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STRONG CONVERGENCE OF A CYCLIC ITERATIVE ALGORITHM FOR SPLIT COMMON FIXED-POINT PROBLEMS OF DEMICONTRACTIVE MAPPINGS

JIAYAN WANG, YAQIN WANG*

Department of Mathematics, Shaoxing University, Shaoxing 312000, China

Abstract. In this paper, A new cyclic iterative algorithm for split common fixed-point problems of demicontractive mappings is investigated. A strong convergence theorem with no compactness assumptions on the spaces or the mappings and with no extra conditions on fixed-point sets is established in real Hilbert spaces.

Keywords. Demicontractive mapping; Fixed point; Strong convergence; Split common fixed-point problem.

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1. Introduction

In 1994, Censor and Elfving [4] first introduced the split feasibility problem (SFP) for modelling inverse problems formulated as follows:

$$\text{Find } x^* \in C \quad \text{such} \quad \text{that } Ax^* \in Q, \tag{1.1}$$

where C and Q are respectively closed convex subsets of Hilbert spaces H_1 and H_2, and $A : H_1 \to H_2$ is a bounded linear mapping. There are a number of significant applications of the SFP in intensity-modulated radiation therapy, signal processing, image reconstruction and so on. An efficient algorithm for solving the SFP (1.1) is the Byrne's CQ algorithm. For any $x_0 \in H_1$, the CQ algorithm generates an iterative sequence as

$$x_{k+1} = P_C(I + \gamma A^*(P_Q - I)A)x_k,$$

where $0 < \gamma < 2/\|A\|^2$, P_C and P_Q are the metric projections from H_1 onto C and from H_2 onto Q, respectively. It is known that the CQ algorithm converges weakly to a solution of the SFP (1.1) if such a solution exists; see [1, 3, 11, 12] and the references therein.

In the case that C and Q in the SFP (1.1) are the intersections of finitely many fixed-point sets of nonlinear operators, problem (1.1) is called by Censor and Segal [5] the split common fixed-point problem (SCFP). More precisely, the SCFP requires to seek an element $x^* \in H_1$ satisfying

$$x^* \in \cap_{i=1}^{p} F(U_i) \text{ and } Ax^* \in \cap_{j=1}^{s} F(T_j), \tag{1.2}$$

where $p, s \geq 1$ are integers, $F(U_i)$ and $F(T_j)$ denote the fixed-point sets of two classes of nonlinear operators $U_i : H_1 \to H_1$ $(i = 1, 2, \cdots, p)$, $T_j : H_2 \to H_2$ $(j = 1, 2, \cdots, s)$. In particular, if $p = s = 1$, then

*Corresponding author.
E-mail address: wangyaqin0579@126.com (Y. Wang).
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problem (1.2) is reduced to find a point x^* with the property:

$$x^* \in F(U) \;\text{ and }\; Ax^* \in F(T), \tag{1.3}$$

which is usually called the two-set SCFP. To solve the two-set SCFP (1.3), Censor and Segal [5] proposed the following iterative method: for any initial guess $x_1 \in H_1$, define $\{x_n\}$ recursively by

$$x_{n+1} = U(x_n - \lambda A^*(I-T)Ax_n),$$

where U and T are directed operators. The further generalization of this algorithm was studied by Moudafi [10] for demicontractive operators. Under suitable conditions, he proved that the sequence $\{x_n\}$ converges weakly to a point of the two-set SCFP (1.3). Recently, many authors investigated problem 1.2 and problem 1.3 with iterative methods in Hilbert or Banach spaces; see [2, 7, 14, 16] and the references therein.

Recently, Wang and Xu [19] proposed the following cyclic algorithm:

$$x_{n+1} = U_{[n]}(x_n - \lambda A^*(I-T_{[n]})Ax_n),$$

where U_i and T_i are directed operators for $i = 1,2,\cdots,p$, $[n] = n\ (mod\ p)$. They proved that the sequence $\{x_n\}$ generated by this algorithm converges weakly to a solution of problem (1.2) if $p = s$.

Very recently, Shehu and Ogbuisi [13] used a modified Mann iterative algorithm

$$\begin{cases} x_1 \in H_1, \\ w_k = (1-\alpha_k)x_k, \\ y_k = w_k + \gamma A^*(T-I)Aw_k, \\ x_{k+1} = (1-\beta_k)y_k + \beta_k U y_k \end{cases} \tag{1.4}$$

to approximate solutions of the two-set SCFP (1.3) for demicontractive mappings in a real Hilbert space. They obtained a strong convergence result with no compactness assumptions; see [13] and the references therein.

