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RELAXED EXTRAGRADIENT-LIKE METHODS FOR SYSTEMS OF GENERALIZED EQUILIBRIA WITH CONSTRAINTS OF MIXED EQUILIBRIA, MINIMIZATION AND FIXED POINT PROBLEMS

LU-CHUAN CENG[1], CHING-FEN WEN[2], YONGHONG YAO[3,*]

[1]*Department of Mathematics, Shanghai Normal University, Shanghai 200234, China*
[2]*Center for Fundamental Science, and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 807, Taiwan*
[3]*Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China*

Dedicated to Professor Anthony To-Ming Lau on the occasions of his 75th birthday

Abstract. In this paper, we introduce two multistep relaxed extragradient-like schemes for finding a solution of the system of generalized equilibrium problems (SGEP) with constraints. Under suitable control conditions, we establish the strong convergence of these two multistep relaxed extragradient-like schemes to a solution of the SGEP with constraints. Our results complement, develop, improve and extend the corresponding ones given by some authors in this area.
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1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Let P_C be the metric projection of H onto C and let $T: C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T and by $\mathbf{R}$ the set of all real numbers. A mapping $A: H \to H$ is called $\bar{\gamma}$-strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma}\|x\|^2, \ \forall x \in H.$$

A mapping $F: C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Fx - Fy\| \leq L\|x - y\|, \ \forall x, y \in C.$$

In particular, if $L = 1$ then F is called a nonexpansive mapping; if $L \in [0, 1)$ then F is called a contraction. A mapping $T: C \to C$ is called k-strictly pseudocontractive (or a k-strict pseudocontraction) if there exists

*Corresponding author.
E-mail addresses: zenglc@hotmail.com (L.-C. Ceng), cfwen@kmu.edu.tw (C.-F. Wen), yaoyonghong@aliyun.com (Y. Yao).
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a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2, \ \forall x,y \in C.$$

Recently, many authors have devoted to study the problem of finding fixed points of nonlinear mappings; see e.g., [3, 4, 5, 6] and the references therein.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, $\mathscr{A} : C \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [16] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x,y) + \varphi(y) - \varphi(x) + \langle \mathscr{A}x, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

We denote the set of solutions of GMEP (1.1) by GMEP$(\Theta,\varphi,\mathscr{A})$. The GMEP (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others. The GMEP (1.1) is further considered and studied; see e.g., [9]. In particular, if $\varphi = 0$, then GMEP (1.1) reduces to the generalized equilibrium problem (GEP) which is to find $x \in C$ such that

$$\Theta(x,y) + \langle \mathscr{A}x, y-x\rangle \geq 0, \quad \forall y \in C.$$

It was considered and studied in [17, 20]. The set of solutions of GEP is denoted by GEP$(\Theta,\mathscr{A})$.

If $\mathscr{A} = 0$, then GMEP (1.1) reduces to the mixed equilibrium problem (MEP) which is to find $x \in C$ such that

$$\Theta(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

It was considered and studied in [2]. The set of solutions of MEP is denoted by MEP(Θ,φ).

If $\varphi = 0$ and $\mathscr{A} = 0$, then GMEP (1.1) reduces to the equilibrium problem (EP) which is to find $x \in C$ such that

$$\Theta(x,y) \geq 0, \quad \forall y \in C.$$

It was considered and studied in [8, 18]. The set of solutions of EP is denoted by EP(Θ).

Given a positive number $r > 0$. Let $T_r^{(\Theta,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem, that is, for each $x \in H$,

$$T_r^{(\Theta,\varphi)}(x) := \{y \in C : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y-x, z-y\rangle \geq 0, \forall z \in C\}.$$

In particular, if $\varphi \equiv 0$ then $T_r^{(\Theta,\varphi)}$ is rewritten as $T_r^{\Theta} : H \to C$, i.e.,

$$T_r^{\Theta}(x) := \{y \in C : \Theta(y,z) + \frac{1}{r}\langle y-x, z-y\rangle \geq 0, \forall z \in C\}.$$

Let $\Phi_1, \Phi_2 : C \times C \to \mathbf{R}$ be two bifunctions, and $B_1, B_2 : C \to H$ be two nonlinear mappings. Consider the following system of generalized equilibrium problems (SGEP): find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \Phi_1(x^*,x) + \langle B_1y^*, x-x^*\rangle + \frac{1}{\nu_1}\langle x^*-y^*, x-x^*\rangle \geq 0, & \forall x \in C, \\ \Phi_2(y^*,y) + \langle B_2x^*, y-y^*\rangle + \frac{1}{\nu_2}\langle y^*-x^*, y-y^*\rangle \geq 0, & \forall y \in C, \end{cases} \tag{1.2}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. It was introduced and studied in [9]. Whenever $\Phi_1 \equiv \Phi_2 \equiv 0$, the SGEP reduces to a system of variational inequalities, which was considered and studied in [10]. It is worth to mention that the system of variational inequalities is a tool to solve the Nash equilibrium problem for noncooperative games. In 2010, Ceng and Yao [9] transformed the SGEP (1.2) into the fixed point problem of the mapping $G = T_{\nu_1}^{\Phi_1}(I - \nu_1 B_1)T_{\nu_2}^{\Phi_2}(I - \nu_2 B_2)$, that is, $Gx^* = x^*$, where $y^* = T_{\nu_2}^{\Phi_2}(I - \nu_2 B_2)x^*$. Throughout this paper, the fixed point set of the mapping G is denoted by SGEP(G).

Furthermore, let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the convex minimization problem (CMP) of minimizing f over the constraint set C

$$\min_{x\in C} f(x) \tag{1.3}$$

(assuming the existence of minimizers). We denote by Ξ the set of minimizers of CMP (1.3).

It is our main purpose in this paper that we introduce one multistep relaxed implicit extragradient-like scheme and another multistep relaxed explicit extragradient-like scheme for finding a solution of the SGEP (1.2) with constraints of several problems: CMP (1.3), finitely many GMEPs and the fixed point problem (FPP) of a strict pseudocontraction in a real Hilbert space H. Under suitable control conditions, we establish the strong convergence of these two multistep relaxed extragradient-like schemes to a solution of the SGEP (1.2) with constraints of CMP (1.3), finitely many GMEPs and the FPP, which is also the unique solution of some VIP. Our results complement, develop, improve and extend the corresponding ones given by some authors recently in this area, e.g., Ceng, Guu and Yao [7], Ceng and Yao [9], Jung [13], Yao, Liou and Marino [23].

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e., $\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}$.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property $\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x,C)$.

The following properties of projections are useful and pertinent to our purpose.

Proposition 2.1. *Given any $x \in H$ and $z \in C$, one has*

(i) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z\rangle \le 0,\ \forall y \in C$;

(ii) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \ge \|P_C x - P_C y\|^2,\ \forall y \in H$.

Definition 2.1. A mapping $T : H \to H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \ge \|Tx - Ty\|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as $T = \frac{1}{2}(I + S)$ where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.2. A mapping $F : C \to H$ is said to be

(i) monotone if $\langle Fx - Fy, x - y\rangle \ge 0,\ \forall x, y \in C$;

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that $\langle Fx - Fy, x - y\rangle \ge \eta\|x - y\|^2,\ \forall x, y \in C$;

(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \alpha \|Fx - Fy\|^2, \ \forall x, y \in C.$$

On the other hand, it is obvious that if $F : C \to H$ is α-inverse-strongly monotone, then F is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. Moreover, we also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\|(I - \lambda F)u - (I - \lambda F)v\|^2 \leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|Fu - Fv\|^2. \tag{2.1}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda F$ is a nonexpansive mapping from C to H.

It was assumed in [16] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)-(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restrictions (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$;
(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;
(A3) Θ is upper-hemicontinuous, i.e., for each $x, y, z \in C$, $\limsup_{t \to 0^+} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y)$;
(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;
(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$, $\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x \rangle < 0$;
(B2) C is a bounded set.

Next we list some elementary conclusions for the MEP.

Proposition 2.2. *(see [2]). Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies (A1)-(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. Then the following hold:

(i) for each $x \in H$, $T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued;
(ii) $T_r^{(\Theta,\varphi)}$ is firmly nonexpansive, that is, $\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\|^2 \leq \langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y \rangle$;
(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;
(iv) $\mathrm{MEP}(\Theta, \varphi)$ is closed and convex;
(v) $\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leq \frac{s-t}{s}\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x \rangle$ for all $s, t > 0$ and $x \in H$.

Definition 2.3. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is, $T \equiv (1 - \alpha)I + \alpha S$ where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged mappings.

Proposition 2.3. *(see [1]) . Let $T : H \to H$ be a given mapping.*

(i) T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.
(ii) If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.
(iii) T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.

Proposition 2.4. *([11]). Let $S, T, V : H \to H$ be given operators.*

(i) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I-T$ is firmly nonexpansive.*

(iii) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N).$$

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1. *Let X be a real inner product space. Then there holds the following inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in X.$$

Lemma 2.2. *Let H be a real Hilbert space. Then the following hold:*

(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all $x, y \in H$;*

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x-y\|^2$ *for all $x, y \in H$ and $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$;*

(c) *If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x-y\|^2, \quad \forall y \in H.$$

Lemma 2.3. *(see* [14, Proposition 2.1]*) Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a mapping.*

(i) *If T is a k-strictly pseudocontractive mapping, then T satisfies the Lipschitzian condition*

$$\|Tx - Ty\| \le \frac{1+k}{1-k}\|x-y\|, \quad \forall x, y \in C.$$

(ii) *If T is a k-strictly pseudocontractive mapping, then the mapping $I-T$ is semiclosed at* 0*, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-T)x_n \to 0$, then $(I-T)\tilde{x} = 0$.*

(iii) *If T is k-(quasi-)strict pseudocontraction, then the fixed-point set* $\mathrm{Fix}(T)$ *of T is closed and convex so that the projection $P_{\mathrm{Fix}(T)}$ is well defined.*

Lemma 2.4. *(see* [22]*). Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma+\delta)k \le \gamma$. Then*

$$\|\gamma(x-y) + \delta(Tx - Ty)\| \le (\gamma+\delta)\|x-y\|, \quad \forall x, y \in C.$$

Lemma 2.5. *(*[12]*). Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive self-mapping on C with* $\mathrm{Fix}(S) \ne \emptyset$*. Then $I-S$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-S)x_n\}$ strongly converges to some y, it follows that $(I-S)x = y$. Here I is the identity operator of H.*

Lemma 2.6. *Let $\mathscr{A} : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 2.1 (i)) implies $u \in \mathrm{VI}(C, \mathscr{A}) \Leftrightarrow u = P_C(u - \lambda \mathscr{A} u)$, $\forall \lambda > 0$.*

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to C$, we define the mapping $T^\lambda : C \to H$ by $T^\lambda x := Tx - \lambda\mu F(Tx)$, $\forall x \in C$ where $F : C \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C; that is, F satisfies the conditions: $\|Fx - Fy\| \leq \kappa\|x - y\|$ and $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, forall x, y \in C$.

Lemma 2.7. *([21]) T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.

Lemma 2.8. *(see [14, Lemma 2.1]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \omega_n)a_n + \omega_n\delta_n + r_n, \quad \forall n \geq 0,$$

where $\{\omega_n\}, \{\delta_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\omega_n\} \subset [0,1]$ and $\sum_{n=0}^\infty \omega_n = \infty$;
(ii) either $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=0}^\infty \omega_n|\delta_n| < \infty$;
(iii) $r_n \geq 0$ for all $n \geq 0$, and $\sum_{n=1}^\infty r_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9. *(see [15]). Assume that A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Let LIM be a Banach limit. According to time and circumstances, we use $\mathrm{LIM}_n a_n$ instead of $\mathrm{LIM} a$ for every $a = \{a_n\} \in l^\infty$. The following lemma was given in [19, Proposition 2].

Lemma 2.10. *Let $a \in \mathbf{R}$ be a real number and let a sequence $\{a_n\} \in l^\infty$ satisfy the condition $\mathrm{LIM}_n a_n \leq a$ for all Banach limit LIM. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

In 2010, Ceng and Yao [9] transformed the SGEP (1.2) into a fixed point problem in the following way:

Lemma 2.11. *(see [9]). Let $\Phi_1, \Phi_2 : C \times C \to \mathbf{R}$ be two bifunctions satisfying conditions (A1)-(A4). Then $(x^*, y^*) \in C \times C$ is a solution of the SGEP (1.2) if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by $Gx = T_{\nu_1}^{\Phi_1}(I - \nu_1 B_1)T_{\nu_2}^{\Phi_2}(I - \nu_2 B_2)x$, $\forall x \in C$ where $y^* = T_{\nu_2}^{\Phi_2}(I - \nu_2 B_2)x^*$.*

In particular, if the mapping $B_j : C \to H$ is ζ_j-inverse-strongly monotone for $j = 1,2$, then the mapping G is nonexpansive provided $\nu_j \in (0, 2\zeta_j]$ for $j = 1,2$. We denote by $\mathrm{SGEP}(G)$ the fixed point set of the mapping G.

