof.

□

Remark 2.1. Three sequences are proved to be strongly convergent to the zero point of the sum of $A_i + B_i$ for $i \in N$. Compared with [5, 7, 8, 9, 10, 11], different iterative scheme is constructed, which leads to different proof techniques. Compared with [12], computational experiment is conducted and $\{u_n\}$ is concerned in the iterative scheme which makes the difference between ours and that in [12] although the proof technique is similar.

Remark 2.2. Next, we shall provide numerical experiments to show our iterative scheme is effective. In our numerical experiment, we consider the case of $H = (-\infty, +\infty)$. Take $f(x) = \frac{x}{2}$, $Tx = \frac{x}{4}$, $A_i x = B_i x =$

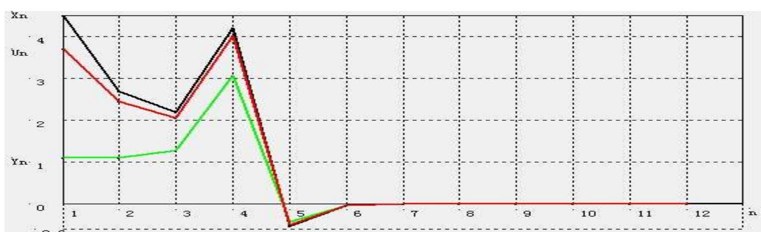
$2^i x$, for $x \in H$ and $i \in N$. Then $\bigcap_{i=1}^{\infty} N(A_i + B_i) = \{0\}$. Take $\alpha_n = \frac{1}{n}, \beta_n = \lambda_n = \frac{1}{2(2^n+2)}, \delta_n = \frac{2^n+1}{2^n+2}, \varepsilon_n = \frac{1}{2^n+2}, k = \frac{1}{2}, \overline{\gamma} = \frac{1}{6}, \eta = \frac{1}{7}, a_{n,i} = \frac{n+1}{(n+2)^i}, r_{n,i} = \frac{1}{2^{n+i}}$ and $\mu_i = \frac{1}{2^i}$, where $i, n \in N$.

Then, for iterative scheme (2.1), if choose $x_1 = 4.5$, we can see the results of convergence of the sequences $\{x_n\}, \{y_n\}, \{u_n\}$ in Table 2.1 and Figure 2.1.

Table 2.1 Numerical Results of $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ with Initial $x_1 = 4.5$

n	x_n	y_n	u_n
1	4.50000000000000	1.09183673469388	3.69642857142857
2	2.68624509511462	1.10580819987444	2.44640178305081
3	2.17785721635672	1.26813714353246	2.04822285824025
4	4.19596077779728	3.05684824566802	4.00864110021705
5	−0.5302647763283	−0.4414336267721	−0.5113267486023
6	−0.0428134368121	−0.0383095512662	−0.0415392273832
7	−0.00279357484097	−0.002562130789	−0.0027233101767
8	−0.00023431533088	−0.000209846019	−0.00022908507796
9	−0.00015530415628	−0.000146987446	−0.00015222272460
10	−0.00002993534751	−0.000028310736	−0.00002940078773
11	−0.00001736305066	−0.000016801977	−0.00001708118296
12	−0.00000372243073	−0.000003603064	−0.00000366703741

Figure 2.1 Convergence of $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ with Initial $x_1 = 4.5$



Remark 2.3. All codes are written in Visual Basic Six in the numerical experiments.

3. APPLICATIONS

In this section, we have two purposes: one is to show that the topic to discuss the zero point of the sum of infinite m-accretive mappings and infinite μ_i-inversely strongly accretive mappings is meaningful. The other is to show the effectiveness of our proposed iterative scheme.

Now, suppose that E is a real Banach space with E^* its dual space. Let $\langle \cdot, \cdot \rangle$ denote the generalized duality pairing between E and E^*.