In this paper, motivated by the above results, we introduce a new cyclic iterative scheme for solving the SCFP (1.2):

$$\begin{cases} x_1 \in H_1, \\ w_n = (1-\alpha_n)x_n, \\ y_n = w_n + \gamma A^*(T_{[n]}-I)Aw_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n U_{[n]} y_n. \end{cases} \tag{1.5}$$

We obtain a strong convergence result with no compactness assumptions on the spaces or the mappings and with no extra conditions on the fixed-point sets in the framework of Hilbert spaces.

2. Preliminaries

Throughout this paper, let N and R be the set of positive integers and real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and norm $\|\cdot\|$. Let $\{x_n\}$ be a sequence in H. We denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$, and use $\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}$ to stand for the weak ω-limit set of $\{x_n\}$. Let T be a mapping of C into H. We denote by $F(T)$ the fixed-point set of T.

In order to facilitate our main results in this paper, we recall some definitions as follows.

Definition 2.1. A mapping $T : H \to H$ is said to be

(i) nonexpansive iff $\|Tx - Ty\| \le \|x - y\|,\ \forall\, x, y \in H$;

(ii) quasi-nonexpansive iff

$$\|Tx - q\| \le \|x - q\|,\ \forall\, (x,q) \in H \times F(T);$$

(iii) directed iff

$$\|Tx - q\|^2 \le \|x - q\|^2 - \|x - Tx\|^2,\ \forall\, (x,q) \in H \times F(T);$$

(iv) μ-demicontractive iff there exists a constant $\mu \in (-\infty, 1)$ such that

$$\|Tx - q\|^2 \le \|x - q\|^2 + \mu\|x - Tx\|^2,\ \forall\, (x,q) \in H \times F(T),$$

which is equivalent to

$$\langle Tx - x, x - q\rangle \le \frac{\mu - 1}{2}\|x - Tx\|^2. \tag{2.1}$$

It is worth noting that the class of demicontractive mappings contains important mappings such as quasi-nonexpansive mappings and directed operators.

Recall that the metric (or nearest point) projection from H onto C is the mapping $P : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property $\|x - P_C x\| = \inf_{y \in C}\|x - y\|$. It is well known [15] that $P_C x$ is characterized by the inequality

$$\langle x - P_C x, y - P_C x\rangle \le 0,\ \ \forall\, y \in C. \tag{2.2}$$

Let us also recall that $I - T$ is said to be demiclosed at origin, if for any sequence $\{x_n\} \subset H$ and $x^* \in H$, we have

$$\left.\begin{array}{l} x_n \rightharpoonup x^* \\ (I - T)x_n \to 0 \end{array}\right\} \Rightarrow x^* = Tx^*.$$

As a special case of the demicloseness principle on uniformly convex Banach spaces given by [6], we know that if C is a nonempty closed convex subset of a Hilbert space H, and $T : C \to H$ is a nonexpansive mapping, then $I - T$ is demiclosed on C.

Now the following question is naturally raised: if $T : C \to H$ is quasi-nonexpansive, is $I - T$ still demiclosed on C? The answer is negative even at 0 as follows.

Example 2.1. ([18]; see Example 2.11). The mapping $T : [0,1] \to [0,1]$ is defined by

$$Tx = \begin{cases} \frac{x}{5}, & x \in [0, \frac{1}{2}], \\ x \sin \pi x, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Then T is a quasi-nonexpansive mapping, but $I - T$ is not demiclosed at 0.

In fact, $F(T) = \{0\}$. For any $x \in [0, \frac{1}{2}]$, we have

$$|Tx - 0| = \left|\frac{x}{5} - 0\right| \le |x - 0|,$$

and for any $x \in (\frac{1}{2}, 1]$, we have

$$|Tx - 0| = |x \sin \pi x - 0| \le |x - 0|.$$

Thus T is quasi-nonexpansive. Taking $\{x_n\} \subset (\frac{1}{2}, 1]$ and $x_n \to \frac{1}{2}(n \to \infty)$, we have

$$|(I - T)x_n| = |x_n[1 - \sin \pi x_n]| \to 0 (n \to \infty).$$

But $T\frac{1}{2} = \frac{1}{10} \neq \frac{1}{2}$, i.e., $(I-T)\frac{1}{2} \neq 0$, so $I-T$ is not demiclosed at 0.

In what follows, we give some lemmas which are needed to prove our results.