3. Main Results

Let C be a nonempty closed convex subset of a real Hilbert space H and let M,N be two integers. Throughout this section, we always assume the following:

$F: C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$, and $f: C \to \mathbf{R}$ is a convex functional with L-Lipschitz continuous gradient ∇f;

Φ_1, Φ_2, $\Theta_i : C \times C \to \mathbf{R}$ are bifunctions satisfying conditions (A1)-(A4) and $\varphi_i : C \to \mathbf{R} \cup \{+\infty\}$ is a proper lower semicontinuous and convex function with restrictions (B1) or (B2) for each $i = 1,...,N$;

$A_i : C \to H$ is η_i-inverse strongly monotone for each $i = 1,...,N$ and $B_j : C \to H$ is ζ_j-inverse strongly monotone for $j = 1,2$; $T : C \to C$ is a k-strictly pseudocontractive mapping, $V : C \to C$ is an l-Lipschitzian mapping and A is a $\bar{\gamma}$-strongly positive bounded linear operator on H with $\bar{\gamma} \in (l, l+1)$; $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \le \gamma < \tau l$ with $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$; $S : C \to C$ is a mapping defined by $Sx = \lambda x + (1-\lambda)Tx$ for $0 \le k < \lambda < 1$; $P_C(I - \lambda_t \nabla f) = s_t I + (1 - s_t)T_t$, where T_t is nonexpansive, $s_t = \frac{2-\lambda_t L}{4} \in (0, \frac{1}{2})$ and $\lambda_t : (0,1) \to (0, \frac{2}{L})$ with $\lim_{t\to 0} \lambda_t = \frac{2}{L}$; $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n)T_n$, where T_n is nonexpansive, $s_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ and $\{\lambda_n\} \subset (0, \frac{2}{L})$ with $\lim_{n\to\infty} \lambda_n = \frac{2}{L}$;

$\Delta_t^N : C \to C$ is a mapping defined by

$$\Delta_t^N x = T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I - r_{N,t}A_N) \cdots T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I - r_{1,t}A_1)x, t \in (0,1)$$

for $\{r_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\quad i = 1,...,N$; $\Delta_n^N : C \to C$ is a mapping defined by $\Delta_n^N x = T_{r_{N,n}}^{(\Theta_N,\varphi_N)}(I - r_{N,n}A_N) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}A_1)x$ with $\{r_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\lim_{n\to\infty} r_{i,n} = r_i$ for each $i = 1,...,N$; $G_t := T_{\nu_{1,t}}^{\Phi_1}(I - \nu_{1,t}B_1)T_{\nu_{2,t}}^{\Phi_2}(I - \nu_{2,t}B_2), t \in (0,1)$ and $G := T_{\nu_1}^{\Phi_1}(I - \nu_1 B_1)T_{\nu_2}^{\Phi_2}(I - \nu_2 B_2)$ with $\{\nu_{j,t}\} \subset [c_j, d_j] \subset (0, 2\zeta_j)$ and $\lim_{t\to 0} \nu_{j,t} = \nu_j$ for $j = 1,2$; $G_n := T_{\nu_{1,n}}^{\Phi_1}(I - \nu_{1,n}B_1)T_{\nu_{2,n}}^{\Phi_2}(I - \nu_{2,n}B_2)$ with $\{\nu_{j,n}\} \subset [c_j, d_j] \subset (0, 2\zeta_j)$ and $\lim_{n\to\infty} \nu_{j,n} = \nu_j$ for $j = 1,2$; $\Omega := \cap_{i=1}^N \mathrm{GMEP}(\Theta_i, \varphi_i, A_i) \cap \mathrm{SGEP}(G) \cap \mathrm{Fix}(T) \cap \Xi \neq \emptyset$ and P_Ω is the metric projection of H onto Ω; $\{\alpha_n\} \subset [0,1]$, $\quad \{s_n\} \subset (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$ and $\{s_t\}_{t\in(0,\gamma^\dagger)} \subset (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$, where $\gamma^\dagger = \min\{1, \frac{l+1-\bar{\gamma}}{\tau l - \gamma}\}$.

Next, put

$$\Delta_t^i = T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I - r_{i,t}A_i)T_{r_{i-1,t}}^{(\Theta_{i-1},\varphi_{i-1})}(I - r_{i-1,t}A_{i-1}) \cdots T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I - r_{1,t}A_1),\ \forall t \in (0,1),$$

$$\Delta_n^i = T_{r_{i,n}}^{(\Theta_i,\varphi_i)}(I - r_{i,n}A_i)T_{r_{i-1,n}}^{(\Theta_{i-1},\varphi_{i-1})}(I - r_{i-1,n}A_{i-1}) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}A_1),\ \forall n \ge 0,$$

for $i = 1,...,N$, and $\Delta_t^0 = \Delta_n^0 = I$, where I is the identity mapping on H. Since ∇f is L-Lipschitzian, it follows that ∇f is $1/L$-ism. By Proposition 2.3 (ii) we know that for $\lambda > 0$, $\quad \lambda\nabla f$ is $\frac{1}{\lambda L}$-ism. So by Proposition 2.3 (iii) we deduce that $I - \lambda\nabla f$ is $\frac{\lambda L}{2}$-averaged. Now since the projection P_C is $\frac{1}{2}$-averaged, it is easy to see from Proposition 2.4 (iv) that the composite $P_C(I - \lambda\nabla f)$ is $\frac{2+\lambda L}{4}$-averaged for $\lambda \in (0, \frac{2}{L})$. Hence we obtain that for each $t \in (0,1)$, $P_C(I - \lambda_t\nabla f)$ is $\frac{2+\lambda_t L}{4}$-averaged for each $\lambda_t \in (0, \frac{2}{L})$. Therefore, we can write

$$P_C(I - \lambda_t\nabla f) = \frac{2 - \lambda_t L}{4} I + \frac{2 + \lambda_t L}{4} T_t = s_t I + (1 - s_t)T_t,$$

where T_t is nonexpansive and $s_t := s_t(\lambda_t) = \frac{2-\lambda_t L}{4} \in (0, \frac{1}{2})$ for each $\lambda_t \in (0, \frac{2}{L})$. It is clear that $\lambda_t \to \frac{2}{L} \Leftrightarrow s_t \to 0$. It is obvious that $\mathrm{Fix}(T_t) = \mathrm{Fix}(T_n) = \Xi$. By Lemma 2.4, we know that S is nonexpansive. It is clear that $\mathrm{Fix}(S) = \mathrm{Fix}(T)$. Since $\{r_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ for each $i = 1,...,N$, utilizing (2.1) and

Proposition 2.2 (ii) we have that for all $x,y\in C$

$$\begin{aligned}\|\Delta_t^N x-\Delta_t^N y\| &= \|T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I-r_{N,t}A_N)\Delta_t^{N-1}x-T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I-r_{N,t}A_N)\Delta_t^{N-1}y\|\\ &\le \|(I-r_{N,t}A_N)\Delta_t^{N-1}x-(I-r_{N,t}A_N)\Delta_t^{N-1}y\|\\ &\le \|\Delta_t^{N-1}x-\Delta_t^{N-1}y\|\\ &\le \|x-y\|,\end{aligned}$$

which implies that $\Delta_t^i : C\to C$ is a nonexpansive mapping for all $t\in(0,1)$.

In this section, we introduce the first multistep relaxed implicit extragradient-like scheme that generates a net $\{x_t\}_{t\in(0,\min\{1,\frac{1+l-\bar\gamma}{\tau l-\gamma}\})}$ in an implicit manner:

$$\begin{cases} u_t = T_{\nu_{1,t}}^{\Phi_1}(I-\nu_{1,t}B_1)T_{\nu_{2,t}}^{\Phi_2}(I-\nu_{2,t}B_2)Sx_t,\\ v_t = T_{r_{N,t}}^{(\Theta_N,\varphi_N)}(I-r_{N,t}A_N)T_{r_{N-1,t}}^{(\Theta_{N-1},\varphi_{N-1})}(I-r_{N-1,t}A_{N-1})\cdots T_{r_{1,t}}^{(\Theta_1,\varphi_1)}(I-r_{1,t}A_1)u_t,\\ x_t = P_C[(I-s_tA)T_tv_t+s_t[Vx_t-t(\mu FVx_t-\gamma T_tv_t)]]. \end{cases} \tag{3.1}$$

We prove the strong convergence of $\{x_t\}$ as $t\to 0$ to a point $\tilde x\in\Omega$ which is a unique solution to the VIP

$$\langle (A-V)\tilde x, p-\tilde x\rangle \ge 0,\quad \forall p\in\Omega. \tag{3.2}$$

For arbitrarily given $x_0\in C$, we also propose the second multistep relaxed explicit extragradient-like scheme, which generates a sequence $\{x_n\}$ in an explicit way:

$$\begin{cases} u_n = T_{\nu_{1,n}}^{\Phi_1}(I-\nu_{1,n}B_1)T_{\nu_{2,n}}^{\Phi_2}(I-\nu_{2,n}B_2)Sx_n,\\ v_n = T_{r_{N,n}}^{(\Theta_N,\varphi_N)}(I-r_{N,n}A_N)T_{r_{N-1,n}}^{(\Theta_{N-1},\varphi_{N-1})}(I-r_{N-1,n}A_{N-1})\cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I-r_{1,n}A_1)u_n,\\ y_n = \alpha_n\gamma T_nv_n+(I-\alpha_n\mu F)Vx_n,\\ x_{n+1} = P_C[(I-s_nA)T_nv_n+s_ny_n],\quad \forall n\ge 0, \end{cases} \tag{3.3}$$

and establish the strong convergence of $\{x_n\}$ as $n\to\infty$ to the same point $\tilde x\in\Omega$, which is also the unique solution to VIP (3.2).

Now, for $t\in(0,\gamma^\dagger)$, and $s_t\in(0,\min\{\frac12,\|A\|^{-1}\})$, consider a mapping $Q_t : C\to C$ defined by

$$Q_tx = P_C[(I-s_tA)T_t\Delta_t^NG_tSx+s_t[Vx-t(\mu FVx-\gamma T_t\Delta_t^NG_tSx)]],\quad \forall x\in C.$$

It is easy to see that Q_t is a contractive mapping with constant $1-s_t(\bar\gamma-l+t(\tau l-\gamma))$. Indeed, by Lemmas 2.7, 2.9 and 2.11, for $\nu_{j,t}\in[c_j,d_j]\subset(0,2\zeta_j)$, $j=1,2$ we have

$$\begin{aligned}\|Q_tx-Q_ty\| &\le \|(I-s_tA)T_t\Delta_t^NG_tSx+s_t[(I-t\mu F)Vx+t\gamma T_t\Delta_t^NG_tSx]\\ &\quad -(I-s_tA)T_t\Delta_t^NG_tSy-s_t[(I-t\mu F)Vy+t\gamma T_t\Delta_t^NG_tSy]\|\\ &\le \|(I-s_tA)T_t\Delta_t^NG_tSx-(I-s_tA)T_t\Delta_t^NG_tSy\|\\ &\quad +s_t\|((I-t\mu F)Vx+t\gamma T_t\Delta_t^NG_tSx)-((I-t\mu F)Vy+t\gamma T_t\Delta_t^NG_tSy)\|\\ &\le (1-s_t\bar\gamma)\|x-y\|+s_t[(1-t\tau)\|Vx-Vy\|+t\gamma\|x-y\|]\\ &= [1-s_t(\bar\gamma-l+t(\tau l-\gamma))]\|x-y\|.\end{aligned}$$

Since $\bar\gamma\in(l,l+1)$, $\quad\tau l-\gamma>0$, and $0<t<\gamma^\dagger\le\frac{l+1-\bar\gamma}{\tau l-\gamma}$, it follows that $0<\bar\gamma-l+t(\tau l-\gamma)<1$ which together with $0<s_t\le\min\{\frac12,\|A\|^{-1}\}<1$ yields $0<1-s_t(\bar\gamma-l+t(\tau l-\gamma))<1$. Hence $Q_t : C\to C$ is a contractive mapping. By the Banach contraction principle, Q_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation (3.1).

We summary the basic properties of $\{x_t\}$. The argument techniques in [7, 1, 13] are extended to develop the new argument ones for these basic properties. We include the argument process for the sake of completeness.

Proposition 3.1. *Let $\{x_t\}$ be defined via (3.1). Then*

(i) $\{x_t\}$ *is bounded for* $t \in (0,\gamma^\dagger)$*;*

(ii) $\lim_{t\to 0}\|x_t - T_t x_t\| = 0$, $\lim_{t\to 0}\|x_t - G_t x_t\| = 0$, $\lim_{t\to 0}\|x_t - \Delta_t^N x_t\| = 0$ *and* $\lim_{t\to 0}\|x_t - Sx_t\| = 0$ *provided* $\lim_{t\to 0}\lambda_t = \frac{2}{L}$ ($\Leftrightarrow$ $\lim_{t\to 0} s_t = 0$)*;*

(iii) $x_t : (0,\gamma^\dagger) \to H$ *is locally Lipschitzian provided* $s_t : (0,\gamma^\dagger) \to (0, \min\{\frac{1}{2}, \|A\|^{-1}\})$ *is locally Lipschitzian,* $r_{i,t} : (0,\gamma^\dagger) \to [a_i, b_i]$ *is locally Lipschitzian and* $\nu_{j,t} : (0,\gamma^\dagger) \to [c_j, d_j]$ *is locally Lipschitzian for each* $i = 1,\dots,N$ *and* $j = 1,2$*;*

(iv) x_t *defines a continuous path from* $(0,\gamma^\dagger)$ *into* H *provided* $s_t : (0,\gamma^\dagger) \to (0,\min\{\frac{1}{2}, \|A\|^{-1}\})$ *is continuous,* $r_{i,t} : (0,\gamma^\dagger) \to [a_i, b_i]$ *is continuous, and* $\nu_{j,t} : (0,\gamma^\dagger) \to [c_j, d_j]$ *is continuous for each* $i = 1,\dots,N$ *and* $j = 1,2$.