Definition 3.1. [3] Recall that $J : E \to 2^{E^*}$ is called the normalized duality mapping if $\forall x \in E$,

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}.$$

Definition 3.2. [3] Recall that $B : E \to E^*$ is said to be a monotone operator if $\forall x_i \in D(B), i = 1, 2$,

$$\langle x_1 - x_2, Bx_1 - Bx_2 \rangle \geq 0.$$

Monotone operator B is said to be maximal monotone if $R(J + rB) = E^*$, $\forall r > 0$.

Definition 3.3. [3] Recall that $B : E \to E^*$ is said to be coercive if $\{x_n\} \subset D(B)$ with $\lim_{n\to\infty} \|x_n\| = +\infty$, then $\lim_{n\to\infty} \frac{\langle x_n, Bx_n\rangle}{\|x_n\|} = +\infty$.

Definition 3.4. [3] $F : D(F) = E \to E^*$ is said to be a hemi-continuous mapping if $F(x+ty) \rightharpoonup Fx$, as $t \to 0$, $\forall x, y \in E$.

Definition 3.5. [3] $\Phi : E \to (-\infty, +\infty]$ is said to be a proper convex functional if there exists $u_0 \in E$ such that $\Phi(u_0) < +\infty$ and $\Phi((1-\lambda)u + \lambda v) \le (1-\lambda)\Phi(u) + \lambda\Phi(v)$, $\forall u, v \in E$ and $\lambda \in [0,1]$. $\Phi : E \to (-\infty, +\infty]$ is said to be lower-semi-continuous: $\liminf_{y\to x} \Phi(y) \ge \Phi(x)$,$\forall x \in E$. $\partial\Phi : E \to E^*$ is called the subdifferential of Φ if

$$\partial\Phi(u) = \{w \in E^* : \Phi(u) - \Phi(v) \le \langle u - v, w\rangle,\ \forall v \in E\}, \quad \forall u \in E.$$

Example 3.1. The following p_i-Laplacian parabolic systems can be found in [12]:

$$\begin{cases} \frac{\partial u^{(i)}(x,t)}{\partial t} - div[(C_i(x,t) + |\nabla u^{(i)}|^2)^{\frac{s_i}{2}} |\nabla u^{(i)}|^{m_i-1} \nabla u^{(i)}] \\ +\varepsilon |u^{(i)}|^{q_i-2} u^{(i)} + g(x, u^{(i)}(x,t), \nabla u^{(i)}(x,t)) = \widetilde{f}(x,t),\ (x,t) \in \Omega \times (0,T), \\ - < \vartheta, (C_i(x,t) + |\nabla u^{(i)}|^2)^{\frac{s_i}{2}} |\nabla u|^{m_i-1} \nabla u^{(i)} > \in \beta_x(u^{(i)}(x,t)),\ (x,t) \in \Gamma \times (0,T), \\ u^{(i)}(x,0) = u^{(i)}(x,T),\ i \in N. \end{cases} \tag{3.1}$$

In (3.1), Ω is bounded conical domain in R^N with $\Gamma \in C^1$. ϑ is the the exterior normal derivative of Γ. ε is a nonnegative constant and T is a positive constant.

$$\nabla u^{(i)} = (\frac{\partial u^{(i)}}{\partial x_1}, \frac{\partial u^{(i)}}{\partial x_2}, \cdots, \frac{\partial u^{(i)}}{\partial x_N}),$$

$x = (x_1, x_2, \cdots, x_N) \in \Omega$. β_x is the subdifferential of φ_x, where $\varphi_x = \varphi(x,\cdot) : R \to R$, $\forall x \in \Gamma$. $0 \le C_i(x,t) \in V_i := L^{p_i}(0,T; W^{1,p_i}(\Omega))$, $0 \ne \widetilde{f}(x,t) \in W := L^2(0,T; L^2(\Omega))$, $m_i + s_i + 1 = p_i$ and $m_i \ge 0$, $i \in N$.

The discussion of (3.1) is based on the following assumptions:

(1) $\{p_i\}_{i=1}^{\infty}$ satisfies $\frac{2N}{N+1} < p_i < +\infty$, $\{\mu_i\}_{i=1}^{\infty} \subset (0,1]$, $\{q_i\}_{i=1}^{\infty}$ satisfies $\frac{2N}{N+1} < q_i \le \min\{p_i, p_i'\} < +\infty$. $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, $\frac{1}{q_i} + \frac{1}{q_i'} = 1$, $i \in N$.