Lemma 2.1. [15] *Assume that $\{a_n\}$ is a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n b_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a real sequence such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} b_n \leq 0$ *or* $\Sigma_{n=1}^{\infty} \gamma_n |b_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. [9] *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a self-mapping of C. If T is a μ-demicontractive mapping (which is also called μ-quasi-strictly-contraction in* [9]*), then the fixed-point set $F(T)$ of T is closed and convex.*

Lemma 2.3. [8] *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in N$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in N$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\}\} \neq \emptyset$. Then, the following hold:
(i) $\tau(n_0) \leq \tau(n_0+1) \leq \cdots$ *and* $\tau(n) \to \infty$;
(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall\, n \geq n_0$.

3. MAIN RESULTS

In this section, let H_1 and H_2 be two real Hilbert spaces. We consider the SCFP (1.2) with $p = s$: find an element $x^* \in H_1$ such that

$$x^* \in \cap_{i=1}^{p} F(U_i) \text{ and } Ax^* \in \cap_{i=1}^{p} F(T_i), \tag{3.1}$$

where p is a positive integer. Denote the solution set of the SCFP (3.1) by Ω, i.e.,

$$\Omega = (\cap_{i=1}^{p} F(U_i)) \bigcap A^{-1}(\cap_{i=1}^{p} F(T_i)).$$

Note that problem (3.1) is a special case of problem (1.2). However, this is not restrictive. In view of the idea in [19], one can easily extend the results to the general case.

For fixed positive integer p and each $n \geq 1$, the p-mod function $[n]$ is defined by

$$[n] = \begin{cases} p, & \text{if } r = 0; \\ r, & \text{if } 0 < r < p. \end{cases}$$

whenever $n = kp + r$ for some $k \geq 0$.

Lemma 3.1. [17] *Let $\{u_k\}$ be a bounded sequence of a Hilbert space H. Let p be a positive integer and $I = \{1, 2, \cdots, p\}$. If $\lim_{k\to\infty} \|u_{k+1} - u_k\| = 0$ and $x^* \in \omega_w(u_k)$, then, for any $i \in I$, there exists a subsequence $\{u_{k_m}\}$ of $\{u_k\}$ such that $[k_m] = i$ and $u_{k_m} \rightharpoonup x^*$.*

Proof. For any $i \in I$, since $\lim_{k\to\infty} \|u_{k+1} - u_k\| = 0$, we have

$$\|u_{k+i} - u_k\| \leq \|u_{k+i} - u_{k+i-1}\| + \|u_{k+i-1} - u_{k+i-2}\| + \cdots + \|u_{k+1} - u_k\| \to 0.$$

It follows from the boundedness of $\{u_k\}$ and $x^* \in \omega_w(u_k)$ that there exists a subsequence $\{u_{t_m}\}$ of $\{u_k\}$ such that $u_{t_m} \rightharpoonup x^*$. Due to $\|u_{k+i} - u_k\| \to 0$, we obtain $u_{t_m+i} \rightharpoonup x^*$ for all $i \in I$. For any $i \in I$, there exists

$$t_1 + i_1 \in \{t_1 + 1, t_1 + 2, \cdots, t_1 + p\}$$

such that $i(t_1 + i_1) = i$. We choose $k_1 = t_1 + i_1$. There exists

$$t_2 + i_2 \in \{t_2 + 1, t_2 + 2, \cdots, t_2 + p\}$$

such that $i(t_2 + i_2) = i$. If $t_2 + i_2 > k_1$, we choose $k_2 = t_2 + i_2$; if $t_2 + i_2 \leq k_1$, we skip it and go to the t_3. Repeating this process continuously, we can choose a subsequence $\{k_m\}$ such that $i(k_m) = i$ and $u_{k_m} \rightharpoonup x^*$. This completes the proof. □

Theorem 3.1. *Let U_i and T_i be τ_i-demicontractive and μ_i-demicontractive, respectively. Let $I - U_i$ and $I - T_i$ be demiclosed at origin for every $i = 1, 2, \cdots, p$. Assume that $\Omega \neq \emptyset$ and $A : H_1 \to H_2$ is a bounded linear operator. Let $\{\alpha_n\} \subset (0,1), \{\beta_n\} \subset [0,1]$ satisfy the following conditions:*
(i) $\beta_n \in (a, 1-\nu)$, *for some* $a > 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\gamma \in (0, \dfrac{1-\mu}{\|A\|^2})$,
where $\nu = \max_{1\leq i\leq p} \tau_i$, $\mu = \max_{1\leq i\leq p} \mu_i$. Then the sequence $\{x_n\}$ defined by (1.5) converges strongly to $P_\Omega(0)$.