Proof. (i) Let $p \in \Omega$. Noting that $\mathrm{Fix}(S) = \mathrm{Fix}(T)$, $\mathrm{Fix}(T_t) = \Xi$, $\Delta_t^i p = p$ and $G_t p = p$ for all $i \in \{1,\dots,N\}$, by the nonexpansivity of S, T_t, G_t and Δ_t^i and Lemmas 2.7, 2.9 and 2.11 we get

$$\begin{aligned}
\|x_t - p\| &\le \|(I - s_t A)T_t\Delta_t^N G_t S x_t + s_t((I - t\mu F)Vx_t + t\gamma T_t\Delta_t^N G_t S x_t) - p\| \\
&\le \|(I - s_t A)T_t\Delta_t^N G_t S x_t - (I - s_t A)T_t\Delta_t^N G_t S p\| + s_t\|(I - t\mu F)Vx_t + t\gamma T_t\Delta_t^N G_t S x_t - Ap\| \\
&\le (1 - s_t\bar\gamma)\|T_t\Delta_t^N G_t S x_t - T_t\Delta_t^N G_t S p\| \\
&\quad + s_t\|(I - t\mu F)Vx_t - (I - t\mu F)Vp + t(\gamma T_t\Delta_t^N G_t S x_t - \mu F V p) + Vp - Ap\| \\
&\le (1 - s_t\bar\gamma)\|x_t - p\| + s_t[(l - t\tau l)\|x_t - p\| + t(\gamma\|x_t - p\| + \|(\gamma I - \mu F V)p\|) + \|(V - A)p\|] \\
&= [1 - s_t(\bar\gamma - l + t(\tau l - \gamma))]\|x_t - p\| + s_t(t\|(\gamma I - \mu F V)p\| + \|(V - A)p\|).
\end{aligned}$$

So, it follows that $\|x_t - p\| \le \frac{\|(V-A)p\| + \|(\gamma I - \mu F V)p\|}{\bar\gamma - l}$. Hence $\{x_t\}$ is bounded and so are $\{Vx_t\}$, $\{u_t\}$, $\{v_t\}$, $\{T_t v_t\}$ and $\{FVx_t\}$.

(ii) By the definition of $\{x_t\}$, we have

$$\begin{aligned}
\|x_t - T_t v_t\| &= \|P_C[(I - s_t A)T_t v_t + s_t((I - t\mu F)Vx_t + t\gamma T_t v_t)] - P_C T_t v_t\| \\
&\le \|(I - s_t A)T_t v_t + s_t((I - t\mu F)Vx_t + t\gamma T_t v_t) - T_t v_t\| \\
&\le s_t\|Vx_t - AT_t v_t\| + t\|\gamma T_t v_t - \mu F V x_t\| \to 0 \text{ as } t \to 0,
\end{aligned}$$

by the boundedness of $\{Vx_t\}, \{T_t v_t\}$ and $\{FVx_t\}$ in the assertion (i). That is,

$$\lim_{t\to 0}\|x_t - T_t v_t\| = 0. \tag{3.4}$$

From (2.1) and Proposition 2.2 (ii) it follows that for all $i \in \{1,\dots,N\}$

$$\begin{aligned}
\|v_t - p\|^2 &\le \|(I - r_{i,t}A_i)\Delta_t^{i-1}u_t - (I - r_{i,t}A_i)p\|^2 \\
&\le \|\Delta_t^{i-1}u_t - p\|^2 + r_{i,t}(r_{i,t} - 2\eta_i)\|A_i\Delta_t^{i-1}u_t - A_i p\|^2 \\
&\le \|u_t - p\|^2 + r_{i,t}(r_{i,t} - 2\eta_i)\|A_i\Delta_t^{i-1}u_t - A_i p\|^2 \\
&\le \|x_t - p\|^2 + r_{i,t}(r_{i,t} - 2\eta_i)\|A_i\Delta_t^{i-1}u_t - A_i p\|^2.
\end{aligned} \tag{3.5}$$

Since $p = G_t p = T^{\Phi_1}_{\nu_{1,t}}(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)p$ and B_j is ζ_j-inverse-strongly monotone with $\nu_{j,t} \in [c_j, d_j] \subset (0, 2\zeta_j)$ for $j = 1,2$, from (2.1) and Proposition 2.2 (ii) we deduce that

$$\begin{aligned}\|u_t - p\|^2 &= \|T^{\Phi_1}_{\nu_{1,t}}(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - T^{\Phi_1}_{\nu_{1,t}}(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)p\|^2\\ &\le \|(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - (I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)p\|^2\\ &\le \|T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)p\|^2\\ &\le \|(I-\nu_{2,t}B_2)Sx_t - (I-\nu_{2,t}B_2)p\|^2\\ &\le \|x_t - p\|^2 + \nu_{2,t}(\nu_{2,t} - 2\zeta_2)\|B_2Sx_t - B_2p\|^2\\ &\le \|x_t - p\|^2.\end{aligned} \tag{3.6}$$

Simple calculations show that

$$\begin{aligned}x_t - p = x_t - w_t &+ (I - s_tA)(T_tv_t - T_tp) + s_t[(I - t\mu F)Vx_t - (I - t\mu F)Vp\\ &+ t\gamma(T_tv_t - T_tp) + t(\gamma I - \mu FV)p] + s_t(V - A)p,\end{aligned} \tag{3.7}$$

where $w_t = (I - s_tA)T_tv_t + s_t(t\gamma T_tv_t + (I - t\mu F)Vx_t)$. For simplicity, we write $\tilde{x}_t = T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t$ and $\tilde{p} = T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)p$. Then $u_t = T^{\Phi_1}_{\nu_{1,t}}(I-\nu_{1,t}B_1)\tilde{x}_t = G_tSx_t$ and $p = T^{\Phi_1}_{\nu_{1,t}}(I-\nu_{1,t}B_1)\tilde{p}$. Then, by the nonexpansivity of T_t, Propositions 2.1 (i), from (3.5)-(3.7) we obtain

$$\begin{aligned}\|x_t - p\|^2 &\le (1 - s_t\bar{\gamma})\|T_tv_t - T_tp\|\|x_t - p\| + s_t[(1 - t\tau)\|Vx_t - Vp\|\|x_t - p\|\\ &\quad + t\gamma\|T_tv_t - T_tp\|\|x_t - p\| + t\|(\gamma I - \mu FV)p\|\|x_t - p\|] + s_t\|(V - A)p\|\|x_t - p\|\\ &\le (1 - s_t(\bar{\gamma} - t\gamma))\frac{1}{2}(\|\Delta^i_tu_t - p\|^2 + \|x_t - p\|^2) + s_t[(l - t\tau l)\|x_t - p\|^2\\ &\quad + t\|(\gamma I - \mu FV)p\|\|x_t - p\|] + s_t\|(V - A)p\|\|x_t - p\|\\ &\le \|x_t - p\|^2 - \frac{1 - s_t(\bar{\gamma} - t\gamma)}{2}[\nu_{2,t}(2\zeta_2 - \nu_{2,t})\|B_2Sx_t - B_2p\|^2\\ &\quad + \nu_{1,t}(2\zeta_1 - \nu_{1,t})\|B_1\tilde{x}_t - B_1\tilde{p}\|^2 + r_{i,t}(2\eta_i - r_{i,t})\|A_i\Delta^{i-1}_tu_t - A_ip\|^2]\\ &\quad + s_t(t\|(\gamma I - \mu FV)p\| + \|(V - A)p\|)\|x_t - p\|.\end{aligned} \tag{3.8}$$

Since $\lim_{t\to 0} s_t = 0$ and $\{x_t\}$ is bounded, for all $i \in \{1, ..., N\}$, we have

$$\lim_{t\to 0}\|B_2Sx_t - B_2p\| = 0,\ \lim_{t\to 0}\|B_1\tilde{x}_t - B_1\tilde{p}\| = 0 \text{ and } \lim_{t\to 0}\|A_i\Delta^{i-1}_tu_t - A_ip\| = 0 \tag{3.9}$$

On the other hand, in terms of the firm nonexpansivity of $T^{\Phi_j}_{\nu_{j,t}}$ and the ζ_j-inverse strong monotonicity of B_j for $j = 1,2$, we obtain from $\nu_{j,t} \in [c_j, d_j] \subset (0, 2\zeta_j)$, $j = 1,2$ and (3.6) that

$$\begin{aligned}\|\tilde{x}_t - \tilde{p}\|^2 &\le \langle (I-\nu_{2,t}B_2)Sx_t - (I-\nu_{2,t}B_2)p, \tilde{x}_t - \tilde{p}\rangle\\ &\le \frac{1}{2}[\|x_t - p\|^2 + \|\tilde{x}_t - \tilde{p}\|^2 - \|(Sx_t - \tilde{x}_t) - \nu_{2,t}(B_2Sx_t - B_2p) - (p - \tilde{p})\|^2]\\ &= \frac{1}{2}[\|x_t - p\|^2 + \|\tilde{x}_t - \tilde{p}\|^2 - \|(Sx_t - \tilde{x}_t) - (p - \tilde{p})\|^2\\ &\quad + 2\nu_{2,t}\langle (Sx_t - \tilde{x}_t) - (p - \tilde{p}), B_2Sx_t - B_2p\rangle - \nu^2_{2,t}\|B_2Sx_t - B_2p\|^2],\end{aligned}$$

and

$$\begin{aligned}\|u_t-p\|^2 &\le \langle (I-\nu_{1,t}B_1)\tilde{x}_t-(I-\nu_{1,t}B_1)\tilde{p},u_t-p\rangle\\ &\le \frac{1}{2}[\|\tilde{x}_t-\tilde{p}\|^2+\|u_t-p\|^2-\|(\tilde{x}_t-u_t)+(p-\tilde{p})\|^2\\ &\quad+2\nu_{1,t}\langle B_1\tilde{x}_t-B_1\tilde{p},(\tilde{x}_t-u_t)+(p-\tilde{p})\rangle-\nu_{1,t}^2\|B_1\tilde{x}_t-B_1\tilde{p}\|^2]\\ &\le \frac{1}{2}[\|x_t-p\|^2+\|u_t-p\|^2-\|(\tilde{x}_t-u_t)+(p-\tilde{p})\|^2+2\nu_{1,t}\langle B_1\tilde{x}_t-B_1\tilde{p},(\tilde{x}_t-u_t)+(p-\tilde{p})\rangle].\end{aligned}$$

Thus, we have

$$\begin{aligned}\|\tilde{x}_t-\tilde{p}\|^2 \le \|x_t-p\|^2&-\|(Sx_t-\tilde{x}_t)-(p-\tilde{p})\|^2+2\nu_{2,t}\langle (Sx_t-\tilde{x}_t)-(p-\tilde{p}),B_2Sx_t-B_2p\rangle\\ &-\nu_{2,t}^2\|B_2Sx_t-B_2p\|^2,\end{aligned} \tag{3.10}$$

and

$$\|u_t-p\|^2 \le \|x_t-p\|^2-\|(\tilde{x}_t-u_t)+(p-\tilde{p})\|^2+2\nu_{1,t}\|B_1\tilde{x}_t-B_1\tilde{p}\|\|(\tilde{x}_t-u_t)+(p-\tilde{p})\|. \tag{3.11}$$

Consequently, from (3.6), (3.8) and (3.10) it follows that

$$\begin{aligned}\|x_t-p\|^2 \le \|x_t-p\|^2&-\frac{1-s_t(\bar{\gamma}-t\gamma)}{2}\|(Sx_t-\tilde{x}_t)-(p-\tilde{p})\|^2+\nu_{2,t}\|(Sx_t-\tilde{x}_t)-(p-\tilde{p})\|\|B_2Sx_t-B_2p\|\\ &+s_t(t\|(\gamma I-\mu FV)p\|+\|(V-A)p\|)\|x_t-p\|.\end{aligned}$$