(2) Green's formula is available.

(3) $\forall x \in \Gamma$, $\varphi_x = \varphi(x,\cdot) : R \to R$ is a proper and lower-semi-continuous mapping and $\varphi_x(0) = 0$.

(4) $0 \in \beta_x(0)$, $\forall t \in R$, $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

(5) $g : \Omega \times R^{N+1} \to R$ satisfies the following:

(5.1) Carathéodory's condition;

(5.2) $|g(x, r_1, \cdots, r_{N+1})|^2 \le |h(x,t)|^2 + b|r_1|^2$, where $(r_1, r_2, \cdots, r_{N+1}) \in R^{N+1}$, $x \in \Omega$, $h(x,t) \in W$, b is a positive constant;

(5.3)$g(x, r_1, \cdots, r_{N+1}) - g(x, t_1, \cdots, t_{N+1}) \ge (r_1 - t_1)$, $\forall x \in \Omega$ and $(r_1, \cdots, r_{N+1}), (t_1, \cdots, t_{N+1}) \in R^{N+1}$.

(5.4) $g(x, 0, \cdots, 0) \equiv 0$, $\forall x \in \Omega$.

Lemma 3.1. [12] *Define $B_i : V_i \to V_i^*$ by*

$$\langle w, B_i u\rangle = \int_0^T \int_\Omega \langle (C_i(x,t) + |\nabla u|^2)^{\frac{s_i}{2}} |\nabla u|^{m_i-1} \nabla u, \nabla w\rangle dxdt$$
$$+\varepsilon \int_0^T \int_\Omega |u|^{q_i-2} uwdxdt, \quad \forall u, w \in V_i.$$

Then B_i is everywhere defined, hemi-continuous, monotone and coercive, for $i \in N$.

Definition 3.6. [12] Define $\Phi_i : V_i \to R$ by $\Phi_i(u) = \int_0^T \int_\Gamma \varphi_x(u|_\Gamma(x,t))\mathrm{d}\Gamma(x)\mathrm{d}t, \forall u(x,t) \in V_i$, $i \in N$.

Definition 3.7. [12] Define $S_i : D(S_i) = \{u(x,t) \in V_i : \frac{\partial u}{\partial t} \in V_i^*, u(x,0) = u(x,T)\} \to V_i^*$ by $S_i u = \frac{\partial u}{\partial t}$ for $i \in N$.

Definition 3.8. [12] Define $A_i : W \to W$ by

$$D(A_i) = \{u \in W \mid \exists\, f \in W \text{ such that } f \in B_i u + \partial\Phi_i(u) + S_i u\}.$$

For $u \in D(A_i)$, $A_i u = \{f \in W \mid f \in B_i u + \partial\Phi_i(u) + S_i u\}$.

Lemma 3.2. [12] *$A_i : W \to W$ is m-accretive, $i \in N$.*

Definition 3.9. [12] Define $C_i : D(C_i) = L^{\max\{p_i, p_i'\}}(0,T; W^{1,\max\{p_i,p_i'\}}(\Omega)) \subset W \to W$ by

$$(C_i u^{(i)})(x,t) = g(x, u^{(i)}, \nabla u^{(i)}) - \widetilde{f}(x,t), \quad \forall u^{(i)}(x,t) \in D(C_i),$$

where $\widetilde{f}(x,t)$ is the same as that in (3.1) for $i \in N$.