Proof. It is obvious that U_i and T_i are ν-demicontractive and μ-demicontractive($1 \leq i \leq p$), respectively. From Lemma 2.2, for every $i \in \{1, 2, \cdots, p\}$, we notice that $F(T_i)$ and $F(U_i)$ are closed and convex. Thus $\bigcap_{i=1}^{p} F(T_i)$ and $\bigcap_{i=1}^{p} F(U_i)$ are also closed and convex. Since A is bounded and linear, $A^{-1}(\cap_{i=1}^{p} F(T_i))$ is closed and convex. Therefore, Ω is closed and convex. Put $x^* = P_\Omega(0)$. It follows from (1.5) and (2.1) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\beta_n)(y_n - x^*) + \beta_n(U_{[n]}y_n - x^*)\|^2 \\
&= (1-\beta_n)^2\|y_n - x^*\|^2 + \beta_n^2\|U_{[n]}y_n - x^*\|^2 \\
&\quad +2\beta_n(1-\beta_n)\langle U_{[n]}y_n - x^*, y_n - x^*\rangle \\
&= (1-\beta_n)^2\|y_n - x^*\|^2 + \beta_n^2\|U_{[n]}y_n - x^*\|^2 \\
&\quad +2\beta_n(1-\beta_n)\langle U_{[n]}y_n - y_n, y_n - x^*\rangle + 2\beta_n(1-\beta_n)\|y_n - x^*\|^2 \\
&\leq (1-\beta_n)^2\|y_n - x^*\|^2 + \beta_n^2\|y_n - x^*\|^2 + \beta_n^2\nu\|y_n - U_{[n]}y_n\|^2 \\
&\quad +\beta_n(1-\beta_n)(\nu-1)\|U_{[n]}y_n - y_n\|^2 + 2\beta_n(1-\beta_n)\|y_n - x^*\|^2 \\
&= \|y_n - x^*\|^2 + \beta_n(\nu - 1 + \beta_n)\|U_{[n]}y_n - y_n\|^2 \qquad (3.2) \\
&\leq \|y_n - x^*\|^2, \qquad (3.3)
\end{aligned}$$

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|w_n - x^* + \gamma A^*(T_{[n]} - I)Aw_n\|^2 \\
&= \|w_n - x^*\|^2 + 2\gamma\langle A^*(T_{[n]} - I)Aw_n, w_n - x^*\rangle + \gamma^2\|A^*(T_{[n]} - I)Aw_n\|^2 \\
&\leq \|w_n - x^*\|^2 + 2\gamma\langle (T_{[n]} - I)Aw_n, Aw_n - Ax^*\rangle + \gamma^2\|A\|^2\|(T_{[n]} - I)Aw_n\|^2 \\
&\leq \|w_n - x^*\|^2 + \gamma(\mu - 1)\|(T_{[n]} - I)Aw_n\|^2 + \gamma^2\|A\|^2\|(T_{[n]} - I)Aw_n\|^2 \\
&= \|w_n - x^*\|^2 + \gamma(\mu - 1 + \gamma\|A\|^2)\|(T_{[n]} - I)Aw_n\|^2 \qquad (3.4) \\
&\leq \|w_n - x^*\|^2. \qquad (3.5)
\end{aligned}$$

By (1.5), we have

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) - \alpha_n x^*\|^2 \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x^*\|^2. \qquad (3.6)
\end{aligned}$$

Substituting (3.4) and (3.6) into (3.2), we obtain from conditions (i) and (iii) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \beta_n(\nu - 1 + \beta_n)\|U_{[n]}y_n - y_n\|^2 \\
&\quad + \alpha_n\|x^*\|^2 + \gamma(\mu - 1 + \gamma\|A\|^2)\|(T_{[n]} - I)Aw_n\|^2 \qquad (3.7) \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x^*\|^2 \\
&\leq \max\{\|x_1 - x^*\|^2, \|x^*\|^2\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. So are $\{y_n\}$ and $\{w_n\}$.

Next, we prove that $x_n \to x^*$. To see this, let us consider two possible cases.