Since $\lim_{t\to 0}s_t=0$ and $\lim_{t\to 0}\|B_2Sx_t-B_2p\|=0$ (due to (3.9)), we deduce from $\nu_{2,t}\in[c_2,d_2]\subset(0,2\zeta_2)$ and the boundedness of $\{x_t\}$ and $\{\tilde{x}_t\}$ that

$$\lim_{t\to 0}\|(Sx_t-\tilde{x}_t)-(p-\tilde{p})\|=0. \tag{3.12}$$

Furthermore, from (3.6), (3.8) and (3.11) it follows that

$$\begin{aligned}\|x_t-p\|^2 &\le (1-s_t(\bar{\gamma}-t\gamma))\frac{1}{2}[\|u_t-p\|^2+r_{i,t}(r_{i,t}-2\eta_i)\|A_i\Delta_t^{i-1}u_t-A_ip\|^2\\ &\quad+\|x_t-p\|^2]+s_t[(l-t\tau l)\|x_t-p\|^2+t\|(\gamma I-\mu FV)p\|\|x_t-p\|]+s_t\|(V-A)p\|\|x_t-p\|\\ &\le \|x_t-p\|^2-\frac{1-s_t(\bar{\gamma}-t\gamma)}{2}\|(\tilde{x}_t-u_t)+(p-\tilde{p})\|^2+\nu_{1,t}\|B_1\tilde{x}_t-B_1\tilde{p}\|\|(\tilde{x}_t-u_t)+(p-\tilde{p})\|\\ &\quad+s_t(t\|(\gamma I-\mu FV)p\|+\|(V-A)p\|)\|x_t-p\|.\end{aligned}$$

Since $\lim_{t\to 0}s_t=0$ and $\lim_{t\to 0}\|B_1\tilde{x}_t-B_1\tilde{p}\|=0$ (due to (3.9)), we deduce from $\nu_{1,t}\in[c_1,d_1]\subset(0,2\zeta_1)$ and the boundedness of $\{x_t\},\{u_t\}$ and $\{\tilde{x}_t\}$ that

$$\lim_{t\to 0}\|(\tilde{x}_t-u_t)+(p-\tilde{p})\|=0. \tag{3.13}$$

Note that

$$\|Sx_t-u_t\|\le\|(Sx_t-\tilde{x}_t)-(p-\tilde{p})\|+\|(\tilde{x}_t-u_t)+(p-\tilde{p})\|.$$

Hence from (3.12) and (3.13) we get

$$\lim_{t\to 0}\|Sx_t-G_tSx_t\|=\lim_{t\to 0}\|Sx_t-u_t\|=0. \tag{3.14}$$

Utilizing Proposition 2.2 (ii) and Lemma 2.2(a), we obtain from (2.1) and $r_{i,t} \in [a_i, b_i] \subset (0, 2\eta_i), i \in \{1, ..., N\}$ that

$$\begin{aligned}\|\Delta_t^i u_t - p\|^2 &\le \langle (I - r_{i,t}A_i)\Delta_t^{i-1}u_t - (I - r_{i,t}A_i)p, \Delta_t^i u_t - p\rangle \\ &\le \frac{1}{2}(\|\Delta_t^{i-1}u_t - p\|^2 + \|\Delta_t^i u_t - p\|^2 - \|\Delta_t^{i-1}u_t - \Delta_t^i u_t - r_{i,t}(A_i\Delta_t^{i-1}u_t - A_i p)\|^2) \\ &\le \frac{1}{2}(\|x_t - p\|^2 + \|\Delta_t^i u_t - p\|^2 - \|\Delta_t^{i-1}u_t - \Delta_t^i u_t - r_{i,t}(A_i\Delta_t^{i-1}u_t - A_i p)\|^2),\end{aligned}$$

which immediately leads to

$$\begin{aligned}\|\Delta_t^i u_t - p\|^2 &\le \|x_t - p\|^2 - \|\Delta_t^{i-1}u_t - \Delta_t^i u_t - r_{i,t}(A_i\Delta_t^{i-1}u_t - A_i p)\|^2 \\ &\le \|x_t - p\|^2 - \|\Delta_t^{i-1}u_t - \Delta_t^i u_t\|^2 + 2r_{i,t}\|\Delta_t^{i-1}u_t - \Delta_t^i u_t\|\|A_i\Delta_t^{i-1}u_t - A_i p\|.\end{aligned} \tag{3.15}$$

Combining (3.8) and (3.15) we conclude

$$\begin{aligned}\|x_t - p\|^2 &\le (1 - s_t(\bar{\gamma} - l + t(\tau l - \gamma)))\|x_t - p\|^2 - \frac{1 - s_t(\bar{\gamma} - t\gamma)}{2}\|\Delta_t^{i-1}u_t - \Delta_t^i u_t\|^2 \\ &\quad + r_{i,t}\|\Delta_t^{i-1}u_t - \Delta_t^i u_t\|\|A_i\Delta_t^{i-1}u_t - A_i p\| + s_t(t\|(\gamma I - \mu FV)p\| + \|(V - A)p\|)\|x_t - p\| \\ &\le \|x_t - p\|^2 - \frac{1 - s_t(\bar{\gamma} - t\gamma)}{2}\|\Delta_t^{i-1}u_t - \Delta_t^i u_t\|^2 + r_{i,t}\|\Delta_t^{i-1}u_t - \Delta_t^i u_t\|\|A_i\Delta_t^{i-1}u_t - A_i p\| \\ &\quad + s_t(t\|(\gamma I - \mu FV)p\| + \|(V - A)p\|)\|x_t - p\|.\end{aligned}$$

Since $\lim_{t\to 0} s_t = 0, \quad \lim_{t\to 0}\|A_i\Delta_t^{i-1}u_t - A_i p\| = 0$ (due to (3.9)) and $\{r_{i,t}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, we deduce from the boundedness of $\{x_t\}$ and $\{\Delta_t^i u_t\}$ that

$$\lim_{t\to 0}\|\Delta_t^{i-1}u_t - \Delta_t^i u_t\| = 0, \quad \forall i \in \{1, ..., N\}. \tag{3.16}$$

Note that

$$\|u_t - v_t\| \le \|\Delta_t^0 u_t - \Delta_t^1 u_t\| + \|\Delta_t^1 u_t - \Delta_t^2 u_t\| + \cdots + \|\Delta_t^{N-1}u_t - \Delta_t^N u_t\|.$$

Hence, from (3.16) we get

$$\lim_{t\to 0}\|u_t - v_t\| = 0. \tag{3.17}$$

Thus, taking into account that $\|Sx_t - v_t\| \le \|Sx_t - u_t\| + \|u_t - v_t\|$, we obtain from (3.14) and (3.17) that

$$\lim_{t\to 0}\|Sx_t - v_t\| = 0. \tag{3.18}$$

Next, let us show that $x_t - Sx_t \to 0$ as $t \to 0$. As a matter of fact, putting $\nu = 1 - \lambda$ and $\mathscr{A} = I - T$, we know that $\mathscr{A}$ is $\frac{1-k}{2}$-inverse-strongly monotone since T is k-strictly pseudocontractive. Observe that $Sx_t = x_t - \nu\mathscr{A}x_t$ which together with (2.1), yields

$$\|Sx_t - p\|^2 = \|x_t - p\|^2 - (1 - \lambda)(\lambda - k)\|x_t - Tx_t\|^2.$$

So, from (3.8) and the last inequality it follows that

$$\begin{aligned}\|x_t-p\|^2 &\le (1-s_t(\bar{\gamma}-t\gamma))\frac{1}{2}(\|v_t-p\|^2+\|x_t-p\|^2)+s_t[(l-t\tau l)\|x_t-p\|^2\\&\quad+t\|(\gamma I-\mu FV)p\|\|x_t-p\|]+s_t\|(V-A)p\|\|x_t-p\|\\&\le (1-s_t(\bar{\gamma}-t\gamma))\frac{1}{2}(\|x_t-p\|^2-(1-\lambda)(\lambda-k)\|x_t-Tx_t\|^2+\|x_t-p\|^2)\\&\quad+s_t[(l-t\tau l)\|x_t-p\|^2+t\|(\gamma I-\mu FV)p\|\|x_t-p\|]+s_t\|(V-A)p\|\|x_t-p\|\\&\le \|x_t-p\|^2-\frac{1-s_t(\bar{\gamma}-t\gamma)}{2}(1-\lambda)(\lambda-k)\|x_t-Tx_t\|^2\\&\quad+s_t(t\|(\gamma I-\mu FV)p\|+\|(V-A)p\|)\|x_t-p\|.\end{aligned}$$

Thus,

$$\lim_{t\to 0}\|x_t-Sx_t\|=(1-\lambda)\|x_t-Tx_t\|=0.$$

Consequently, from (3.4), (3.14), (3.18) and $\|x_t-Sx_t\|\to 0$ we immediately deduce that

$$\lim_{t\to 0}\|x_t-T_tx_t\|=0,\quad \lim_{t\to 0}\|x_t-\Delta_t^N x_t\|=0\quad\text{and}\quad \lim_{t\to 0}\|x_t-G_tx_t\|=0. \tag{3.19}$$

(iii) Let $t,t_0\in(0,\gamma^\dagger)$. Since ∇f is $\frac{1}{L}$-ism, $P_C(I-\lambda_t\nabla f)$ is nonexpansive for $\lambda_t\in(0,\frac{2}{L})$. So, it follows that for any given $p\in\Omega$, $\|P_C(I-\lambda_t\nabla f)v_{t_0}\|\le\|v_{t_0}\|+2\|p\|$. This implies that $\{P_C(I-\lambda_t\nabla f)v_{t_0}\}$ is bounded. Also, observe that

$$\begin{aligned}\|T_tv_{t_0}-T_{t_0}v_{t_0}\| &\le \left\|\frac{4P_C(I-\lambda_t\nabla f)}{2+\lambda_tL}v_{t_0}-\frac{4P_C(I-\lambda_{t_0}\nabla f)}{2+\lambda_{t_0}L}v_{t_0}\right\|+\left\|\frac{2-\lambda_{t_0}L}{2+\lambda_{t_0}L}v_{t_0}-\frac{2-\lambda_tL}{2+\lambda_tL}v_{t_0}\right\|\\&=\left\|\frac{4L(\lambda_{t_0}-\lambda_t)P_C(I-\lambda_t\nabla f)v_{t_0}+4(2+\lambda_tL)(P_C(I-\lambda_t\nabla f)v_{t_0}-P_C(I-\lambda_{t_0}\nabla f)v_{t_0})}{(2+\lambda_tL)(2+\lambda_{t_0}L)}\right\|\\&\quad+\frac{4L|\lambda_t-\lambda_{t_0}|}{(2+\lambda_tL)(2+\lambda_{t_0}L)}\|v_{t_0}\|\\&\le\frac{4L|\lambda_t-\lambda_{t_0}|}{(2+\lambda_tL)(2+\lambda_{t_0}L)}\|v_{t_0}\|+\frac{4(2+\lambda_tL)\|P_C(I-\lambda_t\nabla f)v_{t_0}-P_C(I-\lambda_{t_0}\nabla f)v_{t_0}\|}{(2+\lambda_tL)(2+\lambda_{t_0}L)}\\&\quad+\frac{4L|\lambda_{t_0}-\lambda_t|\|P_C(I-\lambda_t\nabla f)v_{t_0}\|}{(2+\lambda_tL)(2+\lambda_{t_0}L)}\\&\le\lambda_t-\lambda_{t_0}|[L\|P_C(I-\lambda_t\nabla f)v_{t_0}\|+4\|\nabla f(v_{t_0})\|+L\|v_{t_0}\|]\\&\le\widetilde{M}|\lambda_t-\lambda_{t_0}|,\end{aligned}\tag{3.20}$$

where $\sup_{t\in(0,\gamma^\dagger)}\{L\|P_C(I-\lambda_t\nabla f)v_{t_0}\|+4\|\nabla f(v_{t_0})\|+L\|v_{t_0}\|\}\le\widetilde{M}$ for some $\widetilde{M}>0$. So, by (3.20), we have that

$$\|T_tv_t-T_{t_0}v_{t_0}\|\le\|T_tv_t-T_tv_{t_0}\|+\|T_tv_{t_0}-T_{t_0}v_{t_0}\|\le\|v_t-v_{t_0}\|+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|. \tag{3.21}$$

Since $B_j : C \to H$ is ζ_j-inverse strongly monotone and $\nu_{j,t} \in [c_j, d_j] \subset (0, 2\zeta_j)$ for $j = 1, 2$, we know that $T^{\Phi_j}_{\nu_{j,t}}(I - \nu_{j,t}B_j)$ is nonexpansive for $j = 1, 2$. So, from Proposition 2.2 (ii) it follows that

$$\begin{aligned}
\|u_t - u_{t_0}\| &= \|T^{\Phi_1}_{\nu_{1,t}}(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - T^{\Phi_1}_{\nu_{1,t_0}}(I-\nu_{1,t_0}B_1)T^{\Phi_2}_{\nu_{2,t_0}}(I-\nu_{2,t_0}B_2)Sx_{t_0}\| \\
&\le \|T^{\Phi_1}_{\nu_{1,t}}(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - T^{\Phi_1}_{\nu_{1,t_0}}(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t\| \\
&\quad + \|T^{\Phi_1}_{\nu_{1,t_0}}(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - T^{\Phi_1}_{\nu_{1,t_0}}(I-\nu_{1,t_0}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t\| \\
&\quad + \|(I-\nu_{1,t_0}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - (I-\nu_{1,t_0}B_1)T^{\Phi_2}_{\nu_{2,t_0}}(I-\nu_{2,t_0}B_2)Sx_{t_0}\| \\
&\le \nu_{1,t} - \nu_{1,t_0}|[\|B_1T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t\| + \frac{1}{\nu_{1,t}}\|T^{\Phi_1}_{\nu_{1,t}}(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t \\
&\quad - (I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t\|] + \|T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - T^{\Phi_2}_{\nu_{2,t_0}}(I-\nu_{2,t_0}B_2)Sx_{t_0}\|,
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\|T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - T^{\Phi_2}_{\nu_{2,t_0}}(I-\nu_{2,t_0}B_2)Sx_{t_0}\| \\
&\le |\nu_{2,t} - \nu_{2,t_0}|[\|B_2Sx_t\| + \frac{1}{\nu_{2,t}}\|T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - (I-\nu_{2,t}B_2)Sx_t\|] + \|x_t - x_{t_0}\|.
\end{aligned}$$