Lemma 3.3. [12] *$C_i : D(C_i) \subset W \to W$ is continuous and strongly accretive. Moreover, if $g(x, r_1, \cdots, r_{N+1}) \equiv r_1$, then C_i is μ_i-inversely strongly accretive, for $i \in N$.*

Lemma 3.4. [12] *If $g(x, r_1, \cdots, r_{N+1}) \equiv r_1$ and $\widetilde{f}(x,t) \equiv k + \varepsilon k^{q_i - 1} sgnk$, then $u^{(i)}(x,t) \equiv k$ is the unique solution of (3.1). Moreover, $\{k\} = \bigcap_{i=1}^\infty N(A_i + C_i)$.*

Theorem 3.1. *If $g(x, r_1, \cdots, r_{N+1}) \equiv r_1$, $\quad \widetilde{f}(x,t) \equiv k + \varepsilon k^{q_i-1} sgnk$, where k is a constant, then define the following iterative sequence:*

$$\begin{cases} u_1(x,t) \in W \text{ is given arbitrarily}, \\ v_n(x,t) = \alpha_n \eta f(u_n) + (I - \alpha_n T)(u_n(x,t)), \\ y_n(x,t) = \beta_n v_n(x,t) + \delta_n \sum_{i=1}^\infty a_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}C_i)(\frac{v_n + y_n}{2}) + \lambda_n \varepsilon_n, \\ H_1 := W, \\ H_{n+1} = \{v \in H_n : \|y_n - v\|^2 \le \frac{1+\beta_n - \lambda_n}{2-\delta_n}\|v_n - v\|^2 + \frac{2\lambda_n}{2-\delta_n}\|\varepsilon_n - v\|^2\}, \\ u_{n+1}(x,t) = P_{H_{n+1}}(u_1), n \in N. \end{cases}$$

Under the assumptions of Theorem 2.1 and from Lemma 3.4, we know that $\{u_n(x,t)\}$, $\{v_n(x,t)\}$ and $\{y_n(x,t)\}$ converge strongly to $P_{\bigcap_{i=1}^\infty N(A_i+C_i)}(u_1)$ as $n \to \infty$, which is the unique solution of (3.1).

Example 3.2. The following p_i-Laplacian elliptic systems are extended from the p-Laplacian elliptic boundary problem in [10]:

$$\begin{cases} -div[(C_i(x) + |\nabla u^{(i)}|^2)^{\frac{p_i - 2}{2}} \nabla u^{(i)}] + \varepsilon |u^{(i)}|^{q_i - 2} u^{(i)} + g(x, u^{(i)}(x), \nabla u^{(i)}(x)) = h(x),\ x \in \Omega, \\ - < \vartheta, (C_i(x) + |\nabla u^{(i)}|^2)^{\frac{p_i-2}{2}} \nabla u^{(i)} > \in \beta_x(u^{(i)}(x)),\ x \in \Gamma,\ i \in N. \end{cases} \tag{3.2}$$

In (3.2), Ω, Γ, ϑ , ε, β_x and φ_x are the same as those in Example 3.1. $0 \le C_i(x) \in L^{p_i}(\Omega)$, $h(x) \in L^2(\Omega)$. The discussion of (3.2) is based on the following assumptions:

(1) $\{p_i\}_{i=1}^\infty$ satisfies $\frac{2N}{N+1} < p_i < +\infty$, $\{q_i\}_{i=1}^\infty$ satisfies $\frac{2N}{N+1} < q_i < +\infty$, $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, $\frac{1}{q_i} + \frac{1}{q_i'} = 1$, $i \in N$.

(2) Green's formula is available.

(3) $g : \Omega \times R^{N+1} \to R$ satisfies

(3.1) Carathéodory's condition;

(3.2) $|g(x,r_1,\cdots,r_{N+1})-g(x,t_1,\cdots,t_{N+1})|\leq|r_1-t_1|$, $\forall x\in\Omega$ and $(r_1,r_2,\cdots,r_{N+1}),(t_1,\cdots,t_{N+1})\in R^{N+1}$;

(3.3) $(g(x,r_1,\cdots,r_{N+1})-g(x,t_1,\cdots,t_{N+1}))(r_1-t_1)\geq 0$, $\forall x\in\Omega$ and $(r_1,\cdots,r_{N+1}),(t_1,\cdots,t_{N+1})\in R^{N+1}$.