Case I. Suppose that $\{\|x_n - x^*\|\}$ is monotonically decreasing. In this case, $\{\|x_n - x^*\|\}$ must be convergent. From (3.7) and the conditions (i)-(iii), we have

$$\lim_{n\to\infty}\|U_{[n]}y_n - y_n\| = \lim_{n\to\infty}\|(T_{[n]} - I)Aw_n\| = 0. \qquad (3.8)$$

Using (3.8) and the condition (ii), we obtain

$$\begin{aligned}
\|y_n - x_n\| &= \|\gamma A^*(T_{[n]} - I)Aw_n + w_n - x_n\| \\
&\leq \|\gamma A^*(T_{[n]} - I)Aw_n\| + \|\alpha_n x_n\| \\
&\leq \gamma\|A\|\|(T_{[n]} - I)Aw_n\| + \alpha_n\|x_n\| \to 0. \qquad (3.9)
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|y_n - x_n + \beta_n(U_{[n]}y_n - y_n)\| \\
&\leq \|y_n - x_n\| + \beta_n\|U_{[n]}y_n - y_n\| \to 0. \qquad (3.10)
\end{aligned}$$

It follows from (3.3) and (3.5) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|w_n - x^*\|^2 \\
&= \|(1-\alpha_n)(x_n - x^*) - \alpha_n x^*\|^2 \\
&= (1-\alpha_n)^2\|x_n - x^*\|^2 + \alpha_n^2\|x^*\|^2 - 2\alpha_n(1-\alpha_n)\langle x_n - x^*, x^*\rangle \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n(\alpha_n\|x^*\|^2 + 2(1-\alpha_n)\langle x^* - x_n, x^*\rangle). \qquad (3.11)
\end{aligned}$$

Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x^* - x_n, x^*\rangle = \lim_{i\to\infty}\langle x^* - x_{n_i}, x^*\rangle.$$

Without loss of generality, we assume that $\{x_{n_i}\}$ converges weakly to an element $\widetilde{x}$. Then $\widetilde{x}\in\omega_w(x_n)$. Let an index $i\in\{1,2,\cdots,p\}$ be fixed. From the fact that the pool of indexes is finite and (3.10), by Lemma 3.1 we can find a subsequence $\{x_{m_k}\}\subset\{x_n\}$ such that $x_{m_k}\rightharpoonup\widetilde{x}$ and $[m_k]=i$ for all $k\geq 1$. From (3.9), we obtain $y_{m_k}\rightharpoonup\widetilde{x}$. Since

$$\|U_i y_{m_k}-y_{m_k}\|=\|U_{[m_k]}y_{m_k}-y_{m_k}\|\to 0$$

and U_i-I is demiclosed at origin, we obtain that $\widetilde{x}\in F(U_i)$. It follows from $w_n=(1-\alpha_n)x_n$ and the weak continuity of A that $Aw_{m_k}\rightharpoonup A\widetilde{x}$. Furthermore, since $I-T_i$ is demiclosed at origin and $\|(T_i-I)Aw_{m_k}\|\to 0$, we have $A\widetilde{x}\in F(T_i)$. Since the index i is arbitrary, we have $\widetilde{x}\in\cap_{i=1}^{p}F(U_i)$ and $\widetilde{x}\in A^{-1}(\cap_{i=1}^{p}F(T_i))$, i.e., $\widetilde{x}\in\Omega$. Thus by (2.2) and $x^*=P_\Omega(0)$, we obtain

$$\limsup_{n\to\infty}\langle x^*-x_n,x^*\rangle=\langle x^*-\widetilde{x},x^*\rangle\leq 0. \tag{3.12}$$

Now for (3.11) since all the hypotheses of Lemma 2.1 are fulfilled, we conclude that $x_n\to x^*$.

Case II. Assume that $\{\|x_n-x^*\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n=\|x_n-x^*\|^2$. Then there exists a subsequence $\{\Gamma_{n_i}\}\subset\{\Gamma_n\}$ such that $\Gamma_{n_i}<\Gamma_{n_i+1}$, for all $i\in N$. Let $\tau:N\to N$ be a mapping for all $n\geq n_0$ (for some n_0 large enough) by

$$\tau(n)=\max\{k\leq n:\Gamma_k<\Gamma_{k+1}\}.$$

Using Lemma 2.3, we find that $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$, $\Gamma_n\leq\Gamma_{\tau(n)+1}$ and $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$. By (3.7) and the condition (ii) we have

$$\begin{aligned}
&\beta_{\tau(n)}(1-\nu-\beta_{\tau(n)})\|U_{[\tau(n)]}y_{\tau(n)}-y_{\tau(n)}\|^2\\
&+\gamma(1-\mu-\gamma\|A\|^2)\|(T_{[\tau(n)]}-I)Aw_{\tau(n)}\|^2\\
\leq\;&(1-\alpha_{\tau(n)})\Gamma_{\tau(n)}-\Gamma_{\tau(n)+1}+\alpha_{\tau(n)}\|x^*\|^2\\
\leq\;&\alpha_{\tau(n)}\|x^*\|^2\to 0,
\end{aligned} \tag{3.13}$$