Combining the last two inequalities, we get

$$\begin{aligned}
\|u_t - u_{t_0}\| &\le |\nu_{1,t} - \nu_{1,t_0}|\{\|B_1T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t\| + \frac{1}{\nu_{1,t}}\|T^{\Phi_1}_{\nu_{1,t}}(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t \\
&\quad - (I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t\|\} + \|T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - T^{\Phi_2}_{\nu_{2,t_0}}(I-\nu_{2,t_0}B_2)Sx_{t_0}\| \\
&\le \|x_t - x_{t_0}\| + \widetilde{M}_0\sum_{j=1}^{2}|\nu_{j,t} - \nu_{j,t_0}|,
\end{aligned} \tag{3.22}$$

where $\sup_{t\in(0,\gamma^\dagger)}\{\|B_1T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t\| + \frac{1}{\nu_{1,t}}\|T^{\Phi_1}_{\nu_{1,t}}(I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - (I-\nu_{1,t}B_1)T^{\Phi_2}_{\nu_{2,t}}$ $(I-\nu_{2,t}B_2)Sx_t\|\} \le \widetilde{M}_0$ and $\sup_{t\in(0,\gamma^\dagger)}\{\|B_2Sx_t\| + \frac{1}{\nu_{2,t}}\|T^{\Phi_2}_{\nu_{2,t}}(I-\nu_{2,t}B_2)Sx_t - (I-\nu_{2,t}B_2)Sx_t\|\} \le \widetilde{M}_0$ for some $\widetilde{M}_0 > 0$. Also, utilizing Proposition 2.2 (ii), we deduce

$$\begin{aligned}
\|\Delta^N_t u_t - \Delta^N_{t_0}u_{t_0}\| &\le \|T^{(\Theta_N,\varphi_N)}_{r_{N,t}}(I-r_{N,t}A_N)\Delta^{N-1}_t u_t - T^{(\Theta_N,\varphi_N)}_{r_{N,t_0}}(I-r_{N,t_0}A_N)\Delta^{N-1}_t u_t\| \\
&\quad + \|T^{(\Theta_N,\varphi_N)}_{r_{N,t_0}}(I-r_{N,t_0}A_N)\Delta^{N-1}_t u_t - T^{(\Theta_N,\varphi_N)}_{r_{N,t_0}}(I-r_{N,t_0}A_N)\Delta^{N-1}_{t_0}u_{t_0}\| \\
&\le \frac{|r_{N,t} - r_{N,t_0}|}{r_{N,t}}\|T^{(\Theta_N,\varphi_N)}_{r_{N,t}}(I-r_{N,t}A_N)\Delta^{N-1}_t u_t - (I-r_{N,t}A_N)\Delta^{N-1}_t u_t\| \\
&\quad + |r_{N,t} - r_{N,t_0}|\|A_N\Delta^{N-1}_t u_t\| + \|\Delta^{N-1}_t u_t - \Delta^{N-1}_{t_0}u_{t_0}\| \\
&\le |r_{N,t} - r_{N,t_0}|[\|A_N\Delta^{N-1}_t u_t\| + \frac{1}{r_{N,t}}\|T^{(\Theta_N,\varphi_N)}_{r_{N,t}}(I-r_{N,t}A_N)\Delta^{N-1}_t u_t \\
&\quad - (I-r_{N,t}A_N)\Delta^{N-1}_t u_t\|] + \cdots + |r_{1,t} - r_{1,t_0}|[\|A_1\Delta^0_t u_t\| \\
&\quad + \frac{1}{r_{1,t}}\|T^{(\Theta_1,\varphi_1)}_{r_{1,t}}(I-r_{1,t}A_1)\Delta^0_t u_t - (I-r_{1,t}A_1)\Delta^0_t u_t\|] + \|\Delta^0_t u_t - \Delta^0_{t_0}u_{t_0}\| \\
&\le \widetilde{M}_1\sum_{i=1}^{N}|r_{i,t} - r_{i,t_0}| + \|u_t - u_{t_0}\|,
\end{aligned} \tag{3.23}$$

where $\sup_{t\in(0,\gamma^\dagger)}\{\sum_{i=1}^N[\|A_i\Delta_t^{i-1}u_t\|+\frac{1}{r_{i,t}}\|T_{r_{i,t}}^{(\Theta_i,\varphi_i)}(I-r_{i,t}A_i)\Delta_t^{i-1}u_t-(I-r_{i,t}A_i)\Delta_t^{i-1}u_t\|]\}\le\widetilde{M}_1$ for some $\widetilde{M}_1>0$. Consequently, in terms of (3.21)-(3.23) we calculate

$$\begin{aligned}\|T_tv_t-T_{t_0}v_{t_0}\|&\le\|v_t-v_{t_0}\|+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|\\&\le\|x_t-x_{t_0}\|+\widetilde{M}_0\sum_{j=1}^2|\nu_{j,t}-\nu_{j,t_0}|+\widetilde{M}_1\sum_{i=1}^N|r_{i,t}-r_{i,t_0}|+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|,\end{aligned}$$

and hence

$$\begin{aligned}\|x_t-x_{t_0}\|&\le\|(I-s_tA)T_tv_t+s_t((I-t\mu F)Vx_t+t\gamma T_tv_t)-(I-s_{t_0}A)T_{t_0}v_{t_0}-s_{t_0}((I-t_0\mu F)Vx_{t_0}+t_0\gamma T_{t_0}v_{t_0})\|\\&\le|s_t-s_{t_0}|\|A\|\|T_tv_t\|+(1-s_{t_0}\bar{\gamma})\|T_tv_t-T_{t_0}v_{t_0}\|+|s_t-s_{t_0}|\|(I-t\mu F)Vx_t+t\gamma T_tv_t\|\\&\quad+s_{t_0}\|(t-t_0)\gamma T_tv_t+t_0\gamma(T_tv_t-T_{t_0}v_{t_0})-(t-t_0)\mu FVx_t+(I-t_0\mu F)Vx_t-(I-t_0\mu F)Vx_{t_0}\|\\&\le|s_t-s_{t_0}|\|A\|\|T_tv_t\|+(1-s_{t_0}(\bar{\gamma}-l+t_0(\tau l-\gamma)))(\|x_t-x_{t_0}\|+\widetilde{M}_0\sum_{j=1}^2|\nu_{j,t}-\nu_{j,t_0}|\\&\quad+\widetilde{M}_1\sum_{i=1}^N|r_{i,t}-r_{i,t_0}|+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|)+|s_t-s_{t_0}|[\|Vx_t\|+t(\gamma\|T_tv_t\|+\mu\|FVx_t\|)]\\&\quad+s_{t_0}(\gamma\|T_tv_t\|+\mu\|FVx_t\|)|t-t_0|.\end{aligned}$$

So, it follows that

$$\begin{aligned}\|x_t-x_{t_0}\|&\le(1-s_{t_0}(\bar{\gamma}-l+t_0(\tau l-\gamma)))(\|x_t-x_{t_0}\|+\widetilde{M}_0\sum_{j=1}^2|\nu_{j,t}-\nu_{j,t_0}|+\widetilde{M}_1\sum_{i=1}^N|r_{i,t}-r_{i,t_0}|\\&\quad+\frac{4\widetilde{M}}{L}|s_t-s_{t_0}|)+|s_t-s_{t_0}|[\|A\|\|T_tv_t\|+\|Vx_t\|+t(\gamma\|T_tv_t\|+\mu\|FVx_t\|)]\\&\quad+s_{t_0}(\gamma\|T_tv_t\|+\mu\|FVx_t\|)|t-t_0|,\end{aligned}$$

which immediately implies that

$$\begin{aligned}\|x_t-x_{t_0}\|\le&\frac{\|A\|\|T_tv_t\|+\|Vx_t\|+t(\gamma\|T_tv_t\|+\mu\|FVx_t\|)+\frac{4\widetilde{M}}{L}}{s_{t_0}(\bar{\gamma}-l+t_0(\tau l-\gamma))}|s_t-s_{t_0}|+\frac{\gamma\|T_tv_t\|+\mu\|FVx_t\|}{\bar{\gamma}-l+t_0(\tau l-\gamma)}|t-t_0|\\&+\frac{\widetilde{M}_0}{s_{t_0}(\bar{\gamma}-l+t_0(\tau l-\gamma))}\sum_{j=1}^2|\nu_{j,t}-\nu_{j,t_0}|+\frac{\widetilde{M}_1}{s_{t_0}(\bar{\gamma}-l+t_0(\tau l-\gamma))}\sum_{i=1}^N|r_{i,t}-r_{i,t_0}|.\end{aligned}$$

Since $s_t:(0,\gamma^\dagger)\to(0,\min\{\frac{1}{2},\|A\|^{-1}\})$ is locally Lipschitzian, $r_{i,t}:(0,\gamma^\dagger)\to[a_i,b_i]$ is locally Lipschitzian for each $i=1,...,N$ and $\nu_{j,t}:(0,\gamma^\dagger)\to[c_j,d_j]$ is locally Lipschitzian for $j=1,2$, we conclude that $x_t:(0,\gamma^\dagger)\to C$ is locally Lipschitzian.

(iv) From the last inequality in (iii), the result follows immediately. □

We prove the following theorem for strong convergence of the net $\{x_t\}$ as $t\to0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.1. *Let the net $\{x_t\}$ be defined via (3.1). If $\lim_{t\to0}s_t=0$, then x_t converges strongly to a point $\tilde{x}\in\Omega$ as $t\to0$, which solves the VIP (3.2). Equivalently, we have $P_\Omega(I+V-A)\tilde{x}=\tilde{x}$.*

Proof. We first show the uniqueness of solutions of the VIP (3.2), which is indeed a consequence of the strong monotonicity of $A-V$. In fact, since V is an l-Lipschitzian mapping and A is a $\bar{\gamma}$-strongly positive bounded linear operator with $\bar{\gamma}\in(l,l+1)$, we know that $A-V$ is $(\bar{\gamma}-l)$-strongly monotone with constant $\bar{\gamma}-l\in(0,1)$. Suppose that $\tilde{x}\in\Omega$ and $\hat{x}\in\Omega$ both are solutions to the VIP (3.2). Then we have

$$\langle (A-V)\tilde{x},\tilde{x}-\hat{x}\rangle \le 0, \tag{3.24}$$

and

$$\langle (A-V)\hat{x},\hat{x}-\tilde{x}\rangle \le 0. \tag{3.25}$$

Adding up (3.24) and (3.25) yields $\langle (A-V)\tilde{x}-(A-V)\hat{x},\tilde{x}-\hat{x}\rangle \le 0$. The strong monotonicity of $A-V$ implies that $\tilde{x}=\hat{x}$ and the uniqueness is proved.