Lemma 3.5. [10] *Define $B_i: W^{1,p_i}(\Omega)\to(W^{1,p_i}(\Omega))^*$ by*

$$\langle w,B_iu\rangle=\int_\Omega\langle(C_i(x)+|\nabla u|^2)^{\frac{p_i-2}{2}}\nabla u,\nabla w\rangle dx+\varepsilon\int_\Omega|u|^{q_i-2}uwdx,\quad\forall u,w\in W^{1,p_i}(\Omega).$$

Then B_i is everywhere defined, hemi-continuous, monotone and coercive, for $i\in N$.

Definition 3.10. [10] Define $\Phi_i: W^{1,p_i}(\Omega)\to R$ by $\Phi_i(u)=\int_\Gamma\varphi_x(u|_\Gamma(x))d\Gamma(x),\forall u(x,t)\in W^{1,p_i}(\Omega)$, $i\in N$.

Definition 3.11. [10] Define $A_i: L^2(\Omega)\to L^2(\Omega)$ by

$$D(A_i)=\{u\in L^2(\Omega)|\ \exists\ f\in L^2(\Omega)\ \text{such that}\ f\in B_iu+\partial\Phi_i(u)\}.$$

For $u\in D(A_i)$, $A_iu=\{f\in L^2(\Omega)\mid f\in B_iu+\partial\Phi_i(u)\}$.

Lemma 3.6. [10] *$A_i: L^2(\Omega)\to L^2(\Omega)$ is m-accretive, $i\in N$.*

Lemma 3.7. [10] *Define $S_i: D(S_i)\subset L^2(\Omega)\to L^2(\Omega)$ by*

$$(S_iu)(x)=g(x,u,\nabla u)-h(x),\quad\forall u(x)\in D(S_i),$$

where $h(x)$ is the same as that in (3.2) for $i\in N$. Then S_i is μ_i-inversely strongly accretive, for $i\in N$.

Lemma 3.8. [10] *$u^{(i)}(x)\in L^2(\Omega)$ is the solution of (3.2) if and only if $u^{(i)}(x)\in N(A_i+S_i)$, for $i\in N$.*

Lemma 3.9. [10] *For $h(x)\in L^2(\Omega)$, (3.2) has a unique solution $u^{(i)}(x)\in L^2(\Omega)$, for $i\in N$.*

Lemma 3.10. [10] *If $g(x,r_1,\cdots,r_{N+1})\equiv r_1$ and $h(x)\equiv k+\varepsilon k^{q_i-1}sgnk$, then $u^{(i)}(x)\equiv k$ is the unique solution of (3.2). Moreover, $\{k\}=\bigcap_{i=1}^\infty N(A_i+S_i)$.*

Proof. The following proof is similar to that of Lemma 3.4.

In view of Lemma 3.9, if $g(x,r_1,\cdots,r_{N+1})\equiv r_1$ and $h(x)\equiv k+\varepsilon k^{q_i-1}sgnk$, then (3.2) has a unique solution. Since $u^{(i)}(x)\equiv k$ satisfies (3.2) in this case, one sees that $u^{(i)}(x)\equiv k$ is the unique solution of (3.2). It follows from Lemma 3.8 that $u^{(i)}(x)\equiv k\in N(A_i+S_i)$, for $i\in N$, which implies that $k\in\bigcap_{i=1}^\infty N(A_i+S_i)$.

To finish the proof, it suffices to prove that $\bigcap_{i=1}^\infty N(A_i+S_i)$ is a singleton.

In fact, if $A_iu+S_iu=0$ and $A_iv+S_iv=0$, then $A_iu+u=A_iv+v$, which implies that

$$0\leq\langle u-v,A_iu-A_iv\rangle=\langle u-v,v-u\rangle\leq 0.$$

Thus $u(x)=v(x)$, that is, $\bigcap_{i=1}^\infty N(A_i+S_i)$ is a singleton.