which together with the conditions (i) and (iii) implies that $\|U_{[\tau(n)]}y_{\tau(n)}-y_{\tau(n)}\|$ and $\|(T_{[\tau(n)]}-I)Aw_{\tau(n)}\|$ converge to zero. As (3.9), (3.10)and (3.12) are in Case I, we have

$$\lim_{n\to\infty}\|x_{\tau(n)+1}-x_{\tau(n)}\|=0, \tag{3.14}$$

$$\limsup_{n\to\infty}\langle x^*-x_{\tau(n)},x^*\rangle\leq 0. \tag{3.15}$$

It follows from (3.11) and $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$ that

$$\Gamma_{\tau(n)}\leq\alpha_{\tau(n)}\|x^*\|^2+2(1-\alpha_{\tau(n)})\langle x^*-x_{\tau(n)},x^*\rangle,$$

which together with (3.15) and the condition (ii) implies that $\limsup_{n\to\infty}\Gamma_{\tau(n)}=0$. Hence $\Gamma_{\tau(n)}\to 0$. From (3.14) and $\Gamma_{\tau(n)}\to 0$, we get

$$\begin{aligned}
\Gamma_{\tau(n)+1}&\leq|\Gamma_{\tau(n)+1}-\Gamma_{\tau(n)}|+\Gamma_{\tau(n)}\\
&\leq\|x_{\tau(n)+1}-x_{\tau(n)}\|(\|x_{\tau(n)+1}-x^*\|+\|x_{\tau(n)}-x^*\|)+\Gamma_{\tau(n)}\to 0.
\end{aligned}$$

Therefore, it follows from $\Gamma_n\leq\Gamma_{\tau(n)+1}$ that $\Gamma_n\to 0$, i.e., $\{x_n\}$ converges strongly to $x^*=P_\Omega(0)$. □

If $U_i=U,\ T_i=T,\ i=1,2,\cdots,p$ in Theorem 3.1, we obtain the following results.

Corollary 3.1. *Let U and T be ν-demicontractive and μ-demicontractive, respectively. Let $I-U$ and $I-T$ be demiclosed at origin. Assume that $\widetilde{\Omega}=F(U)\bigcap A^{-1}(F(T))\neq\emptyset$ and $A:H_1\to H_2$ is a bounded linear operator. Let $\{\alpha_n\}\subset(0,1),\{\beta_n\}\subset[0,1]$ satisfy the following conditions:*
(i) $\beta_n\in(a,1-\nu)$, *for some* $a>0$;
(ii) $\lim_{n\to\infty}\alpha_n=0,\ \sum_{n=1}^{\infty}\alpha_n=\infty$;
(iii) $\gamma\in(0,\frac{1-\mu}{\|A\|^2})$.
Then the sequence $\{x_n\}$ defined by (1.4) converges strongly to $P_{\widetilde{\Omega}}(0)$.

Corollary 3.2. *Let U and T be quasi-nonexpansive. Let $I-U$ and $I-T$ be demiclosed at origin. Assume that $\widetilde{\Omega}=F(U)\bigcap A^{-1}(F(T))\neq\emptyset$ and $A:H_1\to H_2$ is a bounded linear operator. Let $\{\alpha_n\}\subset(0,1),\{\beta_n\}\subset[0,1]$ satisfy the following conditions:*
(i) $\beta_n\in(a,1)$, *for some* $a>0$;
(ii) $\lim_{n\to\infty}\alpha_n=0,\ \sum_{n=1}^{\infty}\alpha_n=\infty$;
(iii) $\gamma\in(0,\frac{1}{\|A\|^2})$.
Then the sequence $\{x_n\}$ defined by (1.4) converges strongly to $P_{\widetilde{\Omega}}(0)$.

Remark 3.1. (i) Theorem 3.1 extends and improves Theorem 3.1 [13] from the two-set SCFP (1.3) to the SCFP (1.2); (ii) In Theorem 3.1 [13], there is a gap. They obtained a strong convergence result, i.e., the sequence $\{x_n\}$ defined by (1.4) converges strongly to $p\in\widetilde{\Omega}$, however, p should be $P_{\widetilde{\Omega}}(0)$. Corollary 3.1 modifies the gap.

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