Next, we prove that $x_t\to\tilde{x}$ as $t\to 0$. Observing $\mathrm{Fix}(S)=\mathrm{Fix}(T)$ and $\mathrm{Fix}(T_t)=\Xi$, from (3.1), we write, for given $p\in\Omega$,

$$\begin{aligned} x_t-p=&\,x_t-w_t+(I-s_tA)(T_tv_t-T_tp)+s_t[(I-t\mu F)Vx_t-(I-t\mu F)Vp\\ &+t\gamma(T_tv_t-T_tp)+t(\gamma I-\mu FV)p]+s_t(V-A)p, \end{aligned}$$

where $w_t=(I-s_tA)T_tv_t+s_t((I-t\mu F)Vx_t+t\gamma T_tv_t)$. Then, by Proposition (2.1) (i) and the nonexpansivity of T_t, we obtain from (3.5) that

$$\begin{aligned} \|x_t-p\|^2 =&\, \langle x_t-w_t,x_t-p\rangle+\langle (I-s_tA)(T_tv_t-T_tp),x_t-p\rangle+s_t[\langle (I-t\mu F)Vx_t-(I-t\mu F)Vp,x_t-p\rangle\\ &+t\gamma\langle T_tv_t-T_tp,x_t-p\rangle+t\langle(\gamma I-\mu FV)p,x_t-p\rangle]+s_t\langle (V-A)p,x_t-p\rangle\\ \le&\,(1-s_t\bar{\gamma})\|x_t-p\|^2+s_t[(l-t\tau l)\|x_t-p\|^2+t\gamma\|x_t-p\|^2+t\langle(\gamma I-\mu FV)p,x_t-p\rangle]\\ &+s_t\langle (V-A)p,x_t-p\rangle\\ =&\,[1-s_t(\bar{\gamma}-l)+t(\tau l-\gamma))]\|x_t-p\|^2+s_t(t\langle(\gamma I-\mu FV)p,x_t-p\rangle+\langle (V-A)p,x_t-p\rangle). \end{aligned}$$

Therefore,

$$\|x_t-p\|^2\le \frac{1}{\bar{\gamma}-l+t(\tau l-\gamma)}(t\langle(\gamma I-\mu FV)p,x_t-p\rangle+\langle (V-A)p,x_t-p\rangle). \tag{3.26}$$

Since the net $\{x_t\}_{t\in(0,\gamma^\dagger)}$ is bounded (due to Proposition 3.1 (i)), we know that if $\{t_n\}$ is a subsequence in $(0,\gamma^\dagger)$ such that $t_n\to 0$ and $x_{t_n}\rightharpoonup x^*$, then from (3.26), we obtain $x_{t_n}\to x^*$. Let us show that $x^*\in\Omega$. Indeed, by Proposition 3.1 (ii), we know that $\lim_{n\to\infty}\|x_{t_n}-Sx_{t_n}\|=0$, $\lim_{n\to\infty}\|x_{t_n}-T_{t_n}x_{t_n}\|=0$ and $\lim_{n\to\infty}\|x_{t_n}-G_{t_n}x_{t_n}\|=0$. Then by Lemma 2.5 and the nonexpansivity of S we obtain $x^*\in\mathrm{Fix}(S)=\mathrm{Fix}(T)$. Utilizing the arguments similar to those of (3.22), we have

$$\|Gx_{t_n}-x_{t_n}\|\le\|Gx_{t_n}-G_{t_n}x_{t_n}\|+\|G_{t_n}x_{t_n}-x_{t_n}\|\le \overline{N}_0\sum_{j=1}^{2}|\nu_{j,t}-\nu_{j,t_0}|+\|G_{t_n}x_{t_n}-x_{t_n}\|,$$

where $\sup_{t_n\in(0,\gamma^\dagger)}\{\|B_1T_{\nu_2}^{\Phi_2}(I-\nu_2B_2)Sx_{t_n}\|+\frac{1}{\nu_1}\|T_{\nu_1}^{\Phi_1}(I-\nu_1B_1)T_{\nu_2}^{\Phi_2}(I-\nu_2B_2)Sx_{t_n}-(I-\nu_1B_1)T_{\nu_2}^{\Phi_2}(I-\nu_2B_2)Sx_{t_n}\|\}\le\overline{N}_0$ and $\sup_{t_n\in(0,\gamma^\dagger)}\{\|B_2Sx_{t_n}\|+\frac{1}{\nu_2}\|T_{\nu_2}^{\Phi_2}(I-\nu_2B_2)Sx_{t_n}-(I-\nu_2B_2)Sx_{t_n}\|\}\le\overline{N}_0$ for some $\overline{N}_0>0$. So, $\lim_{n\to\infty}\|x_{t_n}-Gx_{t_n}\|=0$. Thus, by Lemma 2.5 and the nonexpansivity of G we get $x^*\in\mathrm{Fix}(G)=:\mathrm{SGEP}(G)$. Also, observe that

$$\|P_C(I-\lambda_{t_n}\nabla f)x_{t_n}-x_{t_n}\|=\|s_{t_n}x_{t_n}+(1-s_{t_n})T_{t_n}x_{t_n}-x_{t_n}\|\le\|T_{t_n}x_{t_n}-x_{t_n}\|,$$

where $s_{t_n}=\frac{2-\lambda_{t_n}L}{4}\in(0,\frac{1}{2})$ for $\lambda_{t_n}\in(0,\frac{2}{L})$. Hence we have

$$\begin{aligned} \|P_C(I-\frac{2}{L}\nabla f)x_{t_n}-x_{t_n}\|\le&\,\|P_C(I-\frac{2}{L}\nabla f)x_{t_n}-P_C(I-\lambda_{t_n}\nabla f)x_{t_n}\|+\|P_C(I-\lambda_{t_n}\nabla f)x_{t_n}-x_{t_n}\|\\ \le&\,(\frac{2}{L}-\lambda_{t_n})\|\nabla f(x_{t_n})\|+\|T_{t_n}x_{t_n}-x_{t_n}\|. \end{aligned}$$

From the boundedness of $\{x_{t_n}\}$, $s_{t_n} \to 0$ ($\Leftrightarrow$ $\lambda_{t_n} \to \frac{2}{L}$) and $\|T_{t_n}x_{t_n} - x_{t_n}\| \to 0$, it follows that

$$\|x^* - P_C(I - \frac{2}{L}\nabla f)x^*\| = \lim_{n\to\infty} \|x_{t_n} - P_C(I - \frac{2}{L}\nabla f)x_{t_n}\| = 0.$$

So, $x^* \in \mathrm{VI}(C, \nabla f) = \Xi$. Furthermore, from (3.14), (3.16) and $\|x_{t_n} - Sx_{t_n}\|$, we have that $u_{t_n} \to x^*$ and $\Delta_{t_n}^m u_{t_n} \to x^*$ where $m \in \{1,\ldots,N\}$. Let us show that $x^* \in \bigcap_{m=1}^N \mathrm{GMEP}(\Theta_m, \varphi_m, A_m)$. As a matter of fact, since $\Delta_{t_n}^m u_{t_n} = T_{r_{m,t_n}}^{(\Theta_m,\varphi_m)}(I - r_{m,t_n}A_m)\Delta_{t_n}^{m-1}u_{t_n}, n \geq 0, m \in \{1,\ldots,N\}$, we have

$$0 \leq \Theta_m(\Delta_{t_n}^m u_{t_n}, y) + \varphi_m(y) - \varphi_m(\Delta_{t_n}^m u_{t_n}) + \langle A_m\Delta_{t_n}^{m-1}u_{t_n}, y - \Delta_{t_n}^m u_{t_n}\rangle + \frac{1}{r_{m,t_n}}\langle y - \Delta_{t_n}^m u_{t_n}, \Delta_{t_n}^m u_{t_n} - \Delta_{t_n}^{m-1}u_{t_n}\rangle.$$

By (A2), we have

$$\Theta_m(y, \Delta_{t_n}^m u_{t_n}) \leq \varphi_m(y) - \varphi_m(\Delta_{t_n}^m u_{t_n}) + \langle A_m\Delta_{t_n}^{m-1}u_{t_n}, y - \Delta_{t_n}^m u_{t_n}\rangle + \frac{1}{r_{m,t_n}}\langle y - \Delta_{t_n}^m u_{t_n}, \Delta_{t_n}^m u_{t_n} - \Delta_{t_n}^{m-1}u_{t_n}\rangle.$$

Let $z_t = ty + (1-t)x^*$ for all $t \in (0,1]$ and $y \in C$. This implies that $z_t \in C$. Then, we have

$$\begin{aligned}\langle z_t - \Delta_{t_n}^m u_{t_n}, A_m z_t\rangle &\geq \varphi_m(\Delta_{t_n}^m u_{t_n}) - \varphi_m(z_t) + \langle z_t - \Delta_{t_n}^m u_{t_n}, A_m z_t\rangle - \langle z_t - \Delta_{t_n}^m u_{t_n}, A_m\Delta_{t_n}^{m-1}u_{t_n}\rangle \\ &\quad - \langle z_t - \Delta_{t_n}^m u_{t_n}, \frac{\Delta_{t_n}^m u_{t_n} - \Delta_{t_n}^{m-1}u_{t_n}}{r_{m,t_n}}\rangle + \Theta_m(z_t, \Delta_{t_n}^m u_{t_n}) \\ &= \langle z_t - \Delta_{t_n}^m u_{t_n}, A_m z_t - A_m\Delta_{t_n}^m u_{t_n}\rangle + \langle z_t - \Delta_{t_n}^m u_{t_n}, A_m\Delta_{t_n}^m u_{t_n} - A_m\Delta_{t_n}^{m-1}u_{t_n}\rangle \\ &\quad + \varphi_m(\Delta_{t_n}^m u_{t_n}) - \varphi_m(z_t) - \langle z_t - \Delta_{t_n}^m u_{t_n}, \frac{\Delta_{t_n}^m u_{t_n} - \Delta_{t_n}^{m-1}u_{t_n}}{r_{m,t_n}}\rangle + \Theta_m(z_t, \Delta_{t_n}^m u_{t_n}).\end{aligned}$$

By (3.16), we have $\|A_m\Delta_{t_n}^m u_{t_n} - A_m\Delta_{t_n}^{m-1}u_{t_n}\| \to 0$ as $n \to \infty$. In the meantime, by the monotonicity of A_m, we obtain $\langle z_t - \Delta_{t_n}^m u_{t_n}, A_m z_t - A_m\Delta_{t_n}^m u_{t_n}\rangle \geq 0$. Then, by (A4) we obtain

$$\langle z_t - x^*, A_m z_t\rangle \geq \varphi_m(x^*) - \varphi_m(z_t) + \Theta_m(z_t, x^*).$$

Utilizing (A1), (A4) and the last inequality, we obtain

$$\begin{aligned}0 &= \Theta_m(z_t, z_t) + \varphi_m(z_t) - \varphi_m(z_t) \\ &\leq t[\Theta_m(z_t, y) + \varphi_m(y) - \varphi_m(z_t)] + (1-t)\langle z_t - x^*, A_m z_t\rangle \\ &= t[\Theta_m(z_t, y) + \varphi_m(y) - \varphi_m(z_t)] + (1-t)t\langle y - x^*, A_m z_t\rangle,\end{aligned}$$

and hence $0 \leq \Theta_m(z_t, y) + \varphi_m(y) - \varphi_m(z_t) + (1-t)\langle y - x^*, A_m z_t\rangle$. Letting $t \to 0$, we have, for each $y \in C$, $0 \leq \Theta_m(x^*, y) + \varphi_m(y) - \varphi_m(x^*) + \langle y - x^*, A_m x^*\rangle$. This implies that $x^* \in \mathrm{GMEP}(\Theta_m, \varphi_m, A_m)$ and hence $x^* \in \cap_{m=1}^N \mathrm{GMEP}(\Theta_m, \varphi_m, A_m)$. Therefore,

$$x^* \in \cap_{m=1}^N \mathrm{GMEP}(\Theta_m, \varphi_m, A_m) \cap \mathrm{SGEP}(G) \cap \mathrm{Fix}(T) \cap \Xi =: \Omega.$$

Next, we prove that $x_t \to \tilde{x}$ as $t \to 0$. First, let us assert that x^* is a solution of the VIP (3.2). As a matter of fact, since

$$x_t = x_t - w_t + (I - s_t A)T_t\Delta_t^N G_t S x_t + s_t((I - t\mu F)Vx_t + t\gamma T_t\Delta_t^N G_t S x_t),$$

we have

$$x_t - T_t\Delta_t^N G_t S x_t = x_t - w_t + s_t(V - A)T_t\Delta_t^N G_t S x_t + s_t(Vx_t - VT_t\Delta_t^N G_t S x_t + t(\gamma T_t\Delta_t^N G_t S x_t - \mu FVx_t)).$$

Since S, T_t, G_t and Δ_t^N are nonexpansive mappings, $I - T_t\Delta_t^N G_t S$ is monotone. So, from the monotonicity of $I - T_t\Delta_t^N G_t S$, it follows that, for $p \in \Omega$,

$$\begin{aligned}
0 &\le \langle (I - T_t\Delta_t^N G_t S)x_t - (I - T_t\Delta_t^N G_t S)p, x_t - p\rangle \\
&= \langle (I - T_t\Delta_t^N G_t S)x_t, x_t - p\rangle \\
&\le s_t\langle (V - A)T_t\Delta_t^N G_t S x_t, x_t - p\rangle + s_t\langle V x_t - V T_t\Delta_t^N G_t S x_t, x_t - p\rangle + s_t t\langle (\gamma T_t\Delta_t^N G_t S x_t - \mu F V x_t), x_t - p\rangle \\
&= s_t\langle (V - A)x_t, x_t - p\rangle + s_t\langle (V - A)T_t v_t - (V - A)x_t, x_t - p\rangle \\
&\quad + s_t\langle V x_t - V T_t v_t, x_t - p\rangle + s_t t\langle (\gamma T_t v_t - \mu F V x_t), x_t - p\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\langle (A - V)x_t, x_t - p\rangle &\le \langle (V - A)T_t v_t - (V - A)x_t, x_t - p\rangle + \langle V x_t - V T_t v_t, x_t - p\rangle \\
&\quad + t\langle (\gamma T_t v_t - \mu F V x_t), x_t - p\rangle \\
&\le (l + \|A\|)\|T_t v_t - x_t\|\|x_t - p\| + l\|x_t - T_t v_t\|\|x_t - p\| \\
&\quad + t(\gamma\|T_t v_t\| + \mu\|F V x_t\|)\|x_t - p\| \\
&= (2l + \|A\|)\|T_t v_t - x_t\|\|x_t - p\| + t(\gamma\|T_t v_t\| + \mu\|F V x_t\|)\|x_t - p\|.
\end{aligned} \tag{3.27}$$

Now, replacing t in (3.27) with t_n and letting $n \to \infty$, noticing the boundedness of $\{\gamma\|T_{t_n}v_{t_n}\| + \mu\|F V x_{t_n}\|\}$ and the fact that $T_{t_n}v_{t_n} - x_{t_n} \to 0$ as $n \to \infty$ (due to (3.4)), we obtain $\langle (A - V)x^*, x^* - p\rangle \le 0$. That is, $x^* \in \Omega$ is a solution of the VIP (3.2); hence $x^* = \tilde{x}$ by uniqueness. In summary, we have proven that each cluster point of $\{x_t\}$ (as $t \to 0$) equals $\tilde{x}$. Consequently, $x_t \to \tilde{x}$ as $t \to 0$.