This completes the proof. □

Theorem 3.2. *If $g(x,r_1,\cdots,r_{N+1})\equiv r_1$, $h(x)\equiv k+\varepsilon k^{q_i-1}sgnk$, where k is a constant, then define the following iterative sequence:*

$$\begin{cases} u_1(x)\in L^2(\Omega) \text{ is given arbitrarily,}\\ v_n(x)=\alpha_n\eta f(u_n)+(I-\alpha_n T)(u_n(x)),\\ y_n(x)=\beta_n v_n(x)+\delta_n\sum_{i=1}^{\infty}a_{n,i}(I+r_{n,i}A_i)^{-1}(I-r_{n,i}S_i)(\frac{v_n+y_n}{2})+\lambda_n\varepsilon_n,\\ H_1:=W,\\ H_{n+1}=\{v\in H_n:\|y_n-v\|^2\le\frac{1+\beta_n-\lambda_n}{2-\delta_n}\|v_n-v\|^2+\frac{2\lambda_n}{2-\delta_n}\|\varepsilon_n-v\|^2\},\\ u_{n+1}(x)=P_{H_{n+1}}(u_1),n\in N.\end{cases}$$

Under the assumptions of Theorem 2.1 and from Lemma 3.10, we know that $\{u_n(x)\}$, $\{v_n(x)\}$ and $\{y_n(x)\}$ converge strongly to $P_{\bigcap_{i=1}^{\infty}N(A_i+S_i)}(u_1)$ as $n\to\infty$.

Acknowledgements

The first and the fourth author were supported by the National Natural Science Foundation of China under Grant No.11071053, Natural Science Foundation of Hebei Province under Grant No. A2014207010, Key Project of Science and Research of Hebei Educational Department under Grant No. ZD2016024 and Key Project of Science and Research of Hebei University of Economics and Business under Grant No. 2016KYZ07, the third author was supported by Science and Technology Foundation of Agricultural University of Hebei under Grant No. LG201612.

REFERENCES

[1] R.P. Agarwal, D. O'Regan, D.R. Sahu, Fixed Point Theory for Lipschitz-type Mappings with and Applications, Springer, New York, 2009.

[2] R.E. Bruck, Properties of fixed-point sets of nonexpansive mappings in Banach spaces, Trans. Amer. Math. Soc. 179 (1973), 251-262.

[3] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff, Leyden, The Netherlands, 1976.

[4] G. Cai, S. Bu, Approximation of common fixed points of a countable family of continuous pseudocontractions in a uniformly smooth Banach space, Appl. Math. Lett. 24 (2011), 1998-2004.

[5] S.Y. Cho, X. Qin, L. Wang, Strong convergence of a splitting algorithm for treating monotone operators, Fixed Point Theory Appl. 2014 (2014), Article ID 94.

[6] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591-597.

[7] X. Qin, S.Y. Cho, L. Wang, A regularity method for treating zero points of the sum of two monotone operators, Fixed Point Theory Appl. 2014 (2014), Article ID 75.

[8] P. Tseng, A modified forward-backward splliting method for maximal monotone mappings, SIAM J. Control Optim. 38 (2000), 431-446.

[9] L. Wei, L. Duan, A new iterative algorithm for the sum of two different types of finitely many accretive operators in Banach space and its connection with capillarity equation, Fixed Point Theory Appl. 2015 (2015), Article ID 25.

[10] L. Wei, A. Shi, Splitting-midpoint method for zeros of the sum of accretive operator and μ-inversely strongly accretive operator in a q-uniformly smooth Banach space and its applications, J. Inequal. Appl. 2015 (2015), Article ID 183.

[11] L. Wei, R.P. Agarwal, A new iterative algorithm for the sum of infinite m-accretive mappings and infinite μ_i-inversely strongly accretive mappings and its applications to integro-differential systems, Fixed Point Theory Appl. 2016 (2016), Article ID 7.

[12] L. Wei, Y. Zhang, Implicit monotone projection scheme for infinite m-accretive mappings and inversely strongly accretive mappings and p-Laplacian systems, Appl. Math. J. Chinese Univ. Ser. A 32 (2017), 229-240.

[13] L. Wei, R. Tan, Strong and weak convergence theorems for common zeros of finite accretive mappings, Fixed Point Theory Appl. 2014 (2014), Article ID 77.