The VIP (3.2) can be rewritten as

$$\langle (I + V - A)\tilde{x} - \tilde{x}, \tilde{x} - p\rangle \ge 0, \quad \forall p \in \Omega.$$

Recalling Proposition 2.1 (i), the last inequality is equivalent to the fixed point equation

$$P_\Omega(I + V - A)\tilde{x} = \tilde{x}.$$

□

Taking $F = \frac{1}{2}I$, $\mu = 2$ and $\gamma = 1$ in Theorem 3.1, we get

Corollary 3.1. *Let $\{x_t\}$ be defined by*

$$\begin{cases}
u_t = T_{\nu_{1,t}}^{\Phi_1}(I - \nu_{1,t}B_1)T_{\nu_{2,t}}^{\Phi_2}(I - \nu_{2,t}B_2)S x_t, \\
v_t = T_{r_{N,t}}^{\Theta_N,\varphi_N}(I - r_{N,t}A_N)T_{r_{N-1,t}}^{\Theta_{N-1},\varphi_{N-1}}(I - r_{N-1,t}A_{N-1})\cdots T_{r_{1,t}}^{\Theta_1,\varphi_1}(I - r_{1,t}A_1)u_t, \\
x_t = P_C[(I - s_t A)T_t v_t + s_t((1 - t)V x_t + t T_t v_t)].
\end{cases}$$

If $\lim_{t\to 0} s_t = 0$, then $\{x_t\}$ converges strongly as $t \to 0$ to a point $\tilde{x} \in \Omega$, which is the unique solution of the VIP (3.2).

First, we prove the following result in order to establish the strong convergence of the sequence $\{x_n\}$ generated by the multistep relaxed explicit extragradient-like scheme (3.3).

Theorem 3.2. *Let $\{x_n\}$ be the sequence generated by the explicit scheme (3.3), where $\{\alpha_n\}$ and $\{s_n\}$ satisfy the following condition: $\{\alpha_n\} \subset [0,1]$, $\{s_n\} \subset (0, \frac{1}{2})$ and $\alpha_n \to 0$, $s_n \to 0$ as $n \to \infty$. Then*

$$\mathrm{LIM}_n\langle (A - V)\tilde{x}, \tilde{x} - x_n\rangle \le 0,$$

where $\tilde{x} = \lim_{t\to 0^+} x_t$ *with* x_t *being defined by*

$$x_t = P_C[(I - s_t A)\Gamma\Delta^N GSx_t + s_t(Vx_t - t(\mu FVx_t - \gamma\Gamma\Delta^N GSx_t))], \tag{3.28}$$

where $\Gamma x = P_C(I - \frac{2}{L}\nabla f)x$ *and* $\Delta^N x = T_{r_N}^{\Theta_N,\varphi_N}(I - r_N A_N)\cdots T_{r_1}^{\Theta_1,\varphi_1}(I - r_1 A_1)x$.

Proof. First, note that from the condition (C1), without loss of generality, we may assume that $0 < s_n \leq \|A\|^{-1}$ for all $n \geq 0$. Let $\{x_t\}$ be the net generated by (3.28). Since Γ, G and Δ^N is are nonexpansive self-mappings on C, by Theorem 3.1 with $T_t = \Gamma$, $G_t = G$ and $\Delta_t^N = \Delta^N$, there exists $\lim_{t\to 0} x_t \in \Omega$. Denote it by $\tilde{x}$. Moreover, $\tilde{x}$ is the unique solution of the VIP (3.2). From Proposition 3.1(i) with $T_t = \Gamma$, $G_t = G$ and $\Delta_t^N = \Delta^N$, we know that $\{x_t\}$ is bounded and so are the nets $\{Vx_t\}$, $\{GSx_t\}$, $\{\Delta^N GSx_t\}$ and $\{FVx_t\}$.

Now, let us show that $\{x_n\}$ is bounded. To this end, take $p \in \Omega$. Then $Sp = p$, $T_n p = p$, $G_n p = p$ and $\Delta_n^N p = p$. Utilizing Lemmas 2.7 and 2.9, we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|(I - s_n A)T_n v_n + s_n(\alpha_n \gamma T_n v_n + (I - \alpha_n \mu F)Vx_n) - p\| \\
&\leq (1 - s_n\bar{\gamma})\|x_n - p\| + s_n[(l - \alpha_n \tau l)\|x_n - p\| + \alpha_n \gamma\|x_n - p\| \\
&\quad + \alpha_n\|(\gamma I - \mu FV)p\|] + s_n\|(V - A)p\| \\
&\leq [1 - s_n(\bar{\gamma} - l + \alpha_n(\tau l - \gamma))]\|x_n - p\| \\
&\quad + s_n(\bar{\gamma} - l + \alpha_n(\tau l - \gamma))\frac{\|(V - A)p\| + \alpha_n\|(\gamma I - \mu FV)p\|}{\bar{\gamma} - l + \alpha_n(\tau l - \gamma)} \\
&\leq \max\{\|x_n - p\|, \frac{\|(V - A)p\| + \|(\gamma I - \mu FV)p\|}{\bar{\gamma} - l}\}.
\end{aligned}$$

By induction, one has

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|(V - A)p\| + \|(\gamma I - \mu FV)p\|}{\bar{\gamma} - l}\}, \ \forall n \geq 0.$$

This implies that $\{x_n\}$ is bounded and so are $\{u_n\}$, $\{v_n\}$, $\{T_n v_n\}$, $\{FVx_n\}$, $\{Vx_n\}$ and $\{y_n\}$. Thus, utilizing the control condition (C1), we get

$$\|x_{n+1} - T_n v_n\| \leq \|(I - s_n A)T_n v_n + s_n y_n - T_n v_n\| = s_n\|y_n - AT_n v_n\| \to 0 \text{ as } n \to \infty.$$

Utilizing the similar arguments to those of (3.21), we have that

$$\|\Gamma\Delta^N GSx_n - T_n\Delta_n^N G_n Sx_n\| \leq \|\Delta^N GSx_n - \Delta_n^N G_n Sx_n\| + \widehat{M}|\frac{2}{L} - \lambda_n|, \tag{3.29}$$

where $\sup_{n\geq 0}\{L\|P_C(I - \frac{2}{L}\nabla f)v_n\| + 4\|\nabla f(v_n)\| + L\|v_n\|\} \leq \widehat{M}$ for some $\widehat{M} > 0$. Utilizing the similar arguments to those of (3.22), we have that $\|GSx_n - G_n Sx_n\| \leq \widehat{M}_0 \sum_{j=1}^{2} |\nu_j - \nu_{j,n}|$, where

$$\begin{aligned}
&\sup_{n\geq 0}\{\|B_1 T_{\nu_2}^{\Phi_2}(I - \nu_2 B_2)Sx_n\| + \frac{1}{\nu_1}\|T_{\nu_1}^{\Phi_1}(I - \nu_1 B_1)T_{\nu_2}^{\Phi_2}(I - \nu_2 B_2)Sx_n \\
&-(I - \nu_1 B_1)T_{\nu_2}^{\Phi_2}(I - \nu_2 B_2)Sx_n\|\} \leq \widehat{M}_0,
\end{aligned} \tag{3.30}$$

and $\sup_{n\geq 0}\{\|B_2 Sx_n\| + \frac{1}{\nu_2}\|T_{\nu_2}^{\Phi_2}(I - \nu_2 B_2)Sx_n - (I - \nu_2 B_2)Sx_n\|\} \leq \widehat{M}_0$ for some $\widehat{M}_0 > 0$. Utilizing the similar arguments to those of (3.23), we have that

$$\|\Delta^N GSx_n - \Delta_n^N G_n Sx_n\| \leq \widehat{M}_1 \sum_{i=1}^{N} |r_i - r_{i,n}| + \|GSx_n - G_n Sx_n\|, \tag{3.31}$$

where $\sup_{n\geq 0}\{\sum_{i=1}^N[\|A_i\Delta^{i-1}GSx_n\|+\frac{1}{r_i}\|T_{r_i}^{(\Theta_i,\varphi_i)}(I-r_iA_i)\Delta^{i-1}GSx_n-(I-r_iA_i)\Delta^{i-1}GSx_n\|]\}\leq \widehat{M}_1$ for some $\widehat{M}_1>0$. Consequently, in terms of (3.29)-(3.31) we calculate

$$\begin{aligned}\|\Gamma\Delta^N GSx_n-T_n\Delta_n^N G_nSx_n\| &\leq \|\Delta^N GSx_n-\Delta_n^N G_nSx_n\|+\widehat{M}|\frac{2}{L}-\lambda_n| \\ &\leq \|GSx_n-G_nSx_n\|+\widehat{M}_1\sum_{i=1}^N|r_i-r_{i,n}|+\widehat{M}|\frac{2}{L}-\lambda_n| \\ &\leq \widehat{M}_0\sum_{j=1}^2|\nu_j-\nu_{j,n}|+\widehat{M}_1\sum_{i=1}^N|r_i-r_{i,n}|+\widehat{M}|\frac{2}{L}-\lambda_n|.\end{aligned}$$

Consequently, it is not hard to find that

$$\begin{aligned}\|\Gamma\Delta^N GSx_t-x_{n+1}\| &\leq \|x_t-x_n\|+\widehat{M}_0\sum_{j=1}^2|\nu_j-\nu_{j,n}|+\widehat{M}_1\sum_{i=1}^N|r_i-r_{i,n}|+\widehat{M}|\frac{2}{L}-\lambda_n|+\|T_nv_n-x_{n+1}\| \\ &= \|x_t-x_n\|+\varepsilon_n,\end{aligned} \tag{3.32}$$

where $\varepsilon_n=\widehat{M}_0\sum_{j=1}^2|\nu_j-\nu_{j,n}|+\widehat{M}_1\sum_{i=1}^N|r_i-r_{i,n}|+\widehat{M}|\frac{2}{L}-\lambda_n|+\|T_nv_n-x_{n+1}\|\to 0$ as $n\to\infty$. Also observing that A is strongly positive, we have

$$\langle Ax_t-Ax_n,x_t-x_n\rangle=\langle A(x_t-x_n),x_t-x_n\rangle\geq\bar{\gamma}\|x_t-x_n\|^2. \tag{3.33}$$

Furthermore, for simplicity, we write

$$w_t=(I-s_tA)\Gamma\Delta^N GSx_t+s_t((I-t\mu F)Vx_t+t\gamma\Gamma\Delta^N GSx_t).$$

By (3.28), we get $x_t=P_Cw_t$ and

$$\begin{aligned}x_t-x_{n+1}=&(I-s_tA)\Gamma\Delta^N GSx_t-(I-s_tA)x_{n+1}+s_t[(I-t\mu F)Vx_t-(I-t\mu F)Vx_{n+1} \\ &+t(\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1})+(V-A)x_{n+1}]+x_t-w_t.\end{aligned}$$

Applying Lemma 2.1 and Proposition 2.1 (i), we have

$$\begin{aligned}\|x_t-x_{n+1}\|^2\leq&\|(I-s_tA)\Gamma\Delta^N GSx_t-(I-s_tA)x_{n+1}\|^2 \\ &+2s_t\langle(I-t\mu F)Vx_t-(I-t\mu F)Vx_{n+1},x_t-x_{n+1}\rangle \\ &+2s_tt\langle\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1},x_t-x_{n+1}\rangle \\ &+2s_t\langle(V-A)x_{n+1},x_t-x_{n+1}\rangle+2\langle x_t-w_t,x_t-x_{n+1}\rangle \\ \leq&(1-s_t\bar{\gamma})^2\|\Gamma\Delta^N GSx_t-x_{n+1}\|^2+2s_tt\mu\|FVx_t-FVx_{n+1}\|\|x_t-x_{n+1}\| \\ &+2s_t\langle Vx_t-Vx_{n+1},x_t-x_{n+1}\rangle+2s_tt\|\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1}\|\|x_t-x_{n+1}\| \\ &+2s_t\langle(V-A)x_{n+1},x_t-x_{n+1}\rangle \\ =&(1-s_t\bar{\gamma})^2\|\Gamma\Delta^N GSx_t-x_{n+1}\|^2+2s_t\langle Vx_t-Vx_{n+1},x_t-x_{n+1}\rangle \\ &+2s_tt(\mu\|FVx_t-FVx_{n+1}\|+\|\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\| \\ &+2s_t\langle(V-A)x_{n+1},x_t-x_{n+1}\rangle.\end{aligned} \tag{3.34}$$

Using (3.32) and (3.33) in (3.34), we obtain

$$\begin{aligned}\|x_t-x_{n+1}\|^2 &\le (1-s_t\bar{\gamma})^2\|\Gamma\Delta^N GSx_t-x_{n+1}\|^2+2s_t\langle Vx_t-Vx_{n+1},x_t-x_{n+1}\rangle\\ &\quad+2s_t t(\mu\kappa l\|x_t-x_{n+1}\|+\|\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\|\\ &\quad+2s_t\langle (V-A)x_{n+1},x_t-x_{n+1}\rangle\\ &\le s_t^2\bar{\gamma}\langle A(x_t-x_n),x_t-x_n\rangle+\|x_t-x_n\|^2+(1-s_t\bar{\gamma})^2(2\|x_t-x_n\|\varepsilon_n+\varepsilon_n^2)\\ &\quad+2s_t t(\mu\kappa l\|x_t-x_{n+1}\|+\|\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\|\\ &\quad+2s_t[\langle (V-A)x_t,x_t-x_{n+1}\rangle+\langle A(x_t-x_{n+1}),x_t-x_{n+1}\rangle-\langle A(x_t-x_n),x_t-x_n\rangle].\end{aligned}\tag{3.35}$$

Applying the Banach limit LIM to (3.35), together with $\lim_{n\to\infty}\varepsilon_n=0$, we have

$$\begin{aligned}\mathrm{LIM}_n\|x_t-x_{n+1}\|^2 &\le s_t^2\bar{\gamma}\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle+\mathrm{LIM}_n\|x_t-x_n\|^2\\ &\quad+2s_t t\mathrm{LIM}_n(\mu\kappa l\|x_t-x_{n+1}\|+\|\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\|\\ &\quad+2s_t[\mathrm{LIM}_n\langle (V-A)x_t,x_t-x_{n+1}\rangle+\mathrm{LIM}_n\langle A(x_t-x_{n+1}),x_t-x_{n+1}\rangle\\ &\quad-\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle].\end{aligned}\tag{3.36}$$

Using the property $\mathrm{LIM}_n a_n=\mathrm{LIM}_n a_{n+1}$ of the Banach limit in (3.36), we obtain

$$\begin{aligned}\mathrm{LIM}_n\langle (A-V)x_t,x_t-x_n\rangle &\le t\mathrm{LIM}_n(\mu\kappa l\|x_t-x_{n+1}\|+\|\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\|\\ &\quad+\mathrm{LIM}_n\langle A(x_t-x_{n+1}),x_t-x_{n+1}\rangle-\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle\\ &\quad+\frac{1}{2s_t}[\mathrm{LIM}_n\|x_t-x_n\|^2-\mathrm{LIM}_n\|x_t-x_{n+1}\|^2]\\ &\quad+\frac{s_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle\\ &=t\mathrm{LIM}_n(\mu\kappa l\|x_t-x_{n+1}\|+\|\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\|\\ &\quad+\frac{s_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle.\end{aligned}\tag{3.37}$$

Since

$$s_t\langle A(x_t-x_n),x_t-x_n\rangle\le s_t\|A\|\|x_t-x_n\|^2\le s_t\|A\|K^2\to 0\quad\text{as}\quad t\to 0,\tag{3.38}$$

where $\|x_t-x_n\|+\|\gamma\Gamma\Delta^N GSx_t-\mu FVx_n\|\le K$,

$$t\|x_t-x_{n+1}\|^2\to 0 \text{ and } t\|\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1}\|\|x_t-x_{n+1}\|\to 0 \text{ as } t\to 0,\tag{3.39}$$

we conclude from (3.37)-(3.39) that

$$\begin{aligned}\mathrm{LIM}_n\langle (A-V)\tilde{x},\tilde{x}-x_n\rangle &\le \limsup_{t\to 0}\mathrm{LIM}_n\langle (A-V)x_t,x_t-x_n\rangle\\ &\le \limsup_{t\to 0}\frac{s_t\bar{\gamma}}{2}\mathrm{LIM}_n\langle A(x_t-x_n),x_t-x_n\rangle\\ &\quad+\limsup_{t\to 0}t\mathrm{LIM}_n(\mu\kappa l\|x_t-x_{n+1}\|+\|\gamma\Gamma\Delta^N GSx_t-\mu FVx_{n+1}\|)\|x_t-x_{n+1}\|\\ &=0.\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let $\{x_n\}$ be the sequence generated by the explicit scheme (3.3), where $\{\alpha_n\}$ and $\{s_n\}$ satisfy the following conditions:*

(C1) $\{\alpha_n\}\subset[0,1]$, $\{s_n\}\subset(0,\frac{1}{2})$ and $\alpha_n\to 0$, $s_n\to 0$ as $n\to\infty$;

(C2) $\sum_{n=0}^{\infty} s_n = \infty$.

If $\{x_n\}$ is weakly asymptotically regular (i.e., $x_{n+1} - x_n \rightharpoonup 0$), then x_n converges strongly to a point $\tilde{x} \in \Omega$, which is the unique solution of the VIP (3.2).

Proof. First, note that from the condition (C1) and $\bar{\gamma} \in (l, l+1)$, without loss of generality, we may assume that $\bar{\gamma} - l + \alpha_n(\tau l - \gamma) < 1$ for all $n \geq 0$. Let x_t be defined by (3.28), that is,

$$x_t = P_C[(I - s_t A)\Gamma\Delta^N GSx_t + s_t(Vx_t - t(\mu FVx_t - \gamma\Gamma\Delta^N GSx_t)],$$

for $t \in (0, \gamma^\dagger)$, where $\Gamma x = P_C(I - \frac{2}{L}\nabla f)x$, $\Delta^N x = T_{r_N}^{\Theta_N,\varphi_N}(I - r_N A_N)\cdots T_{r_1}^{\Theta_1,\varphi_1}(I - r_1 A_1)x$ for $r_i \in [a_i, b_i] \subset (0, 2\eta_i), i = 1, \ldots, N$, and $\lim_{t\to 0} x_t := \tilde{x} \in \Omega$ (due to Theorem 3.1). Then $\tilde{x}$ is the unique solution of the VIP (3.2).

We divide the rest of the proof into several steps.

Step 1. We see that $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|(V-A)p\| + \|(\gamma I - \mu FV)p\|}{\bar{\gamma} - l}\}$, $\forall n \geq 0$ for all $p \in \Omega$ as in the proof of Theorem 3.2. Hence $\{x_n\}$ is bounded and so are $\{u_n\}, \{v_n\}, \{T_n v_n\}, \{FVx_n\}, \{Vx_n\}$ and $\{y_n\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle(V - A)\tilde{x}, x_n - \tilde{x}\rangle \leq 0$. To this end, put $a_n := \langle(A - V)\tilde{x}, \tilde{x} - x_n\rangle$, $\forall n \geq 0$. Then, by Theorem 3.2 we get $\mathrm{LIM}_n a_n \leq 0$ for any Banach limit LIM. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \limsup_{j\to\infty}(a_{n_j+1} - a_{n_j})$$

and $x_{n_j} \rightharpoonup v \in H$. This implies that $x_{n_j+1} \rightharpoonup v$ since $\{x_n\}$ is weakly asymptotically regular. Therefore, we have

$$w - \lim_{j\to\infty}(\tilde{x} - x_{n_j+1}) = w - \lim_{j\to\infty}(\tilde{x} - x_{n_j}) = (\tilde{x} - v),$$

and so

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}\langle(A - V)\tilde{x}, (\tilde{x} - x_{n_j+1}) - (\tilde{x} - x_{n_j})\rangle = 0.$$

Then, by Lemma 2.10, we obtain $\limsup_{n\to\infty} a_n \leq 0$, that is,

$$\limsup_{n\to\infty}\langle(V - A)\tilde{x}, x_n - \tilde{x}\rangle = \limsup_{n\to\infty}\langle(A - V)\tilde{x}, \tilde{x} - x_n\rangle \leq 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$. Indeed, for simplicity, we write $w_n = (I - s_n A)T_n v_n + s_n y_n$ for all $n \geq 0$. Then $x_{n+1} = P_C w_n$. Utilizing (3.3) and $T_n\Delta_n^N G_n S\tilde{x} = T_n\tilde{x} = \tilde{x}$, we have

$$\begin{aligned}x_{n+1} - \tilde{x} = &(I - s_n A)(T_n v_n - T_n\tilde{x}) + s_n[(I - \alpha_n\mu F)Vx_n - (I - \alpha_n\mu F)V\tilde{x} \\ &+ \alpha_n\gamma(T_n v_n - T_n\tilde{x}) + \alpha_n(\gamma I - \mu FV)\tilde{x}] + s_n(V - A)\tilde{x} + x_{n+1} - w_n.\end{aligned}$$

Thus, utilizing Proposition 2.1 (i) and Lemma 2.1, we get

$$
\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &= \|(I-s_nA)(T_nv_n-T_n\tilde{x})+s_n[(I-\alpha_n\mu F)Vx_n-(I-\alpha_n\mu F)V\tilde{x} \\
&\quad +\alpha_n\gamma(T_nv_n-T_n\tilde{x})+\alpha_n(\gamma I-\mu FV)\tilde{x}]+s_n(V-A)\tilde{x}+x_{n+1}-w_n\|^2 \\
&\le [\|(I-s_nA)(T_nv_n-T_n\tilde{x})\|+s_n(\|(I-\alpha_n\mu F)Vx_n-(I-\alpha_n\mu F)V\tilde{x}\| \\
&\quad +\alpha_n\gamma\|T_nv_n-T_n\tilde{x}\|)]^2+2s_n[\alpha_n\|(\gamma I-\mu FV)\tilde{x}\|\|x_{n+1}-\tilde{x}\|+\langle(A-V)\tilde{x},\tilde{x}-x_{n+1}\rangle] \\
&\le [(1-s_n\bar{\gamma})\|x_n-\tilde{x}\|+s_n((l-\alpha_n\tau l)\|x_n-\tilde{x}\|+\alpha_n\gamma\|x_n-\tilde{x}\|)]^2 \\
&\quad +2s_n[\alpha_n\|(\gamma I-\mu FV)\tilde{x}\|\|x_{n+1}-\tilde{x}\|+\langle(A-V)\tilde{x},\tilde{x}-x_{n+1}\rangle] \\
&= [1-s_n(\bar{\gamma}-l+\alpha_n(\tau l-\gamma))]^2\|x_n-\tilde{x}\|^2 \\
&\quad +2s_n[\alpha_n\|(\gamma I-\mu FV)\tilde{x}\|\|x_{n+1}-\tilde{x}\|+\langle(A-V)\tilde{x},\tilde{x}-x_{n+1}\rangle] \\
&\le [1-s_n(\bar{\gamma}-l+\alpha_n(\tau l-\gamma))]\|x_n-\tilde{x}\|^2 \\
&\quad +2s_n[\alpha_n\|(\gamma I-\mu FV)\tilde{x}\|\|x_{n+1}-\tilde{x}\|+\langle(A-V)\tilde{x},\tilde{x}-x_{n+1}\rangle] \\
&= [1-s_n(\bar{\gamma}-l+\alpha_n(\tau l-\gamma))]\|x_n-\tilde{x}\|^2+\langle(A-V)\tilde{x},\tilde{x}-x_{n+1}\rangle] \\
&\quad +s_n(\bar{\gamma}-l+\alpha_n(\tau l-\gamma))\cdot\frac{2}{\bar{\gamma}-l+\alpha_n(\tau l-\gamma)}[\alpha_n\|(\gamma I-\mu FV)\tilde{x}\|\|x_{n+1}-\tilde{x}\| \\
&= (1-\omega_n)\|x_n-\tilde{x}\|^2+\omega_n\delta_n,
\end{aligned}
$$

where $\omega_n=s_n(\bar{\gamma}-l+\alpha_n(\tau l-\gamma))$ and

$$
\delta_n=\frac{2}{\bar{\gamma}-l+\alpha_n(\tau l-\gamma)}[\alpha_n\|(\gamma I-\mu FV)\tilde{x}\|\|x_{n+1}-\tilde{x}\|+\langle(A-V)\tilde{x},\tilde{x}-x_{n+1}\rangle].
$$

It can be readily seen from conditions (C1) and (C2) that $\omega_n\to 0$ $\sum_{n=0}^{\infty}\omega_n=\infty$ and $\limsup_{n\to\infty}\delta_n\le 0$. By Lemma 2.8 with $r_n=0$, we conclude that $\lim_{n\to\infty}\|x_n-\tilde{x}\|=0$. This completes the proof. □

Corollary 3.2. *Let $\{x_n\}$ be the sequence generated by the explicit scheme (3.3). Assume that the sequences $\{\alpha_n\}$ and $\{s_n\}$ satisfy the conditions (C1) and (C2) in Theorem 3.3. If $\{x_n\}$ is asymptotically regular (i.e., $x_{n+1}-x_n\to 0$), then $\{x_n\}$ converges strongly to a point $\tilde{x}\in\Omega$, which is the unique solution of the VIP 3.2.*

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